Luttinger liquid fixed point for a two-dimensional flat Fermi surface

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We consider a system of two-dimensional interacting fermions with a flat Fermi surface. The apparent conflict between Luttinger and non-Luttinger liquid behaviors found through different approximations is resolved by showing the existence of a line of nontrivial fixed points, for the renormalization group (RG) flow, corresponding to Luttinger liquid behavior; the presence of marginally relevant operators can cause flow away from the fixed point. The analysis is nonperturbative and based on the implementation, at each RG iteration, of Ward identities obtained from local phase transformations depending on the Fermi surface side, implying the partial vanishing of the beta function.

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I. INTRODUCTION

The properties of the two-dimensional interacting fermions are still largely unknown, despite the tremendous effort devoted to their understanding in the last years. One of the most debated questions is on the possible existence of a *Luttinger liquid phase*, which was first suggested by Anderson[1] as an explanation of some properties of high T_c superconductors, as observed also in recent experiments, see, e.g., Ref. 2.

It has been proved, in the case of symmetric, smooth, and convex Fermi surfaces (such as in the Jellium model[3] or in the Hubbard model in the non-half-filled case[4]), that the wave function renormalization Z is essentially temperature independent up to exponentially small temperatures. In a Luttinger liquid one expects instead a logarithmic behavior in this regime, i.e., $Z \simeq 1+O(U^2 \log \beta)$, so that such results rule out for sure the possibility of Luttinger liquid behavior.

On the contrary, the presence in the Fermi surface of flat regions can produce non-Fermi liquid behavior. The simplest model with a flat Fermi surface is the 2D Hubbard model at half filling, in which the Fermi surface is a square. It was proved in Ref. 5 that the wave function renormalization is $Z=1+O(U^2 \log^2 \beta)$ up to exponentially small temperatures; the presence of the $\log^2 \beta$ is a consequence of the Van Hove singularities, which are related to the fact that the Fermi velocity is vanishing at the corners of the squared Fermi surface and implies that also in this case there is not Luttinger liquid behavior.

It is important to stress that the results in Refs. 3–5 are rigorous as they are based on expansions, which are *convergent* provided that the temperature is not too low, the finite temperature acting as an infrared cutoffs; however, such expansions cannot give any information on the zero temperature properties.

A lot of attention has been devoted in recent years to the zero temperature properties of Fermi surfaces with flat regions and no corners, which share some features with the Fermi surfaces of some cuprates as seen in photoemission experiments. Parquet method results[6] and perturbative renormalization group (RG) analysis[7] truncated at one loop indicate that for repulsive interactions, there is no indication of a Luttinger liquid phase at zero temperature; the effective couplings flow toward a strong coupling regime that is related to the onset of d-wave superconductivity. In a more recent RG analysis that is truncated at two loops,[8] one still gets a flow to strong coupling, but in some intermediate region, some indication of Luttinger liquid behavior is found.

Apparently, conflicting results are found by applying bosonization: in Refs. 9 and 10, a model of electrons on a square Fermi surface was mapped in a collection of fermions on coupled chains, and it is found that the correlations at zero temperature in momentum space are similar to the ones of the Luttinger model. A related but somewhat different strategies consists in proposing an exactly solvable 2D analog of the Luttinger model; this approach was pursued in Refs. 11 and 12 and again Luttinger liquid behavior up to zero temperature was found.

A possible explanation of such conflicting results was suggested in Ref. 13, postulating the existence for the RG flow, in addition to the trivial fixed point associated to noninteracting fermions, of a *nontrivial* fixed point associated with Luttinger behavior, which could be made instable by the presence of marginally relevant operators. In this paper, we provide a quantitative verification of such hypothesis, explicitly showing the existence of a line nontrivial Luttinger fixed points for the RG flow of a system of 2D interacting fermions with a flat Fermi surface. It would be not possible to directly derive such result from the perturbative expansions, as it is related to cancellations between graphs to all orders of the expansion which are too complex to be seen explicitly; it is well known that even in one dimension, Ward identities (WI) are necessary to prove the existence of a Luttinger liquid fixed point.[14] Our analysis is based on the implementation, in an exact RG approach, of WI with corrections due the the cutoffs introduced in the multiscale analysis, extending a technique already used to establish Luttinger liquid behavior in a large class of 1D fermionic systems[15,16] or 2D spin systems.[17] Such methods are the only ones, which can be applied to *nonexactly solvable* models, such as the model analyzed in this paper.

II. MODEL

We consider a model with a square Fermi surface similar to the one considered in Refs. 6, 8, and 9; the Schwinger

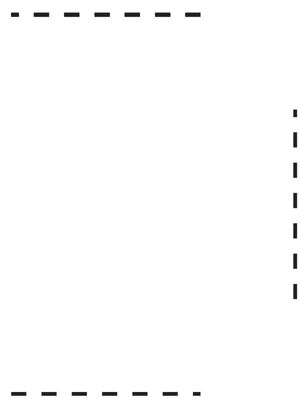

FIG. 1. The Fermi surface corresponding to the singularities of $g_{\mathbf{k}}$; the four sides are labeled by $(\sigma,\omega)=(\pm,\pm)$.

functions are given by functional derivatives of the generating functional,

$$e^{\mathcal{W}(\phi)}=\int P(d\psi)\exp\{\mathcal{V}(\psi)+\int d\mathbf{x}[\psi^+_{\mathbf{x}}\phi^-_{\mathbf{x}}+\psi^-_{\mathbf{x}}\phi^+_{\mathbf{x}}]\},\quad (1)$$

with $\psi^{\pm}_{\mathbf{k}}$ are Grassmann variables, $\mathbf{k}=(k_-,k_+,k_0)$, $k_\pm=\frac{2\pi}{L_\pm}n_\pm$, $k_0=\frac{2\pi}{\beta}(n_0+\frac{1}{2})$, $n_\pm,n_0=0,\pm1,\pm2,\ldots$, and $P(d\psi)$ is the fermionic integration with propagator

$$g_{\mathbf{k}}=\sum_{\sigma=\pm}\sum_{\omega=\pm}\frac{H(k_{-\sigma})C_0^{-1}\{\sqrt{a_0^{-2}[k_0^2+v_F^2(|k_\sigma|-p_F)^2]}\}}{-ik_0+v_F(k_\sigma-\omega p_F)}$$
$$\equiv\sum_{\sigma,\omega=\pm}g_{\sigma,\omega,\mathbf{k}},\quad (2)$$

$H(k_{-\sigma})=\chi(a^{-2}k^2_{-\sigma})$, $\chi(t)=1$ if $t<1$, and 0 otherwise, $C_0^{-1}(t)$ is a smooth compact support function=1 for $t<1$ and=0 for $t\geq\gamma$, $\gamma>1$. We assume, for definiteness, $a\leq\frac{p_F}{4}$ and $a_0\leq\frac{p_F}{20}$ so that the support of $g_{\mathbf{k}}$ is over four disconnected regions; the Fermi surface is defined as the set of momenta in which $g_{\mathbf{k}}$ for $k_0=0$ is singular in the limit $\beta\to\infty$ (Fig. 1).

