Transmission and reflection of phonons and rotons at the superfluid helium-solid interface

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We solve the problem of the transmission and reflection of phonons and rotons at the interface between superfluid helium and a solid for all angles of incidence and in both directions. A consistent solution of the problem is presented, which allows us to rigorously describe the simultaneous creation of phonons and R^- and R^+ rotons in helium by either a phonon from the solid or a helium quasiparticle incident on the interface. The interaction of all He II quasiparticles with the interface, as well as their transmission, reflection, and conversion into each other, is described in a unified way. The angles of propagation and the probabilities of creating quasiparticles are obtained for all of the cases. The Andreev reflection of helium phonons and rotons is predicted. Energy flows through the interface due to phonons and R^- and R^+ rotons are derived. The small contribution of the R^- rotons is due to the small probability of an R^- roton being created by a phonon in the solid, and vice versa. This explains the failure to directly create beams of R^- rotons before the experiments of Tucker and Wyatt in 1999 [Science **283**, 1150 (1999)]. Experiments for creating R^- rotons by beams of high-energy phonons are suggested.

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I. INTRODUCTION

Many physical properties of continuous media at low temperatures can be described in terms of quasiparticles. The quasiparticles of superfluid helium are called phonons, R^- rotons, and R^+ rotons. They have a nonmonotonic dispersion curve and the R^- rotons have a negative group velocity, i.e., their momentum is directed opposite to the group velocity; see Fig. 1. The phonons and rotons are observed in many experiments, such as in neutron scattering in helium[1] and in direct experiments[2,3] wherein beams of superfluid helium quasiparticles are created by a heated solid. The quasiparticles propagate in helium and interact and reflect from different surfaces. Also, they quantum evaporate helium atoms from the free surface. These have been investigated both experimentally and theoretically (see, for example, Refs. 4–8). Interestingly, R^- rotons had not been detected in direct experiments until 1999, when they were finally created by a specially constructed source.[3] They were observed by quantum evaporation. All of the earlier attempts to create R^-, with ordinary solid heaters, were unsuccessful.

The problem of the interaction of rotons in superfluid helium with interfaces (their reflection, transmission, and mode change) was first considered in Ref. 9. However, the method used there did not take into account the simultaneous creation of phonons and R^+ and R^- rotons by a phonon in the solid incident on the interface, and it could not distinguish between the R^+ and R^- rotons. Later, in Ref. 10, it was shown that R^- rotons cannot be created at the interface with a solid by a phonon from the solid, provided we can neglect the possibility of the creation of the other quasiparticles in the same process at the interface. In the current work, a consistent solution is introduced, which allows us to rigorously solve the problem of the simultaneous creation of phonons and rotons. Also, it describes the interaction of all He II quasiparticles with the interface (their transmission, reflection, and conversion into each other) in a unified way. These are the fundamental elementary processes that determine the heat exchange between He II and a solid, and the associated phenomena, such as the Kapitza temperature jump (see, for example, Ref. 11). We investigate all of these phenomena. The probability of the creation of each quasiparticle at the interface is derived for all cases. The failures of attempts to detect R^- rotons before experiments[3] is explained, and predictions are made for experiments on the interaction of phonons and rotons with a solid and the creation of R^- rotons at the interface by high energy phonons (h-phonons).

We describe superfluid helium with its distinctive dispersion relation $\Omega(k)$, with the maxon maximum and roton minimum, within the framework of the theory developed in Ref. 10. The quantum fluid is considered as a continuous medium at all length scales. This model is based on the fact that the thermal de Broglie wavelength of a particle of a quantum fluid exceeds the average interatomic separation. Then, the variables of the continuous medium can only be assigned values at each mathematical point of space in a probabilistic sense.

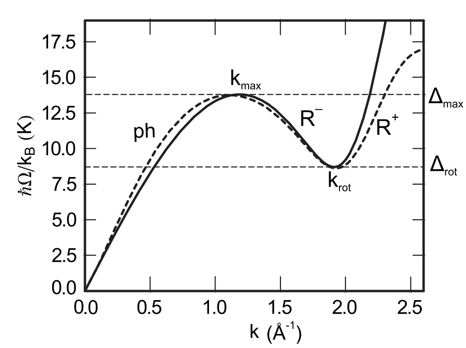

FIG. 1. The solid line is $\Omega(k)$ from Eq. (1) for $s=230.7$ m/s, $k_g=1.9828$ Å^{-1}, and $\lambda=-0.9667$; the dashed line is the measured dispersion curve of superfluid helium (Ref. 1) at the saturated vapor pressure.

The idea to describe superfluid helium as a continuous medium at microscopic scales has been successfully used for decades. Atkins[12] used it in the 1950s to describe the mobility of electrons and ions in He II, when he introduced bubbles and snowballs of microscopic size. Lately, the vortices in superfluid helium with cores of sizes on the scale of interatomic distances are being extensively studied; see, for example, Ref. 13 and the references cited therein. Some recent simulations on the dynamics of atoms in helium nanodroplets[14,15] also affirm that He II is described well as a continuous medium at microscopic scales.

As showed in Ref. 10, application of the methods of the theory of continuous medium at microscopic scales requires the relationships between the variables of the continuous medium to become nonlocal. In Ref. 16, the nonlocal hydrodynamics was introduced to describe small oscillations in superfluid helium, and in Refs. 17 and 18, it was used to describe ripplon-roton hybridization and the dispersion relation of ripplons. The nonlocality allows one to analyze a continuous medium with an arbitrary dispersion relation. This possibility was discussed in Ref. 19. However, the theoretical justification of this approach remained on the intuitive level until the work in Ref. 10.

In this paper, by following Ref. 10, the quasiparticles are described as wave packets propagating in the superfluid. The long wavelength excitations are phonons, while the short wavelength ones are rotons. R^- rotons correspond to the descending part of the dispersion curve and have a negative group velocity, i.e., they propagate in the direction opposite their momentum. This simple model allows us to use the methods of the theory of continuous medium and avoid the difficulties that appear in other phenomenological models, such as those in Refs. 17 and 18.

So, with the help of boundary conditions for the continuous media at the interface, we find the probabilities of the creation of all the quasiparticles when any of them is incident on the interface. These are the energy reflection and transmission coefficients for the corresponding wave packets. The method allows us to obtain the analytical expressions for the probabilities as functions of angle and frequency.

In Sec. II, we formulate the problem and obtain the general solution of the nonlocal equations of the quantum fluid in half-space. We consider a parametrized dispersion relation that is a good approximation of the measured dispersion curve of superfluid helium. The solution is sought in the form that generalizes the solution for a monotonic dispersion of a general form obtained in Ref. 20. The consequences of using the boundary conditions are discussed. These include multiple critical angles, backward refraction, and retroreflection (or Andreev reflection[21]) of phonons and rotons (see also Ref. 18).

In Sec. III, the boundary conditions are used to derive both the amplitude and energy reflection and transmission coefficients for any incident wave for arbitrary incidence angles. The preliminary results for rotons at normal incidence were discussed in the authors' report at a conference.[22]

In Sec. IV, the energy flows through the interface due to phonons and R^- and R^+ rotons are calculated as functions of temperature. The contribution of the R^- rotons to the energy flows in both directions is shown to be small. This means that R^- rotons are hardly created by a solid heater and are poorly detected by a solid bolometer. This explains why R^- rotons could not be detected in direct experiments until the work in Ref. 3. There, they were created by a source made up of two heaters facing each other, which allowed mode changes, and detection was achieved by quantum evaporation.

The results obtained in this work can be used in other fields of physics. In particular, they are important for classical acoustics, wherein the problem of wave transmission through an interface has been solved only for the case wherein the dispersion relations of both adjacent media are strictly linear (see, for example, Ref. 23), let alone nonmonotonic. This problem concerning real media, with nonlinear dispersion, was of interest in the middle of the past century[24] and is still relevant today.[25,26]

II. DERIVATION OF EQUATIONS AND THEIR SOLUTION

A. Problem formulation

Let us consider two continuous media separated by a sharp interface $z=0$. In the region $z<0$, there is an ordinary continuous medium with sound velocity $s_{\rm sol}$ and equilibrium density $\rho_{\rm sol}$. For the solid, we only take into account longitudinal waves.

Transverse waves can be treated in the same framework of the theory of continuous media without any difficulty. However, the calculations become much more cumbersome, while on the whole, the situation does not change. Due to the very small impedance of the solid-helium interface (see Sec. II D and below), the reflection coefficients hardly change at all. For the transmitted waves, additional critical angles appear, corresponding to the sound velocity of the transverse waves. Also, it should be noted that taking into account both the longitudinal and transverse waves in the solid allows one to consider the contribution of Rayleigh waves, which contribute to the transmission coefficients of He II quasiparticles into the solid at fixed incidence angles. For phonons with linear dispersion, this problem was solved in Ref. 27. The problem for the helium-solid interface may be the subject of our next paper.

