Homogenization of acoustic metamaterials of Helmholtz resonators in fluid

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By using a two-step homogenization approach, we derive analytical formulas of effective mass density ρ_e and effective bulk modulus B_e for two- and three-dimensional acoustic metamaterials of Helmholtz resonators (HRs) in fluid. A negative B_e is found at certain frequencies due to the monopolar resonance, leading to a low-frequency acoustic band gap. A unified picture is presented for metamaterials of HRs and three-component metamaterials of negative ρ_e. Our work supports recent observations in a one-dimensional array of HRs [N. Fang *et al.*, Nat. Mater. **5**, 452 (2006)] and presents important high-dimensional extensions for exploring more fascinating phenomena.

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Recently, acoustic and/or elastic metamaterials (artificial structured materials[1,2]) have received considerable interest due to their exotic acoustic and/or elastic properties.[3–17] In particular, their effective mass density and modulus can be singly or simultaneously negative,[3–13] allowing intriguing phenomena for sound waves such as low-frequency band gaps,[3–10] negative refraction, and superlensing.[1,12–14]

Negative effective mass density or modulus can occur at certain frequencies if an appropriate resonance is included into the structures.[3–13] A famous example is the three-component phononic crystal with locally resonant structures,[3] which exhibits a negative effective mass density ρ_e due to a dipolar resonance.[4,5] Very recently, a low-frequency band gap was demonstrated in a one-dimensional (1D) array of Helmholtz resonators (HRs).[6] By using further retrieval analysis of wave-scattering coefficients,[7–9] a negative effective bulk modulus B_e was confirmed in the 1D metamaterials of HRs.[9] Unlike other designs for negative B_e using air bubbles or soft-rubber spheres,[12,13] this metamaterial consists of a rigid body but still possesses a sizable working frequency range. However, high-dimensional metamaterials of HRs, which are essential for more fascinating phenomena,[6,14,15] remain unexplored.

In this Brief Report, we study two-dimensional (2D) and three-dimensional (3D) acoustic metamaterials consisting of cylindrical and spherical HRs in fluid (such as water or air), respectively. By using a two-step homogenization approach, we can derive analytical formulas for the ρ_e and B_e within coherent-potential approximation (CPA).[18] The accuracy of these formulas is confirmed by accurate retrieval results using multiple-scattering techniques. We show that our metamaterials of HRs can exhibit a negative B_e at certain frequencies due to a monopolar resonance. Systematic analyses are done for the frequency range of negative B_e and the low-frequency-limit behavior. Based on our derivations, a unified picture is presented for metamaterials of HRs and three-component metamaterials of negative ρ_e.

Our HRs are cylindrical and/or spherical rigid shells with uniformly distributed slits and/or holes, as shown in Fig. 1(c). The inner and outer radii of the shells are r_2 and r_1, respectively. The background fluid has a mass density ρ_0 and bulk modulus B_0. Our HR becomes a conventional HR if only a single slit and/or hole exists in the shell. It has been shown that for acoustic waves of long wavelength (λ_0), a rigid-body slab with small slits and/or holes (with a size $\Delta < d/4$ and period $d < \lambda_0/4$) is effectively equivalent to a uniform slab of a thickness h/n, density $\rho_0 n^2$, and bulk modulus B_0, where h is the thickness of the rigid-body slab, $n=1/f$, and f is the filling ratio of slits and/or holes ($f=d/\Delta$ for slits).[17] In the following, we will use this conclusion and

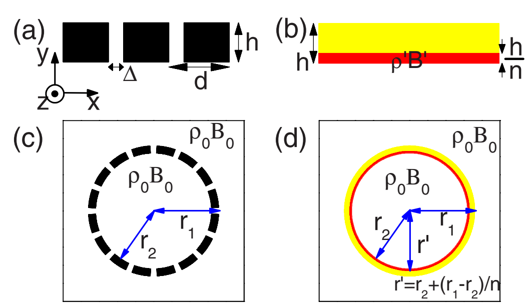

FIG. 1. (Color online) (a) A rigid-body slab with periodic slits (holes) and (b) its equivalence of a two-layer slab. The slabs have a thickness h and are extending in the X-Z plane. The slits and/or holes have a size Δ, period d, and filling ratio f ($f=d/\Delta$ for slits). The top layer has a thickness $h-h/n$ but the same fields at their two surfaces, and the bottom layer has a thickness h/n, density $\rho' = \rho_0 n^2$ with $n=1/f$, and modulus $B' = B_0$. The background fluid has a density ρ_0 and modulus B_0. (c) A 2D (3D) HR of rigid cylindrical (spherical) shell with uniformly distributed slits (holes) and (d) its equivalence of a two-layer shell made of the slabs in (a) and (b), respectively.

