Variational method for solving the Bethe-Salpeter equation for excitons in GaAs-(Ga,Al)As quantum wells in perpendicular magnetic fields

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The Bethe-Salpeter (BS) formalism is used to study the effect of the coupling between the center of mass and the relative internal motions of quantum-well excitons in a constant magnetic field. The BS equation in the case of an in-plane magnetic field is reduced to the well-known Schrödinger equation for magnetoexcitons. In a perpendicular magnetic field, the BS equation has an extra term (BS term) that does not exist in the Schrödinger equation. Within the framework of the variational method, it is shown that (i) the ground-state energy of a heavy-hole magnetoexciton with a zero wave vector in GaAs-(Ga,Al)As quantum wells, which is calculated by means of the BS formalism, is very close to the ground-state energy obtained from the Schrödinger equation by using the same trial function; and (ii) in a strong perpendicular magnetic field, the magnetoexciton dispersion (in-plane magnetoexciton mass) is determined mainly by the BS term rather than the term that describes the electron-hole Coulomb interaction in the Schrödinger equation.

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I. INTRODUCTION

The calculation of the energy levels of a hydrogen atom in superstrong magnetic fields[1] is an important problem in astrophysics and cosmology. A similar problem arises in solid state physics, wherein the magnetoexcitons in three dimensions,[2] in a semiconductor single quantum well (SQW) or coupled quantum wells[3] (CQW's), have been extensively studied over the past decades. The photon momentum is quite small and the finite center-of-mass momentum is usually not important in optical measurements. Thus, many of the studies are primarily focused on the effects of a strong magnetic field on the exciton energy levels assuming the absence of transverse motion of the exciton as a whole. However, even a small transverse exciton velocity (or small transverse wave vector $\mathbf{Q}$) will induce an electric field in the rest frame of the exciton. This electric field will push the electron and the hole apart, so the binding energy must decrease as the transverse velocity increases. In other words, the magnetic field induces a coupling between the center of mass and the relative internal motions and, therefore, the correct description of excitons in a strong magnetic field should take into account this coupling effect.

In what follows, we focus our attention on the excitons in SQW and CQW's in the presence of constant in-plane $\mathbf{B}_\parallel=(B_\parallel,0,0)$ and perpendicular $\mathbf{B}_\perp=(0,0,B_\perp)$ magnetic fields. First, we will consider a SQW of width L $(-L/2\leq z\leq L/2)$ made with a direct-gap semiconductor that has nondegenerate and isotropic bands. All SQW calculations can be trivially generalized to describe magnetoexcitons in CQW's. In both structures, the z axis is parallel to the growth direction and the x-y plane is assumed to be the plane of confinement of the two-dimensional (2D) electron-hole system. Thus, the electron and hole motions in the z direction are quantized into discrete levels due to the presence of confinement potentials $U_{c,v}(z)$ along this direction. The dispersion laws for electrons and holes are $E_c(\mathbf{k},k_z)=E_g+\hbar^2\mathbf{k}^2/2m_c+\hbar^2k_z^2/2m_c$ and $E_v(\mathbf{k},k_z)=\hbar^2\mathbf{k}^2/2m_v+\hbar^2k_z^2/2m_v$, where $\mathbf{k}$ is a 2D wave vector, E_g is the semiconductor band gap, and m_c (m_v) is the electron (hole) effective mass. In what follows, we neglect any electron-hole correlations along the z axis. This approximation takes place when the effective mass of the hole considerably exceeds that of the electron and the slow motion of the hole is separated from the fast motion of the electron. The assumption is applicable for many crystals of $A^{\mathrm{III}}B^{\mathrm{V}}$ type. For simplicity, we shall take into account only the first electron E_{0c} and hole E_{0v} confinement levels. This approximation allows us to write the exciton wave function as $\Psi_{\mathbf{Q}}(\mathbf{r})\varphi_c^0(z_c)\varphi_v^0(z_v)$, where $\mathbf{r}=\mathbf{r}_c-\mathbf{r}_v=(x,y)$. Here, $\mathbf{r}_{c(v)}$ is the 2D electron (hole) position vector, while z_c and z_v are the corresponding z coordinates. $\varphi_c^0(z_c)$ and $\varphi_v^0(z_v)$ are the electron and hole confinement wave functions.

By introducing a pseudomomentum $\hbar\mathbf{Q}=\hbar(Q_x,Q_y,0)$, in principle, we can obtain the magnetoexciton energy $E(\mathbf{Q},\mathbf{B}_{\parallel(\perp)})$ and the corresponding wave functions from the solutions of the Schrödinger equation with the following Hamiltonian:[3,4]

$$\begin{aligned}\hat{H}_\parallel &= \frac{\hbar^2Q_x^2}{2M}-\frac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2+\frac{\imath\hbar eB_\parallel}{c}\left(\frac{z_c}{m_c}+\frac{z_v}{m_v}\right)\frac{\partial}{\partial y}\\ &\quad+\frac{\hbar^2}{2m_c}\left(\frac{m_c}{M}Q_y+\frac{eB_\parallel}{\hbar c}z_c\right)^2+\frac{\hbar^2}{2m_c}\left(\frac{m_v}{M}Q_y-\frac{eB_\parallel}{\hbar c}z_v\right)^2\\ &\quad-\frac{\hbar^2}{2m_c}\frac{\partial^2}{\partial z_c^2}+U_c(z_c)-\frac{\hbar^2}{2m_v}\frac{\partial^2}{\partial z_v^2}+U_v(z_v)-V_C(\mathbf{r}),\end{aligned}\tag{1}$$

$$\begin{aligned}\hat{H}_\perp &= E_g+E_{0c}+E_{0v}-\frac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2+\frac{\imath e\gamma\hbar}{2\mu c}(\mathbf{B}_\perp\times\mathbf{r})\cdot\nabla_{\mathbf{r}}\\ &\quad+\frac{e^2B_\perp^2}{8\mu c^2}\mathbf{r}^2-V_C(\mathbf{r}+\mathbf{R}_0).\end{aligned}\tag{2}$$

