Generalized clustering conditions of Jack polynomials at negative Jack parameter α

B. Andrei Bernevig[1,2] and F. D. M. Haldane[2]
[1]*Princeton Center for Theoretical Physics, Princeton, New Jersey 08544, USA*
[2]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
(Received 1 December 2007; revised manuscript received 18 March 2008; published 6 May 2008)

We present several conjectures on the behavior and clustering properties of Jack polynomials at a *negative* parameter $\alpha=-\frac{k+1}{r-1}$, with partitions that violate the (k,r,N)- admissibility rule of [Feigin *et al.* [Int. Math. Res. Notices **23**, 1223 (2002)]. We find that the "highest weight" Jack polynomials of specific partitions represent the minimum degree polynomials in N variables that vanish when s distinct clusters of $k+1$ particles are formed, where s and k are positive integers. Explicit counting formulas are conjectured. The generalized clustering conditions are useful in a forthcoming description of fractional quantum Hall quasiparticles.

DOI: 10.1103/PhysRevB.77.184502 PACS number(s): 11.25.Hf

I. INTRODUCTION

The Jack polynomials are a family of symmetric homogeneous multivariate polynomials characterized by a dominant partition λ and a rational number parameter α. They appeared in physics in the context of the Calogero–Sutherland model[1] for a positive coupling α.

In a recent paper, Feigin *et al.*[2] initiated the study of Jack polynomials (Jacks) in N variables at a negative rational parameter $\alpha_{k,r}=-\frac{k+1}{r-1}$ ($k+1$, $r-1$ relatively prime), with certain (k,r,N)-admissible partitions λ, $\lambda_i-\lambda_{i+k}\geq r$, by proving that they form a basis of a differential ideal $I_N^{k,r}$ in the space of symmetric polynomials. They showed that the set of Jacks with a parameter $\alpha_{k,2}$ and $(k,2,N)$-admissible λ is a basis for the space of symmetric homogeneous polynomials that vanish when $k+1$ variables z_i coincide. In Ref. 3, we found that these Jacks naturally implement a type of "generalized Pauli principle" on a generalization of Fock spaces for Abelian and non-Abelian fractional statistics.[4] We found that the (bosonic) Laughlin, Moore–Read, and Read–Rezayi fractional quantum Hall (FQH) wave functions (as well as others, such as the state that Simon *et al.*[5] called the "Gaffnian") can be explicitly written as *single* Jack symmetric polynomials. We identified r as the minimum power (cluster angular momentum) with which the admissible Jacks vanish as a cluster of $k+1$ particles come together.

In Ref. 3, we adopted a physical perspective for the Jack problem and uniquely obtained the Abelian and non-Abelian FQH ground states and the admissibility rule on partitions by imposing a highest and a lowest weight condition common in FQH studies on the sphere on an arbitrary Jack polynomial.[6] However, by imposing only the highest weight condition on the Jacks, we obtained another infinite series of polynomials of partitions that violate the admissibility rule of Feigin *et al.*[2] The (k,r,N)-admissible configurations of Ref. 2 do not exhaust the space of well-behaved Jack polynomials at negative $\alpha_{k,r}$. We find an infinite series of Jack polynomials, with partitions characterized by a different integer s, which are still well behaved at negative rational $\alpha_{k,r}$. This paper is devoted to the analysis of the clustering properties and the counting of such polynomials. We note that these polynomials can be interpreted as lowest Landau level (LLL) many-body wave functions and have applications in the construction of FQH quasiparticle excitations.[7]

We obtain a characterization of the symmetric polynomials in N variables $P(z_1,z_2,\ldots,z_N)$ that satisfy the following set of generalized clustering (vanishing) conditions: (i) $P(z_1,z_2,\ldots,z_N)$ vanishes when we form s distinct clusters, each of which has $k+1$ particles, but remains finite when $s-1$ distinct clusters of $k+1$ particles are formed; and (ii) $P(z_1,z_2,\ldots,z_N)$ does not vanish when a large cluster of $s(k+1)-1$ particles is formed. s and k are integers greater than or equal to 1 and the $s=1$ case is the case described by Feigin *et al.*[2] More precisely, let F be the space of all polynomials that satisfy the clustering condition

$$P(z_1=\cdots=z_{k+1},z_{k+2}=\cdots=z_{2(k+1)},\ldots,z_{(s-1)(k+1)+1}=\cdots$$
$$=z_{s(k+1)},z_{s(k+1)+1},z_{s(k+1)+2},\ldots,z_N)=0. \quad (1)$$

Let F_1 be the space of all polynomials that satisfy both Eq. (1) and the clustering condition $P(z_1=\cdots=z_{s(k+1)-1},z_{s(k+1)},z_{s(k+1)+1},\ldots,z_N)=0$. In this paper, we look at the coset space F/F_1 and focus on two problems: (1) we find the generators of the above ideal, which are the *minimum degree* polynomials (for N particles) that satisfy the clustering conditions above; and (2) we give a *conjectured* analytical expression for the number of linearly independent polynomials in N variables, with a momentum (total degree of the polynomial) M and a flux (maximum power in each variable) N_Φ, that span the coset F/F_1. Our motivation is to find new properties of Jack polynomials at *negative* α that are applicable to the study of FQH quasiparticles.[7] Our results are also related to the Cayley–Sylvester problem of coincident loci.[8,9]

II. PROPERTIES OF JACK POLYNOMIALS

The Jacks $J_\lambda^\alpha(z)$ are symmetric homogeneous polynomials in $z\equiv\{z_1,z_2,\ldots,z_N\}$, which are labeled by a partition λ with a length $\ell_\lambda\leq N$ and a parameter α; the partition λ can be represented as a (bosonic) occupation-number configuration $n(\lambda)=\{n_m(\lambda),m=0,1,2,\ldots\}$ of each of the LLL orbitals with an angular momentum $L_z=m\hbar$ (see Fig. 1), where for $m>0$, $n_m(\lambda)$ is the multiplicity of m in λ. When $a\rightarrow\infty$, $J_\lambda^\alpha\rightarrow m_\lambda$, which is the monomial wave function of the free boson state with an occupation-number configuration $n(\lambda)$; a key property of a Jack J_λ^α is that its expansion in terms of

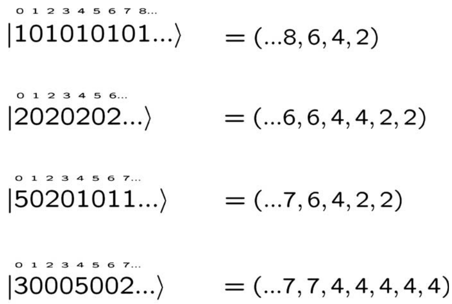

FIG. 1. Examples of occupation to a monomial basis $n(\lambda)\to\lambda$ conversion.

monomials contains only the terms m_μ with $\mu \le \lambda$, where $\mu < \lambda$ means that the partition μ is *dominated* by λ.[10] Jacks are also eigenstates of a Laplace–Beltrami operator $\mathcal{H}_{\rm LB}(\alpha)$ given by

$$\sum_i \left(z_i \frac{\partial}{\partial z_i}\right)^2 + \frac{1}{\alpha}\sum_{i<j} \frac{z_i + z_j}{z_i - z_j}\left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j}\right). \tag{2}$$

A partition λ is "(k,r,N) admissible"[2] if $n(\lambda)$ obeys a generalized Pauli principle where for all $m \ge 0$, $\sum_{j=1}^{r} n_{m+j-1} \le k$, so r consecutive "orbitals" contain no more than k particles.[3]

