Quantum transport of Dirac electrons in graphene in the presence of a spatially modulated magnetic field

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(Received 13 July 2007; revised manuscript received 27 February 2008; published 14 May 2008)

We have investigated the electrical transport properties of Dirac electrons in a monolayer graphene sheet in the presence of a perpendicular magnetic field that is weakly and periodically modulated along one direction. We find that the Landau levels broaden into bands and their width oscillates as a function of the band index and the magnetic field. We determine the σ_{yy} component of the magnetoconductivity tensor for this system, which is shown to exhibit Weiss oscillations. We also analytically determine the asymptotic expressions for σ_{yy}. We compare these results to recently obtained results for electrically modulated graphene, as well as those for a magnetically modulated conventional two-dimensional electron gas (2DEG) system. We find that in the magnetically modulated graphene system considered in this work, Weiss oscillations in σ_{yy} have a reduced amplitude compared to that in the 2DEG but are less damped by temperature, while they have a higher amplitude than in the electrically modulated graphene system. We also find that these oscillations are out of phase by π with those of the electrically modulated system while they are in phase with those in the 2DEG system.

DOI: 10.1103/PhysRevB.77.195421 PACS number(s): 73.50.Dn, 72.20.My, 72.80.Rj, 73.40.−c

I. INTRODUCTION

The successful preparation of the graphene monolayer has allowed the possibility of studying the properties of electrons in graphene.[1] The nature of quasiparticles called Dirac electrons in these two-dimensional (2D) systems is very different from those in the conventional two-dimensional electron gas (2DEG) realized in semiconductor heterostructures. Graphene has a honeycomb lattice of carbon atoms. The quasiparticles in graphene have a band structure in which electron and hole bands touch at two points in the Brillouin zone. At these Dirac points, the quasiparticles obey the massless Dirac equation. In other words, they behave as massless Dirac particles leading to a linear dispersion relation $\epsilon_k = vk$ (with a characteristic velocity $v \simeq 10^6$ m/s). This difference between the natures of quasiparticles in graphene and of those in the conventional 2DEG has given rise to a host of new and unusual phenomena such as anomalous quantum Hall effects and a π Berry phase. Earlier, it was found that if the conventional 2DEG is subjected to artificially created periodic potentials in the submicrometer range, it leads to the appearance of Weiss oscillations in the magnetoresistance. This type of electrical modulation of the 2D system can be carried out by depositing an array of parallel metallic strips on the surface or through two interfering laser beams.[3–5]

Besides the fundamental interest in understanding the electronic properties of graphene, there are also serious suggestions that can serve as the building block for nanoelectronic devices.[6] Since Dirac electrons cannot be confined by electrostatic potentials due to Klein's paradox, it was suggested that magnetic confinement should be considered.[7] The technology for this already exists as the required magnetic field can be created by having ferromagnetic or superconducting layers beneath the substrate.[8]

In conventional 2DEG systems, electron transport in the presence of magnetic barriers and superlattices has continued to be an active area of research.[9] Recently, electrical transport in graphene in the presence of electrical modulation was considered and theoretical predictions were made.[10] Along the same lines, in this work, we investigate low temperature magnetotransport of Dirac electrons in a single graphene layer subjected to a one-dimensional (1D) magnetic modulation. The perpendicular magnetic field is weakly and periodically modulated along one direction.

II. FORMULATION AND ENERGY SPECTRUM

We consider two-dimensional Dirac electrons in graphene moving in the x-y plane. The magnetic field (B) is applied along the z direction perpendicular to the graphene plane. The perpendicular magnetic field B is weakly and periodically modulated along one direction, such that $\vec{\mathbf{B}}=[B+B_0\cos(Kx)]\hat{z}$. Here, B_0 is the strength of the magnetic modulation. In this work, we consider the modulation to be weak, such that $B_0 \ll B$. We consider the graphene layer within the single electron approximation. The low energy excitations are described by the 2D Dirac-type Hamiltonian (here, $\hbar = c = 1$)[1,10,11]

$$H = v\vec{\sigma}\cdot(-i\vec{\nabla}+e\vec{A}). \tag{1}$$

Here, $\vec{\sigma}=\{\vec{\sigma}_x,\vec{\sigma}_y\}$ are the Pauli matrices and v characterizes the electron velocity. We employ the Landau gauge and write the vector potential as $\vec{A}=[0,Bx+(B_0/K)\sin(Kx),0]$, where $K=2\pi/a$ and a is the period of the modulation. The Hamiltonian given by Eq. (1) can be expressed as

$$H = -iv\vec{\sigma}\cdot\vec{\nabla} + ev\vec{\sigma}_y Bx + ev\vec{\sigma}_y\frac{B_0}{K}\sin(Kx). \tag{2}$$