Here, $M=m_c+m_v$ and $\mu=m_cm_v/M$ are the exciton in-plane and reduced masses, respectively. $\gamma=(m_v-m_c)/M$ and $\mathbf{R}_0=R^2\mathbf{Q}_0$, where $\mathbf{Q}_0=(-Q_y,Q_x,0)$ and $R=(\hbar c/eB_\perp)^{1/2}$ is the magnetic length in the case of a perpendicular magnetic field. V_C represents the electron-hole Coulomb attraction. Numerical calculations with the Hamiltonians (1) and (2) show that the binding energy decreases with increasing in-plane pseudomomentum.[4,5]

We may well ask whether the Schrödinger equation correctly describes the nontrivial effects due to the finite inplane pseudomomentum. This question has its origin in the fact that the magnetoexcitons are bound states between two charged fermions in a constant magnetic field and, therefore, the appropriate framework for the description of the bound states is the Bethe-Salpeter (BS) formalism.[6–8] Several nontrivial effects produced by magnetic fields have been recently predicted in quantum field theories. For example, in the massless QED, the BS equation is reduced to the Schrödinger equation and, as a result, it was predicted that the external constant magnetic field generates an energy gap (dynamical mass) in the spectrum of massless fermions for any arbitrary weak attractive interaction between fermions.[9,10] It was shown that the effect is model independent (universal), because the physical reason for this effect lies in the dimensional reduction in the dynamics of fermion pairing in the presence of a constant magnetic field. Later, it was suggested that a similar effect could explain some experimental findings in the physics of high-temperature layered superconductors.[11]

In this paper, we focus our attention on the nontrivial effects produced by magnetic fields in the case of nonrelativistic bound states, such as the excitons in semiconductors. It is known[12] that in the absence of a magnetic field, by using a series of approximations (such as the introduction of an equal-time wave function, in which the BS kernel is assumed to depend only on the difference between the relative momenta), the BS equation for electron-hole bound states can be simplified to the well-known Schrödinger equation for the relative internal motion. The existence of a magnetic field induces a coupling between the center of mass and the relative internal motions, so in the presence of a magnetic field, the simplification of the BS equation to the Schrödinger equation is not trivial. In what follows, we apply the BS formalism to the magnetoexcitons in SQW and CQW structures. It is natural to expect the following results: (i) in the case of an in-plane magnetic field, the BS (after reasonable approximations) and the Schrödinger equations provide exactly the same energies and wave functions; (ii) in the case of a perpendicular magnetic field, there could be terms in the BS equation that do not exist in the Schrödinger equation; (iii) these extra terms should be important only in strong magnetic fields.

The basic assumption in the BS formalism is that the electron-hole bound states are described by the BS wave function (BS amplitude) $\Psi(1;2)=\Psi(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2)$, where the variables 1 and 2 represent the corresponding coordinates and the time variables. This function determines the probability amplitude to find the electron at the point $(\mathbf{r}_c,z_c)$ at the moment t_1 and the hole at the point $(\mathbf{r}_v,z_v)$ at the moment t_2. The BS amplitude satisfies the following equation:

$$\Psi(1;2)=\int d(1',2',1'',2'')G_c(1;1')G_v(2';2)I\begin{pmatrix}1' & 1''\\ 2' & 2''\end{pmatrix}\times\Psi(1'';2''). \tag{3}$$

Here, I is the irreducible BS kernel and $G_{c,v}$ are the electron and hole Green's functions. They can be obtained by solving the Schwinger-Dyson equations, but in the effective-mass approximation, $G_{c(v)}$ satisfies the Schrödinger equation $[\imath\hbar\,\partial/\partial t_1-\hat{H}_{c(v)}(1)]G_{c(v)}(1,1')=\delta(1-1')$, where $\hat{H}_{c(v)}$ is the effective-mass Hamiltonian for an electron (hole) in a magnetic field. By using the ladder approximation for the irreducible kernel, one can rewrite the BS equation (3) in the following form:

$$\begin{aligned}&\left[\imath\hbar\frac{\partial}{\partial t_1}-E_g-\frac{1}{2m_c}\left(-\imath\hbar\nabla_{\mathbf{r}_c}+\frac{e}{c}\mathbf{A}(x_c,y_c,z_c)\right)^2-\frac{\hbar^2}{2m_c}\frac{\partial^2}{\partial z_c^2}-U_c(z_c)\right]\\&\times\left[\imath\hbar\frac{\partial}{\partial t_2}-\frac{1}{2m_v}\left(-\imath\hbar\nabla_{\mathbf{r}_v}-\frac{e}{c}\mathbf{A}(x_v,y_v,z_v)\right)^2-\frac{\hbar^2}{2m_v}\frac{\partial^2}{\partial z_v^2}-U_v(z_v)\right]\Psi(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2)\\&=-\imath I(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2)\Psi(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2).\end{aligned} \tag{4}$$

If the retardation effects are neglected, then the irreducible kernel represents the Coulomb attraction between electrons and holes that constitute the excitons as follows:

$$\begin{aligned}I(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2)&=4\pi e^2\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{dq_z}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{\epsilon^{-1}(\mathbf{q},q_z,\omega)}{\mathbf{q}^2+q_z^2}\\&\times\exp\{\imath[\mathbf{q}\cdot(\mathbf{r}_c-\mathbf{r}_v)+q_z(z_c-z_v)-\omega(t_1-t_2)]\},\end{aligned} \tag{5}$$

where $\epsilon^{-1}(\mathbf{q},q_z,\omega)$ is the Fourier transform of the inverse dielectric function. For simplicity, we neglect the dispersion of the dielectric function by assuming $\epsilon^{-1}(\mathbf{q},q_z,\omega)=1/\epsilon_\infty$. In what follows, we will use 2D relative $\mathbf{r}=\mathbf{r}_c-\mathbf{r}_v$ and center-of-mass $\mathbf{R}=\alpha_c\mathbf{r}_c+\alpha_v\mathbf{r}_v$ coordinates $[\alpha_{c,v}=m_{c,v}/(m_c+m_v)]$.