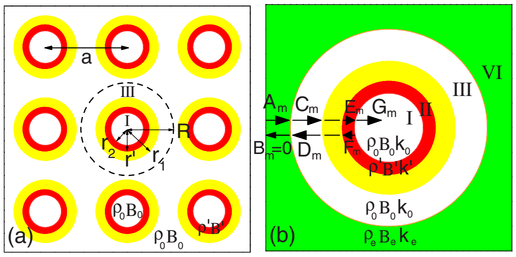

FIG. 2. (Color online) (a) A 2D (3D) acoustic metamaterial consisting of a square (simple cubic) lattice of the cylindrical (spherical) shells in Fig. 1(d) and (b) a coated shell in an effective medium of (ρ_e, B_e).

replace the wall of the HR by its equivalence of a two-layer shell,[19] where the inner layer has a thickness $(r_1-r_2)/n$, density $\rho'=\rho_0 n^2$, and modulus $B'=B_0$, and the outer layer has a thickness $(r_1-r_2)(1-1/n)$ but the same fields at its two surfaces. Here, we introduce the outer layer to make the total wall thickness unchanged.

We first consider a 2D metamaterial consisting of a square array of above cylindrical HRs in fluid, as shown in Fig. 2(a). The filling ratio of HRs is $f_s=\pi r_1^2/a^2$, where a is the lattice constant. Set $\mathbf{r}=(x,y)=(r,\theta)$ in the plane perpendicular to the axes of HRs. For harmonic acoustic waves with angular frequency ω, the pressure field $P(\mathbf{r})e^{i\omega t}$ obeys the two-dimensional Helmholtz equation,

$$(\nabla^2+k^2)P=0, \tag{1}$$

which is subjected to continuities of P and $\frac{1}{\rho}\partial P/\partial r$ at the layer interfaces of each HR (set $\mathbf{r}=0$ at the center of HR). The wave numbers in the background and inner walls of HRs are given by $k_0=\omega\sqrt{\rho_0/B_0}$ and $k'=\omega\sqrt{\rho'/B'}=nk_0$, respectively.

Then, we derive analytical formulas for the ρ_e and B_e of HRs by the CPA method.[18] We consider a "circular unit cell" (coated HR) of radius $R=r_1/\sqrt{f_s}$ and replace the metamaterial outside by a uniform effective medium of ρ_e, B_e, and $k_e=\omega\sqrt{\rho_e/B_e}$ [see Fig. 2(b)]. The pressure field P can be decomposed into incident and scattering cylindrical waves [represented by Bessel (J_m) and Hankel (H_m) functions, respectively]: $P^{\mathrm{I}}=\Sigma_m G_m J_m(k_0 r)e^{im\theta}$ when $r<r_2$, $P^{\mathrm{II}}=\Sigma_m[E_m J_m(k'r)+F_m H_m(k'r)]e^{im\theta}$ when $r_2<r<r'\equiv r_2+(r_1-r_2)/n$, $P^{\mathrm{III}}=\Sigma_m[C_m J_m(k_0 r)+D_m H_m(k_0 r)]e^{im\theta}$ when $r_1<r<R$, and $P^{\mathrm{VI}}=\Sigma_m[A_m J_m(k_e r)+B_m H_m(k_e r)]e^{im\theta}$ when $r>R$. For the outer layer of HR, the fields will be the same at its two surfaces so that $P^{\mathrm{II}}(r')=P^{\mathrm{III}}(r_1)$ and $\frac{1}{\rho'}\frac{\partial P^{\mathrm{II}}(r')}{\partial r}=\frac{1}{\rho_0}\frac{\partial P^{\mathrm{III}}(r_1)}{\partial r}$. Combining $P^{\mathrm{I}}(r_2)=P^{\mathrm{II}}(r_2)$, $\frac{1}{\rho_0}\frac{\partial P^{\mathrm{I}}(r_2)}{\partial r}=\frac{1}{\rho'}\frac{\partial P^{\mathrm{II}}(r_2)}{\partial r}$, $P^{\mathrm{III}}(R)=P^{\mathrm{VI}}(R)$, and $\frac{1}{\rho_0}\frac{\partial P^{\mathrm{III}}(R)}{\partial r}=\frac{1}{\rho_e}\frac{\partial P^{\mathrm{VI}}(R)}{\partial r}$, the mth order scattering coefficient B_m/A_m of the coated HR can be obtained and $B_m/A_m=0$ (which "defines" the effective medium) when

