# Critical dynamics in confined systems with quenched random impurities

H. Chamati[1] and E. Korutcheva[1,2]

[1]*Institute of Solid State Physics, Bulgarian Academy of Sciences, 72 Tzarigradsko Chaussée, 1784 Sofia, Bulgaria*

[2]*Departamento Física Fundamental, Universidad Nacional de Educación a Distancia, c/ Senda del Rey No 9, 28080 Madrid, Spain*

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We study the critical dynamics of a system confined to a hypercubic geometry with periodic boundary conditions in the presence of quenched short-range correlated impurities. By using the random $T_c$ model with purely relaxational dynamics (model A) and the renormalization group method in the vicinity of the upper critical dimension $d_u=4$, we derive to first order in $\epsilon=4-d$ the expression for the relaxation time. Its scaling behavior is discussed in detail both analytically and numerically when the system is quenched from the high-temperature phase toward its bulk critical temperature.

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## I. INTRODUCTION

The effect of quenched random impurities on the critical behavior of magnetic systems has been intensively studied during the last few decades (for a review, see Refs. 1 and 2 and references therein). The interest in these systems stems from the fact that in nature no system is really pure. Indeed, the critical behavior of real systems is affected by the presence of cavities, grain boundaries, or some other kind of disorder. According to the Harris criterion,[3] if in the absence of disorder the specific heat exponent in the vicinity of the critical point is negative, then the critical behavior of the disordered system is the same as the pure one.

Of great interest is the study of the dynamics in order to understand the underlying processes that govern the evolution of the system under consideration in time. In the study of critical dynamics one is interested in the relaxation time of the large-scale fluctuations of the order parameter and other physical quantities near the critical point. Dynamic critical phenomena have been studied extensively both theoretically[4,5] and experimentally[6,7] for different physical systems including disordered spin models.

An important aspect of critical disordered systems is the self-averaging (SA) property[8,9] relevant to the investigation of the physical properties of finite samples. It states that if the system does not exhibit SA, a measurement performed on a single sample does not give a meaningful result and must be repeated on many samples. A numerical study, via computer simulation, of such a system also is quite difficult. As a consequence, the application of finite-size scaling methodology[10,11] is expected to be problematic.[8,9] Recently[12] the problem of self-averaging has been discussed in detail in the framework of the $O(n)$-symmetric $\psi^4$ model in the presence of quenched disorder. These results were confirmed by extensive Monte Carlo (MC) simulations for the *three-dimensional* lattice spin Heisenberg model with quenched impurities.[13] Furthermore, it has been found that the introduction of the scaling corrections are crucial at the critical temperature for the self-averageness of the model for a large range of dilution.[13]

In Ref. 12 a general scheme for the investigation of the critical properties of confined critical disordered $O(n)$ systems has been proposed. Its main idea is to expand the field in Fourier modes and then to treat the zero mode, corresponding to the uniform magnetization, separately from the nonzero ones. The nonzero modes can be treated by the methods developed for bulk systems (e.g., loop expansion), while the zero mode has to be treated exactly. Some physical quantities were obtained up to one-loop order using the renormalization group (RG) method and universal quantities were computed for the *XY* $(n=2)$ and Heisenberg $(n=3)$ cases. The Ising case $(n=1)$ remained beyond the scope of Ref. 12 mainly because of the computational difficulties it presents. This is due to the accidental degeneracy in the recursion relations that needs higher-loop orders.[1,2]

In the present paper we study the critical dynamics of the Landau-Ginzburg $O(n)$ model, describing purely dissipative dynamics of a nonconserved order parameter—model A according to Ref. 4 (for possible experimental realizations of model A, the reader is referred to Ref. 6). We study the case of a finite-size system with periodic boundary conditions in the presence of short-range correlated quenched impurities. This model is extensively used in the literature for the investigation of the dynamic critical behavior of systems with random impurities by means of field-theoretical methods (see, for example, Refs. 14–17 and references therein). The interest in this class of models emanates from the fact that they are simple, yet nontrivial. Their study might turn out to be very useful for more complicated dynamical models. Furthermore, this model is also relevant to computer simulation studies of spin models with Glauber dynamics. We use the $\epsilon=(4-d)$ formalism up to one-loop expansion and derive an explicit form for the linear relaxation time for $1<n<4$ corresponding to the region of stability of the random fixed point. By making an analogy with a corresponding quantum system, we are able to express numerically this dynamic quantity by calculating first the inverse gap of the corresponding quantum-mechanical Hamiltonian. Finally we compare our results to those for pure and disordered systems available in the literature.

The rest of the paper is organized as follows. In Sec. II we introduce the model and derive the corresponding effective action. In Sec. III we report results concerning the finite-size scaling theory applied to our model. Section IV presents our main results concerning critical dynamics and the linear relaxation time. Finally Sec. V contains our conclusions.