In a constant magnetic field, the 2D magnetic pseudomomentum $\hbar\mathbf{Q}=(\hbar Q_x,\hbar Q_y)$ is an exact integral of motion. This vector is the eigenvalue of the magnetic pseudomomentum operator

$$\hbar\hat{\mathbf{Q}}=-\imath\hbar\nabla_{\mathbf{r}_c}-\imath\hbar\nabla_{\mathbf{r}_v}+\frac{e}{c}\mathbf{A}(x_c,y_c,z_c)-\frac{e}{c}\mathbf{A}(x_v,y_v,z_v)-(e/c)\mathbf{B}\times(\mathbf{r}_c-\mathbf{r}_v). \tag{6}$$

By neglecting any electron-hole correlation along the z axis, we can separate the variables by writing the BS amplitude in the form

$$\Psi(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2) = \exp\{\imath[\mathbf{Q}\cdot\mathbf{R}-(E/\hbar)(\alpha_c t_1+\alpha_v t_2)]\}\Phi_{\mathbf{Q}}(\mathbf{r},t_1-t_2;z_c,z_v). \tag{7}$$

Here, $E=E(\mathbf{Q},\mathbf{B})$ is the magnetoexciton energy, and the BS wave function depends on the relative time t_1-t_2 and the center-of-mass time $\alpha_c t_1+\alpha_v t_2$.

An outline of the paper is as follows: In Secs. II and III, we derive the BS equation for an in-plane magnetic field (along the x axis) and for a perpendicular magnetic field (along the z axis). In Secs. IV and V, we apply a variational method to obtain the ground-state energy and the dispersion of the heavy-hole magnetoexcitons in GaAs/$Al_xGa_{1-x}As$ SQW and CQW's, respectively.

II. IN-PLANE MAGNETIC FIELD

In the case of an in-plane magnetic field $\mathbf{B}_\parallel$ applied along the x axis, we use the following vector potential $\mathbf{A}(x,y,z)=\mathbf{A}_\parallel(z)=(0,-zB_\parallel,0)$ (in the gauge div $\mathbf{A}=0$). With this vector potential, we calculate $\hbar\hat{\mathbf{Q}}=-\imath\hbar\nabla_{\mathbf{r}_c}-\imath\hbar\nabla_{\mathbf{r}_v}$. Within the chosen gauge, the single-electron (hole) Hamiltonian $\hat{H}_{c(v)}$ depends on the exact integral of motion Q_y as follows:

$$\hat{H}_\sigma(Q_y,q_y;q_x) = -\frac{\hbar^2}{2m_\sigma}\frac{d^2}{dz_\sigma^2}+U_\sigma(z_\sigma) + \frac{m_\sigma\omega_{\parallel\sigma}^2(z_\sigma-z_\sigma^0)^2}{2}+\frac{\hbar^2q_x^2}{2m_\sigma}+E_g\delta_{c,\sigma}. \tag{8}$$

Here, σ stands for c,v, and the following notations are used: $z_c^0(q_y,Q_y)=\hbar(q_y+\alpha_cQ_y)/m_c\omega_{\parallel c}$ and $z_v^0(q_y,Q_y)=\hbar(q_y-\alpha_vQ_y)/m_v\omega_{\parallel v}$. The quantity $\omega_{\parallel\sigma}=eB_\parallel/cm_\sigma$ denotes the electron (hole) cyclotron frequency. Let

$$\varphi_c^\xi(z_c;q_y,Q_y),$$

$$\varphi_v^\zeta(z_v;q_y,Q_y),$$

$$E_c^\xi(L,q_y,Q_y;q_x)=E_c^\xi(L,q_y,Q_y)+\hbar^2q_x^2/2m_c+E_g$$

and

$$E_v^\zeta(L,q_y,Q_y;q_x)=E_v^\zeta(L,q_y,Q_y)+\hbar^2q_x^2/2m_v$$

be the wave functions and corresponding energies of operators (8), where ξ and ζ denote the quantum indices of the states in the quantum well. For $L\to\infty$, there is no confining potential, and the above quantities are defined as follows:

$$E_c^{\xi=n}(L\to\infty,q_y,Q_y)=\left(n+\frac{1}{2}\right)\hbar\omega_{\parallel c},$$

$$E_c^{\zeta=n'}(L\to\infty,q_y,Q_y)=\left(n'+\frac{1}{2}\right)\hbar\omega_{\parallel v},$$

$$\varphi_\sigma^n(z_\sigma;q_y,Q_y)=\frac{1}{\pi^{1/4}R_\sigma^{1/2}\sqrt{2^nn!}}\exp\left[-\frac{[z_\sigma-z_\sigma^0(q_y,Q_y)]^2}{2R_\sigma^2}\right] \times H_n\left(\frac{z_\sigma-z_\sigma^0(q_y,Q_y)}{R_\sigma}\right),$$

where H_n are the Hermite polynomials, $R_\sigma=\sqrt{\hbar/m_\sigma\omega_{\parallel\sigma}}$, and $n,n'=0,1,2,\ldots$, are the Landau magnetic subband indices.

Let us introduce the Fourier transform $\Phi_{E,\mathbf{Q}}(\mathbf{q},\Omega;z_c,z_v)$ of the exciton wave function $\Phi_{E,\mathbf{Q}}(\mathbf{r},t;z_c,z_v)$ as follows:

$$\Phi_{\mathbf{Q}}(\mathbf{r},t_1-t_2;z_c,z_v) = \int\frac{d^2\mathbf{q}}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}e^{\imath[\mathbf{q}\cdot\mathbf{r}-(t_1-t_2)\Omega]}\Phi_{\mathbf{Q}}(\mathbf{q},\Omega;z_c,z_v). \tag{9}$$

The function $\Phi_{\mathbf{Q}}(\mathbf{q},\Omega;z_c,z_v)$ satisfies the BS equation as follows:

$$\begin{aligned}&\left[\hbar\Omega+\alpha_cE-\frac{m_c\hbar^2Q_x^2}{2M^2}-\frac{\hbar^2}{M}\mathbf{p}\cdot\mathbf{Q}-\hat{H}_c(Q_y,p_y)\right]\\ &\quad\times\left[-\hbar\Omega+\alpha_vE-\frac{m_v\hbar^2Q_x^2}{2M^2}+\frac{\hbar^2}{M}\mathbf{p}\cdot\mathbf{Q}-\hat{H}_v(Q_y,p_y)\right]\\ &\quad\times\Phi_{\mathrm{Q}}(\mathbf{p},\Omega;z_c,z_v)\\ &=-\imath\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{dq_z}{2\pi}\int_{-\infty}^{+\infty}\frac{d\Omega'}{2\pi}\exp[\imath q_z(z_c-z_v)]\\ &\quad\times\frac{4\pi e^2}{\epsilon_\infty[(\mathbf{q}-\mathbf{p})^2+q_z^2]}\Phi_{\mathbf{Q}}(\mathbf{q},\Omega';z_c,z_v).\end{aligned} \tag{10}$$