The region $z>0$ is filled with the quantum fluid with an equilibrium density ρ_0 and a dispersion relation $\Omega(k)$, such that

$$\Omega^2(k) = s^2k^2\left\{1 + 2\lambda\frac{k^2}{k_g^2} + \frac{k^4}{k_g^4}\right\}. \tag{1}$$

Here, s is the sound velocity at zero frequency, k_g is the wave vector that determines the scale of the curve, and λ determines the form of the curve. For a range of parameters, this relation is a good approximation of the measured nonmonotonic dispersion relation of superfluid helium (see Fig. 1). For $\lambda<-1$, there are real k, such that $\Omega^2(k)<0$, which is nonphysical, and for $\lambda>-\sqrt{3}/2$, the curve is monotonic. For $\lambda\in(-1,-\sqrt{3}/2)$, the curve has the roton minimum at $k=k_{\rm rot}$ and the maxon maximum at $k=k_{\rm max}$, as it should. We adopt the following set of values: $s=230.7$ m/s,

$k_g = 1.9828$ Å^{-1}, and $\lambda = -0.9667$. Then, the dispersion curve has the following parameters: the coordinates of roton minimum $k_{rot} = 0.9670 k_g = 1.913$ Å^{-1} and $\Delta_{rot} = \hbar\Omega(k_{rot})/k_B = 8.712$ K (k_B is the Boltzmann constant); the maxon maximum is $\Delta_{max} = \hbar\Omega(k_{max})/k_B = 13.8$ K. These values are the experimentally measured parameters of the superfluid helium dispersion curve at the saturated vapor pressure.[1]

We describe this quantum fluid by nonlocal hydrodynamics, as developed in Ref. 10. Accordingly, the quantum fluid, as well as the ordinary fluid on the other side of the interface, obeys the linearized equations of continuous media:

$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla P, \tag{2}$$

where $\mathbf{v}$ is the hydrodynamic velocity and ρ and P are the deviations of density and pressure from the respective equilibrium values (for brevity, below, we refer to them as just density and pressure). The difference is that the pressure and density in the quantum fluid are related through the following nonlocal relation:

$$\rho(\mathbf{r}) = \int_{z'>0} d^3 r' h(|\mathbf{r} - \mathbf{r}'|) P(\mathbf{r}'), \tag{3}$$

in which the integration domain is the region filled by the quantum fluid.[28]

The suggested model describes well the interface between superfluid helium and a solid because for solids, the relationships are local. In the frequency range of the dispersion curve of He II, the dispersion laws of most solids, such as the heater materials of copper or gold, are very close to linear and they can be described as ordinary continuous media.

Equations (3) and (2) lead to the integrodifferential equation for pressure:

$$\Delta P(\mathbf{r}, t) = \int_{z'>0} d^3 r' h(|\mathbf{r} - \mathbf{r}'|) \ddot{P}(\mathbf{r}', t), \tag{4}$$

which is set for $x, y, t \in (-\infty, \infty), z \in (0, +\infty)$. In the infinite medium, when the integration and definition domains are infinite, the Fourier transform of Eq. (4) gives us the relationship between the Fourier transform of the kernel $h(r)$ and the dispersion relation of the fluid $\Omega(k)$ (Ref. 10):

$$h(k) = \frac{k^2}{\Omega^2(k)}. \tag{5}$$

For dispersion relation (1), we obtain the following from the Fourier transform of Eq. (5):

$$h(r) = \frac{k_g^4}{4\pi s^2 r} \frac{1}{k_+^2 - k_-^2} (e^{ik_+ r} - e^{-ik_- r}), \tag{6}$$

where k_+ and $(-k_-)$ are the poles of $h(k)$ in the upper half-plane $\mathbf{C}_+$:

$$k_\pm = k_g(\sqrt{1-\lambda} \pm i\sqrt{1+\lambda})/\sqrt{2}, \quad k_+ = k_-^* \in \mathbf{C}_+. \tag{7}$$

Here, the asterisk denotes the complex conjugate and $\mathbf{C}_+$ is the upper complex half-plane. Due to the last condition, the kernel [Eq. (6)], despite the complex notation, is real.

There is no convolution product in Eq. (4), either in the sense of one- or two-sided Fourier transform or Laplace transform, because the lower limit by z' is finite while the kernel is symmetrical, $h(\mathbf{r}) = h(r)$.

We consider the problem of waves transferring through the interface. As the relations in Eq. (2) are local and coincide with the notation used in the equations of an ordinary ideal continuous medium, the two boundary conditions on the interface (local) are obtained from their integral forms in the usual way by using the theory of continuous medium:

$$P(x,y,z=-0,t) = P(x,y,z=+0,t),$$

$$V_z(x,y,z=-0,t) = V_z(x,y,z=+0,t). \tag{8}$$

By applying the solutions of the equations of continuous media on both sides of the interface, the boundary conditions in Eq. (8) give us the solution in the whole space and thus provide us with all the coefficients of reflection and transmission. The solution in the solid is well known, and the solution in the quantum fluid is derived in Sec. II B.

B. Solution of Equation (4) in half-space

The equation that determines the relationship between k and ω:

$$\Omega^2(k) = \omega^2, \tag{9}$$

with $\Omega^2(k)$ from Eq. (1) is sixth order with respect to k. Its six roots are functions of ω and are denoted k_μ for $\mu = 1, \ldots, 6$. We note that if we used more terms in the polynomial $\Omega^2(k)$, then the higher order equation for k would give six real roots and the other roots would be imaginary.

In the problem of waves transferring through the interface, the two boundary conditions in Eq. (8) can be satisfied for all x, y, and t only if all of the waves present on both sides of the interface have the same frequency ω and transverse component of wave vector $\mathbf{k}_\tau$. A single monochromatic wave is not a solution of Eq. (4). Therefore, as there are in total six roots of Eq. (9), we search for the solution as a sum of six monochromatic waves, with the same frequency ω and transverse component of wave vector k_τ (the y axis is chosen along $\mathbf{k}_\tau$), i.e., with the following form:

$$P(\mathbf{r}, t) = \sum_{\mu=1}^{6} A_\mu \exp[i(\mathbf{k}_\mu \mathbf{r} - \omega t)]. \tag{10}$$

Here, the vectors $\mathbf{k}_\mu$ are

$$\mathbf{k}_\mu = k_{\mu z} \mathbf{e}_z + k_\tau \mathbf{e}_y, \tag{11}$$

$$k_\mu^2 = k_{\mu z}^2 + k_\tau^2. \tag{12}$$

The transverse component k_τ is real for physical reasons, but k_μ and $k_{\mu z}$ can be either real or complex because we have to ensure the boundedness of our solution only in the half-space $z > 0$.

After substitution of Eq. (10) into Eq. (4), we obtain the system of equations for the amplitudes A_μ and the equations for $k_{\mu z}$ as functions of ω and k_τ. The system for A_μ is

$$\sum_{\mu=1}^{6} A_\mu (k_{\mu z} - k_{+z})^{-1} = 0,$$

$$\sum_{\mu=1}^{6} A_\mu (k_{\mu z} + k_{-z})^{-1} = 0, \tag{13}$$

where

$$k_{\pm z}^2 = k_\pm^2 - k_\tau^2, \quad k_{+z} = k_{-z}^* \in \mathbf{C}_+. \tag{14}$$

The equations for $k_{\mu z}$ are reduced to the following form:

$$\Omega^2(k_\mu^2 = k_{\mu z}^2 + k_\tau^2) = \omega^2 \quad \text{for } \mu = 1, \ldots, 6. \tag{15}$$

The system of two homogeneous equations for the amplitudes A_μ [Eq. (13)] ensures that no (nontrivial) solutions exist with less than three nonzero amplitudes A_μ, i.e., there are no eigensolutions of Eq. (4) in the half-space consisting of less than three monochromatic waves. This is the consequence of the nonlocality, which changes Eq. (4) in the presence of the interface. In infinite space, on the contrary, the domain of integration is the whole space, and the equation is solved by the Fourier transform, and its solution is a superposition of plane waves with dispersion (1).

A solution of Eq. (4), with the smallest possible number of waves being 3, is constructed in form (10), with three terms out of six, by picking a subset of any three different roots $\{k_\alpha, k_\beta, k_\gamma\}$ out of the set of six $\{k_\mu\}_{\mu=1,\ldots,6}$. Then, it can be rewritten with the help of Eq. (13) in the form that contains a single amplitude:

$$P_{\{k_\alpha,k_\beta,k_\gamma\}}(\mathbf{r},t) = P_{\alpha\beta\gamma}^{(0)} \left\{ \frac{(k_{\alpha z} - k_{+z})(k_{\alpha z} + k_{-z})}{(k_{\alpha z} - k_{\beta z})(k_{\alpha z} - k_{\gamma z})} e^{ik_{\alpha z} z} + \frac{(k_{\beta z} - k_{+z})(k_{\beta z} + k_{-z})}{(k_{\beta z} - k_{\gamma z})(k_{\beta z} - k_{\alpha z})} e^{ik_{\beta z} z} + \frac{(k_{\gamma z} - k_{+z})(k_{\gamma z} + k_{-z})}{(k_{\gamma z} - k_{\alpha z})(k_{\gamma z} - k_{\beta z})} e^{ik_{\gamma z} z} \right\} e^{i(k_\tau y - \omega t)}, \tag{16}$$

where $P_{\alpha\beta\gamma}^{(0)}$ is chosen so that $P_{\{k_\alpha,k_\beta,k_\gamma\}}(\mathbf{r}=0, t=0) = P_{\alpha\beta\gamma}^{(0)}$.