Partitions λ can be classified by λ_1, which is their largest part. When J_λ^α is expanded in occupation-number states (monomials), no orbital with $m > \lambda_1$ is occupied and Jacks with $\lambda_1 \le N_\Phi$ form a basis of FQH states on a sphere surrounding a monopole with a charge N_Φ.[6] Uniform states on the sphere satisfy the conditions $L^+\psi=0$ [highest weight (HW)] and $L^-\psi=0$ [lowest weight (LW)], where

$$L^+ = E_0, \quad L^- = N_\Phi Z - E_2, \quad L^z = \frac{1}{2}NN_\Phi - E_1,$$

$$E_n = \sum_i z_i^n \frac{\partial}{\partial z_i}, \tag{3}$$

where $Z \equiv \sum_i z_i$. When both conditions are satisfied, $E_1\psi \equiv M\psi = \frac{1}{2}NN_\Phi\psi$. The L^+, L^-, L^z operators endow the polynomial space with an angular momentum structure, which we use to characterize the polynomials. Any homogeneous polynomial is an eigenstate of the L^z operator; let the L^z eigenvalue of the HW Jacks be $l_z^{\max}$. The HW states then have $\vec{L}^2 = \frac{1}{2}(L^+L^- + L^-L^+) + L^zL^z = l_z^{\max}(l_z^{\max}+1)$; hence, the HW polynomials are the $(l,l_z)=(l_z^{\max}, l_z^{\max})$ states of a $2l_z^{\max}+1$ angular momentum multiplet of linearly independent polynomials. The non-HW states of $(l,l_z)=(l_z^{\max}, l_z^{\max}-i)$, where $i = 1,\ldots,2l_z^{\max}$, can be obtained by successive applications of the lowering operator $(L^-)^i$ on the HW states. They are linearly independent by virtue of having the same N_Φ but different total degrees M. Applying the L^- operator $2l_z^{\max}+1$ times kills the state. This angular momentum structure is extensively used in studies of FQH states on the sphere and we find it to be extremely valuable in the empirical polynomial counting that will be presented in the following sections.

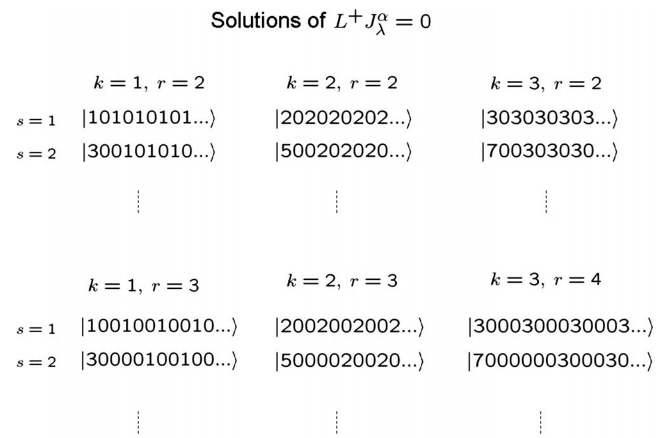

FIG. 2. Solutions to $L^+J_\lambda^\alpha=0$ are parametrized by one integer, $s>0$. Only $s=1$ states are both HW and LW states on the sphere and they satisfy the clustering property that they vanish as the rth power of the distance between $k+1$ particles. The $s>1$ states satisfy generalized clustering conditions.

It is very instructive to find the conditions for a Jack to satisfy the HW condition, $E_0J_\lambda^\alpha=0$. The action of E_0 on a Jack can be obtained from a formula by Lassalle.[11] In Ref. 3, we found that the condition $E_0J_\lambda^\alpha=0$ places severe restrictions on both the Jack parameter α and on the partition λ. We found the conditions $\alpha<0$ and $n_0 \equiv N-\ell_\lambda>0$ (nonzero occupancy of the $m=0$ orbital), as well as

$$N - \ell_\lambda + 1 + \alpha(\lambda_\ell - 1) = 0, \tag{4}$$

where λ_ℓ is the smallest (nonzero) part in λ. This imposes the following two conditions: (i) α is a negative rational, which we can choose to write as $-(k+1)/(r-1)$, where $(k+1)$ and $(r-1)$ are both positive, and is relatively prime; (ii) $\lambda_e = (r-1)s+1$ and $n_0=(k+1)s-1$, where $s>0$ is a positive integer. The remaining HW conditions require that all parts in λ have a multiplicity k, so that the orbital occupation partition is $n(\lambda_{k,r,s}^0)=[n_0 0^{s(r-1)} k 0^{r-1} k 0^{r-1} k \cdots]$, [i.e., the (k,r,N)-admissibility condition is satisfied as an equality for orbitals $m \ge \lambda_\ell$].

We call these Jacks HW (k,r,s,N) states $J_{\lambda_{k,r,s}^0}^{\alpha_{k,r}}$. Non-HW (k,r,s,N) states with n_0 particles in the zeroth orbital can be obtained by inserting zeros (holes) in the partition to the right of the λ_ℓ orbital. This defines a set of partitions whose Jacks satisfy the same clustering properties as the HW Jacks $J_{\lambda_{k,r,s}^0}^{\alpha_{k,r}}$ (see Sec. III). These partitions are $\lambda_{k,r,s}: n(\lambda_{k,r,s}) = [n_0 0^{s(r-1)} n(\lambda_{k,r})]$, where $n(\lambda_{k,r})$ is a $(k,r,N-n_0)$- admissible configuration in the sense of Feigin *et al.*[2] This set of partitions *does not* exhaust the number of polynomials with the clustering property Eq. (1). The $s=1$ case gives the generators of the ideals obtained by Feigin *et al.*[2] and are related to FQH ground states and their quasihole excitations.[3] The $s>1$ cases are new and violate the admissibility conditions of Feigin *et al.*[2] (see Fig. 2).

III. GENERALIZED CLUSTERING CONDITIONS OF JACK POLYNOMIALS

We now present the two generalized clustering conditions satisfied by the (HW and non-HW) Jacks of the (k,r,s,N) partitions $J_{\lambda_{k,r,s}}^{\alpha_{k,r}}$.

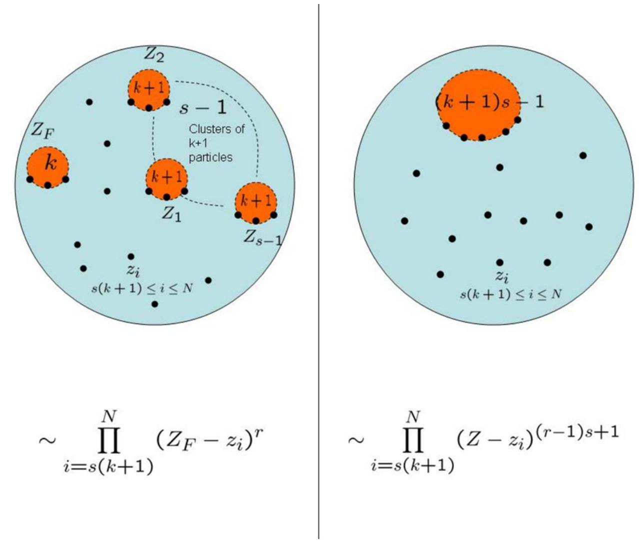

FIG. 3. (Color online) Clustering and vanishing conditions of the polynomials defined by the HW and non-HW Jacks $J^{\alpha_{k,r}}_{\lambda_{k,r,s}}$.