The above Hamiltonian can be written as

$$H = H_0 + H', \tag{3}$$

where H_0 is the unmodulated Hamiltonian given as

$$H_0 = - iv\vec{\sigma} \cdot \vec{\nabla} + ev\vec{\sigma}_y Bx$$

and

$$H' = ev\vec{\sigma}_y \frac{B_0}{K}\sin(Kx).$$

The Landau level energy eigenvalues without modulation are given by

$$\varepsilon(n) = \omega_g \sqrt{n}, \tag{4}$$

where n is an integer and $\omega_g = v\sqrt{2eB}$. As has been pointed out,[10] the Landau level spectrum for Dirac electrons is significantly different from the spectrum for electrons in conventional 2DEG, which is given as $\varepsilon(n) = \omega_c(n+1/2)$, where $\omega_c = eB/m$ is the cyclotron frequency.

The eigenfunctions without modulation are given by

$$\Psi_{n,k_y}(r) = \frac{e^{ik_y y}}{\sqrt{2L_y l}}\begin{pmatrix} - i\Phi_{n-1}[(x + x_0)/l] \\ \Phi_n[(x + x_0)/l] \end{pmatrix}, \tag{5}$$

where

$$\Phi_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}H_n(x), \tag{6}$$

where $l = \sqrt{1/eB}$ is the magnetic length, $x_0 = l^2 k_y$, L_y is the y dimension of the graphene layer, and $H_n(x)$ are the Hermite polynomials. Since we are considering weak modulation $B_0 \ll B$, we can apply the standard perturbation theory to determine the first order corrections to the unmodulated energy eigenvalues in the presence of modulation,

$$\Delta\varepsilon_{n,k_y} = \int_{-\infty}^{\infty} dx \int_0^{L_y} dy \Psi^*_{n,k_y}(r) H'(x) \Psi_{n,k_y}(r), \tag{7}$$

with the result

$$\Delta\varepsilon_{n,k_y} = \frac{\omega_0}{Kl}\cos(Kx_0)\{2\sqrt{n}e^{-u/2}[L_{n-1}(u) - L_n(u)]\}, \tag{8}$$

where $\omega_0 = \frac{evB_0}{K}$, $v = K^2 l^2/2$, and $L_n(v)$ are the Laguerre polynomials. Hence, the energy eigenvalues in the presence of modulation are

$$\varepsilon(n,k_y) = \varepsilon(n) + \Delta\varepsilon_{n,k_y} = \omega_g\sqrt{n} + \omega_0\cos(Kx_0)G_n, \tag{9}$$

with $G_n(u) = \frac{2\sqrt{n}e^{-u/2}}{Kl}[L_{n-1}(u) - L_n(u)]$. We observe that the degeneracy of the Landau level spectrum of the unmodulated system with respect to k_y is lifted in the presence of modulation with the explicit presence of k_y in x_0. The $n=0$ Landau level is different from the rest as the energy of this level is zero and electrons in this level do not contribute to the diffusive conductivity, which will be calculated in Sec. III. The rest of the Landau levels broaden into bands. The Landau bandwidths $\sim G_n$ oscillate as a function of n since $L_n(u)$ are oscillatory functions of the index n.

Before we begin the calculation of electrical conductivity, it is necessary to discuss the regime of the validity of the perturbation theory presented above. For large n, the level spacing given by Eq. (4) goes as $\omega_g[\sqrt{n} - \sqrt{(n-1)}] \to \omega_g \frac{1}{2\sqrt{n}}$ and the width of the nth level given by Eq. (8) goes as $\frac{2\omega_0 n^{1/2}}{Kl}$. There is, therefore, a value of n at which the width becomes equal to the spacing and the perturbation theory is no longer valid. This occurs when $n_{\max} = \sqrt{2}\pi^2 \frac{B'}{B_0}$, where $B' = \frac{1}{ea^2} = 0.0054$ T for $a = 350$ nm. For a fixed electron density and the period of modulation, this suggests the maximum value of the magnetic modulation B_0 above, in which it is necessary to carry out a more sophisticated analysis.