 

## II. MODEL AND EFFECTIVE ACTION

The critical behavior of disordered systems can be described by the $d$-dimensional "random-$T_c$" Ginzburg-Landau-Wilson model. This reads

$$\mathcal{H} = \frac{1}{2}\int_{L^d} d^d x\left[ t_0|\psi|^2 + \varphi|\psi|^2 + |\nabla\psi|^2 + \frac{u_0}{12}|\psi|^4\right], \quad (2.1)$$

where $\psi \equiv \psi(\mathbf{x})$ is an $n$-component field with $|\psi|^2 = \Sigma_{i=1}^{n}\psi_i^2$ and the parameters $t_0$ and $u_0$ are model bare constants. $L$ is the linear size of the sample. The random variable $\varphi \equiv \varphi(\mathbf{x})$, introduced to shift the temperature locally due to the presence of impurities, has a Gaussian distribution

$$P(\varphi) = \frac{1}{\sqrt{2\pi\Delta_0}}\exp\left(-\frac{\varphi^2}{2\Delta_0}\right), \quad (2.2)$$

with zero mean and variance

$$\overline{\varphi(\mathbf{x})\varphi(\mathbf{x}')} = \Delta_0\delta^d(\mathbf{x}-\mathbf{x}'). \quad (2.3)$$

Here we will consider a cubic geometry of volume $L^d$ and periodic boundary conditions. This means that the following expansions take place:

$$\psi(\mathbf{x}) = L^{-d}\sum_{\mathbf{k}}\psi(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}), \quad (2.4a)$$

$$\varphi(\mathbf{x}) = L^{-d}\sum_{\mathbf{k}}\varphi(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{x}), \quad (2.4b)$$

where $\mathbf{k}$ is a discrete vector with components $k_j = 2\pi p_j/L$, $p_j = 0, \pm 1, \pm 2, \ldots$, and $j = 1, \ldots, d$, the cutoff is $\Lambda \sim a^{-1}$ ($a$ is the lattice spacing), and we are interested in the continuum limit—i.e., $a \to 0$. Furthermore, the assumptions $L/a \to \infty$ and $\xi \to \infty$, keeping $\xi/L$ finite, have to be fulfilled.

Hereafter we will be interested in the critical dynamics of the model (2.1), with nonconserved order parameter (model A according to Ref. 4) confined to a fully finite geometry. The dynamics in this case is described by the Langevin equation

$$\frac{\partial\psi_i}{\partial\tau} = -\lambda_0\frac{\partial\mathcal{H}}{\partial\psi_i} + \zeta_i, \quad (2.5)$$

where $\lambda_0$ is the Onsager kinetic coefficient and $\zeta_i \equiv \zeta_i(\mathbf{x},\tau)$ is a Gaussian random-white noise that obeys the conditions of zero mean and variance:

$$\langle\zeta_i(\mathbf{x},\tau)\zeta_j(\mathbf{x}',\tau')\rangle = 2\lambda_0\delta^d(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')\delta_{ij}. \quad (2.6)$$

The critical dynamics of the model under consideration may be investigated by introducing a Martin-Siggia-Rose response field[18] $\tilde{\psi}_i(\mathbf{x},\tau)$, $i = 1, \ldots, n$, to average over the thermal noise. The aim of this procedure is to use a path integral formalism, rather than working directly with the Langevin equation. This approach has the advantage of being appropriate for the application of the field-theoretic renormalization group methods.[19,20] Correlation and response functions are computed as functional integrals with weight $\exp(-F_\varphi[\tilde{\psi},\psi])$. The dynamic functional $F_\varphi[\tilde{\psi},\psi]$ is given by

$$F_\varphi[\tilde{\psi},\psi] = \int d^d\mathbf{x}\, d\tau\left[-\lambda_0\tilde{\psi}^2 + \tilde{\psi}\left(\frac{\partial\psi}{\partial\tau} + \lambda_0(-\nabla^2 + t_0 + \varphi)\psi\right) + \frac{1}{6}\lambda_0 u_0\tilde{\psi}\psi^3\right]. \quad (2.7)$$