A. First clustering property

The Jacks $J^{\alpha_{k,r}}_{\lambda_{k,r,s}}$ allow $s-1$, but not s, different clusters of $k+1$ particles. First, we form $s-1$ clusters of $k+1$ particles, $z_1=\cdots=z_{k+1}(=Z_1)$; $z_{k+2}=\cdots=z_{2(k+1)}(=Z_2)$; $\cdots$; $z_{(s-2)(k+1)+1}=\cdots=z_{(s-1)(k+1)}(=Z_{s-1})$, where the positions of the clusters, $Z_1,\ldots,Z_{s-1}$ can be different. Then, we form a k (*not* $k+1$) particle cluster, $z_{(s-1)(k+1)+1}=\cdots=z_{s(k+1)-1}(=Z_F)$ (a final sth cluster of $k+1$ particles would make the polynomial vanish); see Fig. 3. With the above conditions on the particle coordinates, the clustering condition reads

$$J^{\alpha_{k,r}}_{\lambda_{k,r,s}}(z_1,\ldots,z_N) \propto \prod_{i=s(k+1)}^{N} (Z_F - z_i)^r. \tag{5}$$

Observe that when $s=1$, Eq. (5) reduces to the usual clustering condition satisfied by the (k,r) sequence, which is given in Ref. 3.

B. Second clustering property

The Jacks $J^{\alpha_{k,r}}_{\lambda_{k,r,s}}$ allow a large cluster of $n_0=(k+1)s-1$ particles at the same point. As a particular case of Eq. (5), they cannot allow n_0+1 particles to come at the same point because this would involve the formation of s clusters of $k+1$ particles, which Eq. (5) forbids. By clustering n_0 particles at the same point, $z_1=\cdots=z_{(k+1)s-1}=Z$, we find the following property (see Fig. 345):

$$J^{\alpha_{k,r}}_{\lambda_{k,r,s}}(z_1,\ldots,z_N) \propto \prod_{i=s(k+1)}^{N} (Z - z_i)^{(r-1)s+1}. \tag{6}$$

The HW Jacks of partitions $\lambda^0_{k,r,s}$ satisfy an even more stringent property; with $z_1=\cdots=z_{(k+1)s-1}=Z$, we find

$J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$ generators $k=1,\ r=2,\ s\in Z_+$	Polynomials of minimum degree satisfying the vanishing conditions
$s=1$ HW $\vert 101010101\ldots\rangle$	$P(z_1,z_1,z_2,z_3,\ldots)=0$
$s=2$ HW $\vert 300101010\ldots\rangle$	$P(z_1,z_1,z_2,z_2,z_3,z_4\ldots)=0$ but $P(z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
$s=3$ HW $\vert 5000101010\ldots\rangle$	$P(z_1,z_1,z_2,z_2,z_3,z_3,z_4,z_5\ldots)=0$ but $P(z_1,z_1,z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
$s=4$ HW $\vert 70000101010\ldots\rangle$	$P(z_1,z_1,z_2,z_2,z_3,z_3,z_4,z_4,z_5,z_6\ldots)=0$ but $P(z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
⋮	⋮

FIG. 4. The densest polynomials that satisfy the clustering conditions Eqs. (5) and (6). These are polynomials of flux N^0_Φ and are obtained by applying L^- on the HW Jack. The LW polynomial is also a Jack, but the polynomials in between cannot be expanded in terms of only well-behaved Jack polynomials at $\alpha=-\frac{k+1}{r-1}$.

$$J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N) = \prod_{i=s(k+1)}^{N} (Z-z_i)^{(r-1)s+1} \times J^{\alpha_{k,r}}_{\lambda^0_{k,r}}(z_{s(k+1)},z_{s(k+1)+1},\ldots,z_N), \tag{7}$$

where $n(\lambda^0_{k,r})=[k0^{r-1}k0^{r-1}\cdots k]$ is the maximum density $(k,r,N-n_0)$-admissible partition. For $s=1$, Eq. (7) also reduces to the usual clustering condition satisfied by the (k,r)-admissible sequence.[3] We have performed extensive numerical checks of the above conjectured clustering conditions. We also note that the left hand side (LHS) and right hand side (RHS) of Eq. (7) match in both total momentum M^0 (total degree of the polynomial),

$$E_1 J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}} \equiv M^0 J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}} = [N-(k+1)s+1] \times\left[(r-1)s+1+\frac{1}{2}\frac{r}{k}[N-(k+1)s-k+1]\right] J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}, \tag{8}$$

and in flux (maximum degree in each variable) N^0_Φ,

$$N^0_\Phi = \frac{r}{k}[N-k-(k+1)(s-1)] + (r-1)(s-1). \tag{9}$$

The superscript denotes the fact that we are considering the momentum and flux of Jack polynomials of HW partitions $\lambda^0_{k,r,s}$. Although it is tempting to describe our Jacks as being a subset of the $(k_{\text{new}},r_{\text{new}},N)=[s(k+1)-1,s(r-1)+1,N]$-admissible configuration Jacks of Feigin *et al.*,[2] this is not so because $k_{\text{new}}+1,r_{\text{new}}-1$ have a common divisor equal to s and are not coprime for $s>1$, which is unlike in Ref. 2.

C. Additional clustering condition

We empirically find that the HW Jacks $J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$ satisfy a third type of clustering, which has no correspondence in the $s=1$ case. By forming $s-1$ clusters of $2k+1$ particles together, $z_1=\cdots=z_{2k+1}(=Z_1)$; $z_{(2k+1)+1}=\cdots=z_{2(2k+1)}(=Z_2)$; $\ldots$; $z_{(s-2)(2k+1)+1}=\cdots=z_{(s-1)(2k+1)}=(Z_{s-1})$, we find that the HW Jacks satisfy the clustering

$$J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N) = \prod_{i<j=1}^{s-1}(Z_i-Z_j)^{k(3r-2)}\prod_{i=1}^{s-1}\prod_{l=(s-1)(2k+1)+1}^{N}(Z_i-z_l)^{2r-1} \times J^{\alpha_{k,r}}_{\lambda^0_{k,r}}(z_{(s-1)(2k+1)+1},\ldots,z_N) \quad (10)$$

up to a numerical proportionality constant. Some slightly tedious algebra proves that the total momentum and flux match between the LHS and RHS of Eq. (10).

D. Angular momentum structure

The $J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N)$ are the HW states of an angular momentum multiplet of $l(\lambda^0_{k,r,s})=l_z^{\max}$ and $l_z=l_z^{\max},\ldots,-l_z^{\max}$. The LW states of the multiplet $(L^-)^{2l_z^{\max}}J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N)$ are also single Jack polynomials of the "symmetric" partition to $\lambda^0_{k,r,s}$ in orbital notation (see Fig. 6). The value of $l_z^{\max}$ is

$$L^z J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}} \equiv l_z^{\max} J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}} = \frac{1}{2}\left[[(r-1)s+1][2(k+1)s-2-N] + \frac{r}{k}[(k+1)s-1][N-(k+1)s+1-k]\right]J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}. \quad (11)$$

Most importantly, we find that powers of the operator $L^-=\Sigma_i(z_i^2\frac{\partial}{\partial z_i}-N_\Phi z_i)$ that act on $J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N)$ create linearly independent polynomials with the same vanishing conditions [Eqs. (5) and (6)] as the HW $J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}(z_1,\ldots,z_N)$. There are $2l_z^{\max}+1$ such polynomials.