By using well known properties of Grassmann integrals [see Ref. 18], Eq. (6) allows us to write the Grassmann field as a sum of independent fields,

$$\psi^{\pm}_{\mathbf{k}}=\sum_{\sigma=\pm}\sum_{\omega=\pm}\psi^{\pm}_{\omega,\sigma,\mathbf{k}},\quad (3)$$

with $\psi^{\pm}_{\omega,\sigma,\mathbf{x}}$ independent Grassmann variables with propagator $g_{\omega,\sigma,\mathbf{k}}$. As in Refs. 6, 8, and 9, we can consider only interactions between parallel patches ($V=L_+L_-$),

$$\mathcal{V}=\sum_{\sigma}\sum_{\omega}\frac{1}{(V\beta)^4}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4}U\hat{v}(\mathbf{k}_1-\mathbf{k}_2)$$
$$\times\psi^+_{\omega_1,\sigma,\mathbf{k}_1}\psi^-_{\omega_2,\sigma,\mathbf{k}_2}\psi^+_{\omega_3,\sigma,\mathbf{k}_3}\psi^-_{\omega_4,\sigma,\mathbf{k}_4}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4),\quad (4)$$

with $v(\mathbf{x})$ a short range potential. The two-point Schwinger function is given by

$$S_2(\mathbf{x},\mathbf{y})=\left.\frac{\partial^2\mathcal{W}(\phi)}{\partial\phi_{\mathbf{x}}\partial\phi_{\mathbf{y}}}\right|_{\phi=0}.\quad (5)$$

III. RENORMALIZATION GROUP ANALYSIS

As the interaction does not couple different σ, we can from now on fix $\sigma=+$ for definiteness and forget the index σ. We analyze the functional integral (1) by performing a multiscale analysis by using the methods of constructive quantum field theory (for a general introduction to such methods, see Ref. 18). The propagator (2) can be written as sum of "single slice" propagators in the following way:

$$g_\omega(\mathbf{x}-\mathbf{y})=\sum_{h=-\infty}^{0}e^{i\omega p_F(x_+-y_+)}g^{(h)}_\omega(\mathbf{x}-\mathbf{y}),\quad (6)$$

where

$$g^{(h)}_\omega(\mathbf{x}-\mathbf{y})=\frac{1}{V\beta}\sum_{\mathbf{k}}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{H(k_-)f_h(\mathbf{k})}{-ik_0+\omega v_Fk_+},\quad (7)$$

and $f_h(\mathbf{k})$ has support in a region $O(\gamma^h)$ around each flat side of the Fermi surface, at a distance $O(\gamma^h)$ from it, that is $a_0\gamma^{h-1}\leq\sqrt{k_0^2+v_F^2k_+^2}\leq a_0\gamma^{h+1}$; note that in each term in Eq. (6) the change of variables $k_+\to k_++\omega p_F$ has been performed. The single scale propagator verify the following bound for any integer M,

$$|g^{(h)}_\omega(\mathbf{x})|\leq C_M\left|\frac{\sin ax_-}{x_-}\right|\frac{\gamma^h}{1+[\gamma^h(|x_+|+|x_0|)]^M}.\quad (8)$$

The integration is done iteratively integrating out the fields with momenta closer and closer to the Fermi surface, renormalizing at each step the wave function. After the integration of the fields $\psi^{(0)},\ldots,\psi^{(h+1)}$, we obtain

$$\int P_{Z_h}(d\psi^{(\leq h)})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})},\quad (9)$$

where $P_{Z_h}(d\psi^{(\leq h)})$ is the fermionic integration with propagator $Z_h^{-1}(\mathbf{k})g^{(\leq h)}_{\omega,\mathbf{k}}$, with $g^{(\leq h)}_{\omega,\mathbf{k}}=\sum_{k=-\infty}^{h}g^{(k)}_{\omega,\mathbf{k}}$ and Z_h is iteratively defined starting from $Z_0=1$; moreover, if $\vec{p}_F=(0,p_F,0)$,

$$\mathcal{V}^{(h)}(\psi^{\leq h})=\sum_{n=1}^{\infty}\sum_{\underline{\omega}}\frac{1}{(\beta V)^{2n}}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_{2n}}\delta\left[\sum_i\varepsilon_i(\mathbf{k}_i+\omega_i\vec{p}_F)\right]$$
$$\left[\prod_{i=1}^{2n}\hat{\psi}^{(\leq h)\varepsilon_i}_{\omega_i,\mathbf{k}_i}\right]\hat{W}^{(h)}_{2n}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1}).\quad (10)$$

By using that $\int d\mathbf{k}|g^{(k)}_{\omega,\sigma,\mathbf{k}}|\leq C\gamma^k$ and $|g^{(k)}_{\omega,\sigma,\mathbf{k}}|\leq C\gamma^{-k}$, we see that the kernels $\hat{W}^{(k)}_{2n}$ are $O(\gamma^{-k(n-2)})$; this means that the terms quadratic in the fields have positive scaling dimension and the quartic terms have vanishing scaling dimension, and all the other terms have negative dimension; we have then to properly renormalize the terms with non-negative dimension.

By calling $\bar{\mathbf{k}}=(k_-,0,0)$, we define an $\mathcal{L}$ operator acting linearly on the kernels of the effective potential,

(1) $\mathcal{L}\hat{W}^{(h)}_{2n}=0$ if $n\geq2$.
(2) If $n=1$,

$$\mathcal{L}\hat{W}^h_2(\mathbf{k})=\hat{W}^h_2(\bar{\mathbf{k}})+k_0\partial_{k_0}\hat{W}^h_2(\bar{\mathbf{k}})+k_+\partial_+\hat{W}^h_2(\bar{\mathbf{k}}).\quad (11)$$

(3) If $n=2$,

$$\mathcal{L}\hat{W}_4^h(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\delta_{\Sigma_i\varepsilon_i\omega_i,0}\hat{W}_4^h(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3). \qquad (12)$$

By calling $\partial_0\hat{W}_2^h(\bar{\mathbf{k}})=-iz_h(k_-)$, $\partial_+\hat{W}_2^h(\bar{\mathbf{k}})=\omega z_h(k_-)$ (symmetry considerations are used), and $l_h(k_{-,1},k_{-,2},k_{-,3})=\hat{W}_4^h(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3)$, we obtain

$$\begin{aligned}\mathcal{L}\mathcal{V}^h=&\frac{1}{\beta V}\sum_{\mathbf{k}}[z_h(k_-)\omega k_+-ik_0z_h(k_-)]\hat{\psi}^{+(\le h)}_{\mathbf{k},\omega}\hat{\psi}^{-(\le h)}_{\mathbf{k},\omega}\\&+\sum^*_{\underline{\omega},\underline{\sigma}}\frac{1}{(\beta V)^4}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4}l_h(k_{-,1},k_{-,2},k_{-,3})\hat{\psi}^{+(\le h)}_{\mathbf{k}_1,\omega_1}\hat{\psi}^{-(\le h)}_{\mathbf{k}_2,\omega_2}\hat{\psi}^{+(\le h)}_{\mathbf{k}_3,\omega_3}\\&\times\hat{\psi}^{-(\le h)}_{\mathbf{k}_4,\omega_4}\delta\left(\sum_i\varepsilon_i\mathbf{k}_i\right),\end{aligned} \qquad (13)$$

where $\Sigma^*_{\underline{\omega}}$ is constrained to the condition $\Sigma_i\varepsilon_i\omega_i\vec{p}_F=0$ and we have used that, by symmetry, $W_2^h(\bar{k})=0$.