The average over the disorder variable $\varphi$ can be performed without using the replica trick.[21] This leads to $\exp(-F[\tilde{\psi},\psi]) = \overline{\exp(-F_\varphi[\tilde{\psi},\psi])}$, where the effective action

$$F[\tilde{\psi},\psi] = \int d^d\mathbf{x}\left\{\int d\tau\left[\tilde{\psi}\left(\frac{\partial\psi}{\partial\tau} + \lambda_0(-\nabla^2 + t_0)\psi + \frac{1}{6}\lambda_0 u_0\psi^3\right) - \lambda_0\tilde{\psi}^2\right] - \frac{\lambda_0^2\Delta_0}{2}\left(\int d\tau\,\tilde{\psi}\psi\right)^2\right\} \quad (2.8)$$

and the overbar stands for the average over the distribution of the quenched impurities, given by Eq. (2.3). The Fourier transforms of the Gaussian response and correlation functions for the effective action read

$$R_0(\mathbf{k},\omega) = \frac{1}{-i\omega + \lambda_0(\mathbf{k}^2 + t_0)} \quad (2.9a)$$

and

$$C_0(\mathbf{k},\omega) = \frac{2\lambda_0}{\omega^2 + [\lambda_0(\mathbf{k}^2 + t_0)]^2}, \quad (2.9b)$$

respectively. These two functions are linked through the fluctuation-dissipation theorem which relates the imaginary part of the response function to the correlation function.

The study of the bulk critical behavior of our dynamic model requires the introduction of renormalization constants in order to cure the ultraviolet divergences of the integrals appearing in the perturbation theory of the physical quantities of interest. The theory is renormalized by introducing the scale field amplitudes $Z_\psi$ and $Z_{\tilde{\psi}}$, the renormalization coupling constants $Z_u$ and $Z_\Delta$, and the constant $Z_t$ renormalizing the $\tilde{\psi}\psi$ insertions in the critical theory. The constant $Z_\lambda$ renormalizing the Onsager kinetic coefficient is a combination of both field amplitude factors. In terms of these constants we have

$$t_0 = \mu^2 Z_t t, \quad \lambda_0 = \mu^{-2}Z_\lambda\lambda, \quad Z_\lambda = (Z_\psi/Z_{\tilde{\psi}})^{1/2},$$

$$u_0 = \mu^\epsilon Z_u\hat{u}, \quad \Delta_0 = \mu^\epsilon Z_\Delta\hat{\Delta}, \quad (2.10)$$

where $\mu$ is an external momentum scale and the renormalized parameters $\hat{u}$ and $\hat{\Delta}$ are defined through

$$\hat{u} = u\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}, \quad \hat{\Delta} = \Delta\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}.$$

This renormalization procedure is used in conjunction with the dimensional regularization—i.e., analytic continuation in $d$. The renormalization constants are introduced to absorb the poles at the upper critical dimension. These are known to two-loop order,[14] but here we will need only the first loop terms—i.e.,

$$Z_t = 1 + \frac{n+2}{6\epsilon}\hat{u} - \frac{1}{\epsilon}\hat{\Delta}, \tag{2.11a}$$

$$Z_u = 1 + \frac{n+8}{6\epsilon}\hat{u} - \frac{6\hat{\Delta}}{\epsilon}, \tag{2.11b}$$

$$Z_\Delta = 1 + \frac{n+2}{3\epsilon}\hat{u} - \frac{4}{\epsilon}\hat{\Delta}, \tag{2.11c}$$

$$Z_\lambda = 1 + \frac{\hat{\Delta}}{\epsilon}. \tag{2.11d}$$

The $\beta$ functions associated with $\hat{u}$ and $\hat{\Delta}$ have the form

$$\beta_u = -\hat{u}\epsilon + \frac{n+8}{6}\hat{u}^2 - 6\hat{u}\hat{\Delta}, \tag{2.12a}$$

$$\beta_\Delta = -\hat{\Delta}\epsilon - 4\hat{\Delta}^2 + \frac{n+2}{3}\hat{u}\Delta, \tag{2.12b}$$

and the fixed point values of the parameters $\hat{u}$ and $\hat{\Delta}$ up to one-loop order are[1,2]

$$\frac{\hat{u}^\star}{8\pi^2} = \frac{3}{2(n-1)}\epsilon, \quad \frac{\hat{\Delta}^\star}{8\pi^2} = \frac{4-n}{8(n-1)}\epsilon. \tag{2.13}$$

Note that this fixed point is stable for numbers of components of the order parameter, $n$, belonging to the interval $1<n<4$.[1,2]