IV. $J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$: SMALLEST DEGREE POLYNOMIALS WITH GENERALIZED CLUSTERING

In the remainder of this paper, we will focus on the $r=2$ case. We empirically find the following property: Given N particles [where $N>n_0$ for the clustering condition Eq. (1) to be well defined], we find that the $r=2$ HW Jacks $J^{\alpha_{k,2}}_{\lambda^0_{k,2,s}}(z_1,\ldots,z_N)$ are the *smallest degree* (smallest momentum M [Eq. (8)] and flux N_Φ [Eq. (9)]) polynomials satisfying the clustering conditions Eqs. (5) and (6). There are *exactly* $2l(\lambda^0_{k,2,s})+1$ polynomials in N variables with N_Φ^0 [Eq. (9)] and *unrestricted* total dimensions, satisfying the clustering conditions Eqs. (5) and (6). A basis for this ideal is explicitly

$$(L^-)^m J^{\alpha_{k,2}}_{\lambda^0_{k,2,s}}, \quad m=0,1,\ldots,2\cdot l(\lambda^0_{k,2,s}). \quad (12)$$

We find that $(L^-)^{2\cdot l(\lambda^0_{k,2,s})+1}J^{\alpha_{k,2}}_{\lambda^0_{k,2,s}}=0$. One can easily understand this counting by looking at the orbital occupation numbers of the relevant partitions. The occupation number of the $J^{\alpha_{k,2}}_{\lambda^0_{k,2,s}}$ is $[n_0 0^s k0k0k\cdots k0k]$. This is the lowest weight partition (smallest degree polynomials) where the clustering conditions Eqs. (5) and (6) are satisfied. By interpreting this as an orbital occupation number, there exists a symmetric partition $n(\lambda^{\max}_{k,r,s})=[k0k\cdots k0k0k0^s n_0]$. This is the highest total degree polynomial (bounded by the restriction that each variable separately has at most N_Φ^0 degrees) that satisfies the vanishing condition Eq. (1). It is also a Jack polynomial $J^{\alpha_{k,r}}_{\lambda^{\max}_{k,r,s}}$ (see Fig. 6). Since the L^- operator does not change the value of N_Φ, maintains the clustering property, and implements the angular momentum lowering, we easily count the polynomials as forming the $l=l_z^{\max}$ multiplet. For the $k=1$, $r=2$, $s=2$ case, this counting coincides with the empirical counting observed by Kasatani *et al.*[8]

We remark that the expansion of $(L^-)^m J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$ in the Jack polynomial basis $J^{\alpha_{k,r}}_{\lambda}$ contains ill-behaved Jacks, which diverge at the negative $\alpha_{k,r}$ used. However, their coefficients in the expansion also vanish to give an overall finite contribution. These are the "modified" Jacks introduced by Kasatani *et al.*[8] for the specific $(k,r,s)=(1,2,2)$ case of the problem studied here. As we reach higher k and s integers, the number of modified Jacks that appear in the expansion of $(L^-)^m J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$ grows larger. We therefore prefer to characterize the basis of

$J^{\alpha_{k,r}}_{\lambda^0_{k,r,s}}$ generators $k=2,\ r=2,\ s\in Z_+$	Polynomials of minimum degree satisfying the vanishing conditions
$s=1$ HW $\lvert 202020202\ldots\rangle$	$P(z_1,z_1,z_1,z_2,z_3,\ldots)=0$
$s=2$ HW $\lvert 500202020\ldots\rangle$	$P(z_1,z_1,z_1,z_2,z_2,z_2,z_3,z_4\ldots)=0$ but $P(z_1,z_1,z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
$s=3$ HW $\lvert 8000202020\ldots\rangle$	$P(z_1,z_1,z_1,z_2,z_2,z_2,z_3,z_3,z_3,z_4,z_5\ldots)=0$ but $P(z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
$s=4$ HW $\lvert 11\ 00002020\ldots\rangle$	$P(z_1,z_1,z_1,z_2,z_2,z_2,z_3,z_3,z_3,z_4,z_4,z_4,z_5,z_6\ldots)=0$ but $P(z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_1,z_2,z_3\ldots)\neq 0$
⋮	⋮

FIG. 5. Lowest degree polynomials (generators) of the $(k,r,s)=(1,2,s)$ clusterings.

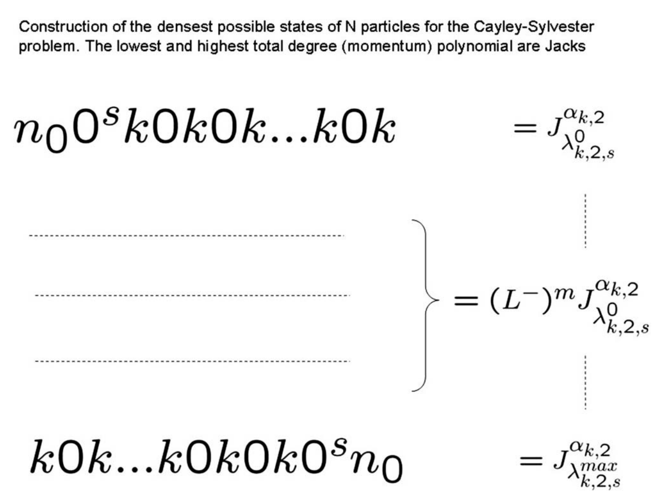

FIG. 6. Lowest degree polynomials (generators) of the $(k,r,s)=(2,2,s)$ clusterings.

these polynomials by the HW Jack and the polynomials that result from it by successive applications of the L^- operator.

We now relax the $N_\Phi = N_\Phi^0$ constraint and focus on the counting of the polynomials that satisfy Eqs. (5) and (6).

V. COUNTING POLYNOMIALS

We want to provide the counting of the number of linearly independent polynomials in the ideal F/F_1. We start by counting the $s=1$ polynomials. These are related to the admissible partitions of Ref. 2 or the generalized Pauli principle of Refs. 3 and 4.

A. Counting of $(k,r,s)=(1,r,1)$ polynomials

We first obtain a counting of linearly independent polynomials in N particle coordinates $z_1,\ldots,z_N$ with a total momentum M, with the degree in each coordinate $\leq N_\Phi$, satisfying the condition $P(z_1,z_2,z_3,z_4,\ldots) \sim (z_i - z_j)^r$. We believe that this result was previously known, although we could not explicitly find it in the literature on symmetric polynomials. From the work of Feigin *et al.*,[2] this number is equal to the number of $(k,r)=(1,r)$-admissible partitions of N_Φ orbitals, which is related by the squeezing rule[3] (so as to keep the partition weight M constant). We call this number $p_{1,r,1}(N,M,N_\Phi)$. From the theory of partitions,[12] such numbers are most easily obtained from a generating function $G(q)$, and we can analytically prove that

$$p_{1,r,1}(N,M,N_\Phi) = \frac{1}{M!} \left. \frac{\partial^M G_{1,r,1}(N,N_\Phi,q)}{\partial q^M} \right|_{q=0}, \qquad (13)$$

where the generating function $G(q)$ reads

$$G_{1,r,1}(N,N_\Phi,q) = \frac{q^{\frac{r}{2}N(N-1)} \prod_{i=1}^{N_\Phi - r(N-1)+N} (1-q^i)}{\prod_{i=1}^{N}(1-q^i) \prod_{i=1}^{N_\Phi - r(N-1)} (1-q^i)}$$

if $N_\Phi \geq r(N-1)$ and $p_{1,r,1}[N,M,N_\Phi < r(N-1)]=0$. $p_{1,r,1}(N,M,N_\Phi)$ represents a building block for future results. Apart from the r-dependent prefactor, this result is given in Ref. 12. We have numerically checked that $p_{1,r,1}(N,M,N_\Phi)$ gives the right polynomial counting by building the null space of polynomials satisfying the clustering condition $P(z_1,z_1,z_2,z_3,\ldots)=0$. In the context of FQH, it reproduces the right counting for quasihole states. For example, it is known that the Laughlin state with x number of quasiholes has $(N+x)!/(N!x!)$ independent states and one can numerically check the identity

$$\binom{N+x}{x} = \sum_{M=\frac{r}{2}N(N-1)}^{\frac{r}{2}N(N-1)+xN} p_{1,r,1}[N,M,N_\Phi = r(N-1)+x].$$