$$-\frac{\frac{1}{\rho_0}k_0 J_m'(k_0R)-J_m(k_0R)\frac{1}{\rho_e}k_e J_m'(k_eR)/J_m(k_eR)}{\frac{1}{\rho_0}k_0 H_m'(k_0R)-H_m(k_0R)\frac{1}{\rho_e}k_e J_m'(k_eR)/J_m(k_eR)}=-\frac{\frac{1}{\rho_0}k_0 J_m'(k_0r_1)-J_m(k_0r_1)U_m}{\frac{1}{\rho_0}k_0 H_m'(k_0r_1)-H_m(k_0r_1)U_m}, \tag{2}$$

where

$$U_m=\frac{1}{\rho'}k'\frac{J_m'(k'r')+H_m'(k'r')P_m}{J_m(k'r')+H_m(k'r')P_m}$$

and

$$P_m=-\frac{\frac{1}{\rho'}k'J_m'(k'r_2)-J_m(k'r_2)\frac{1}{\rho_0}k_0 J_m'(k_0r_2)/J_m(k_0r_2)}{\frac{1}{\rho'}k'H_m'(k'r_2)-H_m(k'r_2)\frac{1}{\rho_0}k_0 J_m'(k_0r_2)/J_m(k_0r_2)}.$$

We note that the first and second terms in Eq. (2) and P_m are just the mth order scattering coefficients of a cylinder with radius R, ρ_e, and B_e in a matrix of ρ_0, B_0, the HR, and the core cylinder, respectively. If we only consider the scattering of cylindrical waves of the two lowest orders $m=0$ and 1, ρ_e and B_e can be numerically calculated after solving $\frac{1}{\rho_e}k_e J_0'(k_eR)/J_0(k_eR)$ and $\frac{1}{\rho_e}k_e J_1'(k_eR)/J_1(k_eR)$ from Eq. (2).

Under the approximation of long wavelength $[k_0a, k_ea \ll 1$, thus $k'(r'-r_2)=k_0(r_1-r_2)\ll 1]$ and small slit size $(n=1/f\gg 1)$, Eq. (2) becomes[20]

$$\frac{1}{B_e}=\frac{1}{B_0}\left(1-f_s+\frac{f_s r_2/r_1}{1-\omega^2/\omega_R^2}\right) \quad \text{for } m=0, \tag{3}$$

$$\rho_e=\rho_0[1+f_s/(L-1)]/(1-f_s) \quad \text{for } m=1, \tag{4}$$

where $L=2$ in the 2D case and the resonant frequency is given by

$$\omega_R=\upsilon_0\sqrt{L/[nr_2(r_1-r_2)]}. \tag{5}$$

Here, $\upsilon_0=\sqrt{B_0/\rho_0}$ is the sound velocity in the background fluid. If only a single slit exists in the wall of HR ($n\approx 2\pi r_2/\Delta$, where Δ is the slit width), Eq. (5) will become the conventional HR frequency in 2D, namely, $\omega_R=\upsilon_0\sqrt{\Delta/[\pi r_2^2(r_1-r_2)]}$.[21] For the case without slits ($n\to\infty$), ω_R will approach zero and Eq. (3) will reduce to the formula of B_e for rigid rods in fluid [where ρ_e is still given by Eq. (4)],[16] namely,

$$B_e=B_0/(1-f_s). \tag{6}$$

From Eq. (3), we can see that the B_e of HRs depends on frequency. At the low-frequency limit ($\omega\to 0$), the effective bulk modulus is given by