When the system is finite in all directions and subject to periodic boundary conditions, it is easy to convince oneself, using the expansion (2.4a), that the ultraviolet singularities of the bulk free propagators are not affected. In this case, the only divergent term in the theory is the one with $\mathbf{k}=\mathbf{0}$ corresponding to the bulk uniform magnetization. Therefore one can use the bulk renormalization constants to study the finite-size scaling properties of the system. An important consequence is that the renormalized cumulants $\mathbf{C}_R^{\tilde{N},N} = Z_{\tilde{\psi}}^{-\tilde{N}/2} Z_{\psi}^{N/2}\mathbf{C}^{\tilde{N},N}$ are subject to the same RG equations as for the bulk system investigated in Ref. 14. Here we will extract the scaling behavior when the system is confined to a finite geometry. Then, we have

$$\left[\mu\partial_\mu + \beta_u\partial_u + \beta_\Delta\partial_\Delta + \eta_t t\,\partial t + \eta_\lambda\lambda\partial_\lambda + \frac{1}{2}\tilde{N}\tilde{\eta}_\psi + \frac{1}{2}N\eta_\psi\right]\mathbf{C}_R^{(\tilde{N},N)} = 0, \tag{2.14}$$

with $\beta_u = \mu\partial_{\mu|0}u$, $\beta_\Delta = \mu\partial_{\mu|0}\Delta$, and $\eta_w = \mu\partial_{\mu|0}\partial_\mu|0 \ln Z_w$ for $w=t,\lambda,\psi$. In these expressions the notation $\partial_\mu|_0$ means that the derivations are performed at the fixed parameters of the bare theory. The linear size $L$ of the system does not renormalize.[11] Using the method of characteristics to solve the above RG differential equations, at the fixed point, one obtains the scaling behavior

$$\mathbf{C}_R^{(\tilde{N},N)}(\{\mathbf{k},\omega\};L,t,u) \approx \lambda^{\tilde{N}} L^{-D_{\tilde{N}N}} \Theta^{(\tilde{N},N)}\left(\left\{\mathbf{k}L, \frac{\omega}{\lambda L^{-z}}\right\}; tL^{1/\nu}\right), \tag{2.15a}$$

with

$$D_{\tilde{N}N} = -(\tilde{N}+N)\beta/\nu - \tilde{N}z. \tag{2.15b}$$

This is the general scaling form of the correlation and response functions and the quantities related to them. In Eqs. (2.15), the parameters $\beta$, $\nu$, and $z$ are the usual critical exponents. From Eqs. (2.15) one can extract the finite-size scaling expressions for the vertex functions of the effective action. In particular for the renormalized coupling constants of the present model we expect to have the scaling forms $\tilde{t} = L^{\eta-2}f_t(tL^{1/\nu})$, $\tilde{u} = L^{d-4+2\eta}f_u(tL^{1/\nu})$, $\tilde{\Delta} = L^{d-4+2\eta}f_\Delta(tL^{1/\nu})$, and $\tilde{\lambda}^{-1} = L^{z-2+\eta}f_\lambda(tL^{1/\nu})$ with the corresponding universal scaling functions $f_w$, $w=t,u,\Delta,\lambda$, and $\eta$ denoting Fisher's critical exponent.

An important quantity for critical dynamics is the linear relaxation time that measures the relaxation of the system toward its equilibrium state. Following Ref. 19, this can be expressed in terms of the correlation and response functions. According to the above RG considerations it is possible to obtain its finite-size scaling form. This is given by

$$\tau_R = L^{-z} f_\tau(tL^{1/\nu}). \tag{2.16}$$

The function $f_\tau(x)$ is a universal scaling function of its argument. In the next section we will check these predictions for our model and determine the expressions for the scaling functions and discuss their properties to the lowest nontrivial order in the $\epsilon$ expansion.

## III. FINITE-SIZE SCALING

In a first approximation we neglect the loop contributions, originating from the nonzero modes, to the effective action. This corresponds to the mean-field case—i.e., $d>4$. Then the zero mode may be treated exactly.[12]

In the case $d<4$, loop corrections are crucial as they are responsible for long-distance singularities. Their effect can be accounted for by decomposing the fields $\tilde{\psi}$ and $\psi$ into zero-momentum components $\tilde{\psi}_0$ and $\psi_0$, which play the role of a uniform magnetization, and orthogonal complements $\tilde{\sigma}$ and $\sigma$, which depend upon the nonzero modes:

$$\tilde{\psi}(\mathbf{x},\tau) = \tilde{\psi}_0(\tau) + \tilde{\sigma}(\mathbf{x},\tau), \quad \psi(\mathbf{x},\tau) = \psi_0(\tau) + \sigma(\mathbf{x},\tau). \tag{3.1}$$

The integration over the nonzero modes $\mathbf{k}\neq 0$ in Eq. (2.8) for the case of a finite cubic geometry of size $L$ with periodic boundary conditions gives the following final expression for the effective functional:

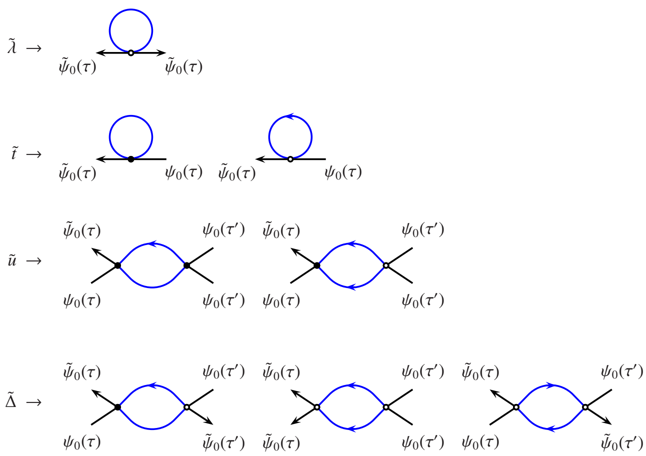


FIG. 1. (Color online) One-loop diagrams showing the relevant contributions to the coupling constants. The solid and open circles correspond to $u_0$ and $\Delta_0$, respectively. The external legs with arrows represent the fields $\tilde{\psi}_0(\tau)$, while the internal propagators with arrows denote the response functions.

$$F[\tilde{\psi}_0,\psi_0]=L^d\left\{\int d\tau\left[\tilde{\psi}_0\left(\frac{\partial\psi_0}{\partial\tau}+\tilde{\lambda}\tilde{t}\psi_0+\frac{1}{6}\tilde{\lambda}\tilde{u}\psi_0^3\right)-\tilde{\lambda}\tilde{\psi}_0^2\right]\right.$$
$$\left.-\frac{1}{2}\lambda^2\tilde{\Delta}\left(\int d\tau\,\tilde{\psi}_0\psi_0\right)^2\right\}, \tag{3.2}$$

where the effective model constants, taking into account the finite-size corrections, are denoted by tildes.

The relevant Feynman diagrams showing the finite-size contributions that shift the bulk critical temperature $t_0$, the Onsager coefficient $\lambda_0$, and the coupling constants $u_0$ and $\Delta_0$ of the bulk theory are presented in Fig. 1. From the first line of this figure one can see that the finite geometry affects $\lambda_0$ already at one-loop order at variance with the case of pure systems, where it is size independent at this order.[22,23] The finite-size contribution $\lambda_L$ to $\tilde{\lambda}$ is purely static and is given by

$$\lambda_L=-\lambda_0\Delta_0L^{-d}\sum_{\mathbf{k}}{}'\frac{1}{(\mathbf{k}^2+t_0)^2}, \tag{3.3}$$

where the prime indicates that the term corresponding to the zero mode is omitted. The first diagram contributing to $\tilde{t}$ is purely static, while the second one and all those corresponding to the coupling constants $u_0$ and $\Delta_0$ generate nonlocal in time interactions. Let us first consider in some details the second diagram associated with $\tilde{t}$. This is

$$\lambda_0\Delta_0\int d\tau\int d\tau'\;\tilde{\psi}_0(\tau)\psi_0(\tau')L^{-d}\sum_{\mathbf{k}}{}'e^{-\lambda_0(\mathbf{k}^2+t_0)(\tau-\tau')}. \tag{3.4}$$

This may be evaluated with the help of the Taylor expansion

$$\psi_0(\tau')=\sum_{s=0}\frac{1}{s!}\psi_0^{(s)}(\tau)(\tau'-\tau)^s \tag{3.5}$$

to get

$$\lambda_0\Delta_0\int d\tau\,\tilde{\psi}_0(\tau)\sum_{j=0}^{\infty}\frac{(-1)^s}{(\lambda_0)^s s!}\psi_0^{(s)}L^{-d}\sum_{\mathbf{k}}{}'\frac{1}{(\mathbf{k}^2+t_0)^{s+1}}. \tag{3.6}$$

Note that corrections involving higher-order derivatives would lead to corrections of order higher than $\epsilon^{1/2}$, since $\psi_0\sim\epsilon^{1/8}$ and consequently $\tilde{\psi}_0\sim\epsilon^{-1/8}$. This can be read off from the behavior of the term $\int dt\,\tilde{\psi}_0\psi_0^3\sim O(1)$ needed to stabilize the effective action at the fixed point. So only the static term corresponding to $s=0$ in Eq. (3.6) is to be kept for the evaluation of the effective action. The same is true for the rest of the Feynman diagrams. Consequently only the static contributions enter the effective action to one-loop order. It is worth mentioning that by virtue of the fluctuation-dissipation theorem only the static finite-size contributions survive at the lowest loop order.