By separating the variables, we write the solution of the above BS equation in the form

$$\Phi_{\mathbf{Q}}(\mathbf{q},\Omega;z_c,z_v)=\sum_{\xi,\zeta}C_{\xi,\zeta}\varphi_c^\xi(z_c;q_y,Q_y)\varphi_v^\zeta(z_v;q_y,Q_y)\Psi_{\mathbf{Q}}^{\xi\zeta}(\mathbf{q},\Omega).$$

As we have mentioned, we take into account only the first electron $\xi=0$ and hole $\zeta=0$ confined levels $E_\sigma^0(L,q_y,Q_y)$. By assuming perfect confinement for the electrons and holes inside the quantum well with a thickness L, the wave functions of the first electron and hole confined levels must obey the boundary conditions $\varphi_\sigma^0(z_\sigma=\pm L/2;q_y,Q_y)=0$. In the first-confined-level approximation, these functions satisfy the following equation:

$$\left[-\frac{d^2}{dZ_\sigma^2}+[Z_\sigma-\Lambda_\sigma(q_y,Q_y)]^2\right]\varphi_\sigma^0(Z_\sigma;q_y,Q_y) = 2\epsilon_\sigma(L,q_y,Q_y)\varphi_\sigma^0(Z_\sigma;q_y,Q_y). \tag{11}$$

Here, we introduce the dimensionless variables $Z_\sigma=z_\sigma/R_\parallel$, $\Lambda_c(q_y,Q_y)=R_\parallel(q_y+\alpha_cQ_y)$, $\Lambda_v(q_y,Q_y)=R_\parallel(q_y-\alpha_vQ_y)$, and $\epsilon_\sigma(L,q_y,Q_y)=E_\sigma^0(L,q_y,Q_y)/\hbar\omega_{\parallel\sigma}$. In this section, we shall use magnetic length $R_\parallel=(\hbar c/eB_\parallel)^{1/2}$ for the unit length, and the exciton cyclotron energy $\hbar\Omega_\parallel=\hbar eB_\parallel/c\mu$ for the unit energy. By writing the wave function as $\varphi_\sigma^0(Z_\sigma;q_y,Q_y)=\exp(-Z_\sigma^2/2)u_\sigma(Z_\sigma;\Lambda_\sigma(q_y,Q_y))$, we find the following equation for u_σ:

$$\frac{d^2u_\sigma}{dZ_\sigma^2} - 2Z_\sigma\frac{du_\sigma}{dZ_\sigma} + [2\Lambda_\sigma(q_y,Q_y)Z_\sigma + 2\epsilon_\sigma - \Lambda_\sigma^2(q_y,Q_y) - 1]u_\sigma(Z_\sigma;\Lambda_\sigma(q_y,Q_y)) = 0, \quad (12)$$

with the following boundary conditions: $u_\sigma(Z_\sigma = \pm L/2R_\parallel;\Lambda_\sigma(q_y,Q_y))=0$. The solution of Eq. (12) is a superposition of two special solutions:

$$u_\sigma(Z_\sigma;\Lambda_\sigma(q_y,Q_y)) = C_1 \exp(\Lambda_\sigma(q_y,Q_y)Z_\sigma)H\left(\frac{-1+2\epsilon_\sigma}{2},Z_\sigma - \Lambda_\sigma(q_y,Q_y)\right) + C_2 \exp(\Lambda_\sigma(q_y,Q_y)Z_\sigma)\,{}_1F_1\left(\frac{1-2\epsilon_\sigma}{4},\frac{1}{2},[Z_\sigma - \Lambda_\sigma(q_y,Q_y)]^2\right). \quad (13)$$

Here, H and ${}_1F_1$ are the Hermite and hypergeometric functions. The boundary conditions provide a set of two linear homogeneous equations for the arbitrary constants C_1 and C_2. The corresponding determinant must be zero, which allows us to calculate the eigenvalues $\epsilon_\sigma(L,q_y,Q_y)$. The normalization condition provides exact expressions for the arbitrary constants C_1 and C_2.

In the first-confined-level approximation, we have

$$\Phi_{\mathbf{Q}}(\mathbf{q},\Omega;z_c,z_v) = \varphi_c^0(z_c;q_y,Q_y)\varphi_v^0(z_v;q_y,Q_y)\Psi_{\mathbf{Q}}(\mathbf{q},\Omega),$$

where $\Psi_{\mathbf{Q}}(\mathbf{q},\Omega)$ is the solution of the following equation:

$$[\hbar\Omega - \Omega_c^0(\mathbf{q},\mathbf{Q},E)][\hbar\Omega - \Omega_v^0(\mathbf{q},\mathbf{Q},E)]\Psi_{\mathrm{Q}}(\mathbf{q},\Omega) = \imath\frac{2\pi e^2}{\varepsilon_\infty}\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{d\Omega'}{2\pi}\frac{F^{q_y,Q_y}(\mathbf{q}-\mathbf{q}')}{|\mathbf{q}-\mathbf{q}'|}\Psi_{\mathbf{Q}}(\mathbf{q}',\Omega'). \quad (14)$$

Here, the following notations are used:

$$F^{q_y,Q_y}(\mathbf{q}) = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} dz_c dz_v \varphi_c^0(z_c;q_y,Q_y)\varphi_c^0(z_c;q_y,Q_y) \times\varphi_v^0(z_v;q_y,Q_y)\varphi_v^0(z_v;q_y,Q_y)\exp(-|\mathbf{q}|\,|z_c - z_v|),$$

$$\Omega_c^0(\mathbf{q},\mathbf{Q},E) = -\alpha_c E + \frac{m_c\hbar^2Q_x^2}{2M^2} + \frac{\hbar^2}{M}\mathbf{q}\cdot\mathbf{Q} + E_c^0(L,q_y,Q_y;q_x),$$

$$\Omega_v^0(\mathbf{q},\mathbf{Q},E) = \alpha_v E - \frac{m_v\hbar^2Q_x^2}{2M^2} + \frac{\hbar^2}{M}\mathbf{q}\cdot\mathbf{Q} - E_v^0(L,q_y,Q_y;q_x). \quad (15)$$