We write Eq. (9) as

$$\int P_{Z_h}(d\psi^{(\le h)})e^{-\mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}, \qquad (14)$$

with $\mathcal{R}=1-\mathcal{L}$. The nontrivial action of $\mathcal{R}$ on the kernel with $n=2$ can be written as

$$\begin{aligned}\mathcal{R}\hat{W}_4^h(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=&[\hat{W}_4^h(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)-\hat{W}_4^h(\bar{\mathbf{k}}_1,\mathbf{k}_2,\mathbf{k}_3)]\\&+[\hat{W}_4^h(\bar{\mathbf{k}}_1,\mathbf{k}_2,\mathbf{k}_3)-\hat{W}_4^h(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\mathbf{k}_3)]\\&+[\hat{W}_4^h(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\mathbf{k}_3)-\hat{W}_4^h(\bar{\mathbf{k}}_1,\bar{\mathbf{k}}_2,\bar{\mathbf{k}}_3)].\end{aligned} \qquad (15)$$

The first addend can be written as

$$k_{0,1}\int_0^1 dt\partial_{k_{0,1}}\hat{W}_4^h(k_{-,1},k_{+,1},tk_{0,1};\mathbf{k}_2,\mathbf{k}_3)+k_{+,1}\times\int_0^1 dt\partial_{k_{+,1}}\hat{W}_4^h(k_{-,1},tk_{+,1},0;\mathbf{k}_2,\mathbf{k}_3). \qquad (16)$$

The factors $k_{0,1}$ and $k_{+,1}$ are $O(\gamma^{h'})$ for the compact support properties of the propagator associated with $\hat{\psi}^{+(\le h)}_{\omega_1,\mathbf{k}_1}$, with $h'\le h$, while the derivatives are dimensionally $O(\gamma^{-h-1})$; hence, the effect of $\mathcal{R}$ is to produce a factor $\gamma^{h'-h-1}<1$, making its scaling dimension negative. Similar considerations can be done for the action of $\mathcal{R}$ on the $n=1$ terms. The effect of the $\mathcal{L}$ operation is to replace in $W_2^h(\mathbf{k})$ the momentum $\vec{k}$ with its projection on the closest flat side of the Fermi surface. Hence, the fact that the propagator is singular over an extended region (the Fermi surface) and not simply in a point has the effect that the renormalization point cannot be fixed but it must be left moving on the Fermi surface.

In order to integrate the field $\psi^{(h)}$ we can write Eq. (14) as

$$\int P_{Z_{h-1}}(d\psi^{(\le h)})e^{-\mathcal{L}\tilde{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}, \qquad (17)$$

where $P_{Z_{h-1}}(d\psi^{(\le h)})$ is the fermionic integration with propagator,

$$\frac{1}{Z_{h-1}(\mathbf{k})}\frac{H(k_-)C_h^{-1}(\mathbf{k})}{-ik_0+\omega v_Fk_+}, \qquad (18)$$

with $C_h^{-1}(\mathbf{k})=\Sigma_{k=-\infty}^h f_k$ and

$$Z_{h-1}(\mathbf{k})=Z_h(k_-)[1+H(k_-)C_h^{-1}(\mathbf{k})z_h(k_-)]. \qquad (19)$$

Moreover, $\mathcal{L}\tilde{\mathcal{V}}^h$ is the second term in Eq. (13).

We rescale the fields by rewriting the right hand side of Eq. (14) as

$$\int P_{Z_{h-1}}(d\psi^{(\le h)})e^{-\mathcal{L}\hat{\mathcal{V}}^h(\sqrt{Z_{h-1}}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}, \qquad (20)$$

where

$$\begin{aligned}\mathcal{L}\hat{\mathcal{V}}^h(\psi)=&\sum^*_{\underline{\omega}}\frac{1}{(\beta V)^4}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4}g_h(k_{-,1},k_{-,2},k_{-,3})\\&\times\hat{\psi}^+_{\mathbf{k}_1,\omega_1}\hat{\psi}^-_{\mathbf{k}_2,\omega_2}\hat{\psi}^+_{\mathbf{k}_3,\omega_3}\hat{\psi}^-_{\mathbf{k}_4,\omega_4}\delta\left(\sum_i\varepsilon_i\mathbf{k}_i\right),\end{aligned} \qquad (21)$$

and the *effective couplings*,

$$g_h(k_{-,1},k_{-,2},k_{-,3})=\left[\prod_{i=1}^4\sqrt{\frac{Z_h(k_{-,i})}{Z_{h-1}(k_{-,i})}}\right]l_h(k_{-,1},k_{-,2},k_{-,3}). \qquad (22)$$

After the integrations of the fields $\psi^{(0)},\psi^{(-1)},\ldots,\psi^{(h)}$, we get an effective theory describing fermions with wave function renormalization Z_h and effective interaction given by Eq. (21). Note that Z_h and g_h are nontrivial functions of the momentum parallel to the Fermi surface.

We write

$$\int P_{Z_{h-1}}(d\psi^{(\le h-1)})\int P_{Z_{h-1}}(d\psi^{(h)})\times e^{-\mathcal{L}\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})-\mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}, \qquad (23)$$

and the propagator of $P_{Z_{h-1}}(d\psi)$ is

$$\hat{g}^h_{\omega,\sigma}(\mathbf{k})=H(k_-)\frac{1}{Z_{h-1}(k_-)}\frac{\tilde{f}_h(\mathbf{k})}{-ik_0+\omega v_Fk_+},$$

and

$$\tilde{f}_h(\mathbf{k})=Z_{h-1}(k_-)\left[\frac{C_h^{-1}(\mathbf{k})}{Z_{h-1}(\mathbf{k})}-\frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}(k_-)}\right], \qquad (24)$$

with $H(k_-)\tilde{f}_h(\mathbf{k})$ having the same support that $H(k_-)f_h(\mathbf{k})$. We integrate then the field $\psi^{(h)}$ and we get

$$\int P_{Z_{h-1}}(d\psi^{(\le h-1)})e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})}, \qquad (25)$$

and the procedure can be iterated.

The above procedure allows us to write $W_{2n}^{(h)}$ as a series in the effective couplings g_k, $k\ge h$, which is *convergent* (see Ref. 18), provided that L_- is finite and $\varepsilon_h=\sup_{k\ge h}\|g_k\|$ small enough; moreover, $\|W_{2n}^{(h)}\|=O(\gamma^{-h(n-2)})$. A similar analysis can be repeated for the two-point function.

However, even if the couplings g_k starts with small values, they can possibly increase iterating the RG and at the end reach the boundary of the (estimated) convergence domain; if this happen, all the above procedure looses its consistency. A finite temperature acts as an infrared cutoff saying that the RG has to be iterated up to a maximum scale $h_\beta = O(\log \beta)$ and, up to exponentially small temperatures, i.e., $\beta \le O(e^{\kappa |U|^{-1}})$, then surely the effective couplings are in the convergence domain; however, in order to get lower temperatures, more information on the effective couplings are necessary.