The finite-size correction terms $t_L,u_L,\Delta_L$, calculated up to one-loop order via the aforementioned diagrammatic technique, read

$$t_L=\left[\frac{n+2}{6}u_0-\Delta_0\right]\frac{1}{L^d}\sum_{\mathbf{k}}{}'\frac{1}{\mathbf{k}^2+t_0}, \tag{3.7a}$$

$$u_L=-\left[u_0^2\frac{n+8}{6}-6u_0\Delta_0\right]\frac{1}{L^d}\sum_{\mathbf{k}}{}'\frac{1}{(\mathbf{k}^2+t_0)^2} \tag{3.7b}$$

$$\Delta_L=-\left[u_0\Delta_0\frac{n+2}{3}-4\Delta_0^2\right]\frac{1}{L^d}\sum_{\mathbf{k}}{}'\frac{1}{(\mathbf{k}^2+t_0)^2}. \tag{3.7c}$$

Finally, the renormalized theory is obtained by expressing the bare constants and the fields by their renormalized counterparts using (2.10). This procedure leads to the following scaling relations expressed in terms of the scaling variable $y=tL^{1/\nu}$ with $\nu^{-1}=2-\frac{3n}{8(n+2)}\epsilon+O(\epsilon^2)$:

$$\tilde{t}L^2=y+\frac{3n\epsilon}{16(n-1)}[y\ln y+4F_{4,2}(y)], \tag{3.8a}$$

$$\tilde{u}L^{\epsilon}=u^{\star}\left[1+\frac{1}{2}(1+\ln y)\epsilon+2\epsilon\frac{\partial}{\partial y}F_{4,2}(y)\right], \tag{3.8b}$$

$$\tilde{\Delta}L^{\epsilon}=\Delta^{\star}\left[1+\frac{1}{2}(1+\ln y)\epsilon+2\epsilon\frac{\partial}{\partial y}F_{4,2}(y)\right], \tag{3.8c}$$

$$\frac{L^{2-z}}{\tilde{\lambda}}=\frac{1}{\lambda}\left[1-\frac{4-n}{16(n-1)}\left(1+4\frac{\partial}{\partial y}F_{4,2}(y)+\ln y\right)\epsilon\right]. \tag{3.8d}$$

Here $z=2+\frac{4-n}{8(n-1)}\epsilon+O(\epsilon^2)$ and

$$F_{d,2}(x)\int_0^{\infty}dz\,\exp\left(-\frac{xz}{(2\pi)^2}\right)\left[\left(\sum_{l=-\infty}^{\infty}e^{-zl^2}\right)^d-1-\left(\frac{\pi}{z}\right)^{d/2}\right]. \tag{3.9}$$

Some particular values of the constant $F_{d,2}(0)$ and a method of its calculation can be found in Ref. 24. Notice that Eqs.

(3.8) verify the finite-size scaling hypothesis as discussed in Sec. II. Let us recall that the Fisher critical exponent $\eta$ is of the order of $O(\epsilon^2)$.

### IV. DYNAMIC CRITICAL PHENOMENA

Following Ref. 15, in order to calculate the relaxation time, one has to transform the Fokker-Planck equation, equivalent to the functional, corresponding to the homogeneous mode, Eq. (3.2) to a Schrödinger equation in imaginary time, by identifying the variable $\psi$ with a coordinate $q$, and calculate the eigenvalues by means of a variational method.

Alternatively,[22] one can integrate over the auxiliary field $\tilde{\psi}$ in Eq. (2.7), which leads to a complex expression in terms of the bare parameters of the theory, $\lambda_0$, $t_0$, $u_0$, and $\Delta_0$, and the dimension of the order parameter $n$. The analysis of the ensuing expression shows that terms depending on the variance of the disorder $\Delta \sim \epsilon$ could be neglected as they contribute with corrections of higher orders in $\epsilon$. To the lowest order in $\epsilon$, the effective functional has formally the same structure as that for the corresponding pure model,[22] *but* now the parameters depend upon the disorder through their fixed point values within the RG analysis.

Both analyses give an effective functional that we can associate to a quantum-mechanical problem in $n$ dimensions, described by the following Hamiltonian:

$$\mathcal{H}[p,q] \equiv \frac{p^2}{2(L^d/2\tilde{\lambda})} + \frac{1}{4}L^d\tilde{\lambda}q^2\left(\tilde{t} + \frac{\tilde{u}}{6}q^2\right)^2 - \frac{1}{12}(n+2)\tilde{\lambda}\tilde{u}q^2 - \frac{1}{2}n\tilde{\lambda}\tilde{t}, \quad (4.1)$$

where $p$ is the momentum conjugated to $q$.

The relaxation time, describing the decay of the correlations with the time, is related to the gap of the Hamiltonian (4.1) and is given by $\tau_R(L)=(E_1-E_0)^{-1}$, where $E_0$ and $E_1$ are the ground-state and first excited-state energies of the quantum-mechanical Hamiltonian.