III. ELECTRICAL CONDUCTIVITY WITH PERIODIC MAGNETIC MODULATION

To calculate the electrical conductivity in the presence of weak magnetic modulation, we use the Kubo formula to calculate the linear response to an applied external field. In a magnetic field, the main contribution to Weiss oscillations comes from the scattering induced migration of the Larmor circle center. This is diffusive conductivity and we shall determine it by following the approach in Refs. 10 and 12, wherein it was shown that the diagonal component of conductivity σ_{yy} can be calculated by the following expression in the case of quasielastic scattering of electrons:

$$\sigma_{yy} = \frac{\beta e^2}{L_x L_y}\sum_{\zeta} f(E_\zeta)[1 - f(E_\zeta)]\tau(E_\zeta)(v_y^\zeta)^2, \tag{10}$$

where L_x and L_y are the dimensions of the layer, $\beta = \frac{1}{k_B T}$ is the inverse temperature with k_B as the Boltzmann constant, $f(E)$ is the Fermi-Dirac distribution function, $\tau(E)$ is the electron relaxation time, and ζ denotes the quantum numbers of the electron eigenstate. The diagonal component of the conductivity σ_{yy} is due to the modulation induced broadening of the Landau bands and, hence, it carries the effects of modulation, in which we are primarily interested in this work. σ_{xx} does not contribute, as the component of velocity here in the x direction is zero. The collisional contribution due to impurities is not taken into account in this work.

The summation in Eq. (10) over the quantum numbers ζ can be written as

$$\frac{1}{L_x L_y}\sum_{\zeta} = \frac{1}{2\pi L_x}\int_0^{L_x/l^2} dk_y \sum_{n=0}^{\infty} = \frac{1}{2\pi l^2}\sum_{n=0}^{\infty}. \tag{11}$$

The component of velocity required in Eq. (10) can be calculated from the following expression:

$$v_y^\zeta = \frac{\partial}{\partial k_y}\varepsilon(n,k_y). \tag{12}$$

Substituting the expression for $\varepsilon(n,k_y)$ obtained in Eq. (9) into Eq. (12) yields

$$v_y^\zeta = \frac{2\omega_0 u}{K}\sin(Kx_0)G_n(u). \tag{13}$$

With the results obtained in Eqs. (11)–(13), we can express the diffusive contribution to the conductivity given by Eq. (10) as

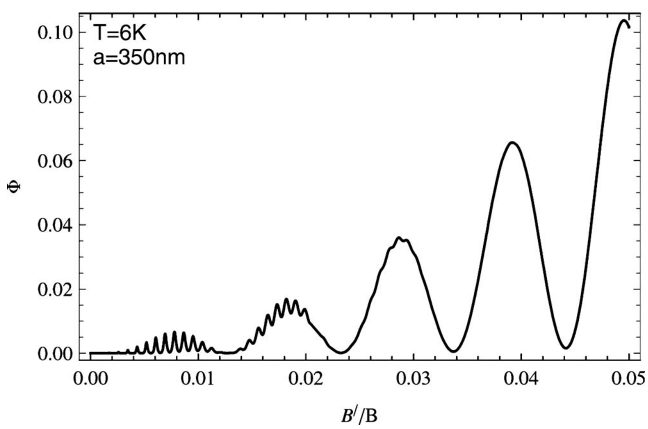

FIG. 1. The dimensionless conductivity vs inverse magnetic field.

$$\sigma_{yy}=A_0\Phi, \tag{14}$$

where

$$A_0=2\omega_0^2e^2\tau\beta, \tag{15}$$

and the dimensionless conductivity Φ is given as

$$\Phi=2e^{-u}\sum_{n=0}^{\infty}\frac{ng(E_n)}{[g(E_n)+1]^2}[L_{n-1}(u)-L_n(u)]^2, \tag{16}$$

where $g(E)=\exp[\beta(E-E_F)]$ and E_F is the Fermi energy.

In Fig. 1, we plot the dimensionless conductivity given by Eq. (16) as a function of inverse magnetic field at the temperature $T=6$ K and electron density $n_e=3\times10^{11}$ cm^{-2}. The dimensionless magnetic field is introduced, which is given as $b=\frac{B}{B'}$, where $B'=\frac{1}{ea^2}$. In the region of high magnetic field, we can see Shubnikov–de Haas (SdH) oscillations superimposed on the Weiss oscillations.