The velocity is obtained from Eqs. (16) and (2):

$$\mathbf{v}_{\{k_\alpha,k_\beta,k_\gamma\}}(\mathbf{r},t) = \frac{P_{\alpha\beta\gamma}^{(0)}}{\rho_0} \left\{ \frac{\mathbf{k}_\alpha}{\omega} \frac{(k_{\alpha z} - k_{+z})(k_{\alpha z} + k_{-z})}{(k_{\alpha z} - k_{\beta z})(k_{\alpha z} - k_{\gamma z})} e^{ik_{\alpha z} z} + \frac{\mathbf{k}_\beta}{\omega} \frac{(k_{\beta z} - k_{+z})(k_{\beta z} + k_{-z})}{(k_{\beta z} - k_{\gamma z})(k_{\beta z} - k_{\alpha z})} e^{ik_{\beta z} z} + \frac{\mathbf{k}_\gamma}{\omega} \frac{(k_{\gamma z} - k_{+z})(k_{\gamma z} + k_{-z})}{(k_{\gamma z} - k_{\alpha z})(k_{\gamma z} - k_{\beta z})} e^{ik_{\gamma z} z} \right\} e^{i(k_\tau y - \omega t)}. \tag{17}$$

As the two conditions in Eq. (13) restrict the number of free amplitudes in Eq. (10) from 6 to 4, any four linear-independent solutions of form (16) constitute the basis set of solutions of Eq. (4) for given ω and k_τ, and any solution consisting of four, five, or six monochromatic waves can be represented as their linear combination.

C. Roots of dispersion equation: In and out solutions

The roots of Eq. (9) with $\Omega^2(k)$ from Eq. (1) with respect to k^2 are $k_i^2 = k_g^2 \xi_i$ for $i=1,2,3$, where ξ_i are the three dimensionless roots of the following cubic equation:

$$\xi^3 + 2\lambda\xi^2 + \xi - \chi^2 = 0. \tag{18}$$

Here, $\chi = \omega/(sk_g)$ is the dimensionless frequency; $\xi_i(\lambda, \chi)$ are some elaborate complex-valued functions.

The most interesting frequency range is $\chi \in (\chi_{\text{rot}}, \chi_{\max})$, where $\chi_{\text{rot}}(\lambda)$ and $\chi_{\max}(\lambda)$ are the dimensionless frequencies that correspond to the roton minimum and maxon maximum, respectively. For such frequencies, there are three types of running waves in the quantum fluid, corresponding to phonons and R^- and R^+ rotons. The branches are numbered in this case in ascending order of the absolute values of their wave vectors k_i: $0 < k_1 < k_{\max} < k_2 < k_{\text{rot}} < k_3$, so that $i=1$ corresponds to phonons, $i=2$ to R^- rotons, and $i=3$ to R^+ rotons.

We now consider the problem of quasiparticle transfer through the interface. The quasiparticles are treated as wave packets that propagate in the two media. Therefore, when we build the solutions in the quantum fluid, we have to take into account that wave packets, as well as quasiparticles, propagate with their group velocities $d\Omega/d\mathbf{k}$.[29] So, a wave packet of the quantum fluid composed of waves with wave vectors close to $\mathbf{k}_0$, with its length $k_0 < k_{\max}$ (so that it is a phonon wave packet) and the zth component $k_{0z} > 0$, propagates away from the interface; however, a wave packet composed of waves with wave vectors close to wave vector with length $k_0 \in (k_{\max}, k_{\text{rot}})$ (so it is an R^- roton packet) and the zth component $k_{0z} > 0$ propagates toward the interface.

Let us construct the solution in the quantum fluid P_{out} (the "out solution") that is realized when a wave in the solid is incident on the interface. This solution should contain only such waves that constitute wave packets traveling away from the interface (i.e., waves with positive group velocity) or waves that are damped at $z \to +\infty$. Picking three out of six vectors $\mathbf{k}_\mu$ is the same as picking their normal components $k_{\mu z}$, because they all have the same k_τ. The six normal components $k_{\mu z}$, which are obtained as solutions of Eq. (15), are grouped into three pairs of roots $\pm\sqrt{k_i^2 - k_\tau^2}$ for $i=1,2,3$. For each pair with the same i, either both roots are real, which occurs for small enough angles $k_\tau < k_i$, or both roots are imaginary for $k_\tau > k_i$. In the first case, one root corresponds to a wave traveling toward the interface and the other corresponds to a wave traveling away from it. In the second case, one root gives a damped wave in $z > 0$, while the other gives an exponentially unbounded wave. So, P_{out} contains no more than three waves (and no less because there are no such solutions) and therefore has the form of Eq. (16) (see Fig. 2). The squared normal components of the three constituent waves are

$$k_{iz}^2 = k_i^2 - k_\tau^2 \quad \text{for } i = 1,2,3. \tag{19}$$

We define the signs of roots k_{iz} for P_{out} to be made up of waves with the normal components of wave vectors equal to k_{1z}, k_{2z}, and k_{3z}. Then, taking into account the negative group velocity of R^- rotons,[29] for the signs of real k_{iz}, we obtain $k_{1z}, k_{3z} > 0$ and $k_{2z} < 0$. If $k_\tau > k_i$ for some i, the correspond-

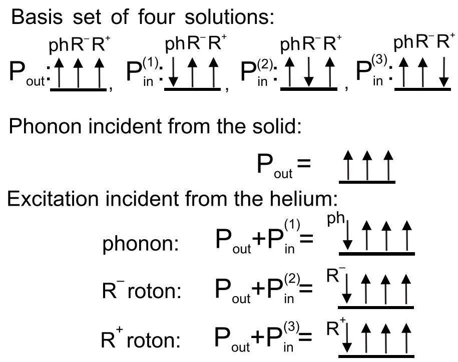

FIG. 2. The basis set of solutions in the superfluid helium in half-space and their sums, which correspond to the different incident excitations.

ing wave, $\sim\exp(ik_{iz}z)$, is bounded in $z>0$ for $\mathrm{Im}\,k_{iz}>0$. Then, in the general case, we have

$$0<k_{1z}<(-k_{2z})<k_{3z}\ \text{ if }\ k_\tau\in(0,k_1),$$

$$0<(-k_{2z})<k_{3z},\quad k_{1z}\in\mathbf{C}_+\ \text{ if }\ k_\tau\in(k_1,k_2),$$

$$0<k_{3z},\quad k_{1z},k_{2z}\in\mathbf{C}_+\ \text{ if }\ k_\tau\in(k_2,k_3). \tag{20}$$

Then, the out solution has the form of Eq. (16), with the three wave vectors picked from the set of six with normal components k_{1z}, k_{2z}, and k_{3z}:

$$P_{\mathrm{out}}=P_{\{k_\alpha,k_\beta,k_\gamma\}}[k_{\alpha z}=k_{1z},k_{\beta z}=k_{2z},k_{\gamma z}=k_{3z}]. \tag{21}$$

In order to solve the problem of a wave transferring through the interface from superfluid helium into the solid, we also need solutions containing waves that are traveling toward the interface (i.e., wave packets comprised of these waves should be traveling toward the interface). We define them in the way that is illustrated in Fig. 2. The solution $P_{\mathrm{in}}^{(1)}$ is constructed of waves with zth components of wave vectors $-k_{1z}$, k_{2z}, and k_{3z}, with the amplitudes related through Eq. (13). Solution $P_{\mathrm{in}}^{(2)}$ is constructed of waves with k_{1z}, $-k_{2z}$, and k_{3z}. The last solution, $P_{\mathrm{in}}^{(3)}$ contains waves with k_{1z}, k_{2z}, and $-k_{3z}$. Then, the three sorts of in solutions, which correspond to the three types of the incident waves, can be written in the following form:

$$P_{\mathrm{in}}^{(i)}=\ P_{\mathrm{out}}\big|_{k_{iz}\to(-k_{iz})}\quad\text{for}\ \ i=1,2,3. \tag{22}$$

The $P_{\mathrm{in}}^{(i)}$ solution corresponds to the incident wave of type i. So, a linear combination of P_{out} and $P_{\mathrm{in}}^{(2)}$ consists of one R^- roton wave ($i=2$) that corresponds to the R^- roton wave packet incident on the interface and all three waves that correspond to the reflected phonon and R^- and R^+ roton wave packets. The amplitudes of the phonon and R^+ roton waves are the sums of the amplitudes of those waves present in both P_{out} and $P_{\mathrm{in}}^{(2)}$. This is the solution in $z>0$ realized when an R^- roton is incident on the interface. The four solutions in Eqs. (21) and (22) are linearly independent due to their structure and can be used as the basis set of solutions, as mentioned at the end of Sec. II B.

