

Supergap anomalies in cotunneling between normal-superconducting and between two superconducting leads via a small quantum dot

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Cotunneling current through a resonant level coupled to either normal and superconducting or to two superconducting leads is studied for the domain of bias voltages V , exceeding the superconducting gap 2Δ . Due to the on-site repulsion in the resonant level, cotunneling of an electron is accompanied by the creation of a quasiparticle in a superconducting lead. Energy conservation imposes a threshold for this inelastic transport channel: $V_c=3\Delta$ for normal-superconductor case and $\tilde{V}_c=4\Delta$ for the superconductor-superconductor (S-S) case. We demonstrate that the behavior of the current near the respective thresholds is nonanalytic, namely, $\delta I^m(V) \propto (V-V_c)^{3/2}\Theta(V-V_c)$ and $\delta I^m(V) \propto (V-\tilde{V}_c)\Theta(V-\tilde{V}_c)$. Stronger anomaly for the S-S leads is the consequence of the enhanced density of states at the edges of the gap. In addition, the enhanced density of states makes the threshold anomalies for two-electron cotunneling processes in the Coulomb-blockaded regions more pronounced than for the normal-normal leads.

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I. INTRODUCTION

Early single-electron-transport devices¹ were based on conducting grains containing large gate-controlled number of electrons. A grain was coupled by tunnel barriers to two macroscopic leads. With number of electrons on the grain being large, superconductivity could be induced in the grain²⁻¹⁰ upon lowering the temperature, while the leads remained either normal^{4,5,8} or also turned into a superconducting state.^{2,3,6,7,9,10} The focus of the early studies was the interplay between the two low-energy³⁻⁹ scales, namely, the charging energy and the superconducting gap. This interplay manifested itself in the Coulomb-blockade oscillations. On the theoretical side, different regimes of transport via superconducting grain²³⁻²⁶ were studied for experimentally relevant situation of a grain containing many electrons.

In the later experiments, the grains have been replaced by much smaller few-electron quantum dots based either on InAs (Refs. 11-13) or carbon nanotubes.¹⁴⁻²² In these devices, there is no superconducting pairing of electrons on the dot. Rather, either one¹⁷ or both^{11-16,18-22} leads are made of superconducting material.

Interesting physics in the superconductor-normal-superconductor (S-N-S) junctions with superconducting leads is due to the fact²⁹ that the Andreev process^{27,28} in these junctions gives rise to a rich subgap structure in the current-voltage characteristics.³⁰⁻³² When the normal (N) region is a small quantum dot (or a single resonant level) coupled by tunneling to the leads, this subgap structure is more pronounced.³³⁻³⁵ In addition, in the latter case, the on-site interaction of two electrons, which in a small dot assumes the role of charging energy, becomes important.³⁶⁻⁴⁴

What makes the S-N-S structures with a resonant level as an N region particularly interesting is a delicate interplay of a new energy scale, Kondo temperature, which is much smaller than the charging energy, and the superconducting gap. This interplay is the focus of the very recent experimental studies.^{12,13,15,17,21,22} The results reported in Refs. 12, 13, and 21 suggest that subgap anomalies in differential conduc-

tance, $G(V)$, at biases $V=\pm\Delta$, where 2Δ is the superconducting gap, are enhanced in the Kondo regime. Another intriguing observation made in Refs. 12, 13, and 21 is that Kondo resonance leads to smearing of the conventional anomalies in $G(V)$ at $V=\pm 2\Delta$.

Therefore, both theoretical³⁶⁻⁴⁴ and experimental^{12,13,15,17,21,22} studies suggest that on-site repulsion affects the subgap structure in the conductance. However, it is commonly believed that for $V>2\Delta$, there is no qualitative difference between the cases when superconducting leads are separated by a barrier or both coupled to a quantum dot. In the present paper, we demonstrate that on-site repulsion manifests itself even for $V>2\Delta$, leading to *supergap* anomalies in $G(V)$. The underlying reason is that, at finite repulsion, *inelastic* electron transitions between normal leads become possible.⁴⁵ These transitions are accompanied by a quasiparticle excitations in the leads. When one of the leads is superconducting, the minimal energy of the excitation is 2Δ . Then, in order for electron tunneling from the normal lead to create the excitation in superconducting lead, the bias should exceed $V_c=3\Delta$. Threshold for inelastic tunneling results in a supergap singularity, $\delta G(V) \propto (V-V_c)^{1/2}$, in the normal-superconductor (N-S) conductance, as demonstrated in Sec. III A. For the same reason, inelastic tunneling between two superconducting leads has a threshold at $\tilde{V}_c=4\Delta$. We show that the supergap anomaly in the superconductor-superconductor (S-S) transport has a steplike form, $\delta G(V) \propto \Theta(V-\tilde{V}_c)$ (Sec. IV); i.e., it is stronger than in the N-S case. This is due to the enhancement of the density of states at the edges of the superconducting gap. In addition, finite temperature T affects the S-S supergap anomaly only via the temperature dependence of Δ , whereas the N-S supergap anomaly is a universal function of $(V-V_c)/T$ (Sec. III A.). In Sec. III B, we also demonstrate that the enhancement of the density of states causes a sharpening of the large-bias transport anomalies⁴⁵ that involve two-electron transitions.

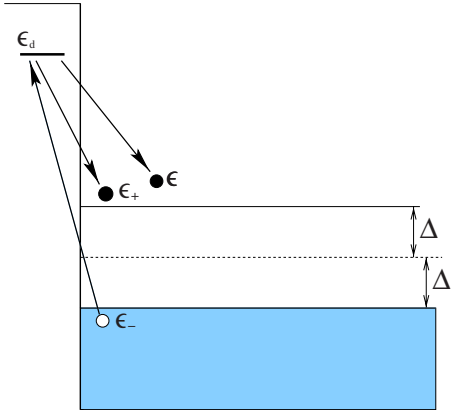


FIG. 1. (Color online) Inelastic correction to the lifetime of the localized state due to excitation of a quasiparticle across the gap is illustrated schematically.

II. ANOMALY IN THE LIFETIME OF A LOCALIZED STATE

In order to illustrate how the on-site repulsion U gives rise to the anomalies in the conductance, $G(V)$, we start from an auxiliary problem of the escape of an electron from the occupied localized state (LS) into a superconductor. This situation is illustrated in Fig. 1. If the energy of the LS ϵ_d lies above the upper boundary of the gap, $\epsilon_d > \Delta$, then the population of the LS, which is occupied at time $t=0$, decays with t as

$$n(t) = \exp(-\Gamma t), \quad (1)$$

where the decay rate Γ is given by the golden-rule expression as

$$\Gamma(\epsilon_d) = \pi \gamma^2 \nu_0 g(\epsilon_d). \quad (2)$$

Here, γ is the tunnel matrix element and

$$\nu_0 g(\epsilon) = \nu_0 \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \quad (3)$$

is the density of states in the superconductor. Equation (3) applies when Γ is much smaller than Δ . Our main point is that for large enough ϵ_d , there exists another inelastic channel of the electron escape into the continuum. Namely, the escape can be accompanied by excitation of a quasiparticle across the gap. This process leads to the threshold anomaly in the dependence $\Gamma(\epsilon_d)$. The position of the threshold $\epsilon_d^{(c)}$ can be found from the following two conditions on the energy ϵ of electron leaving the LS,

$$\epsilon > \Delta, \quad (\epsilon_d - \epsilon) > 2\Delta. \quad (4)$$

The first condition ensures that the state into which electron escapes is empty, while the meaning of the second condition is that the energy loss suffered by escaping electron is sufficient to create a quasiparticle. From Eq. (4), we find the minimal value of ϵ_d

$$\epsilon_d = \epsilon_d^{(c)} = 3\Delta. \quad (5)$$

Inelastic process is enabled by a finite U . To see this, we notice that there are two contributions to the amplitude of the following processes.