We can now provide a formula for the number of polynomials in N variables with a total dimension M of any maximum power in each coordinate (any N_Φ), which satisfy the clustering condition $P(z_1,z_2\ldots.) \sim (z_i - z_j)^r$. We take $N_\Phi \to \infty$ to obtain the simpler expression

$$p_{1,r,1}(N,M) = \frac{1}{M!} \left. \frac{\partial^M G_{1,r,1}(N,q)}{\partial q^M} \right|_{q=0},$$

$$G_{1,r,1}(N,q) = \frac{q^{\frac{r}{2}N(N-1)}}{\prod_{i=1}^{N}(1-q^i)}. \qquad (14)$$

To find the *total* number of symmetric polynomials that satisfy the vanishing $P(z_1,z_1,z_3,z_4,\ldots,z_N)=0$, we must particularize to the lowest vanishing power possible, $r=2$. In this case, Eq. (14) is identical to the formula of Kasatani *et al.*,[8] although Eq. (13) represents a more comprehensive counting of the polynomials as it contains information on the allowed maximum degree in each variable, N_Φ.

Our aim is to conjecture similar expressions for the counting of the dimension space of the polynomials in the coset space F/F_1.

B. Counting of $(k,r,s)=(1,2,s)$ polynomials

By using $p_{1,2,1}(N,M,N_\Phi)$, we now obtain the counting of polynomials in the ideal F/F_1 with $k=1, r=2$, and $s>1$. We first reproduce the result of Kasatani *et al.*,[8] which is the $(k,r,s)=(1,2,2)$ case of our problem. Kasatani *et al.*[8] obtained the dimension of the linear space of polynomials that satisfy the clustering conditions $P(z_1,z_1,z_2,z_2,z_5,z_6,\ldots,z_N)=0$ and $P(z_1,z_1,z_1,z_4,z_5,\ldots,z_N)\neq 0$ with a total dimension M of any allowed maximum degree in each of the coordinates. We then derive the general case, which contains information about N_Φ.

We define $p_{k=1,r=2,s=2}(N,M)=p_{1,2,2}(N,M)=p_{1,2,2}(N,M,N_\Phi \leq \infty)$ as the number of polynomials in N variables of total momentum (degree) M, with any allowed N_ϕ ($\leq M$), satisfying the clustering conditions Eqs. (5) and (6) (with $k=1$, $r=2, s=2$). This number can be found as follows: We start with the partition $n(\lambda_{1,2,2})$, with a total dimension (weight) M, which has all $N-1$ particles maximally pushed to the left of the orbitals, while the Nth particle is pushed as far as needed to the right so that the polynomial has a dimension M. This partition reads

$$[30010101\cdots101 \underbrace{0\cdots0}_{M-2-N(N-4)} 1].$$

Note that by (k,r,s,N) admissibility, we cannot push the first $N-1$ particles farther to the left than they already are. Then, $p_{1,2,2}(N,M)$ is the sum of two terms: First, we can form $(k,r,s,N)=(1,2,2,N)$-admissible partitions by keeping the occupancy of the zeroth orbital to be 3 and by squeezing on the remainder partition

$$[00010101\cdots101 \underbrace{0\cdots0}_{M-2-N(N-4)} 1]$$

to form all the $(k,r)=(1,2)$-admissible partitions. As discussed before, this gives Jack polynomials with the same clustering condition as the HW Jack, and their number is the same as the number of $(k,r)=(1,2)$-admissible partitions of $N-3$ variables and total momentum $M-3(N-3)$, i.e., $p_{1,2,1}[N-3,M-3(N-3)]$. Second, we can form polynomials with the same $(k,r,s,N)=(1,2,2,N)$ clustering by taking some particles out of the zeroth orbital, although these now involve divergent Jacks (with compensating vanishing coefficients). We can form all the polynomials [with a dimension M in N variables, with less than three particles in the zeroth orbital, and that satisfy the clustering conditions $(k,r,s,N)=(1,2,2,N)$] by acting with L^- on *all* the polynomials of dimension $M-1$, which satisfy the same clustering conditions. This number is then $p_{1,2,2}(N,M-1)$, and we find the recursion relation

$$p_{1,2,2}(N,M)=p_{1,2,2}(N,M-1)+p_{1,2,1}(N-3,M-3(N-3)). \tag{15}$$

To find the generating function, multiply Eq. (16) by q^M, sum over M, reshift the variables in the sum, and obtain

$$p_{1,2,2}(N,M)=\frac{1}{M!}\left.\frac{\partial^M G_{1,2,2}(N,q)}{\partial q^M}\right|_{q=0},$$

$$G_{1,2,2}(N,q)=\frac{q^{(N-3)(N-1)}}{(1-q)\prod_{i=1}^{N-3}(1-q^i)}. \tag{16}$$

This reproduces a formula obtained by Kasatani *et al.*[8] through different methods.

We now use the same reasoning to count the dimension of the ideal F/F_1 with $(k=1,r=2)$ and general s. The number of polynomials with N variables of momentum (total degree) M, with unrestricted N_ϕ, that satisfy the clustering conditions Eqs. (5) and (6) with $k=1,r=2$ and any $s>1$ is $p_{1,2,s}(N,M)$:

$$p_{1,2,s}(N,M)=\frac{1}{M!}\left.\frac{\partial^M G_{1,2,s}(N,q)}{\partial q^M}\right|_{q=0},$$

$$G_{1,2,s}(N,q)=\frac{q^{(N-n_0)(N-n_0+s)}}{(1-q)\prod_{i=1}^{N-n_0}(1-q^i)}, \tag{17}$$

where $n_0=2s-1$.

We now separately introduce information on the maximum degree in each coordinate (flux). Let us define $p_{k=1,r=2,s}(N,M,N_\Phi)=p_{1,2,s}(N,M,N_\Phi)$ as the number of polynomials in N variables with a total momentum (degree) M, with a flux $\le N_\phi$, satisfying the clustering conditions Eqs. (5) and (6) with $k=1$ and $s=2$. We briefly present the reasoning used to conjecture a count of these polynomials. The smallest dimension and flux correspond to the partition $[n_0 0^s 10101\cdots101000000\cdots00]$ (with $n_0=2s-1$). The number of zeros on the right is just enough to make $N_\Phi+1$ total number of orbitals. Some of the orbitals to the right might be unoccupied. This "padding" to the right has the effect of allowing L^- to move particles up to the rightmost orbital. By symmetry in orbital space, the highest partition corresponds to $[00\cdots000000101\cdots101010^s n_0]$ (with $N_\Phi+1$ total number of orbitals). We can then immediately see that $p_{1,2,s}(N,M,N_\Phi)=0$ for $M<(s+1)(N-n_0)+(N-n_0)(N-n_0-1)$ or for $N_\phi<s+1+2(N-n_0-1)$. Also, $p_{1,2,s}(N,M,N_\Phi)=0$ for $M>NN_\Phi-(N-n_0)(N-n_0+s)$. There is also an "intermediate" total degree that is important in the counting, which corresponds to the partition $[10101\cdots10100000\cdots00n_0]$ of total degree $n_0N_\Phi+(N-n_0)(N-n_0-1)$ when the rightmost orbital has been occupied by the maximum number of particles possible, n_0. Then, $p_{1,2,s}(N,M,N_\Phi)$ reads