For $\chi\in(\chi_{\mathrm{rot}},\chi_{\max})$, the roots k_{iz} are fully defined by Eqs. (19) and (20). For $\chi\in(0,\chi_{\mathrm{rot}})$, the roots $\xi_{2,3}$ are complex and $\xi_2=\xi_3^*$; then, k_{2z} and k_{3z} are defined so that the roton waves, $\sim\exp(ik_{2z,3z}z)$, are damped, so $k_{2z}=-k_{3z}^*\in\mathbf{C}_+$. In the limit $\chi\to0$, the phonon waves have an almost linear dispersion, $k_1\approx\omega/s$, and it can be shown that $k_{2z}\to k_{+z}$ and $k_{3z}\to-k_{-z}$. Therefore, the amplitudes of all roton waves that contain multipliers $(k_{2z}-k_{+z})$ and $(k_{3z}+k_{-z})$ go to zero [see Eq. (16)], and the general solution tends to the ordinary superposition of incident and reflected phonon waves, with wavelengths much greater than the scale of nonlocality, $|k_\pm|^{-1}$. This limiting case can also be obtained by passing to the long wave limit, $h(r)\to\delta(r)/s^2$, in Eq. (4), thus making it a local wave equation.

Equation (4) was solved earlier in Ref. 20 for the case of an arbitrary but monotonic dispersion relation. The Wiener and Hopf method was used there. The application to this type of problem was suggested in Ref. 28 and developed in Ref. 30. It is much more general and seems to be more rigorous than the method used here, although less straightforward. The solution of Ref. 20 can be generalized to the nonmonotonic case and can be shown to exactly yield solutions (21) and (22) for the dispersion relation (1).

In Ref. 17, the problem analogous to Eq. (4) was considered in order to investigate the hybridization of rotons and ripplons. There, the problem in the half-space was replaced by the one in the infinite medium with a symmetrical extrapolation of the solutions with respect to variable z. However, in contrast to the usual differential (local) equations, for an integral equation such as Eq. (4), the solution cannot be symmetrically extrapolated to $z<0$. Indeed, if we formally consider the solution of Eq. (4) in $z<0$, it would be unambiguously defined by the solution in $z>0$ [from Eq. (16)] through the integral of Eq. (4). Direct substitution shows that the full solution is not an even function. This might be the reason that the method[17] gave wrong results near the surface wave threshold and was then rejected; in the next paper by the authors,[18] another approach was used for that problem.

D. Multiple critical angles and the Andreev reflection

Even before applying the boundary conditions in Eq. (8) and finding the solution in the whole space, we can use the fact that two linear boundary conditions for the variables of continuous media are satisfied on the interface and derive a number of important consequences. First of all, we see that the solution is always constructed in such a way that there are in total four outgoing (i.e., reflected and transmitted) waves: one in the solid and three in the quantum fluid. This is because the four conditions on the wave amplitudes, two from Eq. (8) and two from Eq. (13), can all be satisfied only when there are at least four outgoing waves. On the other hand, they can be at most four either because in the formulation of the problem there is only one incident wave or because of the requirement that the solution is bounded when some of k_{iz} are complex.

Furthermore, the two boundary conditions in Eq. (8) imply that all of the waves constituting the full solution have the same frequency ω and tangential component of wave vector k_τ. When wave i is propagating at an angle θ_i to the normal to the interface, $k_\tau=k_i\sin\theta_i$. Then, if one of the

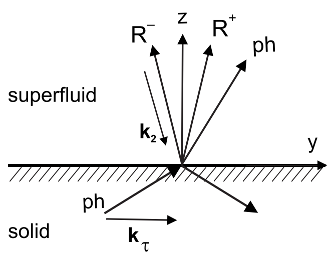

FIG. 3. When a phonon in the solid is incident on the interface, three quasiparticles, a phonon, an R^- roton, and an R^+ roton, are created in superfluid helium with the same ω and k_τ. The created R^- roton propagates backward in the transverse direction (i.e., retrorefracted).

waves is incident, the corresponding angle and k_τ are set and all the other transmission and reflection angles are determined by the generalization of Snell's law:

$$\frac{\sin\,\theta_{\rm sol}}{s_{\rm sol}} = \frac{\sin\,\theta_1}{s_1} = \frac{\sin\,\theta_2}{s_2} = \frac{\sin\,\theta_3}{s_3}. \qquad (23)$$

Here, $s_i=\omega/k_i(\omega)$ are the phase velocities of the corresponding waves that depend on frequency; $\mathbf{k}_{\rm sol}=k_{{\rm sol}\,z}\mathbf{e}_z+k_\tau\mathbf{e}_y$ is the wave vector of the wave in the solid and $s_{\rm sol}=\omega/k_{\rm sol}(\omega)$ $=$const. The reflection angle for the wave of the same type as the incident one is equal to the incidence angle.

From now on, we will consider $s_{\rm sol}>s$, as is the case when superfluid helium is adjacent to a solid. Usually, even the strong inequality holds. Then, we have

$$s_{\rm sol} > s_1 > s_2 > s_3 > 0. \qquad (24)$$

If a wave from the solid is incident at $\theta_{\rm sol}$, it is reflected at the same angle and the three waves are transferred into helium at angles $\theta_i<\theta_{\rm sol}$. The R^- roton wave, as opposed to the others, due to its negative group velocity, propagates backward in the tangential direction (i.e., in the direction $y\to-\infty$, see Fig. 3).

Assume a wave i is incident from helium at θ_i. Then, according to Eq. (23), the transmitted wave in the solid has $\sin\,\theta_{\rm sol}<1$ for $\sin\,\theta_i<s_i/s_{\rm sol}$, and if the incidence angle is greater than the critical value, $k_{{\rm sol}\,z}$ is imaginary and the wave in the solid is exponentially damped. Thus, we obtain the three angles of full internal reflection:

$$\sin\,\theta_i^{\rm cr} = s_i/s_{\rm sol} \quad \text{for } i=1,2,3. \qquad (25)$$

In the same way, there are three new critical angles defined for $i>j$ (so that $s_i<s_j$):

$$\sin\,\theta_{ij}^{\rm cr} = s_i/s_j < 1 \quad \text{for}\{i,j\}=\{2,1\},\{3,2\},\{3,1\}. \qquad (26)$$

If a wave i is incident and $\theta_i<\theta_{ij}^{\rm cr}$, then wave j (with $j<i$) has $\theta_j\in(\theta_i,\pi/2)$ and $k_{jz}\in\mathbf{R}$. For $\theta_i>\theta_{ij}^{\rm cr}$ we have $k_{jz}^2<0$, the jth wave is damped and the corresponding quasiparticle is not created.

In an ordinary fluid, the group velocity of a wave packet is the same as the sound velocity and is constant. When a wave is incident on the interface, the reflected wave has the same wave number and due to preservation of the transverse component of the wave vector $\mathbf{k}_\tau$, the wave is reflected forward. This qualitative picture is maintained in the majority of all known physical systems. However, when the reflected wave (or quasiparticle) is qualitatively different from the incident, another possibility can be realized. So, when an electron of a normal metal is incident on the interface with a superconductor, it can be retroreflected and converted into the hole of a negative effective mass that travels back along the same line as the incident electron. This effect was discovered by Andreev (see Ref. 21) and is called the Andreev reflection or retroreflection.

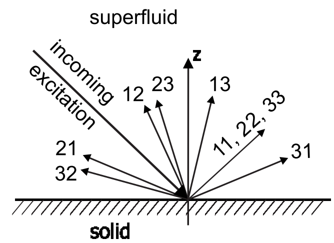

FIG. 4. When a single beam of quasiparticles is incident on the interface, a set of reflected beams is created. The beam marked ij consists of quasiparticles of type j created by incident quasiparticles of type i. Beams ii are specularly reflected, while all others propagate in different directions. Beams 12, 21, 23, and 32 are retroreflected.

In our case, when a helium quasiparticle is incident on the interface, three different quasiparticles are created in helium, with corresponding probabilities. If, for example, an R^- roton is created on the interface, along with other quasiparticles, when a phonon is incident, for the sake of brevity, we will refer to this as "the phonon is reflected into R^- roton." While phonons and R^+ rotons behave like ordinary quasiparticles in these processes, R^- rotons propagate in the direction opposite their wave vector due to the negative group velocity, as was already mentioned above. Therefore, when a phonon or R^+ roton is incident on the interface, the phonons and R^+ rotons are reflected forward, while R^- rotons are reflected backward, or *retroreflected*. In this way, the transverse components of wave vectors $\mathbf{k}_\tau$ are all equal. Likewise, when an R^- roton is incident, the phonon and R^+ roton are retroreflected. Thus, we have described the effect of the Andreev reflection of helium phonons and rotons.