(1) (i) An electron from the LS tunnels into the state $\epsilon > \Delta$, (ii) another electron from the occupied state ϵ_- enters the LS (ii), and subsequently tunnels into the empty state ϵ_+ . These steps are illustrated in Fig. 1.

(2) Initial and final states are the same as in (1), while the intermediate steps (i) and (ii) are interchanged. As a result, after the first step, the LS is doubly occupied. In the absence of the on-site repulsion, the two amplitudes, 1 and 2, would cancel each other identically. At finite U , this cancellation does not happen. Note that, for large $U \gg \epsilon_d$, the energy denominator corresponding to $\epsilon_- \rightarrow \epsilon_d$ contains U so that the second amplitude can be neglected.

The above reasoning is quite similar to that in Ref. 45, where another inelastic process, i.e., occupation of the LS in the course of cotunneling between normal leads, has been considered.

The amplitude $A_{\epsilon_d, \epsilon_-}^{\epsilon, \epsilon_+}$ of the three-step process in Fig. 1 is $\propto \gamma^3$. Taking into account that the energies of the intermediate states are ϵ and $\epsilon + \epsilon_d - \epsilon_-$, the analytical expression for this amplitude reads

$$A_{\epsilon_d, \epsilon_-}^{\epsilon, \epsilon_+} = \frac{\gamma^3}{(\epsilon_d - \epsilon)(\epsilon_- - \epsilon)}. \quad (6)$$

Note that this expression is valid when the states ϵ and ϵ_+ correspond to the opposite spin projections⁴⁵ so that these states are distinguishable. On the contrary, for parallel spins of the states ϵ and ϵ_+ , the amplitude [Eq. (6)] vanishes.⁴⁶

The expression for inelastic correction to the rate Γ follows from Eq. (6),

$$\begin{aligned} \delta\Gamma(\epsilon_d) &= 2\pi \int_{\Delta}^{\infty} d\epsilon \nu(\epsilon) \int_{\Delta}^{\infty} d\epsilon_+ \nu(\epsilon_+) \\ &\times \int_{-\infty}^{-\Delta} d\epsilon_- \nu(\epsilon_-) |A_{\epsilon_d, \epsilon_-}^{\epsilon, \epsilon_+}|^2 \delta[\epsilon_d + \epsilon_- - (\epsilon + \epsilon_+)]. \end{aligned} \quad (7)$$

It is seen from Eq. (7) that the argument of the δ function turns to zero for $\epsilon_d = 3\Delta$ at $\epsilon_- = -\Delta$ and $\epsilon = \epsilon_+ = \Delta$. To establish the form of the anomaly near $\epsilon_d = \epsilon_d^{(c)} = 3\Delta$, we introduce the new variables as

$$E = \epsilon - \Delta, \quad E_+ = \epsilon_+ - \Delta, \quad E_- = -\epsilon_- - \Delta \quad (8)$$

in Eq. (7). Now, it is sufficient to set $\epsilon_- = -\Delta$ and $\epsilon = \epsilon_+ = \Delta$ in the denominator of Eq. (7) and replace $\nu(\epsilon)$, $\nu(\epsilon_+)$, and $\nu(\epsilon_-)$ by $\nu_0 \sqrt{\Delta/2E}$, $\nu_0 \sqrt{\Delta/2E_+}$, and $\nu_0 \sqrt{\Delta/2E_-}$, respectively. Upon this replacement, Eq. (7) simplifies to

$$\begin{aligned} \delta\Gamma(\epsilon_d) &= \frac{\Gamma^3}{2^{9/2} \pi^2 \Delta^{5/2}} \int_0^{\infty} \frac{dE}{\sqrt{E}} \int_0^{\infty} \frac{dE_+}{\sqrt{E_+}} \int_0^{\infty} \frac{dE_-}{\sqrt{E_-}} \\ &\times \delta[\epsilon_d - \epsilon_d^{(c)} - (E + E_+ + E_-)]. \end{aligned} \quad (9)$$

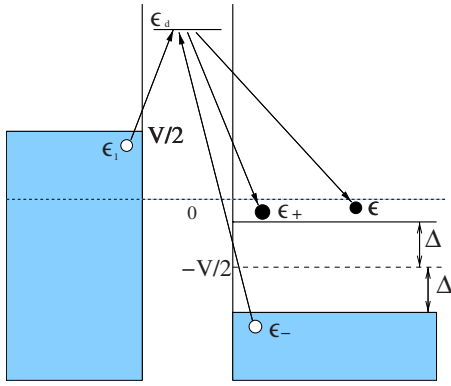


FIG. 2. (Color online) Origin of the anomaly at $V_c = 3\Delta$ in cotunneling between the N and S leads is illustrated schematically.

The above integral is proportional to $(\epsilon_d - \epsilon_d^{(c)})^{1/2}$; the numerical factor can be easily expressed through the surface area of the unit sphere. The final form of the threshold anomaly is the following:

$$\frac{\delta\Gamma(\epsilon_d)}{\Gamma} = \frac{\Gamma^2}{2^{7/2}\pi\Delta^{5/2}}[\epsilon_d - \epsilon_d^{(c)}]^{1/2}\Theta(\epsilon_d - \epsilon_d^{(c)}). \quad (10)$$

In deriving Eq. (10), we assumed that the intrinsic width Γ is much smaller than Δ . This guarantees that the relative correction $\delta\Gamma/\Gamma$ is small. The anomaly [Eq. (10)] is much stronger than the threshold anomaly for two-electron ionization of the LS in Ref. 45. The origin of this enhancement is the divergence of the density of states [Eq. (3)] at edges of the gap.

In the above calculation, we treated the states ϵ_- , ϵ_+ , and ϵ as electron states in a normal metal and took superconductivity into account only via the energy dependence of the density of states, $\nu(\epsilon)$. This is justified when the tunneling amplitude is calculated to the lowest order in the matrix element γ . However, the anomaly [Eq. (10)] emerges in the third order in γ . The proof of the validity of Eq. (7) for $\delta\Gamma(\epsilon_d)$, starting from the BCS Hamiltonian, is presented in the Appendix.

III. SUPERGAP ANOMALIES IN THE NORMAL-SUPERCONDUCTING COTUNNELING

A. Single-electron transport

Passage of current from a metal to a superconductor by single-electron transitions, involving the LS, is illustrated in Fig. 2. Position V_c of the anomaly, at which the cotunneling from the normal lead can be accompanied by creation of a quasiparticle in the superconducting lead, can be found from the similar reasoning, as in Sec. I. The only difference is that electron enters the superconducting lead with energy close the Fermi energy of the normal lead so that

$$V_c = 3\Delta. \quad (11)$$

The magnitude of the anomaly is, however, weaker than for the electron escape considered in Sec. I. This is due to the fact that while the energy of the LS is fixed to ϵ_d , the energy

of the electron in the normal lead is simply restricted to the domain below $V/2$ —the Fermi level in the normal lead.

The elastic cotunneling conductance is given by

$$G_{NS}^{el} = \frac{4e^2}{\pi\hbar} \frac{\Gamma_L\Gamma_R^s}{(\epsilon_d - V/2)^2}, \quad (12)$$

where we assumed $(\epsilon_d - V/2) \ll V$. The widths $\Gamma_{L,R} = \pi\nu_{L,R}\gamma_{L,R}^2$ are defined in a usual way; due to the enhancement of the density of states in the superconductor, the width Γ_R^s , which enters into Eq. (12), becomes $\Gamma_R^s = \pi\nu_{Rg}(V)\gamma_R^2 \approx 3\Gamma_R/\sqrt{8}$.