$$B_e=B_0/(1-f_s+f_s r_2/r_1). \tag{7}$$

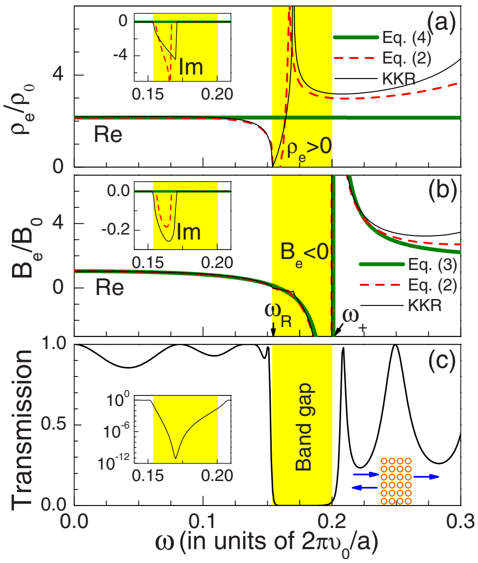

FIG. 3. (Color online) (a) The effective mass density and (b) bulk modulus for the 2D acoustic metamaterial in Fig. 2(a), and (c) the transmission spectrum for normal incidence of a plane wave on a four-layer metamaterial. The parameters are $r_1=0.36a$, $r_2=0.32a$, and $f=6.39\times10^{-3}$ (thus $n=156.5$, $r'=0.320\,256a$, $\rho'=2.45\times10^4\rho_0$, and $B'=B_0$).

Here, B_e will be zero at the resonant frequency ω_R. For a frequency range above ω_R ($\omega_R<\omega<\omega_+$), B_e becomes negative, where the upper frequency edge is given by

$$\omega_+=\omega_R\sqrt{1+f_s r_2/[r_1(1-f_s)]}. \tag{8}$$

Since the effective density ρ_e will always be positive [by Eq. (4)], an acoustic band gap (with an imaginary k_e) will be formed in the frequency range with negative B_e.

Similar CPA derivations can also be done for 3D metamaterials consisting of HRs of spherical shells with uniformly distributed holes. In 3D,[22] the Bessel (J_m) and Hankel (H_m) functions in Eq. (2) should be replaced by spherical Bessel (j_m) and Hankel (h_m) functions, respectively. Under similar approximation, Eqs. (3)–(8) will also be obtained where f_s becomes the volume ratio of spherical HRs ($f_s=\frac{4}{3}\pi r_1^3/a^3$ in a simple cubic lattice) and $L=3$ in the 3D case. For HRs of single hole ($n\approx4\pi r_2^2/S_h$, where S_h is the area of the hole), Eq. (5) will become the conventional formula of HR frequency, namely, $\omega_R=\upsilon_0\sqrt{S_h/[\frac{4}{3}\pi r_2^3(r_1-r_2)]}$.[21] When no holes exist in the shells ($n\rightarrow\infty$), ω_R will approach zero and Eqs. (4) and (6) can be obtained for rigid spheres in fluid.

To check the validity of the above analytic formulas, we apply the multiple-scattering or Korringa-Kohn-Rostoker (KKR) method (including high-order cylindrical waves)[23] to simulate normal incidence of a plane wave on a slab of above 2D acoustic metamaterial [in Fig. 2(a)]. Then, the effective parameters (B_e,ρ_e) can be retrieved from the transmission and reflection coefficients.[7–9] The retrieving is easy for a one-layer slab but the results will be the same for more layers. In Fig. 3, we show the accurate KKR results of (B_e,ρ_e) and those from Eqs. (2)–(4) for a square lattice of cylindrical HRs in fluid where $r_1=0.36a$, $r_2=0.32a$, and $f=6.39\times10^{-3}$ (thus, $n=156.5$, $r'=0.320\,256a$, $\rho'=2.45\times10^4\rho_0$, and $B'=B_0$). A frequency range [$\tilde{\omega}\equiv\omega_0 a/(2\pi\upsilon_0)\in(0.154,0.2)$] with a negative B_e and positive ρ_e can be found above the resonant frequency. As a result, a low-frequency stop band can be observed in the transmission spectrum for a metamaterial slab [see Fig. 3(c)]. We note that the transmission dip at $\tilde{\omega}=0.170$ corresponds to a peak of $\rho_e=9.77-i4.45$ (with $B_e=-0.194-i0.09$ which will not bring a gain or loss; KKR results). This peak of ρ_e cannot be captured by Eq. (4) with the long-wavelength approximation and it is usually understood as a result of band folding. However, we find that this feature can still be obtained by numerically solving Eq. (2) for the lowest 2 orders of cylindrical waves. For higher frequencies ($\tilde{\omega}>0.25$), the scattering of high-order cylindrical waves will be important and more accurate KKR calculations are needed.