$$\begin{aligned}p_{1,2,s}(N,M,N_\Phi)&=0 \quad \text{if} \quad M<(s+1)(N-n_0)+(N-n_0)(N-n_0-1) \quad \text{or} \quad N_\phi<s+1+2(N-n_0-1)\\ &=\sum_{i=0}^{M-(s+1)(N-n_0)} p_{1,2,1}[N-n_0,i,N_\Phi-(s+1)] \quad \text{if} \quad M\le n_0N_\Phi+(N-n_0)(N-n_0-1)\\ &=\sum_{i=0}^{NN_\Phi-(N-n_0)(s+1)-M} p_{1,2,1}[N-n_0,i,N_\Phi-(s+1)] \quad \text{if} \quad n_0N_\Phi+(N-n_0)(N-n_0-1)\\ &<M\le NN_\Phi-(N-n_0)(N-n_0+s)=0 \quad \text{if} \quad M>NN_\Phi-(N-n_0)(N-n_0+s),\end{aligned}$$

$p_{1,2,1}(N,M,N_\Phi)$ was explicitly given in Sec. V A, and $n_0=2s-1$. For $s=2$ and $N_\Phi\to\infty$, this reduces to the same equation as the one obtained by iterating Eq. (16).

By summing the previous expression over all M, we can find the number of polynomials of N variables, with a degree in each variable N_Φ and an unrestricted momentum (total degree), $p_{1,2,s}(N,N_\Phi)=\Sigma_{M=0}^{\infty}p_{1,2,s}(N,M,N_\Phi)$, satisfying the clustering conditions Eqs. (5) and (6) with $k=1$ and s arbitrary integers. However, by applying an empirical rule that we observed, based on the multiplet nature of these polynomials, we find an alternate simpler formula, which is not obviously equal to $\Sigma_{M=0}^{\infty}p_{1,2,s}(N,M,N_\Phi)$; extensive numerical checks have, however, confirmed their equivalence:

$$p_{1,2,s}(N,N_\Phi)=0 \quad \text{if} \quad N_\phi < s+1+2(N-n_0-1)$$
$$=[NN_\Phi-2(s+1)N+2n_0(s+1)+1]$$
$$\times\sum_{i=0}^{NN_\Phi} p_{1,2,1}[N-n_0,i,N_\Phi-(s+1)]$$
$$-2\sum_{i=0}^{NN_\Phi} i\cdot p_{1,2,1}[N-n_0,i,N_\Phi-(s+1)] \quad (18)$$

$p_{1,2,1}(N,M,N_\Phi)$ was explicitly given in Sec. IVA, and $n_0=2s-1$.

C. Counting of $(k,r,s)=(k,2,1)$ polynomials

We now move to the $k>1$ case. We first obtain a count of the polynomials that satisfy the $(k,r)=(k,2)$ statistics, i.e., of the Read–Rezayi Z_k states. We want to count the number of polynomials in N variables, with a momentum (total degree) M and a maximum flux (maximum degree in each coordinate) N_Φ, that vanish when $k+1$ particles come together. We call this number $p_{k,2,1}(N,M,N_\Phi)$. As we know,[2] this is equal to the number of $(k,2)$-admissible partitions with a weight M, which are made out of at most N parts and where $\lambda_1 \leq N_\Phi$. We can derive this by performing a slight modification of a formula by Feigin and Loktev[13] (see also Andrews[12]). $p_{k,2,1}(N,M,N_\Phi)=0$ for $N_\Phi<\frac{2}{k}(N-k)$ or for $M<\frac{1}{k}N(N-k)$ but otherwise is

$$p_{k,2,1}(N,M,N_\Phi)=\frac{1}{M!}\frac{1}{N!}\frac{\partial^N}{\partial z^N}\frac{\partial^M}{\partial q^M}[q^{-N}G_{k,2,1}(N_\phi,q,z)]|_{q=0,z=0},$$

where the generating function $G_{k,2,1}(N_\phi,q,z)$ is[12,13]

$$G_{k,2,1}(N_\phi,q,z)=\sum_{\substack{m_1,n_1=0\\ m_1+n_1\leq\frac{N_\Phi+1}{2}}}^{\frac{N_\Phi+1}{2}}\frac{z^{(k+1)(m_1+n_1)}q^{k(m_1^2-n_1^2+n_1(N_\Phi+2))+m_1(3m_1-1)/2+n_1(N_\Phi+3-n_1)}}{\prod_{i=1}^{m_1}(1-q^i)(zq^{i+m_1-1}-1)\prod_{i=2m_1+1}^{N_\Phi-2n_1+1}(1-zq^i)\prod_{i=1}^{n_1}(q^i-1)(zq^{N_\Phi-i-n_1+3}-1)} \quad (19)$$

We have numerically performed extensive checks of the compatibility of the formula above in the $k=1$ case with the simpler expression of $p_{1,2,1}(N,M,N_\Phi)$ that was obtained earlier. The formula above also correctly gives the dimension of the quasihole Hilbert space in the Z_k parafermion sequence. For one quasihole, this is known to be

$$\binom{N/k+k}{k}=\sum_{i=N(N-k)/k}^{N(N-k)/k+N}p_{k,2,1}\left(N,i,N_\phi=\frac{2}{k}(N-k)+1\right).$$

Extensive numerical checks prove the above identity. Moreover, for the $k=2$ Read–Moore state with two quasiparticles:

$$\binom{N/2+4}{4}+\binom{(N-2)/2+4}{4}=\sum_{i=N(N-k)/k}^{N(N-k)/k+2N}p_{k,2,1}\left(N,i,N_\phi=\frac{2}{k}(N-k)+2\right).$$

Equation (19) for $(k,2)$-admissible partitions found by Feigin and Loktev[13] and, before them, by Andrews[12] gives the most information possible about the counting of the Read–Rezayi wave functions and quasiholes. It provides information about the total degree of the polynomial (multiplet structure), which the usual counting[14] of quasiholes does not since it sums over all the possible total dimensions of the polynomials subject to a flux N_Φ upper bound. The above formulas, which were obtained by using the theory of partitions, give the same counting as that previously obtained by Ardonne[15] in the context of conformal field theory.

D. Counting of the $(k,r,s)=(k,2,s)$ polynomials

By using $p_{k,2,1}(N,M,N_\Phi)$, we obtain the counting of polynomials in the F/F_1 ideal with arbitrary k and s. Following a line of reasoning similar to the one used in the $k=1$ case, we find that the number $p_{k,2,s}(N,M,N_\Phi)$ of polynomials in N variables, with a momentum (total degree) M, with a flux N_Φ, that have the clustering conditions Eqs. (5) and (6) for general k and $s>1$ integers reads

$$
\begin{aligned}
p_{k,2,s}(N,M,N_\Phi) &= 0 \quad \text{if} \quad M < (s+1)\cdot(N-n_0) + \frac{1}{k}(N-n_0)(N-n_0-k) \quad \text{or} \quad N_\Phi < s+1+\frac{2}{k}(N-n_0-k) \\
&= \sum_{i=0}^{M-(s+1)(N-n_0)} p_{k,2,1}(N-n_0,i,N_\Phi-(s+1)); \quad \text{if} \quad 0 \le M \le n_0 N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k) \\
&= \sum_{i=0}^{NN_\Phi-(N-n_0)(s+1)-M} p_{k,2,1}(N-n_0,i,N_\Phi-(s+1)); \quad \text{if} \\
& n_0 N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k) < M \le NN_\Phi - (N-n_0)\left(s+\frac{N-n_0}{k}\right) \\
&= 0 \quad \text{if} \quad M > NN_\Phi - (N-n_0)\left(s+\frac{N-n_0}{k}\right),
\end{aligned} \tag{20}
$$

where $n_0=(k+1)s-1$.