In order to calculate the inelastic correction, $\delta G^{in}(V)$, to the conductance, one cannot simply modify Γ_R^s according to Eq. (10). This is because, in the course of cotunneling, the electron occupies the LS only *virtually*. The correct procedure of finding $\delta G^{in}(V)$ requires the calculation of inelastic correction, $\delta I^{in}(V)$, to the current, taking into account that electron, transferred from the normal into superconducting lead, which can excite a quasiparticle in this lead. Then, we have

$$\begin{aligned} \delta I^{in}(V) &= \frac{4\pi e}{\hbar} \nu_L \nu_R^3 \int_{-\infty}^{\infty} d\epsilon_1 f(\epsilon_1 - V/2) \int_{\Delta-V/2}^{\infty} d\epsilon g(\epsilon + V/2) \\ &\times \int_{\Delta-V/2}^{\infty} d\epsilon_+ g(\epsilon_+ + V/2) \int_{-\infty}^{-\Delta-V/2} d\epsilon_- g(\epsilon_- + V/2) \\ &\times |A_{\epsilon_1, \epsilon_-}^{\epsilon, \epsilon_+}|^2 \delta[\epsilon_1 + \epsilon_- - (\epsilon + \epsilon_+)]. \end{aligned} \quad (13)$$

where $f(\epsilon)$ is the Fermi function. The expression for the transition amplitude $(\epsilon_1, \epsilon_-) \rightarrow (\epsilon, \epsilon_+)$ differs from Eq. (6) by an extra γ_L , namely,

$$A_{\epsilon_1, \epsilon_-}^{\epsilon, \epsilon_+} = \frac{\gamma_L \gamma_R^3}{(\epsilon_d - \epsilon_1)(\epsilon - \epsilon_1)(\epsilon_d - \epsilon_+)} + \frac{\gamma_L \gamma_R^3}{(\epsilon_d - \epsilon_-)(\epsilon_+ - \epsilon_-)(\epsilon_d - \epsilon)}. \quad (14)$$

As in Sec. I, in Eq. (14), we had excluded the virtual states with doubly occupied LS. This implies the on-site repulsion U is much bigger than both ϵ_d and Δ . In fact, for $U \sim \epsilon_d$, the result does not change qualitatively. Explicit calculation shows that, with finite U , the amplitude Eq. (14) must be multiplied by a factor $U/(2\epsilon_d - \epsilon_1 - \epsilon_- + U)$, which vanishes in the absence of on-site interaction, as it should be.

Two terms in Eq. (14) account for two different sequences, in which the transition $(\epsilon_1, \epsilon_-) \rightarrow (\epsilon, \epsilon_+)$ takes place. The first term corresponds to the electron from the normal lead entering the LS at the first step. The second term describes the virtual occupation of the LS by electron from superconductor with energy ϵ_- at the first step, followed by its escape into ϵ_+ and subsequent cotunneling of electron from the normal lead. Note that there is no analog of the second contribution in the amplitude Eq. (6). This is because Eq. (6) describes the process in which the LS was occupied in the initial state.

In order to extract the anomaly, upon substituting Eq. (14) into Eq. (13), we introduce the new variables as

$$E_1 = \epsilon_1 - V/2, \quad E = \epsilon + V/2 - \Delta,$$

$$E_+ = \epsilon_+ + V/2 - \Delta, \quad E_- = -\epsilon_- - V/2 - \Delta. \quad (15)$$

For bias V close to $V_c = 3\Delta$, the characteristic values of E , E_1 , E_+ , and E_- are much smaller than Δ . This allows us to set $\epsilon_1 = V_c/2$ and $\epsilon = \epsilon_+ = -V_c/2 + \Delta$ in the denominators of Eq. (14). We can also use the near-gap-edge asymptotes for the densities of states in the superconducting leads. After these simplifications, Eq. (13) assumes the form

$$\delta I^{in}(V) = \frac{2^{5/2} e}{\pi^3 \hbar} \frac{\Gamma_L \Gamma_R^3 (\Delta T)^{3/2}}{[(\epsilon_d - 3\Delta/2)((\epsilon_d + 3\Delta/2)^2 - \Delta^2)]^2} F\left(\frac{V - V_c}{T}\right), \quad (16)$$

where the dimensionless function F of a single argument, $(V - V_c)/T$, is defined as

$$F\left(\frac{V - V_c}{T}\right) = \int_{-\infty}^{\infty} \frac{dE_1 f(E_1)}{T^{3/2}} \int_0^{\infty} \frac{dE}{\sqrt{E}} \int_0^{\infty} \frac{dE_+}{\sqrt{E_+}} \int_0^{\infty} \frac{dE_-}{\sqrt{E_-}} \times \delta[V - V_c + E_1 - (E + E_+ + E_-)]. \quad (17)$$

Note that the threefold integration over E , E_+ , and E_- has already been carried out in Sec. I. It yields $2\pi(V - V_c - E_1)^{1/2} \Theta[V - V_c + E_1]$. As a result, the bias dependence of δI^{in} is given by a single integral as

$$F\left(\frac{V - V_c}{T}\right) = 2\pi \int_0^{\infty} dx \frac{\sqrt{x}}{\exp\left[x - \frac{V - V_c}{T}\right] + 1}. \quad (18)$$

Inelastic correction, $\delta G^{in}(V)$, to the differential conductance is thus described by the derivative, dF/dV . The asymptotic behavior of $\delta G^{in}(V)$ at low, $(V - V_c) \ll T$, and high, $(V - V_c) \gg T$, temperatures can be easily found from Eq. (18). We present the results for a dimensionless ratio, $\delta G^{in}/G^{el}$, of inelastic and elastic contributions to the conductance as

$$\frac{\delta G^{in}}{G^{el}} = \begin{cases} \frac{2\Delta^2 \Gamma_R^2}{3\pi[(\epsilon_d + 3\Delta/2)^2 - \Delta^2]^2} \sqrt{\frac{V - V_c}{\Delta}} & (V - V_c) \gg T \\ \frac{2\Delta^2 \Gamma_R^2}{3\pi^{1/2}[(\epsilon_d + 3\Delta/2)^2 - \Delta^2]^2} \sqrt{\frac{T}{\Delta}} & (V - V_c) \ll T, \end{cases} \quad (19)$$

where $\alpha = 2^{-1/2} \int_0^{\infty} dx x^{1/2} / \cosh^2 x \approx 0.536$. It is seen from Eq. (19) that, at $V > V_c$, differential conductance acquires a correction $\propto (V - V_c)^{1/2}$. Correspondingly, the second derivative, $d^2 I/dV^2$, has an asymmetric peak of a width $\sim T$ centered at $V = V_c$. The shape of the peak is given by the second derivative of the function F . In Fig. 3, this derivative, which is calculated numerically from Eq. (18), is plotted versus dimensionless deviation, $(V - V_c)/T$.

B. Two-electron transport

1. Ionization of the localized state

In terms of the Coulomb-blockade stability diagram in the (ϵ_d, V) plane [Fig. 4], the anomalies at $V = \pm 3\Delta$ correspond to horizontal lines, which start from the points $(-\Delta/2, 3\Delta)$,

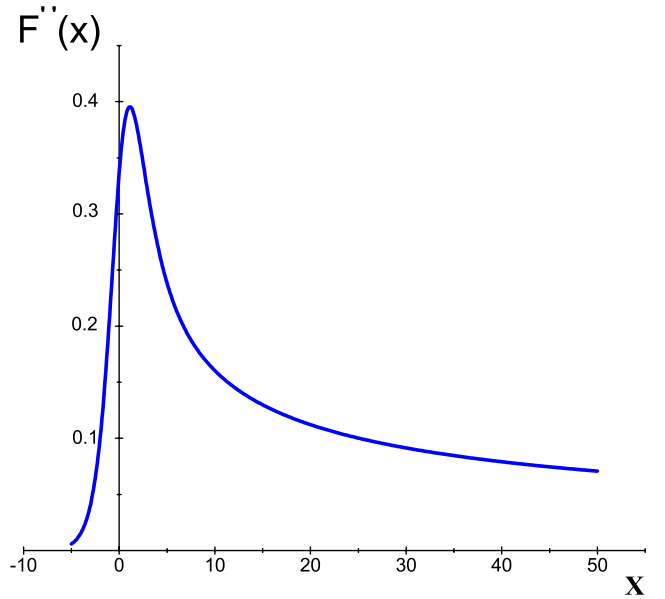


FIG. 3. (Color online) The shape of the peak in the derivative, dG/dV , of the N-S differential conductance is plotted from Eq. (18) versus dimensionless deviation $x = (V - V_c)/T$.