IV. RENORMALIZATION GROUP FLOW AND LUTTINGER LIQUID FIXED POINT

The RG analysis seen in the previous section implies that the effective coupling g_h verifing a flow equation of the form

$$g_{h-1} = g_h + \beta_g^{(h)}(g_h; \dots ; g_0), \tag{26}$$

where the right hand side of the above equation is called *beta function*, which is expressed by a convergent expansion in the couplings if ε_h is small enough. The first nontrivial contribution to $\beta_g^{(h)}$, which is called $\beta_g^{(2)(h)}$, is quadratic in the couplings and it is given by

$$\beta_g^{(2)(h)} = \beta_h^{(a)} + \beta_h^{(b)}, \tag{27}$$

where

$$\beta_h^{(a)} = \int d\mathbf{p} H(k_{1,-} - p_-) H(k_{3,-} + p_-) g_h(k_{1,-}, k_{1,-} - p_-, k_{3,-}) \times g_h(k_{1,-} - p_-, k_{2,-}, k_{3,-} + p_-) \frac{f_h(\mathbf{p}) C_h(\mathbf{p})}{p_0^2 + v_F^2 p_+^2}, \tag{28}$$

$$\beta_h^{(b)} = -\int d\mathbf{p} H(k_{2,-} - p_-) H(k_{3,-} - p_-) \times g_h(k_{1,-}, k_{2,-} - p_-, k_{3,-} - p_-) \times g_h(k_{2,-} - p_-, k_{2,-}, k_{3,-}) \frac{f_h(\mathbf{p}) C_h(\mathbf{p})}{p_0^2 + v_F^2 p_+^2}. \tag{29}$$

The above expression essentially coincides with the one found in Refs. 6 and 8; it is indeed well known that the lowest order contributions to the beta function are essentially independent by RG procedure one follows.

The flow equation (26) encodes most of the physical properties of the model, but its analysis is extremely complex. Some insights can be obtained by truncating the beta function at second order and by the numerical analysis of the resulting flow by discretization of the Fermi surface; it is found (see Refs. 6 and 8) that $g_h(k_{-,1}, k_{-,2}, k_{-,3})$ has a flow which, for certain values of $k_{-,1}, k_{-,2}, k_{-,3}$ increases and reach the estimated domain of convergence of the series for $W_n^{(k)}$. While this increasing can be interpreted as a sign of instability, mathematically speaking this means that the truncation procedure becomes inconsistent.

A basic question is about the fixed points of the flow equation (26); in particular, if there is in addition to the trivial fixed point $g_h = 0$, a nontrivial fixed point corresponding to Luttinger liquid behavior. Note first that the set

$$g_h(k_{-,1}, k_{-,2}, k_{-,3}) = \frac{1}{L_-} \delta(k_{1,-} - k_{2,-}) \delta_{\omega_1,\omega_2} \delta_{\omega_3,\omega_4} \lambda_h, \tag{30}$$

with λ_h constant in $\mathbf{k}$, is invariant under the RG flow, in the sense that if $\mathcal{L}\mathcal{V}^{(k)}$ has the form, for $k \ge h$

$$\frac{1}{(\beta V)^3} \sum_{\substack{\mathbf{k},\mathbf{k}',\mathbf{p} \\ \omega,\omega'}} \lambda_h \delta_{p_-,0} \hat{\psi}^+_{\omega,\mathbf{k}} \hat{\psi}^-_{\omega,\mathbf{k}+\mathbf{p}} \hat{\psi}^+_{\omega',\mathbf{k}'} \hat{\psi}^-_{\omega',\mathbf{k}'-\mathbf{p}}, \tag{31}$$

the same is true for $\mathcal{L}\mathcal{V}^{(h-1)}$. This can be checked by the graph expansion. In the graphs contributing to $W_4^{(h)}$, the external lines either come out from a single point or are connected by a chain of propagators with the same ω, k_-. Moreover, in each Feynman graph, the only dependence from the momenta of the external lines is through the function $H(k_-)$, which are 1 in the support of the external fields $\int d\mathbf{k} H(k_-) \psi^{\pm}_{\mathbf{k}} = \int d\mathbf{k} \psi^{\pm}_{\mathbf{k}}$. For the same reasons, Z_h is also independent of k_-.

The crucial point is that in the set given by Eq. (30), some dramatic cancellation are present, implying the following *asymptotic vanishing of the beta function* (which will be proved in the subsequent sections),

$$\beta_g^{(h)} = O(\gamma^h \varepsilon_h^2), \tag{32}$$

saying that there is a *cancellations* between the graphs with four external lines and the graphs with two lines contributing to the square of Z_h [see Eq. (22)]; such graphs are $O(1)$ but there are cancellations making the size of the sum of them $O(\gamma^h)$. At the second order, Eq. (32) can be verified from Eqs. (29) and (30); at third order, it is compatible with Eqs. (A16), (A15), and (4.8) of Ref. 8.

The validity of Eq. (32) immediately implies the existence of a line of nontrivial fixed points for Eq. (26) of the form

$$g_{-\infty}(k_{-,1}, k_{-,2}, k_{-,3}) = \frac{1}{L_-} \delta(k_{1,-} - k_{2,-}) \delta_{\omega_1,\omega_2} \delta_{\omega_3,\omega_4} \lambda_{-\infty}, \tag{33}$$

with $\lambda_{-\infty}$ $\mathbf{k}$ independent and continuous function of U, $\lambda_{-\infty} = \lambda_0 + O(U^2)$ and $\lambda_0 = cU$ for a suitable constant c.

Note also that to such fixed point is associated Luttinger liquid behavior, as from Eq. (19) $Z_h \simeq \gamma^{2\eta h}$, with $\eta = a\lambda_0^2 + O(U^3)$ and $\bar{\mathbf{p}} = (0, \bar{p}_+, \bar{p}_0)$,

$$a = \lim_{h \to -\infty} \sum_{\omega'} \frac{\lambda_0^2}{L_-} \frac{1}{h} \int d\mathbf{k}' d\bar{p}_+ d\bar{p}_0 \frac{H(k'_-) C_h(\mathbf{k}')}{-ik'_0 + \omega' v_F k'_+} \times \frac{H(k'_-) C_h(\mathbf{k}' + \bar{\mathbf{p}})}{-i(k'_0 + \bar{p}_0) + \omega' v_F (k'_+ + \bar{p}_+)} \frac{\partial}{\partial k_+} \times \left. \frac{H(k_-) C_h^{-1}(\mathbf{k} - \bar{\mathbf{p}})}{-i(k_0 - \bar{p}_0) + \omega v_F (k_+ - \bar{p}_+)} \right|_{k_0 = k_+ = 0}. \tag{34}$$

Indeed, the two-point Schwinger function can be written as

$$S_2(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{i\omega p_F(x_+-y_+)}\frac{1}{V\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{g_\omega^{(h)}(\mathbf{k})}{Z_h}[1+A^{(h)}(\mathbf{k})], \tag{35}$$

with $A^{(h)}(\mathbf{k})=O(\varepsilon_h)$ so that

$$S_2(\mathbf{x},\mathbf{y}) = \sum_{\omega=\pm} e^{i\omega p_F(x_+-y_+)} \times \frac{1}{V\beta}\sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{H(k_-)C_0^{-1}(\mathbf{k})}{-ik_0+\omega v_F k_+}\frac{1+A(\mathbf{k})}{|k_0^2+v_F^2k_+^2|^\eta}, \tag{36}$$

with $|A(\mathbf{k})|\leq C|U|$. This means that to the fixed point is associated Luttinger liquid behavior, as the wave function renormalization vanishes at the Fermi surface as a powerlike with a nonuniversal critical index; the Luttinger liquid behavior is found only if L_- is finite. Note also that the cancellation in Eq. (32) reduce to the one in 1D if $k_-=k_-'$ in Eq. (31).