We are looking for the solution of the BS equation (14) of the form

$$\Psi_{\mathbf{Q}}(\mathbf{q},\Omega) = \frac{g_{\mathbf{Q}}(\mathbf{q})}{[\hbar\Omega - \Omega_c^0(\mathbf{q},\mathbf{Q},E) + \imath 0^+][\hbar\Omega - \Omega_v^0(\mathbf{q},\mathbf{Q},E) - \imath 0^+]}, \quad (16)$$

where $g_{\mathbf{Q}}(\mathbf{q})$ is an unknown function. Let us define the function

$$\Psi_{\mathbf{Q}}(\mathbf{q}) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\Psi_{\mathbf{Q}}(\mathbf{q},\omega) = \int d\mathbf{r}\exp(-\imath\mathbf{q}\cdot\mathbf{r})\Psi_{\mathbf{Q}}(\mathbf{r}), \quad (17)$$

which is the Fourier transform of the equal-time BS amplitude (or exciton wave function) $\Psi_{\mathbf{Q}}(\mathbf{r})$. Thus, by taking into account the analytic properties of $\Psi_{\mathbf{Q}}(\mathbf{q},\omega)$, we obtain the following BS equation for determining the exciton energy E and the Fourier transform of the exciton wave function $\Psi_{\mathbf{Q}}(\mathbf{q})$:

$$\left[E - E_g - \frac{\hbar^2Q_x^2}{2M} - \frac{\hbar^2q_x^2}{2\mu} - E_c^0(L,q_y,Q_y) - E_v^0(L,q_y,Q_y)\right]\Psi_{\mathbf{Q}}(\mathbf{q}) + \frac{2\pi e^2}{\epsilon_\infty}\int\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{F^{q_y,Q_y}(\mathbf{q}-\mathbf{p})}{|\mathbf{q}-\mathbf{p}|}\Psi_{\mathbf{Q}}(\mathbf{q}) = 0. \quad (18)$$

In position representation, the last equation is equivalent to the well-known Schrödinger equation for magnetoexcitons in an in-plane magnetic field $\hat{H}_\parallel\Phi_{\mathbf{Q}}(\mathbf{r},z_c,z_v)=(E-E_g)\Phi_{\mathbf{Q}}(\mathbf{r},z_c,z_v)$, where $\Phi_{\mathbf{Q}}(\mathbf{r},z_c,z_v)=\Psi_{\mathbf{Q}}(x,y)\varphi_c^0(z_c)\varphi_v^0(z_v)$ and the Hamiltonian is given by Eq. (1). Thus, in the case of an in-plane magnetic field, the BS equation in the ladder approximation and the Schrödinger equation provide exactly the same energies and wave functions (for a numerical solution of the Schrödinger equation in the case of CQW's and an in-plane magnetic field, see, e.g., Ref. 4 and references therein).

III. PERPENDICULAR MAGNETIC FIELD

Though the BS equation in this case is not new,[13] we tread the subject in detail for the sake of completeness.

We assume that the magnetic field is along the z axis and is given by $\mathbf{B}_\perp=\mathrm{rot}\,\mathbf{A}(\mathbf{r})=B_\perp\mathbf{z}$, where $\mathbf{z}=(0,0,1)$. In the gauge div $\mathbf{A}(\mathbf{r})=0$, the vector potential of the magnetic field $\mathbf{B}_\perp$ is defined by $\mathbf{A}_\perp(\mathbf{r})=(1/2)\mathbf{B}_\perp\times\mathbf{r}$. Because of the confinement potentials, the electron and hole motions along the z direction are quantized into discrete levels. The corresponding wave functions φ and ϕ are the solutions of the corresponding one-particle Schrödinger equations as follows:

$$-\frac{\hbar^2}{2m_{cz}}\frac{d^2\varphi_\lambda}{dz_c^2}+U_c(z_c)\varphi_\lambda(z_c)=E_{\lambda c}\varphi_\lambda(z_c),$$

$$-\frac{\hbar^2}{2m_{vz}}\frac{d^2\phi_\xi}{dz_v^2}+U_v(z_v)\phi_\xi(z_v)=E_{\xi v}\phi_\xi(z_v).$$

Here, $E_{\lambda c}$ ($E_{\xi v}$) is the electron (hole) confinement energy, while λ and ξ denote the quantum numbers of the states in the confinement potential. In our calculations, we take into account only the first electron E_{0c} and hole E_{0v} confinement levels.

By means of the chosen vector potential, we calculate for the magnetic pseudomomentum operator $\hbar\hat{\mathbf{Q}}=-\imath\hbar\nabla_{\mathbf{r}_c}-\imath\hbar\nabla_{\mathbf{r}_v}-(e/2c)\mathbf{B}_\perp\times(\mathbf{r}_c-\mathbf{r}_v)$. By separating the variables, we write the BS amplitude in the following form:

$$\Psi(\mathbf{r}_c,z_c,t_1;\mathbf{r}_v,z_v,t_2)=\exp\left\{\imath\left[\mathbf{Q}\cdot\mathbf{R}-\frac{e}{c\hbar}\mathbf{r}\cdot\mathbf{A}_\perp(\mathbf{R})-\frac{E}{\hbar}(\alpha_c t_1+\alpha_v t_2)\right]\right\}\Phi_{\mathbf{Q}}(\mathbf{r};t_1-t_2)\varphi_0(z_c)\phi_0(z_v).$$

Here, $E=E(\mathbf{Q},\mathbf{B}_\perp)$ is the magnetoexciton energy. It is useful to introduce the Fourier transform of the BS amplitude as follows:

$$\Phi_{\mathbf{Q}}(\mathbf{r};t_1-t_2)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\times\exp\{\imath[\mathbf{q}\cdot\mathbf{r}-\Omega(t_1-t_2)]\}\Phi_{\mathbf{Q}}(\mathbf{q};\Omega).$$