$(3\Delta/2, 3\Delta)$ and $(-3\Delta/2, -3\Delta)$, $(\Delta/2, -3\Delta)$. These lines extend into the blockaded region. In Ref. 45, it was demonstrated that, without superconductivity, there exists an additional weak structure within the Coulomb-blockade diamond, along the lines $V = \pm 2\epsilon_d/3$. The origin of this structure is the *two-electron* ionization of the LS, namely, the process, in

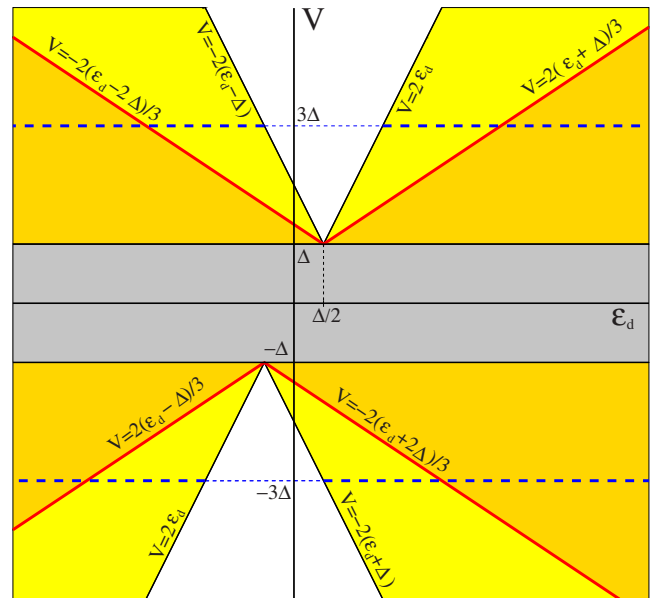


FIG. 4. (Color online) Stability diagram for transport between N and S leads via a localized state. The white region corresponds to the sequential tunneling transport. The horizontal dashed blue (black) lines, $V = \pm 3\Delta$, correspond to the single-electron supergap anomaly, which is illustrated in Fig. 2. The red (gray) lines, $V = 2/3(\epsilon_d \pm \Delta)$ and $V = 2/3(\epsilon_d \pm \Delta)/3$, are the positions of the two-electron resonance. The subgap resonances at $|V| < \Delta$ lie in the shaded region.

which one electron from the left lead is transferred to the right lead, while the other electron from the left lead occupies the LS. The position of the boundary, $V=2\epsilon_d/3$, expresses the threshold for this two-electron transfer, which follows from the energy conservation. In this section, we point out that, in the presence of the superconductivity, the boundaries for two-electron ionization are modified in an *asymmetric* fashion. For positive bias, $V>0$, the boundaries are located at

$$V_+(\epsilon_d) = \pm \frac{2}{3} \left(\epsilon_d - \frac{\Delta}{2} \right) + \Delta, \quad (20)$$

while for negative bias, they are located at

$$V_-(\epsilon_d) = \mp \frac{2}{3} \left(\epsilon_d + \frac{\Delta}{2} \right) - \Delta. \quad (21)$$

These modified boundaries are shown in Fig. 4. More importantly, as we demonstrate below, superconductivity leads to the *strengthening* of the ionization anomaly. The underlying mechanism for this strengthening is, again, the enhancement of the density of states at the boundaries of the gap.

Energy dependence of the density of states can be easily incorporated into the expression from Ref. 45 for ionization rate. Consider first the situation when the initial states of two electrons with energies ϵ_1 and ϵ_2 are in the normal lead, while one of the finite states (with energy ϵ) is in the superconducting lead and the other is on the LS. The ionization rate for $T=0$ is given by

$$\Gamma_{\text{ion}}^{N \rightarrow S}(V) = \frac{\Gamma_L \Gamma_R}{(2\pi)^2} \int_{-\infty}^{V/2} d\epsilon_1 \int_{-\infty}^{V/2} d\epsilon_2 \int_{\Delta-V/2}^{\infty} d\epsilon g(\epsilon + V/2) \times \frac{1}{(\epsilon_d - \epsilon_1)^2 (\epsilon_d - \epsilon_2)^2} \delta[\epsilon_d + \epsilon - \epsilon_1 - \epsilon_2]. \quad (22)$$

Near the threshold, $V=V_+(\epsilon_d)$, one can set $\epsilon_1 = \epsilon_2 = V_+/2$ in the denominator of Eq. (22). Upon measuring the energies ϵ_1 , ϵ_2 , and ϵ from their respective boundaries, as in Eq. (15), we can simplify Eq. (22) to

$$\Gamma_{\text{ion}}^{N \rightarrow S}(V) = \frac{\Gamma_L \Gamma_R \Delta^2}{(2\pi)^2 (\epsilon_d - V_+/2)^4} H_+ \left[\frac{V - V_+}{\Delta} \right], \quad (23)$$

where the one-parameter function H_+ is defined as

$$H_+(x) = \int_0^{\infty} \frac{dE_1}{\Delta} \int_0^{\infty} \frac{dE_2}{\Delta} g \left(\frac{3}{2} \Delta x + \Delta - E_1 - E_2 \right) \times \Theta \left[\frac{3}{2} x - \frac{E_1}{\Delta} - \frac{E_2}{\Delta} \right]. \quad (24)$$

This integral is easily calculable. Its analytic form is

$$H_+(x) = \frac{3x+2}{8} \sqrt{9x^2+12x} - \frac{1}{2} \ln \left[\frac{3}{2} x + 1 + \sqrt{\frac{9}{4} x^2 + 3x} \right]. \quad (25)$$

The large- x and small- x asymptotes of H_+ are

$$H_+(x) = \begin{cases} \sqrt{3} x^{3/2}, & x \ll 1 \\ \frac{9}{8} x^2, & x \gg 1, \end{cases} \quad (26)$$

Consider now $V<0$. The ionization rate is given by

$$\Gamma_{\text{ion}}^{S \rightarrow N}(V) = \frac{\Gamma_L \Gamma_R^2}{(2\pi)^2} \int_{-\infty}^{-V/2-\Delta} d\epsilon_1 g(\epsilon_1 + V/2) \times \int_{-\infty}^{-V/2-\Delta} d\epsilon_2 g(\epsilon_2 + V/2) \times \int_{V/2}^{\infty} d\epsilon \frac{\delta(\epsilon_d + \epsilon - \epsilon_1 - \epsilon_2)}{(\epsilon_d - \epsilon_1)^2 (\epsilon_d - \epsilon_2)^2}, \quad (27)$$

which differs from Eq. (22) by additional density of superconducting states in the integrand. When the bias voltage is near the critical, $V=V_-$, one can replace the values of ϵ_1 and ϵ_2 by their boundary value $-V_-/2-\Delta$ in the denominator of Eq. (27). Then, the ionization rate $\Gamma_{\text{ion}}^{S \rightarrow N}$ can be expressed as

$$\Gamma_{\text{ion}}^{S \rightarrow N}(V) = \frac{\Gamma_L \Gamma_R^2 \Delta^2}{(2\pi)^2 (\epsilon_d + V_-/2 + \Delta)^4} H_- \left[\frac{V_- - V}{\Delta} \right], \quad (28)$$

where we have absorbed all the integrals of Eq. (27) into the new one-parameter function H_- , which are defined as follows:

$$H_-(x) = \int_0^{\infty} \frac{dE_1}{\Delta} g(E_1 + \Delta) \int_0^{\infty} \frac{dE_2}{\Delta} g(E_2 + \Delta) \times \Theta \left[\frac{3}{2} x - \frac{E_1}{\Delta} - \frac{E_2}{\Delta} \right]. \quad (29)$$