If a monochromatic beam of phonons and rotons is incident on the interface at some angle Θ, there will be up to seven reflected beams (see Fig. 4). We denote them as ij, which means "the beam of quasiparticles of type j created on the interface by the incident quasiparticles of type i." Beams 11, 22, and 33 are reflected forward at the incidence angle θ and, therefore, constitute a single beam ii. Beams 13 and 31 are also reflected forward, while 12, 21, 32, and 23 are reflected *backward*. Due to the relations in Eq. (23), beams 32 and 21 are reflected at angles greater than θ and beams 12 and 23 are reflected at angles less than θ.

The reflection angles depend on s_i, which are functions of frequency. Therefore, with an incident beam that is non-monochromatic, the reflected beams all become angularly diffused,[20] except for beam ii. However, the relations in Eq. (24) hold at all frequencies, and therefore, the qualitative picture is not modified. If we place a detector on the same side from the normal as the source at greater angles, it should register the R^- rotons of beam 32 and phonons of beam 21, which were retroreflected. Likewise, the detector at smaller angles should register the R^- rotons of beam 12 and R^+ rotons of beam 23. Such an experiment could be carried out in order to qualitatively verify the current theory.

A successful experiment would very much depend on the intensities of the beams to be detected and, therefore, on the different creation probabilities for the quasiparticles at the interface. The derivation of these probabilities is the subject of Sec. III.

III. REFLECTION AND TRANSMISSION COEFFICIENTS

A. Phonon in the solid incident on the interface

When a phonon in the solid is incident on the interface, it is reflected and three quasiparticles of different types are created in helium, which travel away from the interface (see Fig. 3). The probability of quasiparticle creation is the fraction of incident energy that is reflected or transmitted as the corresponding wave packet. Amplitude reflection and transmission coefficients can be derived in the approximation of plane waves.[30]

In this approximation, we consider a plane wave with frequency ω and wave number, $k_{\mathrm{sol}}(\omega)=\omega/s_{\mathrm{sol}}$, incident on the interface at angle θ_{sol} to the normal. Then, the solution in the solid is the sum of the incident and reflected waves The solution in the quantum fluid is P_{out} from Eq. (21), which consists of three waves. All the waves have the same frequency ω and transverse component of the wave vector, $k_\tau = k_{\mathrm{sol}} \cos \Theta_{\mathrm{sol}}$. The pressure amplitude of each wave P_i is the full coefficient multiplying the exponent $\exp(ik_{iz}z)$ in the out solution. With the help of the boundary conditions in Eq. (8), the amplitudes of all the waves are expressed through the amplitude of the incident wave. Then, after some transformations, the amplitude reflection coefficient, which is defined as the ratio of pressure amplitudes in the reflected and incident waves, can be expressed in the following form:

$$r_{\rightarrow} = \frac{f_z - Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}}. \tag{27}$$

Here, the following notations are used. Z is a real generalization of impedance:

$$Z = Z_g \cos \theta_{\mathrm{sol}}, \quad Z_g = Z_0 \chi, \tag{28}$$

where $Z_0=(\rho_0 s)/(\rho_{\mathrm{sol}} s_{\mathrm{sol}})$ is the ordinary impedance of the interface at zero frequency; $\chi=\omega/sk_g$ is the dimensionless frequency as introduced in Eq. (18); $\tilde{\delta}$ is a dimensionless constant:

$$\tilde{\delta}(\lambda) = \frac{k_{+z} - k_{-z}}{ik_g} \in \mathbf{R}, \tag{29}$$

which is real due to Eq. (14);

$$f_z = f_{3z}/(k_g f_{2z}), \tag{30}$$

where

$$f_{nz} = k_{1z}^n (k_{2z} - k_{3z}) + k_{2z}^n (k_{3z} - k_{1z}) + k_{3z}^n (k_{1z} - k_{2z}) \quad \text{for } n = 2,3.$$

As $s_{\mathrm{sol}} > s_i$, the transmission angles for all the waves θ_i are less then θ_{sol}, so $k_{iz} \in \mathbf{R}$ and f_z is a dimensionless real function of χ and k_τ/k_g.

The full transmission coefficient is $t_{\rightarrow}=1+r_{\rightarrow}$; the partial transmission coefficients $t_i^{\rightarrow}$ are defined as the ratios of the pressure amplitudes of each of the three waves in helium, P_i, to the pressure amplitude of the incident wave. They are obtained in the same way as $r_{\rightarrow}$:

$$t_i^{\rightarrow} = t_{\rightarrow} \frac{\psi_i}{(k_{iz} - k_{jz})(k_{iz} - k_{kz})}, \tag{31}$$

where

$$\psi_i = (k_{iz} - k_{+z})(k_{iz} + k_{-z}),$$

and the subscripts take values $\{i,j,k\}=\{1,2,3\}$+perm (perm is for permutations). Henceforth, the subscripts $\{i,j,k\}$ in the expressions of the kind of Eq. (31) take the same set of values, unless stated otherwise.

All the amplitude coefficients are complex-valued functions of frequency and incidence angle. Therefore, there are always nontrivial phase shifts between the incident, reflected, and transmitted waves.

The energy reflection and transmission coefficients are the normal components of energy density flux, which are expressed as fractions of the incident energy flux, that are reflected or transmitted into helium. The energy density flux in a wave packet in the quantum fluid, as shown in Ref. 29, equals the average energy density multiplied by the group velocity. It was shown in Ref. 30 that the average energy density in a wave packet or plane wave in the quantum fluid with velocity amplitude V_i is given by the same relation as in the ordinary liquid, $\rho_0|V_i|^2$, and from Eq. (17), $V_i=P_i/(\rho_0 s_i)$. The group velocity of wave i can be obtained from Eq. (1):

$$|u_i| = \frac{s^2}{k_g^4} \frac{k_i}{\omega} |(k_i^2 - k_j^2)(k_i^2 - k_k^2)|. \tag{32}$$

Taking all of this into account and with the help of Eq. (31), after some transformations, we obtain the fractions of the normal component of the incident wave packet's energy flux that are carried by waves of each type $i=1,2,3$ in helium. Those are the partial energy transmission coefficients:

$$D_i^{\rightarrow} = \frac{4Z}{(Z+f_z)^2 + \tilde{\delta}^2} \frac{k_{iz}}{k_g} \frac{(k_{iz}+k_{jz})(k_{iz}+k_{kz})}{(k_{iz}-k_{jz})(k_{iz}-k_{kz})}. \tag{33}$$

We note that $D_i^{\rightarrow} > 0$ for all $i=1,2,3$. The full energy transmission coefficient can be expressed in the following form:

$$D_{\rightarrow} = \sum_{i=1}^{3} D_i^{\rightarrow} = \frac{4Zf_z}{(Z+f_z)^2 + \tilde{\delta}^2}. \tag{34}$$

The energy reflection coefficient is $R_{\rightarrow}=|r_{\rightarrow}|^2$. Then, from Eqs. (27) and (34), after some algebraic transformations, we can explicitly show that

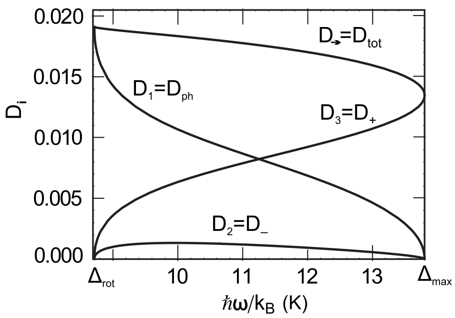

FIG. 5. Energy dependence of the transmission coefficients at $\theta=0$, with energy in temperature units, for the parametrized dispersion curve (see Fig. 1) and $Z_0=0.01$. The R^- roton creation probability D_2 is small.

$$R_{\rightarrow}+D_{\rightarrow}=1, \tag{35}$$

and so, energy is conserved when waves go through the interface. This also verifies that the coefficients $D_i^{\rightarrow}$ and $R_{\rightarrow}$ are the probabilities of the creation of the corresponding quasiparticles at the interface.