By summing the previous expression over all M, we can find the number of polynomials with N variables, with a flux N_Φ and an unrestricted momentum (total degree) $p_{k,2,s}(N,N_\Phi)=\Sigma_{M=0}^{\infty}p_{k,2,s}(N,M,N_\Phi)$, satisfying the clustering conditions Eqs. (5) and (6) with k and s arbitrary integers. However, by using the angular momentum multiplet structure of these polynomials, we find an alternate formula which is not obviously equal to $\Sigma_{M=0}^{\infty}p_{k,2,s}(N,M,N_\Phi)$; extensive numerical checks have, however, confirmed their equivalence:

$$
\begin{aligned}
p_{k,2,s}(N,N_\Phi) &= 0 \quad \text{if} \quad N_\phi < s+1+\frac{2}{k}(N-n_0-k) \\
&= [NN_\Phi - 2(s+1)N + 2n_0(s+1)+1] \\
&\quad \times \sum_{i=0}^{NN_\Phi} p_{k,2,1}[N-n_0,i,N_\Phi-(s+1)] \\
&\quad - 2\sum_{i=0}^{NN_\Phi} i\cdot p_{k,2,1}[N-n_0,i,N_\Phi-(s+1)],
\end{aligned} \tag{21}
$$

$p_{k,2,1}(N,M,N_\Phi)$ is given in Sec. V C, and $n_0=(k+1)s-1$.

VI. SUBIDEALS OF F

So far, we have focused on the ideal F/F_1. We can systematically characterize the ideal F of polynomials $P(z_1,z_2,\ldots,z_N)$ that vanish when we form s clusters of $k+1$ particles in the following way: Let the *subideals* F_i be the polynomials that satisfy Eq. (1) but that also vanish when $(k+1)s-i$ particles are brought at the same point. These subideals were first defined in Ref. 16. Then, $F=\cup_{i=0}^{k+1}F_i/F_{i+1}$, where $F_0=F$ and F_{k+1} is the ideal of polynomials that vanishes when $s-1$ clusters of $k+1$ particles are formed ($F_{k+2}=\emptyset$). Hence, the polynomial ideal F_i is defined by the two clustering conditions

$$
P(z_1=\cdots=z_{k+1},z_{k+2}=\cdots=z_{2(k+1)},\ldots,z_{(s-1)(k+1)+1}=\cdots = z_{s(k+1)},z_{s(k+1)+1},z_{s(k+1)+2},\ldots,z_N)=0, \tag{22}
$$

and

$$
P(z_1=\cdots=z_{s(k+1)-i},z_{s(k+1)-i+1},z_{s(k+1)-i+2},\ldots,z_N)=0. \tag{23}
$$

We have not found the generators for the F_i/F_{i+1} ideals nor were we able to find their counting rules for the general case. However, we have solved the problem for several specific cases, which we will present below.

A. Subideals of the $(k,r,s)=(k,2,2)$

We now give the partitions for the generators (smallest degree highest weight polynomials) for the subideals F_i/F_{i+1} for the infinite series $(k,r,s)=(k,2,2)$. The first smallest degree polynomials that vanish when two distinct clusters of $k+1$ particles are formed but does not vanish when one large cluster of $2k+1$ particles is formed is dominated by the root partition

$$
|2k+100k0k0k0k\cdots k0k\rangle:
$$

$$
P(\underbrace{z_1,\ldots,z_1}_{k+1},\underbrace{z_2,\ldots,z_2}_{k+1},z_3,z_4,\ldots)=0,
$$

$$
P(\underbrace{z_1,\ldots,z_1}_{2k+1},z_2,z_3,\ldots)\neq 0. \tag{24}
$$

The above polynomial, as well as the ones we will introduce below, can be written as a linear combination of monomials of partitions dominated by the root partition above, with coefficients that are *uniquely* defined by the HW and clustering conditions. These are, of course, the Jacks. Then, the smallest degree polynomial that vanishes when either two distinct clusters of $k+1$ particles are formed or when a large single cluster of $2k+1$ particles is formed but does not vanish when one large cluster of $2k$ particles is formed is dominated by the partition (see Fig. 7)

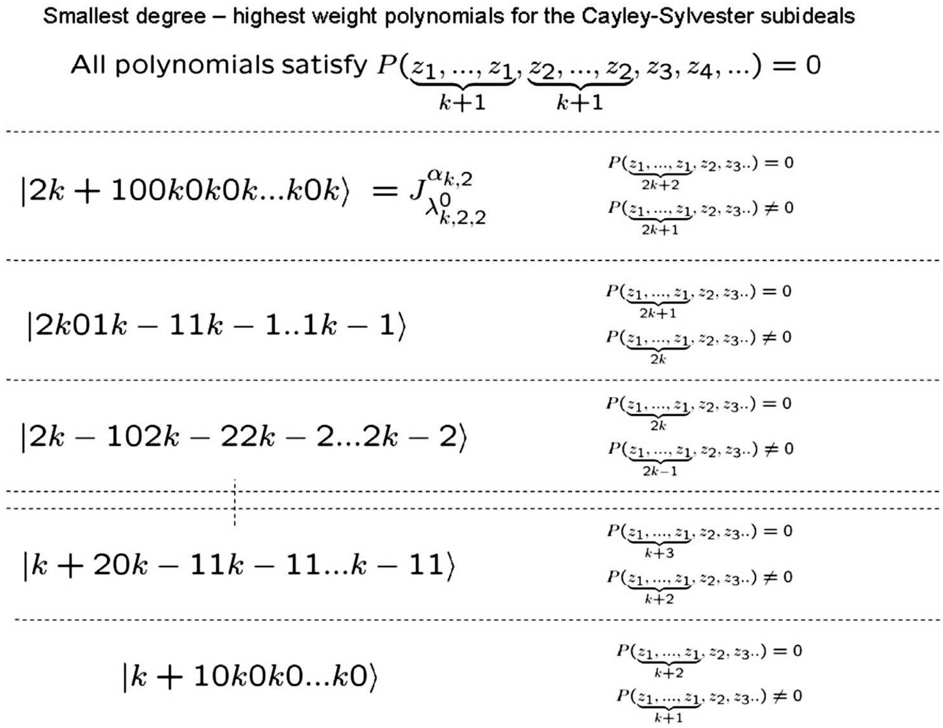

FIG. 7. Cayley–Sylvester subideals for polynomials that satisfy $P(z_1,\ldots,z_1,z_2,\ldots,z_2,z_3,z_4,\ldots)=0$.

$$|2k01k-11k-11k-1\cdots 1k-1\rangle:$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+1},\underbrace{z_2,\ldots,z_2}_{k+1},z_3,z_4,\ldots)=0,$$

$$P(\underbrace{z_1,\ldots,z_1}_{2k+1},z_2,z_3,\ldots)=0, P(\underbrace{z_1,\ldots,z_1}_{2k},z_2,z_3\ldots)\neq 0. \tag{25}$$