One of the integrations in Eq. (29) can be performed explicitly. The final form of the function H_- is the following:

$$H_-(x) = \int_0^{3x/2} dz \frac{(z+1) \sqrt{3x/2-z} \sqrt{3x/2+2-z}}{\sqrt{z^2+2z}}. \quad (30)$$

The easiest way to find the behavior of $H_-(x)$ at large and small x is to set, respectively, $g(\epsilon)=1$ and $g(\epsilon)=\sqrt{\Delta/2\epsilon}$ in the integrand of the definition Eq. (29). This yields

$$H_-(x) = \begin{cases} \frac{3\pi}{4} x \left(1 + \frac{9}{16} x \right), & x \ll 1 \\ \frac{9}{8} x^2, & x \gg 1, \end{cases} \quad (31)$$

Upon populating the LS, the electron rapidly, within the time $(\Gamma_L + \Gamma_R)^{-1}$, escapes either to the left or to the right lead. In terms of contributions to inelastic current, these two channels of escape are different.⁴⁵ For escape to the left, the net charge transfer is e , while for escape to the right, it is $2e$. As a result, the inelastic contribution to the current is equal to $\delta I^{\text{in}}(V) = 2e \Gamma_{\text{ion}}(V) (2\Gamma_R + \Gamma_L) / (\Gamma_L + \Gamma_R)$. The threshold behavior of δI^{in} near V_+ and V_- is determined by the functions H_+ and H_- , respectively. As seen from Eqs. (26) and (31), these behaviors coincide when $V-V_+$ and $V-V_-$ are much bigger than Δ . This is natural since for large deviations from the thresholds, superconducting gap drops out from δI^{in} . Note,

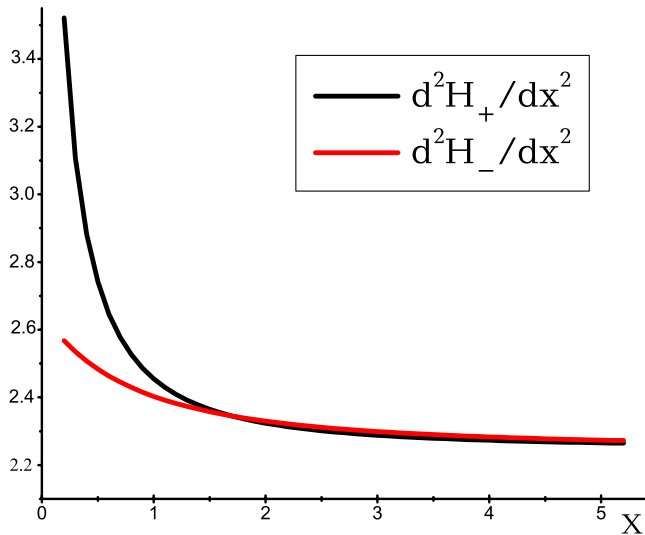


FIG. 5. (Color online) Shapes of the anomalies in d^2I/dV^2 near $V=V_+$ and $V=V_-$ versus dimensionless deviations $x=(V-V_{\pm})/\Delta$ calculated, respectively, from Eq. (25) (black line) and Eq. (30) [red (gray) line]. Both curves approach the value $9/4$ at $x \rightarrow \infty$. For small x , d^2H_+/dx^2 diverges as $3^{3/2}/4x^{1/2}$, while the $x=0$ value of d^2H_-/dx^2 is $27\pi/32$. Note that $H_+(x)$ and $H_-(x)$ are zero for $x < 0$.

however, that in the immediate vicinities of V_+ and V_- , the threshold behaviors are *different*, namely, δI^{in} is more singular near V_- than near V_+ . The origin of this asymmetry is that the inelastic process $N \rightarrow S$ involves only one state near the superconducting gap, while the inelastic process $S \rightarrow N$ involves two such states. Without superconductivity, threshold anomaly at $V = \pm 2\epsilon_d/3$ shows up in the third derivative of the current with respect to V . Our results, Eqs. (22) and (28), suggest that, within the interval $\sim \Delta$ from the thresholds V_+ and V_- , the singularities of current are more pronounced: they show up already in the second derivative d^2I/dV^2 . This is illustrated in Fig. 5, where d^2H_+/dV^2 and d^2H_-/dV^2 are plotted.

In fact, the singular behavior of inelastic current near $V = V_-$ shows up already on the level of differential conductance, $\delta G^{\text{in}} = d\delta I^{\text{in}}/dV$, as a step $\propto \Theta(V_- - V)$. Combining Eqs. (28) and (31), we get the following magnitude of the step:

$$\delta G^{\text{in}}(V) = \frac{3e^2}{16\pi\hbar} \frac{\Gamma_L \Gamma_R^2 \Delta}{(\epsilon_d + V_-/2 + \Delta)^4} \Theta(V_- - V). \quad (32)$$

2. Two-electron tunneling

As seen from Eqs. (23) and (24), the relative correction, $\delta I^{\text{in}}(V)/I^{\text{el}}$, to the elastic current due to ionization of the LS changes on the scale $V \sim \Delta$; the magnitude of correction at $V \sim \Delta$ is $\sim \Gamma_R \Delta / \epsilon_d^2 \ll 1$. Although small, this correction is distinguishable by virtue of its threshold dependence on bias. Indeed, both H_+ and H_- are zero for $V < V_+$ and $V > V_-$, respectively. Another fact that distinguishes the transport at biases near V_+ and V_- is that the inelastic current, $\delta I^{\text{in}}(V)$, has a precursor with singular dependence on deviation $V - V_{\pm}$ and on the temperature T . The origin of this precursor⁴⁵ is direct cotunneling of *two* electrons via the LS. This process

differs from ionization of the LS since, in the course of this two-electron cotunneling, the LS is populated only virtually. As a result, the corresponding contribution to the current, $I_{\pm}^{2e}(V)$, contains extra power Γ_L (or Γ_R). On the other hand, this contribution is more singular in deviation, $V - V_{\pm}$, and has a peculiar T dependence. As all other corrections to the elastic cotunneling calculated above $I_{\pm}^{2e}(V)$ is enabled by a finite on-site repulsion. The golden-rule expression for $I_{\pm}^{2e}(V)$,

$$\begin{aligned} \delta I_{+}^{2e}(V) &= \frac{e}{\hbar} \frac{\Gamma_L^2 \Gamma_R (2\Gamma_R + \Gamma_L)}{(2\pi)^2 (\epsilon_d - V_+/2)^2 (V_+ + \Delta)^2} \\ &\times \int_{-\infty}^{\infty} d\epsilon_1 f(\epsilon_1 - V/2) \int_{-\infty}^{\infty} d\epsilon_2 f(\epsilon_2 - V/2) \\ &\times \int_{-V/2+\Delta}^{\infty} dE_1 g(E_1 + V/2) \\ &\times \int_{-\infty}^{\infty} dE_2 \frac{\delta(\epsilon_1 + \epsilon_2 - E_1 - E_2)}{(\epsilon_d - E_2)^2}, \end{aligned} \quad (33)$$

contains energy denominators that correspond to virtual states; in these states, the LS is occupied by first and then by second tunneling electron. Note that the δ function in Eq. (33) ensures the conservation of the *total* energy, $\epsilon_1 + \epsilon_2$, of two electrons in the initial and final states, while individual energies get redistributed. The sensitivity of $\delta I_{+}^{2e}(V)$ to $V = V_+$ comes from the domain of integration in Eq. (33) with E_1 near the Fermi edge, $E_1 \approx -V_+/2 + \Delta$, and $E_2 \approx \epsilon_d$. For this reason, the nonresonant energy denominators are extracted from the integrand of Eq. (33).