V. AUXILIARY MODEL

There is essentially no hope of proving a property such as Eq. (32) directly from the graph expansion, as the algebra of the graphs is too cumbersome (except than at one loop in which it is easy to check). We will follow instead the same strategy for proving the asymptotic vanishing of the Beta function in 1D followed in Refs. 15 and 16, considering an *auxiliary model* with the same beta function, up to irrelevant terms, but verifying extra symmetries, from which a set of Ward identities can be derived. In the present case, such identities are related to the invariance under local phase transformations depending on the Fermi surface side, which in the model (1) is broken by the cutoff function $C_0^{-1}(\mathbf{k})$.

We consider an *auxiliary model* whose generating function is

$$\int \mathcal{D}\psi \exp\left\{\sum_{\omega=\pm}\int d\mathbf{k}H^{-1}(k_-)C_{h,N}(\mathbf{k})(-ik_0+\omega k_+v_F)\hat{\psi}^+_{\omega,\mathbf{k}}\hat{\psi}^-_{\omega,\mathbf{k}} + \bar{\mathcal{V}}(\psi) + \sum_{\omega,\varepsilon=\pm}\int d\mathbf{x}\psi^\varepsilon_{\omega,\mathbf{x}}\phi^{-\varepsilon}_{\omega,\mathbf{x}} + \sum_{\omega=\pm}\int d\mathbf{x}J_{\mathbf{x},\omega}\rho_{\omega,\mathbf{x}}\right\}, \tag{37}$$

with $C_{h,N}^{-1}(\mathbf{k})=\sum_{k=h}^N f_k(k_0^2+v_F^2k_+^2)$, $h\leq 0$, $\rho_{\omega,\mathbf{x}}=\psi^+_{\omega,\mathbf{x}}\psi_{\omega,\mathbf{x}}$,

$$\bar{\mathcal{V}} = \frac{U}{L_-}\sum_{\omega,\omega'}\int d\mathbf{x}d\mathbf{y}v(x_0-y_0,x_+-y_+)\psi^+_{\omega,\mathbf{x}}\psi^-_{\omega,\mathbf{x}}\psi^+_{\omega',\mathbf{y}}\psi^-_{\omega',\mathbf{y}}$$
$$= \frac{U}{L_-}\frac{1}{(\beta V)^2}\sum_{\mathbf{k},\mathbf{k}'}\frac{1}{\beta L_+}\sum_{p_0,p_+}\hat{v}(p_0,p_+) \times \hat{\psi}^+_{\omega,\mathbf{k}}\hat{\psi}^-_{\omega,\mathbf{k}+(0,p_+,p_0)}\hat{\psi}^+_{\omega',\mathbf{k}'}\hat{\psi}^-_{\omega',\mathbf{k}'-(0,p_+,p_0)}, \tag{38}$$

with $v(x_0,x_+)$ a short range interaction. The above functional integral is very similar to the previous one, with the difference that there is an ultraviolet cutoff γ^N on the + variables, which will be removed at the end; such features are present

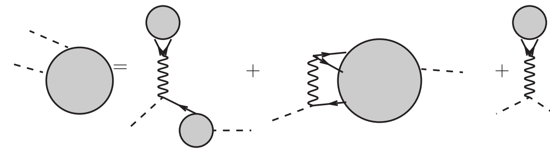

FIG. 2. Graphical representation of Eq. (39); the blobs represent $W^{(k)}_{n,m}$ and the wiggly lines represent v the lines $g^{(k,N)}$.

also in the models introduced in Refs. 9 and 12.

Again, Eq. (37) can be analyzed by a multiscale integration based on a decomposition similar to Eq. (6), with the difference that the scale are from h to N. In the integration of the scales between N and 0, the *ultraviolet scales*, there is no need of renormalization; apparently, the terms with two or four external lines have positive or vanishing dimension but one can use the non locality of the interaction to improve their scaling dimension. We integrate (with $\mathcal{L}=0$) the fields $\psi^{(N)},\psi^{(N-1)},\ldots,\psi^{(k)}$ and we call $W^{(k)}_{2n,m}$ the kernels in the effective potential multiplying $2n$ fermionic fields and a number m of fields of type J. Again, the dimension is $\gamma^{-k(n+m-2)}$, $k\geq 0$, and we have to improve the bounds by using the nonlocality of the interaction. We can write

$$W^{(k)}_{2,0}(\mathbf{x},\mathbf{y}) = \int d\mathbf{y}_1 U\frac{v(x_0-y_{1,0},x_+-y_{1,+})}{L_-} \times W^{(k)}_{0,1}(\mathbf{y}_1)g^{(k,N)}(\mathbf{x}-\mathbf{y}_2)W^{(k)}_{2,0}(\mathbf{y}_2;\mathbf{y}) + U\int d\mathbf{y}_2\frac{v(x_0-y_{0,1},x_+-y_{+,1})}{L_-}g^{(k,N)}(\mathbf{x}-\mathbf{y}_2) \times W^{(k)}_{2,1}(\mathbf{y},\mathbf{y}_2;\mathbf{y}_1) + U\delta(\mathbf{x}-\mathbf{y}) \times \int d\mathbf{y}_1\frac{v(x_0-y_{1,0},x_+-y_{1,+})}{L_-}W^{(k)}_{0,1}(\mathbf{y}_1). \tag{39}$$

The first and the third addend of Fig. 2 are vanishing by the symmetry $g(k_0,k_+,k_-)=-g(-k_0,-k_+,k_-)$; hence, by using that $\|g^{(j)}\|_1\leq\tilde{C}\gamma^{-j}$ and that $W^{(k)}_{2,1}$ is $O(U)$ (by induction), we obtain the following bound:

$$\|W^{(k)}_{2,0}\| \leq C\frac{|U|}{L_-}\|W^{(k)}_{2,1}\|\cdot\sum_{j=k}^N\|g^{(j)}\|_1 \leq C|U|\gamma^k L_-^{-1}\gamma^{-2k}. \tag{40}$$

Note that we have a gain $O(L_-^{-1}\gamma^{-2k})$ due to the fact that we are integrating over a fermionic instead than over a bosonic line. Similar arguments can be repeated for $W^{(k)}_{0,2}$, which can be decomposed as in Fig 3.

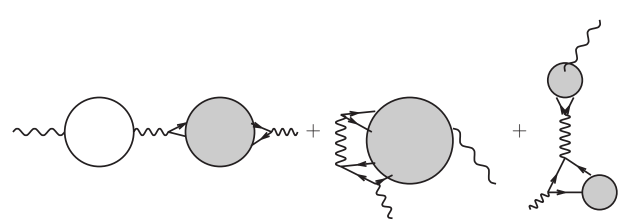

FIG. 3. Decomposition of $W^{(k)}_{2,0}$.