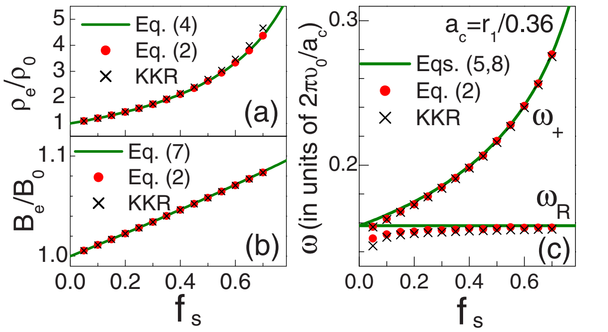

FIG. 4. (Color online) (a) The effective mass density and (b) bulk modulus when $\omega r_1/\upsilon_0\rightarrow0$, and (c) the edges of the frequency range with a negative B_e as functions of the filling fraction $f_s\equiv\pi r_1^2/a^2$ for the 2D acoustic metamaterials in Fig. 2(a). The parameters of r_1/r_2 and f are as in Fig. 3.

Figure 4 shows the ρ_e and B_e at the low-frequency limit ($\omega r_1/\upsilon_0\ll1$) and the frequency edges of negative B_e with varying the filling fraction f_s of HRs. The frequency range of negative B_e is found wider with increasing the filling ratio of HRs. Excellent agreement is found between our analytical formulas and KKR results. Recently, a better impedance match and higher refractive index were found in 2D HRs than rigid rods by finite-difference time-domain simulations.[7] This result was explained by a smaller B_e of HRs ($B_e=B_0$) than rods. We note that our new Eq. (7) can present a better understanding for this phenomenon.[20]

By now, we have shown that our metamaterials of HRs exhibit a negative effective bulk modulus at certain frequencies due to a monopolar ($m=0$) resonance. For the HR, the fluid inside the heavy shell ($\rho'=\rho_0 n^2\gg\rho_0$) will retain a cylindrical and/or spherical shape. However, its volume can still periodically vary (since $B'=B_0$) giving rise to the monopolar resonance. Strong monopolar resonances can also exist in air bubbles[11,13] or soft-rubber spheres[12] in water, leading to a negative B_e.[12,13] Such monopolar resonances can still survive even when a solid matrix is used (e.g., water-coated air bubbles in solid).[13] So if a solid-based structure with a negative B_e is desired, fluid-coated HRs in a solid matrix can be similarly applied. By combining our HRs with the metamaterials of a negative ρ_e, a double negative metamaterial can be constructed,[13] enabling negative refraction and superlensing of acoustic waves. We note that our HRs can also exhibit an arbitrary positive B_e $[0<B_e<B_0/(1-f_s+f_s r_2/r_1)$ or $B_e>B_0/(1-f_s)]$ but normal ρ_e [by Eq. (4)] at certain frequencies ($\omega<\omega_R$ or $\omega>\omega_+$). When an additional metamaterial with an anisotropic ρ_e but normal B_e is provided, a fascinating acoustic cloaking may also be realized.[15]

A coated-cylinder and/or coated-sphere structure exists in our acoustic metamaterials of HRs and the locally resonant sonic materials with soft rubber.[3] However, a negative effective mass density was found in the latter.[4,5] To present a unified picture for these two metamaterials, we mathematically consider metamaterials of soft shells in fluid [by using $B'=B_0/n^2 \ll B_0$ and $\rho'=\rho_0$ in Fig. 1(d)]. Under the long-wavelength approximation $[k_0 a, k_e a \ll 1$, thus $k'(r'-r_2)\ll 1]$, Eq. (2) becomes[20]

$$\frac{1}{B_e}=\frac{1}{B_0}\left[1+f_s(Ln-1)\left(1-\frac{r_2}{r_1}\right)\right], \tag{9}$$

$$\frac{\rho_e-\rho_0}{\rho_e+\rho_0}=-f_s+f_s\frac{Lr_2/(r_1+r_2)}{1-\omega^2/\omega_R'^2}, \tag{10}$$

for $m=0$ and 1, respectively, where the resonant frequency is given by

$$\omega_R'=v_0\sqrt{(L-1+r_1/r_2)/[nr_1(r_1-r_2)]}. \tag{11}$$

From Eq. (10), a negative effective mass density can be seen due to a dipolar ($m=1$) resonance, which agrees with recent explanations.[4,5] For the soft shell, the fluid inside will retain the shape and volume and will vibrate in a certain direction, giving rise to the dipolar resonance.