In order to capture the dependence of δI_{+}^{2e} on $(V - V_+)$ and T , we introduce the dimensionless function K_+ , which is defined as

$$\begin{aligned} K_+(x) &= \int_{-\infty}^{\infty} \frac{dz_1}{e^{z_1} + 1} \int_{-\infty}^{\infty} \frac{dz_2}{e^{z_2} + 1} \int_0^{\infty} \frac{dz_3}{\sqrt{z_3}} \int_{-\infty}^{\infty} \frac{dz_4}{z_4} \\ &\times \delta \left[\frac{3}{2}x + z_1 + z_2 - z_3 - z_4 \right]. \end{aligned} \quad (34)$$

Then $\delta I_{+}^{2e}(V)$ can be presented in the form

$$\delta I_{+}^{2e}(V) = \frac{e}{\hbar} \frac{\Gamma_L^2 \Gamma_R (2\Gamma_R + \Gamma_L)}{(2\pi)^2 (\epsilon_d - V_+/2)^2 (V_+ + \Delta)^2} \left(\frac{\Delta T}{2} \right)^{1/2} K_+ \left[\frac{V - V_+}{T} \right]. \quad (35)$$

As before, in Eq. (38), we used the near-gap-edge asymptote of $g(\epsilon)$ so that Eq. (38) applies in the interval $\Gamma_L, \Gamma_R \ll |V - V_+|, T \ll \Delta$.

The fourfold integration in Eq. (34) can be reduced to a single integral by using the Fourier representation for the δ function and the fact that the Fourier transform of the Fermi function is equal to $\tilde{f}(\omega) = \pi / \sinh(\pi\omega)$. We will present the result for the second derivative, d^2K_+/dx^2 , which describes the near-threshold behavior of $d^2I_{+}^{2e}/dV^2$. It reads

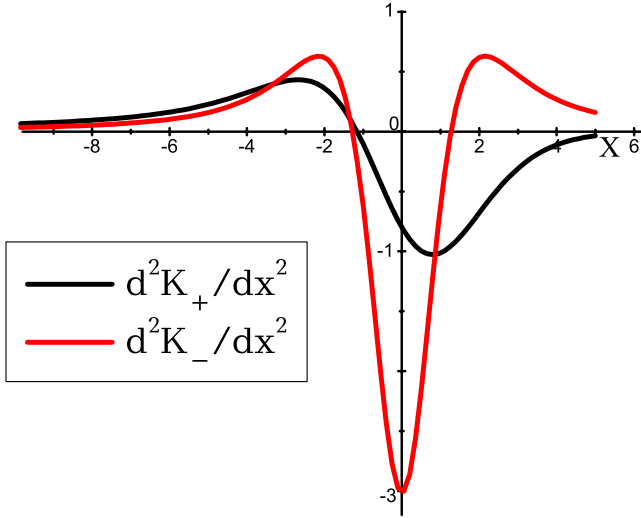


FIG. 6. (Color online) Shapes of the anomalies in $d^2 I_{\pm}^{2e} / dV^2$ near $V=V_+$ and $V=V_-$ versus dimensionless deviations $x=(V-V_{\pm})/T$ calculated, respectively, from Eq. (34) (black line) and Eq. (39) [red (gray) line].

$$\frac{d^2 K_+}{dx^2} = -\frac{9\pi^{5/2}}{2^{5/2}} \int_0^{\infty} ds \frac{s^{5/2}}{\sinh^2(\pi s)} \left[\cos\left(\frac{3}{2}sx\right) + \sin\left(\frac{3}{2}sx\right) \right]. \quad (36)$$

Consider first the limiting case of vanishing T . To realize that the temperature drops out from the expression Eq. (38), we notice that the asymptotic behavior of $d^2 K_+ / dx^2$ at large negative x is $\propto |x|^{-3/2}$. This yields $d^2 \delta I_+^{2e} / dV^2 \propto (V-V_+)^{-3/2}$. The divergence is stronger than $1/(V-V_+)$ in Ref. 45. Another remarkable feature of $d^2 I_+^{2e} / dV^2$ is that, at finite T , it exhibits a fine structure. This is seen from Fig. 6, where the function $d^2 K_+ / dx^2$ is plotted. Asymptotic behavior of $d^2 K_+ / dx^2$ at large positive x is $\propto \exp(-3x/2)$. This suggests that for $V > V_+$, ionization current dominates over δI_+^{2e} .

Calculation of the two-electron current, $\delta I_-^{2e}(V)$, near $V=V_-$ is quite similar to Eqs. (33) and (38). Namely, the golden-rule expression,

$$\begin{aligned} \delta I_-^{2e}(V) &= \frac{e}{\hbar} \frac{\Gamma_L \Gamma_R^2 (\Gamma_R + 2\Gamma_L) \Delta}{(2\pi)^2 (\epsilon_d + V_-/2 + \Delta)^2 (V_- + \Delta)^2} \\ &\times \int_{-\infty}^{-V/2-\Delta} d\epsilon_1 g(\epsilon_1 + V/2) \int_{-\infty}^{-V/2-\Delta} d\epsilon_2 g(\epsilon_2 + V/2) \\ &\times \int_{-\infty}^{\infty} dE_1 [1 - f(E_1 - V/2)] \\ &\times \int_{-\infty}^{\infty} dE_2 \frac{\delta(\epsilon_1 + \epsilon_2 - E_1 - E_2)}{(\epsilon_d - E_2)^2}, \end{aligned} \quad (37)$$

is cast into the form

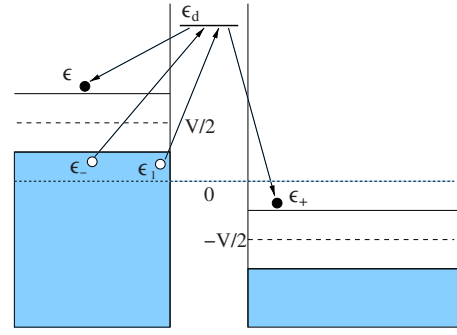


FIG. 7. (Color online) One of the possible inelastic channels in the S-S transport is illustrated schematically. Electron transfer is accompanied by creation of the excitation in the left lead.

$$\delta I_-^{2e}(V) = \frac{e}{\hbar} \frac{\Gamma_L \Gamma_R^2 (\Gamma_R + 2\Gamma_L) \Delta}{8\pi^2 (\epsilon_d + V_-/2 + \Delta)^2 (V_- + \Delta)^2} K_- \left[\frac{V_- - V}{T} \right]. \quad (38)$$

The dimensionless function $K_-(x)$ is a fourfold integral over the electron energies (in the units of T) in the initial and final states,

$$\begin{aligned} K_-(x) &= \int_0^{\infty} \frac{dz_1}{\sqrt{z_1}} \int_0^{\infty} \frac{dz_2}{\sqrt{z_2}} \int_{-\infty}^{\infty} \frac{dz_3}{1 + e^{-z_3}} \int_{-\infty}^{\infty} \frac{dz_4}{z_4^2} \\ &\times \delta \left[\frac{3}{2}x + z_1 + z_2 + z_3 + z_4 \right]. \end{aligned} \quad (39)$$

Three out of four integrations in Eq. (39), over z_1 , z_2 , and z_4 , can be carried out explicitly. Then, we get

$$\frac{d^2 K_-}{dx^2} = -\frac{9\pi^2}{4} \int_0^{\infty} ds \frac{s^2}{\sinh(\pi s)} \cos\left(\frac{3}{2}sx\right). \quad (40)$$

The second derivative, $d^2 K_- / dx^2$, is plotted in Fig. 6. It shows that $d^2 I_-^{2e} / dV^2$ also exhibits a fine structure at $(V_- - V) \sim T$. The low- T behavior of $d^2 I_-^{2e} / dV^2$ is even more singular than that of $d^2 I_+^{2e} / dV^2$. This follows from the large- x asymptote, $\propto 1/x^2$, of the integral Eq. (40). Thus, the below-threshold behavior of $d^2 I_-^{2e} / dV^2$ is $\propto 1/(V_- - V)^2$.