The second term in the figure is bounded by $O(|U|L_-^{-1}\gamma^{-2k})$. A similar bound is found for the third term in Fig. 3; regarding the first term, we can rewrite it as

$$\int d\mathbf{x}d\bar{\mathbf{z}}[g^{(k,N)}(\mathbf{z}-\mathbf{x})]^2\frac{U}{L_-}v(x_0-\bar{z}_0,x_+-\bar{z}_+)W_{0,2}^{(k)}(\bar{\mathbf{z}},\mathbf{y})$$
$$=\int d\mathbf{x}d\bar{\mathbf{z}}U\frac{v(\bar{z}_0-z_0,\bar{z}_+-z_+)}{L_-}[g^{(k,N)}(\mathbf{x}-\mathbf{z})]^2W_{0,2}^{(k)}(\bar{\mathbf{z}},\mathbf{y})$$
$$+\int d\mathbf{x}d\bar{\mathbf{z}}\frac{U}{L_-}[v(\bar{z}_0-x_0,\bar{z}_+-x_+)-v(\bar{z}_0-z_0,\bar{z}_+-z_+)]$$
$$\times[g^{(k,N)}(\mathbf{x}-\mathbf{z})]^2W_{0,2}^{(k)}(\bar{\mathbf{z}},y_0), \tag{41}$$

and using that

$$\int d\mathbf{x}[g^{(k,N)}(\mathbf{x}-\mathbf{z})]^2=\int dk_-H(k_-)\int dk_0dk_+\frac{C_{k,N}^{-2}(\mathbf{k})}{(-ik_0+k_+)^2}=0, \tag{42}$$

the first addend is vanishing; the second addend, by using the interpolation formula for $v(\bar{z}_0-x_0,\bar{z}_+-x_+)-v(\bar{z}_0-z_0,\bar{z}_+-z_+)$, can be bounded by $C|U|\gamma^{-k}$, as by induction $\|W_{0,2}^{(k)}\|\le C|U|$. A similar analysis proves the bound for $W_{4,0}^{(k)}$.

After the integration of the fields $\psi^{(N)},\psi^{(N-1)},\ldots,\psi^{(-1)}$, we get a Grassmann integral very similar to Eq. (1); the integration of the remaining fields $\psi^{(0)},\psi^{(-1)},\ldots$ is done following the same procedure as in Sec. III, with the effective coupling of form (30) and $\mathcal{L}\mathcal{V}^{(k)}$ of form (31). The crucial point is that the beta function *coincides* with the beta function for model (1) up to $O(\gamma^h)$ terms; hence, it is enough to prove the validity of (32) in the auxiliary model.

VI. WARD IDENTITIES

We derive now a set of Ward identities, relating the Schwinger functions of the auxiliary model (37); by performing the change of variables,

$$\psi_{\omega,\mathbf{x}}^{\pm}\to e^{\pm i\alpha_{\omega,\mathbf{x}}}\psi_{\omega,\mathbf{x}}^{\pm}, \tag{43}$$

and making a derivative with respect to $\alpha_{\mathbf{x},\omega}$ and to the external fields, we obtain

$$\int d\mathbf{k}'[H^{-1}(k'_-+p_-)C_{h,N}(\mathbf{k}'+\mathbf{p})[-i(k'_0+p_0)+\omega v_F(k'_++p_+)]$$
$$-H^{-1}(k'_-)C_{h,N}(\mathbf{k}')(-ik'_0+\omega v_Fk'_+)]$$
$$\times\langle\hat{\psi}^+_{\omega,\mathbf{k}'+\mathbf{p}}\hat{\psi}^-_{\omega,\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}-\mathbf{p}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle$$
$$=\delta_{\omega,\omega'}[\langle\hat{\psi}^+_{\omega',\mathbf{k}-\mathbf{p}}\hat{\psi}^-_{\omega',\mathbf{k}-\mathbf{p}}\rangle-\langle\hat{\psi}^+_{\omega',\mathbf{k}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle], \tag{44}$$

where $\langle\hat{\psi}^+_{\omega,\mathbf{k}'+\mathbf{p}}\hat{\psi}^-_{\omega,\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}-\mathbf{p}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle$ is the derivative with respect to $J_{\mathbf{p},\omega},\phi^+_{\omega',\mathbf{k}-\mathbf{p}},\phi^-_{\omega',\mathbf{k}}$ of Eq. (37). Computing Eq. (44) for $p_-=0$, we get if $\bar{\mathbf{p}}=(0,\bar{p}_+,\bar{p}_0)$

$$(-i\bar{p}_0+\omega v_F\bar{p}_+)\langle\rho_{\bar{\mathbf{p}},\omega}\hat{\psi}^+_{\mathbf{k},\omega'}\hat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega'}\rangle+\Delta(\mathbf{k},\bar{\mathbf{p}})$$
$$=\delta_{\omega,\omega'}[\langle\hat{\psi}^+_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\rangle-\langle\hat{\psi}^+_{\omega',\mathbf{k}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle], \tag{45}$$

and

$$\Delta(\mathbf{k},\bar{\mathbf{p}})=\int d\mathbf{k}'C(\mathbf{k}',\bar{\mathbf{p}})\langle\hat{\psi}^+_{\omega,\mathbf{k}'+\bar{\mathbf{p}}}\hat{\psi}^-_{\omega,\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle, \tag{46}$$

with

$$C(\mathbf{k},\bar{\mathbf{p}})=(-ik_0+\omega v_Fk_+)[C_{h,N}(\mathbf{k}+\bar{\mathbf{p}})-C_{h,N}(\mathbf{k})]+(-i\bar{p}_0$$
$$+\omega v_F\bar{p}_+)[C_{h,N}(\mathbf{k}+\bar{\mathbf{p}})-1]. \tag{47}$$

In deriving the above equation, we have used that

$$\int d\mathbf{k}'H^{-1}(k'_-)\langle\hat{\psi}^+_{\omega,\mathbf{k}'+\bar{\mathbf{p}}}\hat{\psi}^-_{\omega,\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\hat{\psi}^-_{\omega,\mathbf{k}}\rangle$$
$$=\int d\mathbf{k}'\langle\hat{\psi}^+_{\omega,\mathbf{k}'+\bar{\mathbf{p}}}\hat{\psi}^-_{\omega,\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle \tag{48}$$

for the compact support properties of the fields $\psi_{\mathbf{k}}$ and $H^2=H$; note also the crucial role of the condition $p_-=0$ in the above derivation.

The presence of the term $\Delta(\mathbf{k},\bar{\mathbf{p}})$ in the Ward identity (45) is related to the presence of the ultraviolet cutoff; as in 1D, such a term is not vanishing even in the limit $N\to\infty$ and it is responsible of the anomalies (see Refs. 15 and 16). The following correction identity holds, which is similar to the one in the 1D case:

$$\Delta(\mathbf{k},\bar{\mathbf{p}})=\nu(-i\bar{p}_0-\omega v_F\bar{p}_+)\sum_{\omega''=\pm}\langle\rho_{\bar{\mathbf{p}},\omega''}\hat{\psi}^+_{\omega',\mathbf{k}-\bar{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}}\rangle$$
$$+R_\omega^{2,1}(\mathbf{k},\bar{\mathbf{p}}), \tag{49}$$

with $R_\omega^{2,1}$ a small correction. Indeed, $R_\omega^{2,1}$ can be written as functional derivative, with respect to ϕ^+, ϕ^-, and J, of

$$e^{\mathcal{W}_\Delta(J,\phi)}=\int P(d\psi)\exp\Bigg\{-V(\psi)+\sum_\omega\int d\mathbf{z}[\psi^+_{\omega,\mathbf{z}}\phi^-_{\omega,\mathbf{z}}$$
$$+\phi^+_{\omega,\mathbf{z}}\psi^-_{\omega,\mathbf{z}}]+T_0(J,\psi)-T_-(J,\psi)\Bigg\}, \tag{50}$$