The function $\Phi_{\mathbf{Q}}(\mathbf{q};\Omega)$ satisfies the BS equation as follows:

$$\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\int d^2\mathbf{r}\,\exp[\imath(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}][\hbar\Omega-\Omega_c(\mathbf{q}',\mathbf{Q})-\Omega_c^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q}';\mathbf{r})][\hbar\Omega-\Omega_v(\mathbf{q}',\mathbf{Q})-\Omega_v^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q}';\mathbf{r})]\Phi_{\mathrm{Q}}(\mathbf{q}';\Omega)$$
$$=-\imath\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\frac{2\pi e^2 f(|\mathbf{q}-\mathbf{q}'|)}{\epsilon_\infty|\mathbf{q}-\mathbf{q}'|}\int_{-\infty}^{+\infty}\frac{d\Omega'}{2\pi}\Phi_{\mathbf{Q}}(\mathbf{q}';\Omega'). \tag{19}$$

The following notations are used:

$$\Omega_c(\mathbf{q},\mathbf{Q})=E_c(\mathbf{q}+\alpha_c\mathbf{Q})+E_{0c}-\alpha_c E,$$
$$\Omega_v(\mathbf{q},\mathbf{Q})=-E_v(\mathbf{q}-\alpha_v\mathbf{Q})-E_{0v}+\alpha_v E, \tag{20a}$$

$$\Omega_c^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q};\mathbf{r})=\frac{e\hbar}{2Mc}(\mathbf{B}_\perp\times\mathbf{r})\cdot\mathbf{Q}+\frac{e\hbar}{2m_c c}(\mathbf{B}_\perp\times\mathbf{r})\cdot\mathbf{q}+\frac{e^2B_\perp^2}{8m_c c^2}\mathbf{r}^2, \tag{20b}$$

$$\Omega_v^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q};\mathbf{r})=\frac{e\hbar}{2Mc}(\mathbf{B}_\perp\times\mathbf{r})\cdot\mathbf{Q}-\frac{e\hbar}{2m_v c}(\mathbf{B}_\perp\times\mathbf{r})\cdot\mathbf{q}+\frac{e^2B_\perp^2}{8m_v c^2}\mathbf{r}^2. \tag{20c}$$

Here, $E_{c,v}(\mathbf{q})=E_{c,v}(\mathbf{q},q_z=0)$, and the structure factor f is given by

$$f(|\mathbf{q}|)=f(q)=\int_{-\infty}^{+\infty}dz_c\int_{-\infty}^{+\infty}dz_v\times\exp\{-q(z_c-z_v)]\}\varphi_{0c}^2(z_c)\phi_{0v}^2(z_v). \tag{21}$$

By comparing the left-hand sides of Eqs. (10) and (19), one can see that the perpendicular magnetic field is nonlocally created in momentum (or position) representation terms.

We rewrite Eq. (19) in the following form:

$$\Phi_{\mathrm{Q}}(\mathbf{q};\Omega)-\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\frac{\Omega_c^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')[\hbar\Omega-\Omega_v(\mathbf{q}',\mathbf{Q})]}{[\hbar\Omega-\Omega_c(\mathbf{q},\mathbf{Q})+\imath0^+][\hbar\Omega-\Omega_v(\mathbf{q},\mathbf{Q})-\imath0^+]}\times\Phi_{\mathrm{Q}}(\mathbf{q}';\Omega)-\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\frac{\Omega_v^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')[\hbar\Omega-\Omega_c(\mathbf{q}',\mathbf{Q})]}{[\hbar\Omega-\Omega_c(\mathbf{q},\mathbf{Q})+\imath0^+][\hbar\Omega-\Omega_v(\mathbf{q},\mathbf{Q})-\imath0^+]}$$
$$\times\Phi_{\mathrm{Q}}(\mathbf{q}';\Omega)+\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\frac{\Omega_{cv}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')}{[\hbar\Omega-\Omega_c(\mathbf{q},\mathbf{Q})+\imath0^+][\hbar\Omega-\Omega_v(\mathbf{q},\mathbf{Q})-\imath0^+]}\Phi_{\mathrm{Q}}(\mathbf{q}';\Omega)$$
$$=\frac{-\imath}{[\hbar\Omega-\Omega_c(\mathbf{q},\mathbf{Q})+\imath0^+][\hbar\Omega-\Omega_v(\mathbf{q},\mathbf{Q})-\imath0^+]}\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\frac{2\pi e^2 f(|\mathbf{q}-\mathbf{q}'|)}{\epsilon_\infty|\mathbf{q}-\mathbf{q}'|}\int_{-\infty}^{+\infty}\frac{d\Omega'}{2\pi}\Phi_{\mathbf{Q}}(\mathbf{q}';\Omega'), \tag{22}$$

where the nonlocal terms are defined as follows:

$$\Omega_{\sigma}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}') = \int d^2\mathbf{r}\,\exp[\imath(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}]\Omega_{\sigma}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q}';\mathbf{r}), \tag{23}$$

$$\Omega_{cv}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}') = \int d^2\mathbf{r}\,\exp[\imath(\mathbf{q}'-\mathbf{q})\cdot\mathbf{r}]\Omega_{c}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q}';\mathbf{r}) \times\Omega_{v}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q}';\mathbf{r}). \tag{24}$$

We are looking for the solution of Eq. (22) of the form

$$\Phi_{\mathbf{Q}}(\mathbf{q};\Omega) = \frac{g_{\mathbf{Q}}(\mathbf{q})}{[\hbar\Omega-\Omega_c(\mathbf{q},\mathbf{Q})+\imath 0^+][\hbar\Omega-\Omega_v(\mathbf{q},\mathbf{Q})-\imath 0^+]}, \tag{25}$$

where $g_{\mathbf{Q}}(\mathbf{q})$ is a function to be determined. Let us define the function $\Phi_{\mathbf{Q}}(\mathbf{q})$, which is the Fourier transform of the equal-time BS amplitude (or exciton wave function) $\Phi_{\mathbf{Q}}(\mathbf{r}) = \Phi_{\mathbf{Q}}(\mathbf{r};t_1-t_2=0)$, as follows:

$$\Phi_{\mathbf{Q}}(\mathbf{q}) = \int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\Phi_{\mathbf{Q}}(\mathbf{q};\Omega). \tag{26}$$