IV. ANOMALY IN THE SUPERCONDUCTOR-SUPERCONDUCTOR COTUNNELING

Energy diagram for transport between two superconducting leads via an LS is shown in Fig. 7 for bias $V > 2\Delta$. Similar to the case of normal and superconducting leads, electron cotunneling can be accompanied by the excitation of a quasiparticle across the gap. It is easy to see from Fig. 7 that the threshold bias for this process is

$$\tilde{V}_c = 4\Delta. \quad (41)$$

Clearly, at such large bias, the Andreev processes, i.e., multiple subgap Andreev reflections, are already irrelevant. The difference from the N-S case is that, at the threshold, electron tunnels from the edge of the gap rather than from the Fermi level of the metal. A more significant qualitative difference

from the N-S geometry is that a quasiparticle can be excited in *both* leads. Besides, as we will see below, the anomaly is stronger in the S-S than in the N-S case. This is due to the divergence of the density of states in both leads. More specifically, instead of the fourfold integral Eq. (13), the near-threshold expression for inelastic contribution to the current reads

$$\begin{aligned} \delta I^{in}(V) &= \frac{e}{\hbar} \frac{4\Gamma_R^3\Gamma_L}{\pi^3[(\epsilon_d + 3\Delta)(\epsilon_d^2 - \Delta^2)]^2} \int_{-\infty}^{V/2-\Delta} d\epsilon_1 g(\epsilon_1 - V/2) \\ &\times \int_{\Delta-V/2}^{\infty} d\epsilon g(\epsilon + V/2) \int_{\Delta-V/2}^{\infty} d\epsilon_+ g(\epsilon_+ + V/2) \\ &\times \int_{-\infty}^{-\Delta-V/2} d\epsilon_- g(\epsilon_- + V/2) \delta[\epsilon_1 + \epsilon_- - (\epsilon + \epsilon_+)]. \end{aligned} \quad (42)$$

Similar to Eq. (13), in order to calculate the integral Eq. (42), we introduce the same variables E , E_+ , and E_- as in Eq. (15), and also $\tilde{E}_1 = -(\epsilon_1 - V/2 + \Delta)$. Upon taking the near-gap asymptotes for the density of states, Eq. (42) assumes the form

$$\begin{aligned} \delta I^{in}(V) &= \frac{e}{\hbar} \frac{\Gamma_R^3\Gamma_L\Delta^2}{\pi^3[(\epsilon_d + 3\Delta)(\epsilon_d^2 - \Delta^2)]^2} \int_0^{\infty} \frac{d\tilde{E}_1}{\sqrt{\tilde{E}_1}} \int_0^{\infty} \frac{dE}{\sqrt{E}} \\ &\times \int_0^{\infty} \frac{dE_+}{\sqrt{E_+}} \int_0^{\infty} \frac{dE_-}{\sqrt{E_-}} \delta[V - \tilde{V}_c - (\tilde{E}_1 + E + E_+ + E_-)]. \end{aligned} \quad (43)$$

After rescaling all variables to $(V - \tilde{V}_c)$, this integral reduces to the surface area of a unit sphere in four dimensions, and we obtain

$$\delta I^{in}(V) = \frac{e}{\hbar} \frac{\Gamma_R^3\Gamma_L\Delta^2}{12\pi^2[(\epsilon_d + 3\Delta)(\epsilon_d^2 - \Delta^2)]^2} (V - \tilde{V}_c) \Theta[V - \tilde{V}_c]. \quad (44)$$

Contribution Eq. (44) describes cotunneling accompanied by excitation of a quasiparticle in the right lead. Similar calculation for inelastic channel, with excitation of a quasiparticle, as depicted in Fig. 7, results in

$$\delta I^{in}(V) = \frac{e}{\hbar} \frac{\Gamma_R\Gamma_L^3\Delta^2}{12\pi^2[(\epsilon_d - 3\Delta)(\epsilon_d^2 - \Delta^2)]^2} (V - \tilde{V}_c) \Theta[V - \tilde{V}_c]. \quad (45)$$

Here, we would like to emphasize that both calculations leading to Eqs. (44) and (45) take into account that quasiparticle can be created at the first as well as at the last step of the cotunneling process, and the corresponding *amplitudes interfere*, as in Eq. (14). Taking this interference into account results in the extra factor $\sim \Delta^2/\epsilon_d^2$ in Eqs. (44) and (45). Obviously, the threshold anomaly [Eq. (44)] in the current results in the jump in the V dependence of the differential conductance. Within a numerical factor and assuming $\epsilon_d \gg \Delta$, the magnitude of the jump can be presented as

$$\left(\frac{\delta G^{in}}{G^{el}} \Big|_{\tilde{V}_c^+} - \frac{\delta G^{in}}{G^{el}} \Big|_{\tilde{V}_c^-} \right) \sim \frac{(\Gamma_L^2 + \Gamma_R^2)\Delta^2}{\epsilon_d^4}. \quad (46)$$

Here, the sum $\Gamma_L^2 + \Gamma_R^2$ accounts for the contributions of the two channels of inelastic current, mentioned above. We note that the step Eq. (46) is abrupt; its temperature smearing is $\propto \exp(-\Delta/T)$ rather than $\sim T$, as in the case of tunneling between N and S leads.

Overall, the stability diagram for superconducting leads differs from Fig. 4 in two respects. First, the positions of the supergap anomalies are $\tilde{V}_c = \pm 4\Delta$. Second, the stability diagram is *symmetric* with respect to $V \rightarrow -V$. Namely, the boundaries of the two-electron ionization anomaly in this case are located at $V_c^* = \pm \frac{2}{3}(\epsilon_d + 3\Delta)$. Regarding the “strength” of two-electron anomaly, the threshold behavior of the differential conductance can be found from the calculation similar to Eqs. (22) and (27), and within a prefactor yields

$$\delta G^{in}(V) \propto (V - V_c^*)^{-1/2}, \quad (47)$$

i.e., the threshold behavior is more singular than Eq. (32). Again, the divergence of δG^{in} is limited by $(V - V_c^*) \sim \Gamma_{L,R}$ rather than by temperature.

V. CONCLUDING REMARKS

Let us list the assumptions adopted in the above consideration:

- (i) energy position ϵ_d of the LS is well outside the superconducting gap Δ ;
- (ii) on-site repulsion U is the largest energy scale, $U \gg |\epsilon_d|$; and
- (iii) the widths, Γ_L and Γ_R , are the smallest energy scales so that

$$\Gamma_L, \Gamma_R \ll \Delta \ll \epsilon_d \ll U. \quad (48)$$

One of the consequences of Eq. (48) is that the Kondo temperature, $T_K \propto \exp[-\pi|\epsilon_d|/2(\Gamma_L + \Gamma_R)]$, is much smaller than Δ . This means that the Kondo effect will not be developed fully but rather manifest itself as an enhancement $\propto \ln^{-2} \Delta/T_K$ of the conductance at small bias.