with

$$T_0(\psi)=\int\frac{d\bar{p}_+}{(2\pi)}\frac{d\bar{p}_0}{(2\pi)}\frac{d\mathbf{k}}{(2\pi)^3}C(\mathbf{k},\bar{\mathbf{p}})J_{\bar{\mathbf{p}}}\hat{\psi}^+_{\mathbf{k}+\bar{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k},\omega}, \tag{51}$$

$$T_-(\hat{\psi})=\sum_{\omega'}\int\frac{d\bar{p}_+}{(2\pi)}\frac{d\bar{p}_0}{(2\pi)}\frac{d\mathbf{k}}{(2\pi)^3}$$
$$\times\nu J_{\bar{\mathbf{p}}}(-i\bar{p}_0-\omega v_F\bar{p}_+)\hat{\psi}^+_{\mathbf{k}+\bar{\mathbf{p}},\omega'}\hat{\psi}^-_{\mathbf{k},\omega'}. \tag{52}$$

Equation (50) can be evaluated by a multiscale integration similar to the previous one, the only difference being that $\int J\rho$ is replaced by T_0-T_-. The terms with vanishing scaling dimension of the form $J\psi^+\psi^-$ can be obtained from the contraction of T_0 and T_-; in the first case, we can perform a decomposition similar to the one in Fig. 3 (see Fig. 4). Regarding the second and third terms, we can exactly proceed as in Sec. V, the main difference being that at least one of the two fields in Eq. (51) have scale N so that they obey to the bound $O(\gamma^{-2k}\gamma^{-(1/2)(N-k)})$. This follows from the fact that

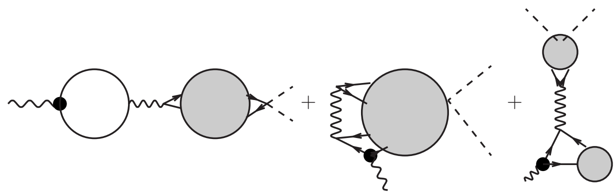

FIG. 4. Terms obtained from the contraction of T_0; the black dot represents $C(\mathbf{k},\bar{\mathbf{p}})$

when C is multiplied by two propagators, we get

$$C(\mathbf{k},\bar{\mathbf{p}})g^{(i)}(\mathbf{k})g^{(j)}(\mathbf{k}+\bar{\mathbf{p}}) = \frac{f_i(\mathbf{k})}{-ik_0+\omega v_F k_+}\left[\frac{f_j(\mathbf{k}+\bar{\mathbf{p}})}{C_{h,N}^{-1}(\mathbf{k}+\bar{\mathbf{p}})} - f_j(\mathbf{k}+\bar{\mathbf{p}})\right] - \frac{f_j(\mathbf{k}+\bar{\mathbf{p}})}{-i(k_0+\bar{p}_0)+\omega v_F(k_+ + \bar{p}_+)}\left[\frac{f_i(\mathbf{k})}{C_{h,N}^{-1}(\mathbf{k})} - f_i(\mathbf{k})\right], \tag{53}$$

which is nonvanishing only if one among i or j are equal to h or N.

The main difference with the analysis in the previous section is in the first term of Fig. 4; the "bubble" in Fig. 3 was vanishing, while here it is not. We choose ν in Eq. (50) equal to the value of this bubble in order to cancel it. The value of the bubble is given by

$$\nu = U v(\bar{p}_0,\bar{p}_+)\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{C(\mathbf{k},\bar{\mathbf{p}})}{-i\bar{p}_0-\omega v_F\bar{p}_+} g_\omega^{(\le N)}(\mathbf{k})g_\omega^{(\le N)}(\mathbf{k}+\bar{\mathbf{p}}), \tag{54}$$

and in the limit $N\to\infty$, $\nu = Uv(\bar{p}_0,\bar{p}_+)\frac{a}{4\pi^2}$. Hence, for $k_0,k_+ = O(\gamma^h)$,

$$|R_\omega^{2,1}(\mathbf{k},\bar{\mathbf{p}})| \le C\varepsilon_h^2\gamma^{-2h}. \tag{55}$$

VII. SCHWINGER–DYSON EQUATION

An immediate consequence of the analysis in Sec. VI is that for momenta computed at the infrared scale $|\mathbf{k}|=|\mathbf{k}'| = |\mathbf{k}+\hat{\mathbf{p}}| = |\mathbf{k}'-\hat{\mathbf{p}}| = |\hat{\mathbf{p}}| = \gamma^h$,

$$\langle \hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}+\hat{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}'-\hat{\mathbf{p}}}\rangle = \frac{1}{(Z_h)^2}\frac{\lambda_h}{L_-} g^{(h)}_{\omega,\mathbf{k}} g^{(h)}_{\omega',\mathbf{k}'} g^{(h)}_{\omega,\mathbf{k}'+\hat{\mathbf{p}}} g^{(h)}_{\omega',\mathbf{k}'-\hat{\mathbf{p}}} \times[1+O(\varepsilon_h)]\langle\hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}}\rangle = \frac{g^{(h)}_{\omega,\mathbf{k}}}{Z_h}[1+O(\varepsilon_h)]. \tag{56}$$

This says that relations between the effective couplings at a certain scale $h\le 0$ can be obtained from the relations between the Schwinger functions of the auxiliary model [Eq. (37)] computed at the infrared cutoff scale. The starting point for deriving such relations is the Schwinger-Dyson equation for the four-point function, given by computing the external momenta at the infrared scale and calling $\hat{\mathbf{p}}=(0,\hat{p}_+,\hat{p}_0)$ and $\bar{\mathbf{p}}=(0,\bar{p}_+,\bar{p}_0)$ (Fig. 5)

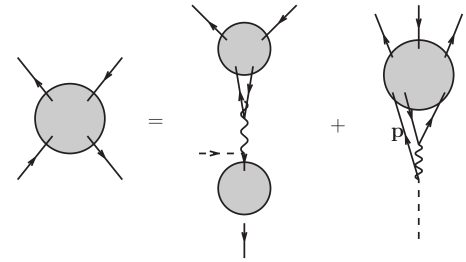

FIG. 5. Graphical representation of Eq. (57); the dotted line represent the free propagator.

$$\langle \hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}'+\hat{\mathbf{p}}}\rangle = \sum_{\omega''}\left\{\frac{U}{L_-}v(\hat{p}_0,\hat{p}_+)g_{\omega',\mathbf{k}'+\hat{\mathbf{p}}}\langle\hat{\psi}^-_{\omega',\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}'}\rangle\langle\rho_{\hat{\mathbf{p}},\omega''}\hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\rangle + \frac{U}{L_-}g_{\omega',\mathbf{k}'+\hat{\mathbf{p}}}\int\frac{d\bar{p}_0}{(2\pi)}\frac{d\bar{p}_+}{(2\pi)}v(\bar{p}_0,\bar{p}_+) \times\langle\rho_{\bar{\mathbf{p}},\omega''}\hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\hat{\psi}^-_{\omega',\mathbf{k}'}\hat{\psi}^+_{\omega',\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}}}\rangle\right\}. \tag{57}$$