IV. ASYMPTOTIC EXPRESSIONS

To get a better understanding of the results of Sec. III, we will consider an asymptotic expression of conductivity wherein analytic results in terms of elementary functions can be obtained by following Ref. 10. We shall compare the asymptotic results for the dimensionless conductivity obtained in this section to the results obtained for a magnetically modulated conventional 2DEG system. We shall also compare these results to those of the graphene that was subjected only to the electric modulation.

The asymptotic expression of dimensionless conductivity can be obtained by using the following asymptotic expression for the Laguerre polynomials:

$$\exp^{-u/2}L_n(u)\rightarrow\frac{1}{\sqrt{\pi\sqrt{nu}}}\cos\left(2\sqrt{nu}-\frac{\pi}{4}\right). \tag{17}$$

Note that the asymptotic results are valid when many Landau levels are filled. We now take the continuum limit

$$n\rightarrow\frac{1}{2}\left(\frac{lE}{v}\right)^2,\quad \sum_{n=0}^{\infty}\rightarrow\left(\frac{l}{v}\right)^2\int_0^{\infty}EdE \tag{18}$$

to express the dimensionless conductivity in Eq. (16) as the following integral:

$$\Phi=\frac{4}{\sqrt{2u}\pi}\left(\frac{l}{v}\right)^3\int_0^{\infty}dE\frac{E^2g(E)}{[g(E)+1]^2}\times\sin^2(1/2\sqrt{u/n})\sin^2\left(2\sqrt{nu}-\frac{\pi}{4}\right), \tag{19}$$

where $u=2\pi^2/b$.

Now, assuming that the temperature is low such that $\beta^{-1}\ll E_F$ and replacing $E=E_F+s\beta^{-1}$, we rewrite the above integral as

$$\Phi=\frac{4p^2a}{uvb^2\beta}\sin^2\left(\frac{\pi}{p}\right)\int_{-\infty}^{\infty}\frac{dse^s}{(e^s+1)^2}\sin^2\left(\frac{2\pi p}{b}-\frac{\pi}{4}+\frac{2\pi a}{vb\beta}s\right), \tag{20}$$

where $p=\frac{E_Fa}{v}=k_Fa=\sqrt{2\pi n_e}a$ is the dimensionless Fermi momentum of the electron. To obtain an analytic solution, we have also replaced E by E_F in the above integral except in the sine term in the integrand.

The above expression can be expressed as

$$\Phi=\frac{8p^2a}{vb^2\beta}\sin^2\left(\frac{\pi}{p}\right)\int_{-\infty}^{\infty}\frac{ds}{\cosh^2(s/2)}\sin^2\left(\frac{2\pi p}{b}-\frac{\pi}{4}+\frac{2\pi a}{vb\beta}s\right). \tag{21}$$

The above integration can be performed by using the following identity:[2]

$$\int_0^{\infty}dx\frac{\cos ax}{\cosh^2\beta x}=\frac{a\pi}{2\beta^2\sinh(a\pi/2\beta)}, \tag{22}$$

with the result

$$\Phi=\frac{p^2T}{\pi^2buT_D}\sin^2\left(\frac{\pi}{p}\right)\left\{1-A\left(\frac{T}{T_D}\right)+2A\left(\frac{T}{T_D}\right)\sin^2\left[2\pi\left(\frac{p}{b}-\frac{1}{8}\right)\right]\right\}, \tag{23}$$

where $k_BT_D=\frac{bv}{4\pi^2a}$, $\frac{T}{T_D}=\frac{4\pi^2a}{vb\beta}$, and $A(x)=\frac{x}{\sinh(x)}\overset{(x\rightarrow\infty)}{\longrightarrow}2xe^{-x}$.

V. COMPARISON TO MAGNETICALLY MODULATED CONVENTIONAL TWO-DIMENSIONAL ELECTRON GAS SYSTEM

We start by comparing the energy spectrum and velocity expression obtained in Eqs. (9) and (13) to similar expressions for the conventional 2DEG wherein the electron spectrum is parabolic.[9] For the energy spectrum, we find that the Landau level spectrum is significantly different from that of standard electrons in the conventional 2DEG. The first term in Eq. (9), $\omega_g\sqrt{n}$, where $\omega_g=v\sqrt{2eB}$, has to be compared to $\omega_c(n+1/2)$, where $\omega_c=eB/m$, for standard electrons. The