By taking into account the analytic properties of $\Phi_{\mathbf{Q}}(\mathbf{q};\omega)$, we obtain the following BS equation to determine the exciton energy $E'=E(\mathbf{Q},\mathbf{B}_\perp)-E_g-E_{0c}-E_{0v}$ and the Fourier transform of the exciton wave function $\Phi_{\mathbf{Q}}(\mathbf{q})$:

$$\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\left[\left(\frac{\hbar^2\mathbf{Q}^2}{2M}+\frac{\hbar^2\mathbf{q}^2}{2\mu}\right)\delta(\mathbf{q}-\mathbf{q}')+\Omega_c^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}') + \Omega_v^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}') - \frac{2\pi e^2}{\epsilon_\infty}\frac{f(|\mathbf{q}-\mathbf{q}'|)}{|\mathbf{q}-\mathbf{q}'|}\right]\Phi_{\mathbf{Q}}(\mathbf{q}') - \int\frac{d^2\mathbf{q}'}{(2\pi)^2}V_{\mathrm{BS}}(\mathbf{q},\mathbf{q}';\mathbf{Q},E')\Phi_{\mathbf{Q}}(\mathbf{q}') = E'\Phi_{\mathbf{Q}}(\mathbf{q}), \tag{27}$$

where the BS term, V_{BS}, is defined as follows:

$$\begin{aligned} V_{\mathrm{BS}}(\mathbf{q},\mathbf{q}';\mathbf{Q},E') &= \frac{[E_v(\mathbf{q}'-\alpha_v\mathbf{Q})-E_v(\mathbf{q}-\alpha_v\mathbf{Q})]\Omega_c^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')}{E'-E_c(\mathbf{q}'+\alpha_c\mathbf{Q})-E_v(\mathbf{q}-\alpha_v\mathbf{Q})} \\ &+ \frac{[E_c(\mathbf{q}'+\alpha_c\mathbf{Q})-E_c(\mathbf{q}+\alpha_c\mathbf{Q})]\Omega_v^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')}{E'-E_c(\mathbf{q}+\alpha_c\mathbf{Q})-E_v(\mathbf{q}'-\alpha_v\mathbf{Q})} \\ &+ \Omega_{cv}^{\mathbf{B}_\perp}(\mathbf{Q},\mathbf{q},\mathbf{q}')\left[\frac{1}{E'-E_c(\mathbf{q}'+\alpha_c\mathbf{Q})-E_v(\mathbf{q}-\alpha_v\mathbf{Q})} + \frac{1}{E'-E_c(\mathbf{q}+\alpha_c\mathbf{Q})-E_v(\mathbf{q}'-\alpha_v\mathbf{Q})}\right]. \end{aligned} \tag{28a}$$

In position representation, the function $V_{\mathrm{BS}}(\mathbf{q},\mathbf{q}';\mathbf{Q},E')$ is related to a nonlocal potential that depends on the energy E' as follows:

$$V_{\mathrm{BS}}(\mathbf{r},\mathbf{r}';\mathbf{Q},E') = \int\frac{d^2\mathbf{q}}{(2\pi)^2}\int\frac{d^2\mathbf{q}'}{(2\pi)^2}V_{\mathrm{BS}}(\mathbf{q},\mathbf{q}';\mathbf{Q},E') \times\exp[\imath(\mathbf{q}\cdot\mathbf{r}-\mathbf{q}'\cdot\mathbf{r}')]. \tag{28b}$$

The solution of Eq. (27) can be written as

$$\Phi_{\mathbf{Q}}(\mathbf{q}) = \exp(-\imath\mathbf{q}\cdot\mathbf{R}_0)\Psi(\mathbf{q}-\mathbf{Q}_0),$$

where $\mathbf{R}_0=R^2\mathbf{Q}_0$, $\mathbf{Q}_0=(-Q_y,Q_x,0)$, $R=(\hbar c/eB_\perp)^{1/2}$ is the magnetic length, and the function $\Psi(\mathbf{q})$ satisfies the following equation:

$$\begin{aligned} &\frac{\hbar^2\mathbf{q}^2}{2\mu}\Psi(\mathbf{q}) - \imath\frac{\gamma\hbar e}{2\mu c}(\mathbf{B}_\perp\times\mathbf{q})\cdot\nabla_{\mathbf{q}}\Psi(\mathbf{q}) - \frac{\hbar\Omega}{8R^2}\nabla_{\mathbf{q}}^2\Psi(\mathbf{q}) \\ &- \frac{2\pi e^2}{\epsilon_\infty}\int\frac{d^2\mathbf{q}'}{(2\pi)^2}\exp[\imath(\mathbf{q}-\mathbf{q}')\cdot\mathbf{R}_0]\frac{f(|\mathbf{q}-\mathbf{q}'|)}{|\mathbf{q}-\mathbf{q}'|}\Psi(\mathbf{q}') \\ &- \int\frac{d^2\mathbf{q}'}{(2\pi)^2}\exp[\imath(\mathbf{q}-\mathbf{q}')\cdot\mathbf{R}_0]V_{\mathrm{BS}}\left(\mathbf{q}+\frac{\gamma}{2}\mathbf{Q}_0,\mathbf{q}' + \frac{\gamma}{2}\mathbf{Q}_0;\mathbf{Q},E'\right)\Psi(\mathbf{q}') = E'\Psi(\mathbf{q}). \end{aligned} \tag{29}$$

Here, $\hbar\Omega=\hbar eB_\perp/\mu c$ is the exciton cyclotron energy and $\gamma=(m_v-m_c)/(m_v+m_c)$. The BS equation (29) differs from the Schrödinger equation. If we neglect the BS term in the left-hand side of Eq. (29), we obtain the Schrödinger equation for magnetoexcitons with the Hamiltonian (2). The Coulomb term is the only term in the right-hand side of Eq. (2), which depends on the exciton wave vector $\mathbf{Q}$ as follows:

$$V_C(\mathbf{r}+\mathbf{R}_0) = \frac{2\pi e^2}{\epsilon_\infty}\int\frac{d\mathbf{q}}{(2\pi)^2}\exp[\imath\mathbf{q}\cdot(\mathbf{r}+\mathbf{R}_0)]\frac{f(|\mathbf{q}|)}{|\mathbf{q}|}. \tag{30}$$

In other words, according to the Schrödinger equation, the magnetoexciton dispersion is determined by the Coulomb term only, while according to the BS equation, the effective potential (28b) also contributes to the magnetoexciton dispersion.