Superfluid helium has a very small density and sound velocity, such that at the interface with a solid, the strong inequalities $s \ll s_{\mathrm{sol}}$ and $\rho_0 \ll \rho_{\mathrm{sol}}$ hold. Then, taking into account Eq. (24), we have a set of small parameters:

$$Z_0 \ll 1, \quad s_i/s_{\mathrm{sol}} \ll 1. \tag{36}$$

It can be shown that due to the first condition of Eq. (36), in the sums of Eqs. (27), (34), and (33), the quantity Z can be neglected in comparison with the other terms. The second condition of Eq. (36) implies that due to Eq. (23), all the transmission angles into the helium are very small, as is indeed well known for the interfaces between He II and solids. Then, $k_\tau^2 \ll k_i^2, k_\pm^2$ and due to Eqs. (7) and (14), we have $k_{\pm z} \approx k_\pm$. In this approximation, we obtain $f_z \approx f \equiv f_z(\chi, \Theta_{\mathrm{sol}}=0)$ and $\tilde{\delta} \approx \delta \equiv (k_+ - k_-)/ik_g$. Then, the dependence of $D_i^{\rightarrow}$ on the incidence angle, from Eq. (33), is factorized out and is reduced to the multiplier $\sim\cos\theta_{\mathrm{sol}}$:

$$D_i^{\rightarrow}(\chi,\theta_{\mathrm{sol}}) \approx \frac{4Z_g \cos\theta_{\mathrm{sol}}}{f^2+\delta^2}\left.\left(\frac{k_{iz}}{k_g}\frac{(k_{iz}+k_{jz})(k_{iz}+k_{kz})}{(k_{iz}-k_{jz})(k_{iz}-k_{kz})}\right)\right|_{\theta_{\mathrm{sol}}=0}. \tag{37}$$

The frequency dependence of the transmission factors at normal incidence, $\theta_{\mathrm{sol}}=0$, is shown in Fig. 5. The relative creation probabilities of phonons and R^- and R^+ rotons are determined by the multipliers in parenthesis in Eq. (37). They can be rewritten in terms of k_i while taking care of the signs: k_{1z} and k_{3z} at $\theta_{\mathrm{sol}}=0$ are equal to k_1 and k_3, but k_{2z} at $\theta_{\mathrm{sol}}=0$ is equal to $(-k_2)$ (because of the negative group velocity of R^- rotons). Then, for $i=2$, we obtain

$$D_2^{\rightarrow} \propto \frac{k_2-k_1}{k_2+k_1}\frac{k_3-k_2}{k_3+k_2}k_2. \tag{38}$$

Both the first and second multipliers here are less than unity. In the analogous expressions for $D_{1,3}^{\rightarrow}$, one of the two corresponding multipliers is reversed. So, for the ratio $D_2^{\rightarrow}/D_{1,3}^{\rightarrow}$, the effect is squared and we obtain

$$D_2^{\rightarrow} \ll D_{1,3}^{\rightarrow}. \tag{39}$$

Near the roton minimum, when $\chi \rightarrow \chi_{\mathrm{rot}}$, the R^- and R^+ roton branches merge Their group velocities tend to zero, and so do their creation probabilities $D_{2,3} \rightarrow 0$ (because they are proportional to the energy density fluxes, which are proportional to the group velocities). In Eq. (38), the multiplier (k_3-k_2) comes from the group velocity. In the same way, $D_{1,2}^{\rightarrow} \rightarrow 0$ near the maxon maximum $\chi \rightarrow \chi_{\max}$, where the phonon and R^- roton branches merge. Thus, $D_2^{\rightarrow}$ becomes zero at both ends of the frequency interval in which $D_2^{\rightarrow}$ is defined. Both the strong inequality in Eq. (39) and the asymptotic behavior of D_2 at $\chi \rightarrow \chi_{\mathrm{rot}}, \chi_{\max}$ (see Fig. 5) are the consequences of the simple relations $0 < k_{1z} < (-k_{2z}) < k_{3z}$ from Eq. (20), which reflect the *qualitative* behavior of the dispersion curve for superfluid helium, as shown in Fig. 1.

The creation probability of R^- rotons at the interface is very small for all energies. It should also be noted that, at low temperatures, the main contribution to the energy flow through the interface is due to phonons of energies less than the roton gap (i.e., with $\chi < \chi_{\mathrm{rot}}$), which are not yet taken into account. Thus, we have a convincing explanation why R^- rotons were not detected in experiments that created beams of quasiparticles in helium by a solid heater, as, for example, in Ref. 2.

The expressions for $D_i^{\rightarrow}$ [Eqs. (33) and (37)] and $D_{\rightarrow}$ [Eq. (34)] are written as functions of the incidence angle or k_τ. What can be experimentally measured are the energy flows as functions of transmission angles. If phonons are isotropically incident on the interface, then as the transmission angle for each wave is defined by Eq. (23), the quasiparticles of each type are transmitted in a narrow cone with the cone angle twice the θ_i^{cr}. Thus, the phonons are injected into the helium in the widest cone and R^+ rotons in the narrowest cone. For the total transmission coefficient as a function of transmission angle θ, we obtain

$$D_{\rightarrow}(\chi,\Theta) = \sum_{i=1}^{3} D_i^{\rightarrow}(\chi, k_{\tau i}=k_i \sin\theta), \tag{40}$$

where

$$D_i^{\rightarrow}(\theta)=0 \quad \text{for} \quad \theta > \theta_i^{\mathrm{cr}},$$

and $k_{\tau i}$ are the transverse components of the wave vectors of wave i transmitted at angle θ.

B. Phonon or roton in the helium incident on the interface

Now, let us consider one of the quasiparticles of superfluid helium (phonon or roton) incident on the interface. In terms of plane waves, a wave i of frequency ω and wave vector of length $k_i(\omega)$ is incident. The solution in the solid consists of only one transmitted wave, and the solution in the quantum fluid consists of one incident wave i and three reflected waves $j=1,2,3$; it can be represented as a sum of solutions P_{out} and $P_{\mathrm{in}}^{(i)}$ from Eqs. (21) and (22) (see Fig. 2). The boundary conditions in Eq. (8) enable us to express all the amplitudes through the amplitude of the incident wave and, thus, to obtain the nine amplitude reflection coefficients

r_{ij}. The coefficient r_{ij} is the ratio of the pressure amplitudes of the reflected wave j to the incident wave i for $i,j=1,2,3$:

$$r_{ii} = -\frac{\psi_i}{\psi_i^*}\frac{f_{-2z}^{(i)}}{f_{2z}}\frac{f_{-z}^{(i)} + Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}}, \quad (41)$$

$$r_{ij} = 2\frac{\psi_j}{\psi_i^*}\frac{k_{iz}(k_i^2 - k_k^2)}{f_{2z}}\frac{k_{kz}/k_g + Z - i\tilde{\delta}}{f_z + Z - i\tilde{\delta}}\varepsilon_{ijk}. \quad (42)$$

Here, the subscripts take values $\{i,j,k\}=\{1,2,3\}+$perm; ε_{ijk} is the Levi–Civita symbol, which is equal to 1 if $\{i,j,k\}=\{1,2,3\},\{2,3,1\}$, or$\{3,1,2\}$ and to (-1) if $\{i,j,k\}=\{2,1,3\},\{1,3,2\}$, or$\{3,2,1\}$; $Z=-Z_g k_{\mathrm{sol}\,z}/k_{\mathrm{sol}}$ is the generalization of definition (28) For incidence angles less than critical, $k_\tau^2 < k_{\mathrm{sol}}^2$ and $k_{\mathrm{sol}\,z}<0$, and Z is given by Eq. (28). For greater incidence angles, the new notation must be used because $\cos\theta$ is not defined. Then, $k_{\mathrm{sol}\,z}\in\mathbf{C}_-$ for the wave to be damped in $z<0$, and therefore, $Z=i|Z|$. The constructions $f_{-nz}^{(i)}$ are

$$f_{-z}^{(i)} = f_{-3z}^{(i)}/(k_g f_{-2z}^{(i)}),$$

$$f_{-nz}^{(i)} = f_{nz}[k_{1z},k_{2z},k_{3z}]|_{k_{iz}\to(-k_{iz})} \quad \text{for } i=1,2,3;\ n=2,3. \quad (43)$$

The amplitude coefficient of transmission $t_i^{\leftarrow}$ for the incident wave of type i is

$$t_i^{\leftarrow} = \frac{(k_{iz}+k_{jz})(k_{iz}+k_{kz})}{\psi_i^*}\frac{2k_{iz}/k_g}{f_z + Z - i\tilde{\delta}}. \quad (44)$$

Then, the energy transmission coefficient $D_i^{\leftarrow}$ for wave i can be calculated as the fraction of the energy of the incident wave packet that is transmitted into the solid. It is explicitly shown that

$$D_i^{\leftarrow}(\chi,k_\tau) = D_i^{\rightarrow}(\chi,k_\tau). \quad (45)$$

This important relation ensures thermodynamic equilibrium between the solid and helium at equal temperatures on both sides of the interface. Due to Eq. (45), from now on, we can omit the arrows in the sub- and superscripts of D_i and D.

The reflection coefficients for $i=j$ are just $R_{ii}=|r_{ii}|^2$ and from Eq. (41), we obtain

$$R_{ii} = \left|\frac{(Z-i\tilde{\delta})k_g f_{-2z}^{(i)} + f_{-3z}^{(i)}}{(Z-i\tilde{\delta})k_g f_{2z} + f_{3z}}\right|^2. \quad (46)$$

For $i\neq j$, we have to take into account that energy flows for all waves are proportional to group velocities [Eq. (32), and then from Eqs. (42) and (43), we derive

$$R_{ij} = R_{ji} = 4k_g^2|k_{iz}k_{jz}(k_i^2-k_k^2)(k_j^2-k_k^2)|\left|\frac{Z-i\tilde{\delta}+k_{kz}/k_g}{(Z-i\tilde{\delta})k_g f_{2z}+f_{3z}}\right|^2. \quad (47)$$

The quantity R_{ij} is the probability of quasiparticle j of being created at the interface when quasiparticle i is incident, so the R_{ij} can also be called "conversion coefficients."