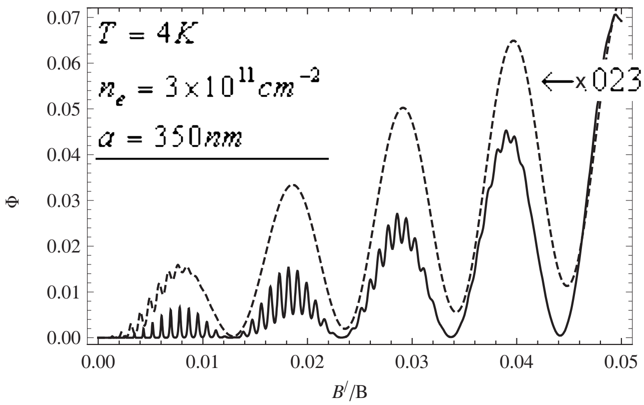

FIG. 2. The dimensionless conductivity vs inverse magnetic field. Magnetically modulated graphene (solid curve); conventional 2DEG (dashed curve).

modulation effects are in the second term, where the essential difference is in the structure of the function $G_n(u) = \frac{\sqrt{2n}e^{-u/2}}{\sqrt{u}}[L_{n-1}(u)-L_n(u)]$. We find that there are essentially two basic differences: First, for Dirac electrons, we have a difference of two successive Laguerre polynomials, whereas we had the sum of the Laguerre polynomials in the corresponding term for standard electrons in 2DEG. Second, the expression for Dirac electrons is multiplied by the square root of the Landau band index $\sqrt{n}$ that was absent in the expression for standard electrons. The above mentioned differences in the $G_n(u)$ function cause the velocity expression for the Dirac electrons given by Eq. (13) to be different from that of the standard electrons.

As expected, these differences in the energy spectra and velocities lead to different results for the diffusive conductivity in the two cases. We now compare the results obtained for the asymptotic expression of the diffusive conductivity σ_{yy}. To make this comparison possible, we first express Φ given by Eq. (23) as

$$\Phi = \frac{4k_F^2 k_B T}{ve^2B^2a}\sin^2\left(\frac{\pi}{p}\right)F, \tag{24}$$

where $F=\{1-A(\frac{T}{T_D})+2A(\frac{T}{T_D})\sin^2[2\pi(\frac{p}{b}-\frac{1}{8})]\}$. We now compare the results for dimensionless conductivity obtained in Eq. (24) to those presented in Eq. (18) of the first work cited in Ref. 9. We find that the result for the graphene (Dirac electrons) system differs from that of the conventional 2DEG (standard electrons) system by a factor of $\frac{p^2}{\pi^2 ub}\sin^2(\frac{\pi}{p})$. For the system under consideration $p\sim 50$ and, in this limit, if we take $\sin^2(\frac{\pi}{p})\rightarrow(\frac{\pi}{p})^2$, it yields the factor $\frac{1}{2\pi^2}$. Hence, we conclude that the amplitude of the oscillations in the conductivity will be reduced by this factor in the magnetically modulated graphene system compared to the conventional 2DEG under the same conditions. For the parameters considered in this work, the conductivity is larger by a factor of ≈ 30 (at magnetic field of 0.5 T) in the 2DEG system compared to that for graphene. In Fig. 2, we plot the dimensionless conductivity versus inverse magnetic field for magnetically modulated graphene and 2DEG. Note that in Fig. 2, the dimensionless conductivity for conventional 2DEG is rescaled by a factor of 0.023.

The temperature scale for the damping of Weiss oscillations in graphene can be obtained from Eq. (23) and is characterized by the T_D given above, while the characteristic temperature for 2DEG is given in the first work cited in Ref. 9 as $k_B T_a = (\hbar\omega_c/4\pi^2)ak_F$. Comparing T_D and T_a, we obtain $\frac{T_a}{T_D}=\frac{v_F}{v}$ where v_F is the Fermi velocity in 2DEG. This shows that Weiss oscillations in the graphene are less damped with temperature compared to 2DEG due to the difference in Fermi velocities in the two systems.

VI. COMPARISON TO ELECTRICALLY MODULATED GRAPHENE SYSTEM

We will now compare the results obtained in this work to the results obtained in Ref. 10 for the case of the electrically modulated graphene system. We will first compare the energy spectra in the two cases. The difference in the energy spectra due to the modulation effects was obtained in Eq. (8). If we compare this result to the corresponding expression for the electrically modulated case, we find the following differences: First, in the magnetic modulation case, we have a difference of two successive Laguerre polynomials, whereas we had the average of two successive Laguerre polynomials in the electric case. Second, in the magnetic modulation case, the energy eigenvalues are multiplied by the square root of the Landau band index $\sqrt{n}$ that was absent in the expression for the electric case. These differences cause the velocity expression for the Dirac electrons given by Eq. (13) to be different from that of the electrons in the electrically modulated system.