After a suitable rescaling of the variables $q \to L^{-d/4}\tilde{u}^{-1/4}q$ and $p \to L^{d/4}\tilde{u}^{1/4}p$, we can identify the relaxation time $\tau_R(L)$:

$$\tau_R(L) = \frac{1}{2\lambda}\tilde{u}^{-1/2}L^{d/2}g_n\left(\frac{1}{2}\tilde{u}^{-1/2}L^{d/2}\tilde{t}\right), \quad (4.2)$$

where the function $g_n$ is the inverse gap of the quantum-mechanical Hamiltonian (4.1). The subscript $n$ indicates the number of components of the model.

After renormalization of the parameters entering in the expression of the relaxation time (4.2) and after performing some rearrangement, we get the following finite-size scaling form for $\tau_R$:

$$\tau_R = \frac{1}{2}L^z\frac{L^{2-z}}{\tilde{\lambda}}\frac{1}{(\tilde{u}L^{\epsilon})^{1/2}}g_n\left(\frac{1}{2}\frac{\tilde{t}L^2}{(\tilde{u}L^{\epsilon})^{1/2}}\right). \quad (4.3)$$

The function $g_n$, depending on the parameter $\rho=\tilde{t}L^{d/2}/\sqrt{\tilde{u}}$, is the inverse gap of the quantum-mechanical Hamiltonian

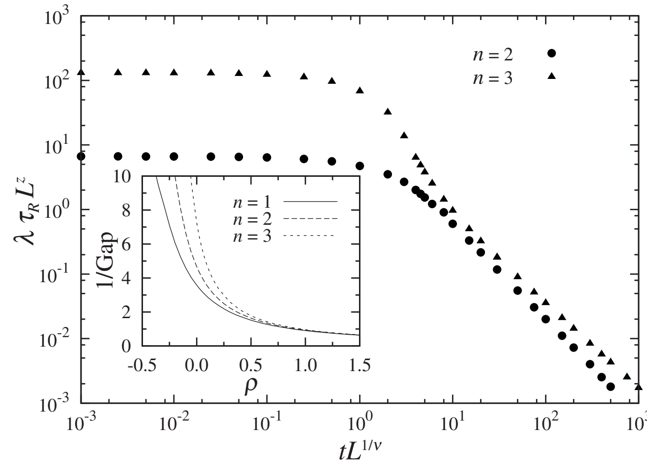


FIG. 2. Behavior of the universal scaling function $\lambda\tau_R L^{-z}$ associated with the linear relaxation time as a function of the scaling variable $tL^{1/\nu}$ in logarithmic scale. In the inset we show the behavior of the inverse gap of the quantum-mechanical Hamiltonian (4.4) as a function of $\rho$.

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}q^2\left(\frac{\rho}{2} + \frac{1}{12}q^2\right)^2 - \frac{(n+2)}{4!}q^2 - \frac{n}{4}\rho. \quad (4.4)$$

Here $p^2=-\frac{d^2}{dq^2}$ and $\rho$ is given at the fixed point and up to one-loop order by[12]

$$\rho^{\star} = \frac{1}{\sqrt{u^{\star}}}\left\{y - \frac{1}{4}y\left[1 - \frac{4-n}{4(n-1)}\ln y\right]\epsilon + \frac{3n}{4(n-1)}F_{4,2}(y)\epsilon - y\frac{\partial}{\partial y}F_{4,2}(y)\epsilon\right\}. \quad (4.5)$$

To obtain the behavior of the gap between the ground state and first excited state of the quantum-mechanical system (4.4) we follow the procedure devised in Ref. 25 for the pure Ising model by choosing for the first excited state a variational wave function of the form $\Psi_1=N_1 q\exp[-\alpha(q^2+\beta)^2]$, where $\alpha$ and $\beta$ are variational parameters. Computing $\langle\Psi_1|\mathcal{H}|\Psi_1\rangle/\langle\Psi_1|\Psi_1\rangle$ and minimizing it with respect to $\alpha$ and $\beta$, we obtain an estimate of the first state energy and hence the value of the gap. The behavior of the inverse gap is shown in the inset of Fig. 2 for $n=1,2,3$. The gap for the case corresponding to the Ising model—i.e., $n=1$—coincides with the result reported in Ref. 25.

The correct scaling form of the relaxation time in Eq. (4.3) is given by

$$\tau_R(y) = \frac{L^z}{\lambda}\frac{1}{4\pi}\sqrt{\frac{n-1}{3\epsilon}}\left[1 - \frac{3n}{16(n-1)} \times\epsilon\left(1 + \ln y + 4\frac{\partial}{\partial y}F_{4,2}(y)\right)\right]g_n\left(\frac{1}{2}\rho(y)\right). \quad (4.6)$$

This expression shows that the dynamic finite-size scaling hypothesis, predicted in Eq. (2.16), holds for this class of models with random impurities. It has a universal scaling form in the sense that it is independent of the microscopic details of the model.