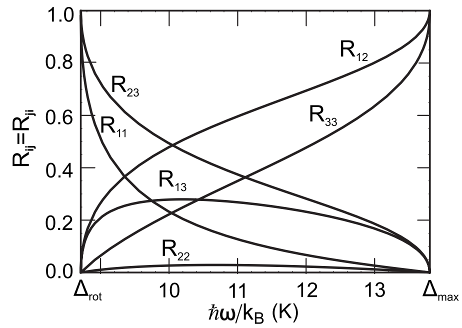

FIG. 6. Functions $R_{ij}(\omega,\theta_i=0)$ for $i,j=1,2,3$.

Their dependence on frequency at normal incidence is shown in Fig. 6. We see, in particular, that at the roton minimum, $\chi\to\chi_{\mathrm{rot}}$, where the R^- and R^+ roton branches merge, these quasiparticles are reflected into each other with probability that tends to unity, $R_{23}\to 1$. The same effect is present for phonons and R^- rotons at the maxon maximum.

The angular dependence of R_{ij} is most easily analyzed in terms of k_τ instead of the three angles of incidence. The values of k_τ equal to $k_{\mathrm{sol}}(\omega)$ or $k_i(\omega)$ correspond to different critical angles of incidence. So, when $k_\tau\in(0,k_{\mathrm{sol}})$, the quantities Z and $f_{\pm nz}^{(i)}$ (i.e., f_{nz} and $f_{-nz}^{(i)}$ for $i=1,2,3$) are all real, so all the waves are traveling waves and $D\neq 0$. When $k_\tau\in(k_{\mathrm{sol}},k_1)$, the wave in the solid is damped, $Z=i|Z|$, and $D=0$, but $f_{\pm nz}^{(i)}\in\mathbf{R}$ and all the waves in the helium are still reflected into each other. When $k_\tau\in(k_1,k_2)$, the phonon wave in helium is damped, $k_{1z}=i|k_{1z}|$, and no longer gives a traveling wave packet, and $f_{\pm nz}^{(i)}$ also become complex. This corresponds to $R^\pm$ rotons incident at angles greater than $\theta_{31,21}^{\mathrm{cr}}$ and reflecting into themselves or into each other. When $k_\tau\in(k_2,k_3)$, the quantities $f_{\pm nz}^{(i)}$ are also complex but the structure is different; this case corresponds to R^+ rotons incident at angles greater than $\theta_{32}^{\mathrm{cr}}$ and reflecting into R^+ rotons, again with probability of 1.

In all the cases, energy conservation can be explicitly verified but it takes different forms:

$$\sum_{j=1}^{3} R_{ij} = 1 - D_i \quad \text{for } i=1,2,3 \text{ if } k_\tau < k_{\mathrm{sol}}(\omega),$$

$$\sum_{j=1}^{3} R_{ij} = 1 \quad \text{for } i=1,2,3 \text{ if } k_{\mathrm{sol}}(\omega) < k_\tau < k_1(\omega),$$

$$R_{22} = R_{33} = 1 - R_{23} \text{ if } k_1(\omega) < k_\tau < k_2(\omega),$$

$$R_{33} = 1 \text{ if } k_2(\omega) < k_\tau < k_3(\omega). \quad (48)$$

For the interface between helium and a solid, the limit $Z_0 \ll 1$ is a good approximation, and in Eqs. (41) and (42) and Eqs. (46) and (47), Z can be neglected (in this limiting case, $D\to 0$ and $\Theta_i^{\mathrm{cr}}\to 0$). However, the angles are not small anymore, as was the case for D_i, and the angular dependence of the coefficients is strong. This can be clearly seen in Fig. 7, where the graphs of R_{1j} and R_{2j} are shown for $\hbar\omega/k_B=10$ K. The coefficient R_{21} becomes zero at the critical

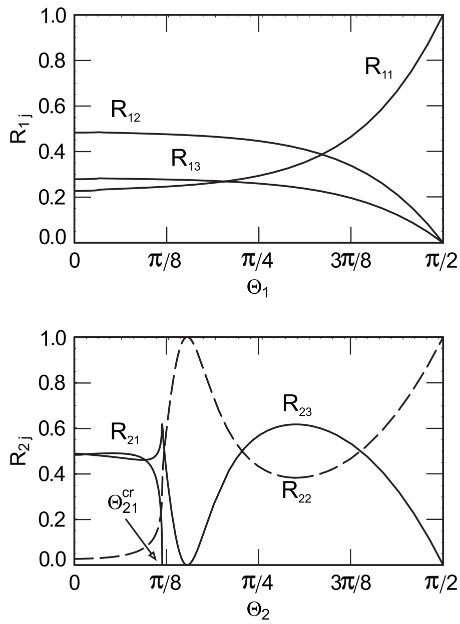

FIG. 7. [(a) and (b)] Reflection coefficients R_{1j} and R_{2j} for $j=1,2,3$ as functions of incidence angle θ at $\chi=0.2871$ ($\hbar\omega/k_B=10$ K).

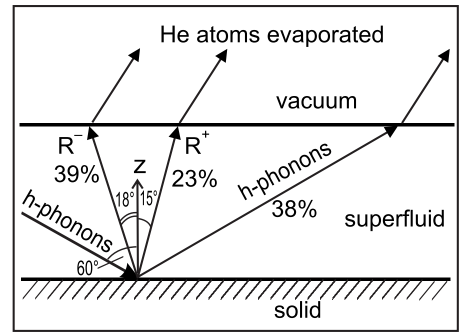

FIG. 8. The predicted creation of R^- rotons by h-phonons incident on the interface with a solid. There should be backward reflection and quantum evaporation with backward refraction.

angle θ_{21}^{cr}. The peak of R_{22} and minimum of R_{23} correspond to angles above critical, where k_{1z} is imaginary and the damping depth of the phonon wave becomes roughly half of the damping depth of the nonlocality kernel $h(r)$; then, the imaginary part of the numerator of Eq. (47) turns to zero, and as Z is small, R_{23} has a deep minimum.

A more extensive analysis of the functions $R_{ij}(\chi,\theta_i)$ for the case $Z_0 \ll 1$ allows us to state the following: the main processes near the roton minimum, $\chi \to \chi_{\text{rot}}+0$, for all angles are the conversion of R^- and R^+ rotons into each other and reflection of phonons into themselves; near the maxon maximum, phonons and R^- rotons are converted into each other and the reflection of R^+ rotons into themselves. For phonons and R^- rotons, when the incidence angle becomes close to $\pi/2$, the probabilities of reflection into themselves, $R_{11,22}$, tend to unity; for R^+ rotons, this happens at $\theta_3 \to \theta_{32}^{\text{cr}}-0$, and at greater angles, $R_{33}=1$ exactly. The conversion coefficients $R_{1j,j1}$ for $j=2,3$ are monotonically decreasing functions of the angles of incidence; R_{1j} becomes zero at $\theta_j \to \pi/2$ as $\sqrt{\pi/2-\theta_1}$ and R_{j1} becomes zero at $\theta_j \to \theta_{j1}^{\text{cr}}$ as $\sqrt{\theta_{j1}^{\text{cr}}-\theta_j}$. A little above θ_{j1}^{cr}, the coefficients R_{22} and R_{33} have high sharp peaks, and R_{23} has a corresponding minimum, as described above.

Then, for the case depicted in Fig. 4, the most powerful beam will always be beam *ii* (basically because of phonons with energies less than χ_{rot}). We have shown that R^- rotons are hardly created by a solid heater [Eq. (39)], and the probability of R^+ roton creation is also quite small at frequencies near χ_{rot} (see Fig. 5) if the incident beam mainly consists of low energy phonons. It was shown in Refs. 7 and 8 that in a phonon beam, low energy phonons (l-phonons) are converted into phonons with energy of about 10 K (h-phonons). The fraction of the energy in the initial beam that is converted to the h-phonons can be up to 50%.[8] The conversion coefficient of these phonons to R^- rotons is given by R_{12} at $\hbar\omega/k_B \approx 10$ K. It is much greater than that that at the roton minimum and almost reaches 1/2 at normal incidence, which is more than R_{11}, see Fig. 7.

We suggest the experimental setup depicted in Fig. 8. The heater injects a phonon beam, in which h-phonons are created. The h-phonons, which are incident on the solid-helium interface, are reflected into three beams of phonon and R^- rotons and R^+ rotons of comparable intensities (the R^- rotons are reflected backward). These beams propagate toward the free surface of helium and quantum evaporate atoms from it (the R^- rotons evaporate atoms backward[3]), which are then detected. Thus, the energy is transported from the heater to the interface by phonons and then to the detector by R^- rotons along a Z-shaped trajectory, with retroreflection at the point of creation of R^- rotons and retrorefraction on the surface. The angles and fractions of the initial beam's energy, which is transferred to different reflected beams, are shown for the h-phonon part of the incident beam. The l-phonons for the most part are directly reflected and are not shown.