We now compare the expressions for dimensionless conductivity Φ given by Eq. (23) to the electrically modulated case [Eq. (22) in Ref. 10]. We find that in the magnetically modulated case, we have $\sin^2 x$ functions in place of $\cos^2 x$ functions for the electric case, which results in the oscillations being out of phase in the two cases. We also find that the amplitude of the oscillations in the magnetic case are larger by a factor of $\frac{4\pi^2}{ub}$ compared to the electrically modulated case. For the parameters considered in this work, the conductivity is larger by a factor of ≈ 2 (at magnetic field of 0.5 T) in the magnetically modulated graphene system compared to the electrically modulated one.

An exact expression of the dimensionless conductivity Φ for the electric and magnetic modulated graphene systems is shown in Fig. 3 as a function of the inverse magnetic field at temperature $T=6$ K, electron density $n_e=3\times 10^{11}$ cm^{-2}, and period of modulation $a=350$ nm. In Fig. 3, we can clearly see that the Weiss oscillations in the dimensionless conductivity are enhanced, in which they have a larger amplitude, in the magnetically modulated case compared to the electrically modulated case for the same parameter values. Furthermore, we note that the oscillations in the magnetic and electric modulated cases have a π phase shift. We also observe that in the region of high magnetic field, SdH oscillations are superimposed on the Weiss oscillations. The oscillations are periodic in $1/B$ and the period depends on electron density as $\sqrt{n_e}$ in both the magnetic and electric modulated cases.

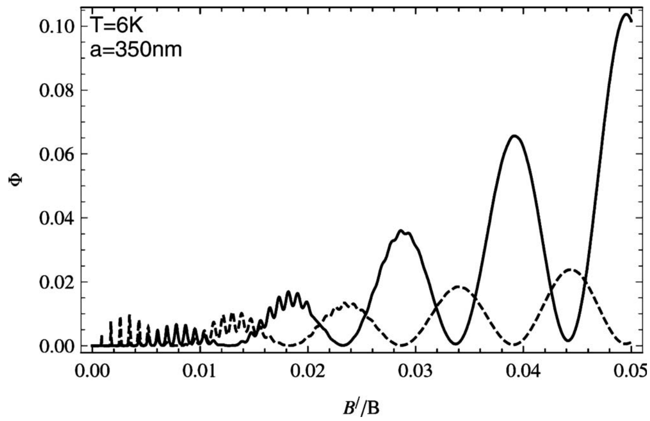

FIG. 3. The dimensionless conductivity vs inverse magnetic field. Magnetic modulation (solid curve); electric modulation (dashed curve).

The characteristic damping temperature is the same for both the systems.

To better understand the increase in amplitude of Weiss oscillations in the magnetically modulated graphene system compared to the electrically modulated one, we consider the difference in bandwidths in the two cases.[13] An important feature is the additional $\sqrt{n}$ factor in the perturbed energy eigenvalues for the magnetically modulated case, which is absent in the electrically modulated case. The result is that the bandwidth in the magnetically modulated case is approximately greater by a factor of $\sqrt{2}$ compared to the electrically modulated graphene system.

VII. CONCLUSIONS

In this work, we have investigated the electrical transport properties of Dirac electrons in a monolayer graphene sheet in the presence of a perpendicular magnetic field that is weakly and periodically modulated along one direction. Our primary focus has been the study of Weiss oscillations in the diffusive magnetoconductivity σ_{yy} of this system. We have compared the results obtained to those obtained for the magnetically modulated conventional 2DEG system and to those of the graphene system subjected only to the electric modulation. We find that in the magnetically modulated graphene system, Weiss oscillations in the magnetoconductivity have a reduced amplitude compared to the conventional 2DEG but are more robust with respect to temperature. In comparison to the electrically modulated graphene case, we find that the conductivity is larger in amplitude. We also find that the oscillations in the magnetoconductivity in graphene are π phase shifted with respect to the electrically modulated case, whereas they are in phase with the conventional 2DEG subjected to magnetic modulation.

ACKNOWLEDGMENTS

One of us (K.S.) would like to acknowledge the support of the Pakistan Science Foundation (PSF) through Project No. C-QU/Phys (129). M.T. would like to acknowledge the support of the Pakistan Higher Education Commission (HEC).

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