The smallest degree polynomial that vanishes when either two distinct clusters of $k+1$ particles are formed or when a large single cluster of $2k$ particles is formed but does not vanish when one large cluster of $2k-1$ particles is formed is dominated by the partition (see Fig. 4)

$$|2k-102k-22k-22k-2\cdots 2k-2\rangle:$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+1},\underbrace{z_2,\ldots,z_2}_{k+1},z_3,z_4,\ldots)=0$$

$$P(\underbrace{z_1,\ldots,z_1}_{2k},z_2,z_3,\ldots)=0,\quad P(\underbrace{z_1,\ldots,z_1}_{2k-1},z_2,z_3,\ldots)\neq 0, \tag{26}$$

and so on. At last, the smallest degree polynomial that vanishes when either two distinct clusters of $k+1$ particles are formed or when a single cluster of $k+2$ particles is formed but does not vanish when one cluster of $k+1$ particles is formed is dominated by the partition (see Fig. 7)

$$|k+10k0k0k\cdots k0\rangle:$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+1},\underbrace{z_2,\ldots,z_2}_{k+1},z_3,z_4,\ldots)=0,$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+2},z_2,z_3,\ldots)=0,\quad P(\underbrace{z_1,\ldots,z_1}_{k+1},z_2,z_3,\ldots)\neq 0. \tag{27}$$

The polynomials of the last subideal [Eq. (29)] are related to the quasiparticle excitations of Abelian and non-Abelian FQH states.[7] They perform well under $k+2$-body repulsive interactions. We can also find the smallest weight partitions of polynomials that vanish when s clusters of $k+1$ particles come together and when $k+2$ particles come together, but do not vanish when $k+1$ particles form a cluster:

$$|\underbrace{k+10k+10\cdots k+10}_{s-1}k0k0k\cdots 0k\rangle:$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+1},\ldots,\underbrace{z_s,\ldots,z_s}_{k+1},z_{s+1},z_{s+2},\ldots)=0,$$

$$P(\underbrace{z_1,\ldots,z_1}_{k+2},z_2,z_3,\ldots)=0,\quad P(\underbrace{z_1,\ldots,z_1}_{k+1},z_2,z_3,\ldots)\neq 0. \tag{28}$$

B. Counting of the F_k/F_{k+1} subideal

The counting of the dimension of the above subideals is a rather difficult (but tractable) problem. The "easy" exceptions are the first subideal, whose generator is $|2k+100k0k0k\cdots k0k\rangle$ and whose counting formulas we have already conjectured in the body of this paper, and the last subideal, whose generator is $|k+10k0k0k\cdots k0k\rangle$, and whose counting formula we will conjecture below. The number $p_{k,2,2}^{F_k/F_{k+1}}(N,M,N_\Phi)$ of polynomials, satisfying the clusterings of Eq. (29), with N variables, with a momentum (total degree M) and of flux (maximum separate degree in any variable) N_Φ is

$$\begin{aligned}
p_{k,2,2}^{F_k/F_{k+1}}(N,M,N_\Phi) &= 0 \quad \text{if} \quad M < 2\cdot(N-n_0) + \frac{1}{k}(N-n_0)(N-n_0-k) \quad \text{or} \quad N_\Phi < 2 + \frac{2}{k}(N-n_0-k)\\
&= \sum_{i=0}^{M-2(N-n_0)} p_{k,2,1}(N-n_0,i,N_\Phi-2); \quad \text{if} \quad 0 \le M \le n_0 N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k)\\
&= \sum_{i=0}^{NN_\Phi-(N-n_0)2-M} p_{k,2,1}(N-n_0,i,N_\Phi-2);\ n_0N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k) < M \le NN_\Phi - (N-n_0)\left(1+\frac{N-n_0}{k}\right)\\
&= 0 \quad \text{if} \quad M > NN_\Phi - (N-n_0)\left(1+\frac{N-n_0}{k}\right),
\end{aligned} \tag{29}$$

where $n_0=k+1$.

By summing the previous expression over all M, we can find the number of polynomials of N variables, with a degree in each variable at most N_Φ and with an unrestricted momentum (total degree) $p_{k,2,2}^{F_k/F_{k+1}}(N,N_\Phi)=\sum_{M=0}^{\infty}p_{k,2,2}^{F_k/F_{k+1}}(N,M,N_\Phi)$. By applying a rule based on the multiplet nature of these polynomials, we find an alternate formula that is not obviously equal to $\sum_{M=0}^{\infty}p_{k,2,2}^{F_k/F_{k+1}}(N,M,N_\Phi)$; extensive numerical checks have, however, confirmed their equivalence:

$$\begin{aligned}
p_{k,2,2}^{\text{subideal}}(N,N_\Phi) &= 0 \quad \text{if} \quad N_\phi < 2 + \frac{2}{k}(N-n_0-k)\\
&= (NN_\Phi - 4N + 4n_0 + 1)\\
&\quad \times \sum_{i=0}^{NN_\Phi} p_{k,2,1}(N-n_0,i,N_\Phi-2)\\
&\quad - 2\sum_{i=0}^{NN_\Phi} i\cdot p_{k,2,1}(N-n_0,i,N_\Phi-2),
\end{aligned} \tag{30}$$

where $n_0=k+1$.

VII. CONCLUSIONS

In this paper, we have made several new conjectures about the behavior of Jack polynomials at a negative Jack parameter α. By applying a HW condition, we find that the (k,r)-admissible partitions of Feigin *et al.*[2] do not exhaust the space of partitions for which the Jack polynomials are well behaved. We find a new infinite series of Jacks, which is described by a positive integer s that vanishes when s distinct clusters of $k+1$ particles are formed but does not vanish when a large cluster of $s(k+1)-1$ particles is formed. We conjecture an empirical counting of polynomials with such clustering properties. We also find the dominant partitions and counting of polynomials that vanish when either s distinct clusters of $k+1$ particles are formed or a cluster of $k+2$ particles is formed but do not vanish when a large cluster of $k+1$ particles is formed. These results will be of physical use in the description of the quasiparticle excitations of the Abelian and non-Abelian fractional quantum Hall states.[7]

[1] B. Sutherland, Phys. Rev. A **4**, 2019 (1971).

[2] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Notices **23**, 1223 (2002).

[3] B. A. Bernevig and F. D. M. Haldane, arXiv:0707.3637 (unpublished).

[4] F. D. M. Haldane, Bull. Am. Phys. Soc. **51**, 633 (2006).

[5] S. H. Simon, E. H. Rezayi, N. R. Cooper, and I. Berdnikov, Phys. Rev. B **75**, 075317 (2007).

[6] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[7] B. A. Bernevig and F. D. M. Haldane (unpublished).

[8] M. Kasatani, T. Miwa, A. N. Sergeev, and A. P. Veselov, *Jack, Hall-Littlewood and Macdonald Polynomials*, Contemporary Mathematics Vol. 417 (American Mathematical Society, Providence, RI, 2006), pp. 207–225.

[9] J. V. Chipalkatti, Arch. Math. **83**, 422 (2004).

[10] R. P. Stanley, Adv. Math. **77**, 76 (1989).

[11] M. Lassalle, J. Funct. Anal. **158**, 289 (1998).

[12] G. E. Andrews, *Theory of Partitions* (Cambridge University Press, Cambridge, 1998).

[13] B. Feigin and S. Loktev, Funct. Anal. Appl. **35**, 44 (2001).

[14] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).

[15] E. Ardonne, J. Phys. A **35**, 447 (2002).

[16] A. V. Stoyanovsky and B. L. Feigin, Funct. Anal. Appl. **28**, 55 (1994).