Recent experimental papers Refs. 12, 13, 15, 17, 21, and 22 are focused on the domain of parameters $T_K \sim \Delta$, where the two prominent regimes of transport compete with each other. This competition is due to the fact that antiparallel spins of electrons in the Cooper pairs cannot mediate the spin-flip processes that are responsible for the Kondo effect. Experimentally, in the case of normal leads, the Kondo effect manifests itself on the stability diagram in the (ϵ_d, V) plane as enhanced zero-bias conductance in the valley $\epsilon_d < 0$, where LS is occupied. It has no effect on the valley $\epsilon_d > 0$. On the other hand, with superconducting leads, conductance is suppressed in the entire domain of biases $V < 2\Delta$ in both valleys. A nontrivial result of interplay between the Kondo effect and superconductivity is that the peaks at $V = \pm \Delta$ emerge in the Kondo valleys, whereas the conventional peaks at $V = \pm 2\Delta$ are suppressed.^{12,13,21} This implies that the Andreev transport process is facilitated by the Kondo reso-

nance. Conversely, in the non-Kondo valleys, the peaks $V = \pm \Delta$ do not show up, while $V = \pm 2\Delta$ peaks are strong and exhibit a well-known threshold behavior, reflecting the BCS density of states.

In the present paper, we predict additional anomalies both outside the Kondo regime and above the gap. Nevertheless, the origin of the new anomalies is intimately related to the Kondo physics. To clarify this relation, we recall that, in a *bulk* metal with magnetic impurities the energy exchange between electrons is possible even without direct electron-electron interaction. This was first demonstrated by Kaminski and Glazman.⁴⁷ Obviously, such an exchange is impossible in the case of nonmagnetic impurities. The reason is that the mechanism, which is responsible for an impurity (LS) being magnetic, is a finite on-site repulsion U . As a result, the interaction between two electrons in metal, leading to the energy exchange, takes place when they *virtually* visit the LS. The energy exchange occurs between electrons with opposite spins, and in the case of magnetic impurity, involves spin flips.⁴⁷ Thus, the mechanism in Ref. 47 represents the most elementary manifestation of the Kondo physics and even does not require the presence of the Fermi sea.

As was demonstrated in Ref. 45, the mechanism⁴⁷ can be extended to the transport between two normal leads, which is coupled to the LS. Then, for two electrons tunneling between the leads, the magnitude of the energy exchange is limited by the applied bias V . This leads to the anomalies in conductance at $V = \pm 2\epsilon_d/3$. The main message of the present paper is that, in the case when one or both leads are superconducting, the gap 2Δ sets the threshold for inelastic process of one-electron transfer accompanied by a quasiparticle excitation in the superconducting lead. The ensuing anomalies at $V_c = \pm 3\Delta$ (for N and S leads) and at $\tilde{V}_c = \pm 4\Delta$ (for S-S leads) are independent of the gate voltage ϵ_d . The anomaly near $V = \tilde{V}_c$ is not smeared by temperature and manifests itself as a sharp peak in the second derivative $d^2I(V)/dV^2$. Although the papers on transport through Coulomb-blockaded dots report the data on first derivative, i.e., the differential conductance, $dI(V)/dV$, the second derivative was previously measured for single-electron transport through a molecule.⁴⁸ In Ref. 48, the second derivative was required to resolve a fine structure in the $I(V)$ dependence, which is related to the vibrational satellites.

As a final remark, we note that higher-order, in parameters, Γ_L/Δ and Γ_R/Δ , processes will lead to anomalies at even larger biases due to creation of more than one quasiparticle by a tunneling electron. For the case of the S-S leads, additional anomalies can be expected at biases $V_c^{(n)} = 2\Delta(2+n)$. The estimate for the behavior of inelastic current $(V - V_c^{(n)}) \ll \Delta$ can be easily found by extending the fourfold integral in Eq. (43) to higher n . This yields $\delta I_n^{in}(V) \propto (V - V_c^{(n)})^{n+1} \Theta(V - V_c^{(n)})$.

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APPENDIX

The fact that the superconductivity manifests itself in expression (2) for the tunneling rate $\Gamma(\epsilon_d)$ *only* through the density of states [Eq. (3)] is well known. However, it is not obvious that higher-order, in the tunnel matrix element γ , corrections to $\Gamma(\epsilon_d)$ can be expressed solely through $g(\epsilon)$ and do not contain coherence factors. Indeed, in our calculations, we treated the amplitude $A_{\epsilon_d, \epsilon_{\pm}}^{\epsilon, \epsilon_{\pm}}$ [Eq. (6)] as a number determined only by the energies,

$$\epsilon(\xi) = \pm \sqrt{\Delta^2 + \xi^2}, \quad (\text{A1})$$

of initial and final states *in superconductor* and ignored the fact that the real amplitude contains contributions of positive and negative bare energies ξ . This contributions enter into the amplitude with different weights, namely,

$$u(\xi) = \frac{1}{\sqrt{2}} \left[1 + \frac{\xi}{\sqrt{\Delta^2 + \xi^2}} \right]^{1/2} \quad (\text{A2})$$

for the upper branch in Eq. (A1) and

$$v(\xi) = \frac{1}{\sqrt{2}} \left[1 - \frac{\xi}{\sqrt{\Delta^2 + \xi^2}} \right]^{1/2} \quad (\text{A3})$$

for the lower branch in Eq. (A1). Then, when performing summation over states corresponding to, say, upper branch, one has to take into account contributions $\propto u(\xi)$ and $\propto u(-\xi)$ since they correspond to the same energy $\epsilon = \sqrt{\Delta^2 + \xi^2}$. Now, the fact that the main contribution $\Gamma(\epsilon_d)$ [Eq. (2)] to the lifetime does not contain coherence factors can be formally interpreted as a consequence of the identity $u^2(\xi) + u^2(-\xi) = 1$.

Turning to the third-order amplitude [Eq. (6)], the correct way to write one particular contribution to $A_{\epsilon_d, \epsilon_{\pm}}^{\epsilon, \epsilon_{\pm}}$ is

$$A_{\epsilon_d, \epsilon_{\pm}}^{\epsilon, \epsilon_{\pm}}(\xi_-, \xi, \xi_+) = \frac{\gamma^3 v(\xi_-) u(\xi) u(\xi_+)}{(\epsilon_d - \sqrt{\Delta^2 + \xi^2})(-\sqrt{\Delta^2 + \xi^2} - \sqrt{\Delta^2 + \xi^2})}. \quad (\text{A4})$$

Then, the correction $\delta\Gamma(\epsilon_d)$ is, actually, the sum of all possible contributions, i.e.,

$$\delta\Gamma(\epsilon_d) \propto \sum_{\substack{\xi_- > 0 \\ \xi_- < 0}} \sum_{\substack{\xi > 0 \\ \xi < 0}} \sum_{\substack{\xi_+ > 0 \\ \xi_+ < 0}} |A_{\epsilon_d, \epsilon_{\pm}}^{\epsilon, \epsilon_{\pm}}(\xi_-, \xi, \xi_+)|^2. \quad (\text{A5})$$

From Eq. (A5), it becomes apparent that coherence factors in the numerators of eight contributions can be combined into the product $[u^2(\xi_-) + u^2(-\xi_-)][u^2(\xi) + u^2(-\xi)][u^2(\xi_+) + u^2(-\xi_+)]$, which is an identical unity. Note, that this conclusion rests on the assumption that the matrix element γ is independent of ξ .

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