By the WI [Eqs. (45) and (49)],

$$(-i\hat{p}_0+\omega' v_F\hat{p}_+)\langle\rho_{\hat{\mathbf{p}},\omega'}\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega} = A_{\omega,\omega'}(\mathbf{p})[\langle\hat{\psi}^-_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\hat{\psi}^+_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\rangle - \langle\hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}}\rangle] + H^{(2,1)}_{\omega,\omega'}(\mathbf{k},\mathbf{p}), \tag{58}$$

with $A_{\omega,\omega}(\mathbf{p})=1+O(\varepsilon_h)$, $A_{\omega,-\omega}(\mathbf{p})=O(\varepsilon_h)$, and even in $\mathbf{p}$; moreover, $H_{\omega,\omega'}(\mathbf{k},\mathbf{p})$ is a linear combination of the $R^{2,1}$ functions in Eq. (49) with bounded coefficients. The WI for the four-point function is given by

$$(-i\bar{p}_0+\omega v_F\bar{p}_+)\langle\rho_{\bar{\mathbf{p}},\omega}\hat{\psi}^-_{\omega,\mathbf{k}}\hat{\psi}^+_{\omega,\mathbf{k}-\hat{\mathbf{p}}}\hat{\psi}^-_{-\omega,\mathbf{k}'}\hat{\psi}^+_{-\omega,\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}}}\rangle = \langle\hat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',-\omega}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle - \langle\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}}+\bar{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',-\omega}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle + \nu\sum_{\bar{\omega}}(-i\bar{p}_0-\omega v_F\bar{p}_+) \times\langle\rho_{\bar{\mathbf{p}},\bar{\omega}}\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',-\omega}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},-\omega}\rangle + R^{4,1}_\omega(\mathbf{k},\mathbf{k}',p_0), \tag{59}$$

and similar ones so that the second addend of the left hand side of (57) is given by

$$\int d\bar{p}_0 d\bar{p}_+\chi_\varepsilon(\bar{\mathbf{p}})\frac{\hat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0+\omega'' v_F\bar{p}_+} [A_1(\bar{\mathbf{p}})\langle\hat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',\omega'}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle + A_2(\bar{\mathbf{p}})\langle\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}}+\bar{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',\omega'}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle + A_3(\bar{\mathbf{p}})\langle\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}'-\bar{\mathbf{p}},\omega'}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},\omega'}\rangle + A_4(\bar{\mathbf{p}})\langle\hat{\psi}^-_{\mathbf{k},\omega}\hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega}\hat{\psi}^-_{\mathbf{k}',\omega'}\hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}},\omega'}\rangle + H^{4,1}(\mathbf{k},\mathbf{k}',\bar{\mathbf{p}})], \tag{60}$$

where $\chi_\varepsilon(\bar{\mathbf{p}})$ is a compact support function vanishing for $\bar{\mathbf{p}}$

$=0$, and such that it becomes the identity in the limit $\varepsilon \to 0$; moreover, the functions $A_i(\mathbf{p})$ are bounded and even in $\mathbf{p}$, and $H^{4,1}$ is a linear combination of the $R^{4,1}$ functions in Eq. (59) with bounded coefficients.

We have now to bound all the sums in the right hand side of Eq. (60). Note first that by parity

$$\int d\bar{p}_0 d\bar{p}_+ \chi_\varepsilon(\bar{\mathbf{p}}) A_4(\bar{\mathbf{p}}) \frac{\hat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + \omega'' v_F \bar{p}_+} \times \langle \hat{\psi}^-_{\mathbf{k},\omega} \hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega} \hat{\psi}^-_{\mathbf{k}',\omega'} \hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}},\omega'} \rangle = 0. \tag{61}$$

Moreover, the first term in the right hand side of Eq. (60) verifies

$$\left| \int d\bar{p}_0 d\bar{p}_+ \frac{\hat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + v_F \omega'' \bar{p}_+} A_1(\bar{\mathbf{p}}) \chi_\varepsilon(\bar{\mathbf{p}}) \times \langle \hat{\psi}^-_{\mathbf{k}-\bar{\mathbf{p}},\omega} \hat{\psi}^+_{\mathbf{k}-\hat{\mathbf{p}},\omega} \hat{\psi}^-_{\mathbf{k}',\omega'} \hat{\psi}^+_{\mathbf{k}'+\hat{\mathbf{p}}-\bar{\mathbf{p}},\omega'} \rangle \right| \leq C\varepsilon_h \frac{\gamma^{-3h}}{Z_h^2}, \tag{62}$$

and a similar bound is true for the second and third term. Finally, as in Ref. 16,

$$\left| \int d\bar{p}_0 d\bar{p}_+ \frac{\hat{v}(\bar{p}_0,\bar{p}_+)}{-i\bar{p}_0 + v_F \omega'' \bar{p}_+} H^{4,1}(\mathbf{k},\mathbf{k}',\bar{\mathbf{p}}) \right| \leq C\varepsilon_h \frac{\gamma^{-3h}}{Z_h^2}. \tag{63}$$

By inserting Eqs. (56), (58), and (60)–(62) in Eq. (57), we get $\lambda_h = \lambda_0 + O(U^2)$, which means that the effective interaction remain close to initial value for any RG iteration; a contradiction argument shows that this can be true only if the beta function is asymptotically vanishing: as this beta function is the same of the 2D model (1) with effective couplings (30) up to $O(\gamma^h)$ terms, then Eq. (32) follows.

VIII. CONCLUSIONS

We have shown that the RG flow for a system of spinless fermions with flat Fermi surface has, in addition to the trivial fixed point, a line of Luttinger liquid fixed points, corresponding to vanishing wave function renormalization and anomalous exponents in the two-point function; such fixed point is in the invariant set (30). This makes quantitative the analysis in Ref. 10, in which the existence of a Luttinger fixed point in 2D was postulated on the basis of bosonization. With respect to previous perturbative RG analysis, the key novelty is the implementation of WI at each RG iteration, which is in analogy to what is done in 1D.

Of course the other effective interactions should cause flows away from this fixed point; a similar phenomenon happens in the 1D (spinning) Hubbard model, in which there is a Luttinger liquid fixed point in the invariant set obtained setting all but the backscattering and umklapp scattering terms equal to zero (that is, the set $g_{1,h}=g_{3,h}=0$ in the g-ology notation, see Ref. 19), but *attractive* backscattering interaction produces a flow to a strong coupling regime. We can in any case expect, as in 1D, that even if the Luttinger fixed point in 2D is not stable its presence has an important role in the physical properties of the system.

Fermi surfaces with flat or almost flat pieces and no van Hove singularities are found in the Hubbard model with next to nearest neighbor interactions or in the Hubbard model close to half filling, and it is likely that our results can be extended, at least partially, to such models. Note, however, that in such models the sides of the Fermi surface are not perfectly flat so that one expects a renormalization of the shape of the Fermi surface, as in Ref. 3, which is absent in the case of flat sides by symmetry. Another simplifying property of the model considered here is that the modulation of the Fermi velocity is taken constant along the Fermi surface, which is contrary to what happens in more realistic models; a momentum dependent Fermi velocity produces extra terms in the WI, as it is evident from Eq. (43) [v_F should be replaced by $v_F(k'_-)$ in the first line and $v_F(k'_- + p_-)$ in the second line], and their effect deserves further analysis.

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