If the source of quasiparticles has more rotons in the incident beam, as the one used in Ref. 3, then beams 32 and 23 may also be detectable.

The main contribution to the energy flow through the interface at low temperatures can be expected to be made by phonons below the roton gap, i.e., with $\chi \in (0,\chi_{\text{rot}})$. The problem of transmission through interfaces by phonons with anomalous dispersion was solved in Refs. 20 and 30. In the current work, the dispersion relation in Eq. (1) that is used is more general than the one used in the previous works. It is nonmonotonic and normal below the roton gap.

When $\chi < \chi_{\text{rot}}$, the roots $k_{2z,3z}$ are defined so that $k_{1z}>0$ and $k_{3z}=-k_{2z}^* \in \mathbf{C}_+$, as shown in Sec. II C. With these k_{iz}, the out and in solutions are constructed [Eqs. (21) and (22)]. So, the amplitude coefficients are still defined by Eqs. (27) and (31), but the quantities $f_{\pm nz}^{(i)}$ are now complex. The only valid reflection coefficient R_{11} is defined by Eq. (46), with the complex k_{iz} as introduced above. The transmission coefficient is $D_{\text{ph}}=1-R_{11}$. It can be shown that in the limit of small frequencies, $\chi \to 0$, when the dispersion is almost linear, the expression for D_{ph} approaches the standard one for linear dispersion. At $\chi \to \chi_{\text{rot}}-0$, it rapidly decreases to less than half because of the increasing influence of the roton waves. The curve $D(\chi)$ is continuous at χ_{rot} but has a kink.

IV. ENERGY FLOWS THROUGH THE INTERFACE

When a phonon in the solid, with frequency ω and wave vector $\mathbf{k}_{\text{sol}}$, is incident on the interface at angle θ_{sol}, the average energy transferred into helium is $\hbar\omega D(\omega,k_\tau)$, where D is given by Eq. (34) and $k_\tau = k_{\text{sol}}\cos\theta_{\text{sol}}$. Let the phonons in the solid be in thermodynamic equilibrium at temperature T. Then, the normal component of the density of energy flow through the interface is (see, for example, Ref. 11)

$$Q(T) = \int \frac{d^3k_{\text{sol}}}{(2\pi)^3}\hbar\omega n_T(\omega)s_{\text{sol}}\cos\theta_{\text{sol}}D, \tag{49}$$

where n_T is the Bose–Einstein distribution function and the integration domain is the half-space $k_{\text{sol}\,z}>0$. The parts of this energy flow that are transferred into helium by either phonons, R^- rotons, or R^+ rotons of helium that are created at the interface by the incident phonons are obtained in the same way. However, instead of the full coefficient D, we now use the partial transmission coefficients D_i. These are the corresponding creation probabilities of the quasiparticles. After changing the integration variables to the arguments of $D_i(\omega,k_\tau)$, the partial energy flows can be expressed in the following form:

$$Q_i^{\rightarrow}(T) = \int \frac{d\omega}{8\pi^2}\hbar\omega n_T(\omega)\int_0^{k_i^2(\omega)} dk_\tau^2 D_i(\omega,k_\tau). \tag{50}$$

Here, the upper limit by k_τ^2 corresponds to the maximum transmission angle of quasiparticles of type i, which is equal to θ_i^{cr} from Eq. (25). The quantities $Q_i^{\rightarrow}$ for $i=1,2,3$ are the individual contributions of phonons and R^- and R^+ rotons to the energy flux from the solid into helium.

Their contributions to the energy flux in the opposite direction, $Q_i^{\leftarrow}(T)$, are the normal components of the energy fluxes from helium into the solid. These are realized by helium quasiparticles of type i incident on the interface. The average energy transferred into the solid per incident quasiparticle of type i is due to Eq. (45), $\hbar\omega D_i(\omega,k_\tau)$. Then, the energy flux is derived in the same way as Eq. (49), with the difference that instead of s_{sol} in the integral, we have $|u_i|$ because the number of quasiparticles incident on the interface per unit of time is proportional to their group velocity:

$$Q_i^{\leftarrow}(T) = \int \frac{d^3k_i}{(2\pi)^3}\hbar\omega n_T(\omega)|u_i|\cos\Theta_i D_i. \tag{51}$$

When changing the integration variables to (ω,k_τ), we use the Jacobian determinant and we explicitly obtain that $Q_i^{\leftarrow}(T)=Q_i^{\rightarrow}(T)$.

Figure 9(a) shows the ratio between the contributions to the energy flux through the interface of all quasiparticles above the roton gap (i.e., with $\hbar\omega/k_B>\Delta$) $Q_> = Q_1+Q_2+Q_3$, and the contribution of phonons below the roton gap $Q_<$, which is obtained by using Eq. (50) with D_{ph}. We see that at temperatures $T<1$ K, the phonons are dominant. However, at $T\approx 2.5$ K, the two contributions are equal, and at higher temperatures, the phonons and rotons above the roton gap play the main role in heat exchange with the solid, see Fig. 9(b). The contribution of the R^+ rotons to $Q_>$ increases with temperature and at $T\approx 3$ K, their contribution

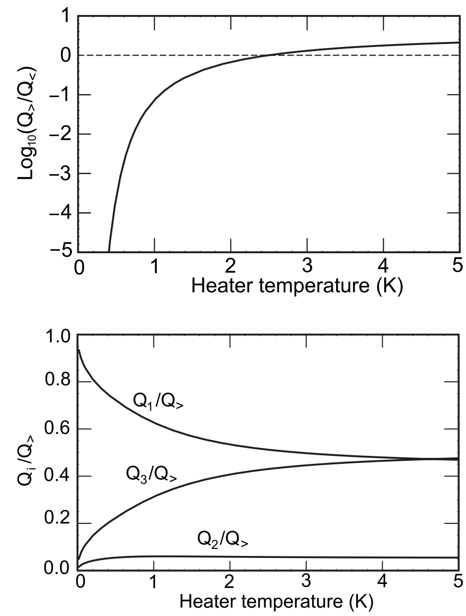

FIG. 9. (a) The ratio of the contributions to the energy flow through the interface by quasiparticles above roton minimum, $Q_>$, to the phonons below roton minimum, $Q_<$, as a function of temperature (on a logarithmic scale). (b) The contributions of phonons, R^- rotons, and R^+ rotons to the energy flow created by quasiparticles above the roton minimum as functions of T.

surpasses that of the phonons, see Fig. 9(b). The contribution of the R^- rotons is approximately constant and is no greater than 6%. This is due to the low creation probability D_2 for all frequencies [Eq. (39)]. At $T=3$ K, the contribution of the R^- rotons to the full energy flow is 3%.

When there is energy flow through the interface, it induces the Kapitza temperature jump at the interface (see, for example, Ref. 11). The contributions of quasiparticles of each type to this jump are obtained by differentiating Eq. (50) with respect to T.

V. CONCLUSION

In this work, we have solved the problem of the interaction of He II quasiparticles, i.e., phonons, R^- rotons, and R^+ rotons, with the interface between helium and a solid. These excitations have the nonmonotonic dispersion curve shown in Fig. 1. The consistent solution of the problem has been introduced, which allows us to rigorously describe the simultaneous creation of the three types of He II quasiparticles by any one of them or by a phonon in the solid that is incident on the interface.

When a phonon in the solid is incident on the interface, it is reflected with some probability and a phonon, R^- roton, or R^+ roton is created with corresponding probabilities in the helium. It is shown that the created R^- roton, due to its negative group velocity, is refracted backward (Fig. 3). When some quasiparticles of helium are incident, all the quasiparticles with the same frequency and transverse wave vectors are created. The set of six critical angles as functions of frequency is introduced [Eqs. (25) and (26)]. These separate the intervals of angles of incidence for the different quasiparticles, from which other quasiparticles can be created. It is shown that when a phonon or R^+ roton is incident, the R^- roton is retroreflected (i.e., reflected backward), and likewise, when an R^- roton is incident, the phonon and R^+ rotons are retroreflected (Fig. 4). This effect is the Andreev reflection of phonons and rotons.

The probabilities of creation of all quasiparticles at the interface when any quasiparticle is incident are derived as functions of frequency and incidence angles [Eqs. (33), (46), and (47) and Figs. 5–7]. It is shown that the creation probability of an R^- roton by a phonon in the solid, and vice versa, is very small for all angles and frequencies [Eqs. (38) and (39)]. This means that R^- rotons are as badly created by a solid heater as they are poorly detected by a solid bolometer. This explains the failure to detect R^- rotons in direct experiments until 1999.[3] New predictions are made for experiments with beams of phonons and rotons interacting with the solid interface and, in particular, creating R^- rotons at the interface by a beam of h-phonons.

The full energy flow through the interface is also calculated as a function of temperature of the solid, as well as the individual contributions of the phonons and R^+ and R^- rotons [Eq. (50)] to it, see Fig. 9. The contribution of the R^- rotons is shown to be very small.

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