In summary, we have proposed a two-step homogenization approach to derive analytical formulas of ρ_e and B_e for 2D and 3D acoustic metamaterials of HRs in fluid. A negative B_e was found at certain frequencies due to the monopolar resonance, leading to a low-frequency acoustic band gap. A unified picture was also presented for metamaterials of HRs and three-component metamaterials of negative ρ_e. Our work supported recent observations in 1D array of HRs and presented important high-dimensional extensions for exploring more fascinating acoustic phenomena.

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[1] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968); J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).

[2] For a recent review, see D. R. Smith, J. B. Pendry, and M. Wiltshire, Science **305**, 788 (2004).

[3] Z. Liu *et al.*, Science **289**, 1734 (2000).

[4] Z. Liu, C. T. Chan, and P. Sheng, Phys. Rev. B **71**, 014103 (2005).

[5] Y. Wu, Y. Lai, and Z. Q. Zhang, Phys. Rev. B **76**, 205313 (2007).

[6] N. Fang *et al.*, Nat. Mater. **5**, 452 (2006).

[7] X. Hu, C. T. Chan, and J. Zi, Phys. Rev. E **71**, 055601(R) (2005).

[8] V. Fokin, M. Ambati, C. Sun, and Xiang Zhang, Phys. Rev. B **76**, 144302 (2007).

[9] Z. G. Wang *et al.*, J. Phys.: Condens. Matter **20**, 055209 (2008).

[10] Z. Hou, J. Liu, W. Kuang, Y. Liu, and S. Wu, Phys. Rev. E **75**, 026608 (2007).

[11] M. Kafesaki, R. S. Penciu, and E. N. Economou, Phys. Rev. Lett. **84**, 6050 (2000).

[12] J. Li and C. T. Chan, Phys. Rev. E **70**, 055602(R) (2004).

[13] Y. Ding, Z. Liu, C. Qiu, and J. Shi, Phys. Rev. Lett. **99**, 093904 (2007).

[14] M. Ambati, N. Fang, C. Sun, and X. Zhang, Phys. Rev. B **75**, 195447 (2007).

[15] W. Milton, M. Briane, and J. R. Willis, New J. Phys. **8**, 248 (2006); S. A. Cummer and D. Schurig, *ibid.* **9**, 45 (2007); H. Chen and C. T. Chan, Appl. Phys. Lett. **91**, 183518 (2007); S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, J. Pendry, M. Rahm, and A. Starr, Phys. Rev. Lett. **100**, 024301 (2008).

[16] X. Hu and C. T. Chan, Phys. Rev. Lett. **95**, 154501 (2005); D. Torrent, A. Hakansson, F. Cervera, and J. Sanchez-Dehesa, *ibid.* **96**, 204302 (2006); J. Mei, Z. Liu, W. Wen, and P. Sheng, *ibid.* **96**, 024301 (2006).

[17] F. Cai, F. Liu, Z. He, and Z. Liu, Appl. Phys. Lett. **91**, 203515 (2007); (to be published).

[18] P. Sheng, *Introduction to Wave Scattering Localization and Mesoscopic Phenomena* (Academic, New York, 1995).

[19] Based on the coordinate transformation method, the two-layer slab can be transformed to a one-layer slab with the same slab thickness. However, anisotropic effective parameters will be found. Here, we use this two-layer slab for convenience of analytical derivation.

[20] See EPAPS Document No. E-PRBMDO-77-015817 for derivation of Eqs. (3), (4), (9), and (10) and comparison of our formulas with FDTD results. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html

[21] L. E. Kinsler, *Fundamentals of Acoustics* (Wiley, New York, 1982).

[22] M. Kafesaki and E. N. Economou, Phys. Rev. B **60**, 11993 (1999).

[23] L. C. Botten *et al.*, J. Opt. Soc. Am. A **17**, 2165 (2000).