At the transition temperature—i.e., $y=0$—$\tau_R(y)$ is finite, which can be demonstrated by using the asymptotic behavior

$$F_{4,2}(y) = -8\ln 2 + \frac{1}{4}y\mathcal{C} - \frac{1}{4}y\ln y + O(y^2), \tag{4.7}$$

with

$$\mathcal{C} = \int_0^\infty \frac{du}{u}\left[\exp\left(-\frac{u}{4\pi^2}\right) - \frac{u^2}{\pi^2}\left(\sum_{l=-\infty}^{\infty} e^{-ul^2}\right)^4 + \frac{u^2}{\pi^2}\right] = 2.2064\ldots \quad . \tag{4.8}$$

$\tau_R(0)$ thus reads

$$\tau_R(0) = \frac{L^z}{\lambda}\frac{1}{4\pi}\sqrt{\frac{n-1}{3\epsilon}}\left[1 - \frac{3n\mathcal{C}}{16(n-1)}\epsilon\right] \times g_n\left(-\frac{n\ln 2}{2\pi}\sqrt{\frac{3\epsilon}{n-1}}\right). \tag{4.9}$$

For the particular cases of three-dimensional $XY$ ($n=2$) and Heisenberg ($n=3$) models with random impurities, we have the universal amplitudes ($T=T_c$)

$$\lambda\frac{\tau_R(0)}{L^z} = \begin{cases} 0.14, & n=2, \\ 2.46, & n=3. \end{cases} \tag{4.10}$$

For $y \gg 1$, corresponding to the limit $L\to\infty$, with fixed $t$, we recover the bulk critical behavior—i.e., $\tau \sim t^{-z\nu}$.

Figure 2 shows the behavior of the rescaled relaxation time $\lambda\tau_R L^{-z}$ as a function of the scaling parameter $y=tL^{1/\nu}$ in a log-log scale. For a three-dimensional system this is obtained numerically from Eq. (4.6) with $\epsilon=1$. In order to plot Eq. (4.6) we have omitted the constant $\frac{1}{4\pi}\sqrt{\frac{n-1}{3}}$, which contributes only to a shift along the vertical axis in the graph of the log-log plot. Furthermore, we have replaced the $\epsilon$-dependent prefactor by $\exp\{\epsilon\frac{3n}{n-1}[1+\ln y+4\frac{\partial}{\partial y}F_{4,2}(y)]\}$, since it is the same to the lowest order in $\epsilon$. The figure represents the relaxation time for values of the number of the components of the order parameter $n=2,3$, where the "random" fixed point is stable. It shows a universal behavior, which is independent on the microscopic details of the model.

The critical dynamic behavior has been widely discussed in the literature such as, for example, in the case of pure $XY$ systems,[26] the Heisenberg model,[27] or the three-dimensional Heisenberg spin glasses.[28] The behavior of the relaxation time, in the case corresponding to $XY$ models, shows a nonmonotonic character with a pronounced maximum for some value of the scaling parameter, which is not present in our system with quenched impurities. The results obtained for the pure Heisenberg model are consistent with those shown in Fig. 2 at least qualitatively. A quantitative comparison would require the use of more accurate techniques. In the case of the spin-glass Heisenberg model, off-equilibrium Monte Carlo simulations, based on the finite-size scaling analysis, show a qualitatively similar behavior as that obtained in the present analysis.

## V. DISCUSSION

Our main result is the expression of the critical relaxation time, Eq. (4.6), for finite systems with short-range correlated quenched impurities, which gives the correct asymptotic behavior characteristic of the bulk limit. Up to first order in $\epsilon=4-d$, the direct effect of the presence of quenched impurities is through the fixed-point values of the parameters $u$ and $\Delta$. The correct treatment of the finite-size scaling is performed by taking into account the corrections originating from the shift of the Onsager kinetic coefficient $\lambda$.

The numerical estimates of the relaxation time for different values of the number of the components of the order parameter $n$ are derived by using the analogy with the corresponding quantum-mechanical system. For $1<n<4$, where the "random" fixed point, corresponding to the critical behavior of the disordered system, is stable, we give the numerical values (Fig. 2) for the relaxation time in terms of the scaling variable $tL^{1/\nu}$ by calculating the inverse gap of the corresponding Hamiltonian. The relaxation time shows a plateau for small values of the scaling parameter before decreasing away from the critical region.

A possible extension of the approach might be the calculation of the distribution of the relaxation times by using the representation of the correlation functions in terms of quantum-mechanical eigenfunctions. Due to the presence of impurities, a continuous spectrum of the distribution of the relaxation time is expected.

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