Since the Bethe-Salpeter term plays an important role in determining the magnetoexciton dispersion (see Secs. IV and V), one may well ask a question about the physical meaning of this term. The answer is not trivial, and the only thing we know for sure is that a similar phenomenon was predicted in QED, where the dynamical mass (energy gap) can be defined by considering the BS equation for composed fermion-antifermion pairs (Nambu-Goldstone bosons[9]) or the Schwinger-Dyson equation for the dynamical mass function.[10] Our calculations are similar to the first approach, and the only difference is that we applied a nonrelativistic version of the BS equation for electron-hole pairs in a constant magnetic field.

IV. SINGLE GaAs/Al$_x$Ga$_{1-x}$As QUANTUM WELL: VARIATIONAL CALCULATIONS

In this section, we calculate first the ground-state energy of a heavy-hole magnetoexciton with a zero wave vector ($\mathbf{Q}=0$), assuming a single GaAs quantum well with a thickness L sandwiched between two $Al_xGa_{1-x}As$ layers. The electron in-plane mass m_c and the electron z mass m_{cz} are chosen to be $m_c = m_{cz} = 0.067m_0$, where m_0 is the bare electron mass. The in-plane heavy-hole mass m_v and the hole z mass m_{vz} are expressed in terms of the Luttinger parameters γ_1 and γ_2: $m_v = m_0/(\gamma_1+\gamma_2)$ and $m_{vz} = m_0/(\gamma_1 - 2\gamma_2)$. It is known that the difference between the band-gap energies of GaAs and $Al_xGa_{1-x}As$ provides a finite potential well, which confines the electron-hole pairs in the GaAs quantum well. We assume that the potentials are square-well potentials of finite depths $V_c = 0.6\Delta Eg(x)$ and $V_v = 0.4\Delta Eg(x)$, respectively. The energy-band-gap discontinuity[14] is assumed to be $\Delta Eg(x) = (1.555x + 0.37x^2)$ meV. The confinement energy levels E_{c0} and E_{v0} are obtained by solving the following transcendental equations:

$$\tan\left(\frac{L}{2a_B}\sqrt{\frac{m_{cz}E_{c0}}{\mu E_B}}\right) = \sqrt{\frac{V_c}{E_{c0}} - 1}, \tag{31}$$

$$\tan\left(\frac{L}{2a_B}\sqrt{\frac{m_{vz}E_{v0}}{\mu E_B}}\right) = \sqrt{\frac{V_v}{E_{v0}} - 1}. \tag{32}$$

Here, $a_B = \varepsilon_\infty \hbar^2/\mu e^2$ and $E_B = \hbar^2/2\mu a_B^2$ are the effective Bohr radius and the exciton Bohr energy, respectively. Structure factor (21) is calculated by means of the following wave functions:

$$\psi^0_{c,v}(z) = A_{c,v}\exp\left[z\frac{L}{a_B}\sqrt{\frac{m_{cz,vz}(V_{c,v} - E_{c0,v0})}{\mu E_B}}\right], \quad -\infty < z < -1/2,$$

$$\psi^0_{c,v}(z) = B_{c,v}\cos\left(z\frac{L}{a_B}\sqrt{\frac{m_{cz,vz}E_{c0,v0})}{\mu E_B}}\right), \quad -1/2 < z < 1/2,$$

$$\psi^0_{c,v}(z) = A_{c,v}\exp\left[-z\frac{L}{a_B}\sqrt{\frac{m_{cz,vz}(V_{c,v} - E_{c0,v0})}{\mu E_B}}\right], \quad 1/2 < z < \infty,$$

$$B_{c,v} = \left[\frac{1}{2} + a_B/\left(L\sqrt{\frac{m_{cz,vz}(V_{c,v} - E_{c0,v0})}{\mu E_B}}\right)\right]^{-1/2},$$

$$A_{c,v} = B_{c,v}\exp\left[\frac{L}{2a_B}\sqrt{\frac{m_{cz,vz}(V_{c,v} - E_{c0,v0})}{\mu E_B}}\right] \times\cos\left(\frac{L}{2a_B}\sqrt{\frac{m_{cz,vz}E_{c0,v0})}{\mu E_B}}\right). \tag{33}$$

Since the BS equation (29) is rather complicated, we shall obtain numerical results for the ground-state energy within the framework of the variational approach. In the case of weak magnetic fields, i.e., $\hbar\Omega \ll E_B$, we use a hydrogenlike trial function with a variational parameter β as follows:

$$\psi_\beta(r) = \frac{2\sqrt{2}\beta}{\sqrt{\pi}a_B}\exp\left(-\frac{2r\beta}{a_B}\right). \tag{34}$$

With this trial function, we calculate the following magnetoexciton energy:

$$E = E_g + E_{c0} + E_{v0} - E(\beta)E_B,$$

where $E(\beta)$ is defined by the solution of the following equation:

$$E(\beta) = -4\beta^2 + 128\beta^3\int_0^\infty dx\frac{f\left(x\frac{L}{a_B}\right)}{(16\beta^2 + x^2)^{3/2}} - \frac{3}{128\beta^2}\left(\frac{\hbar\Omega}{E_B}\right)^2 + V_{BS}(\beta,E,B). \tag{35}$$

With trial function (34), the BS contribution $V_{BS}(\beta,E,B)$ to the ground state is analytically calculated as follows:

$$V_{BS}(\beta,E,B) = \frac{\hbar\Omega}{E_B}\frac{a_B^2(1-\gamma^2)}{2^{12}E^2\beta^4(a_B^2E - 2\beta^2)^7}\Big\{(a_B^2E - 2\beta^2)\{15a_B^{14}E^7 - 162a_B^{12}E^6\beta^2 + 8a_B^8E^4\beta^6(-195 + 896E^2 - 36\gamma^2) - 4a_B^{10}E^5\beta^4(-173 + 128E^2 + 4\gamma^2) + 64a_B^4E^2\beta^{10}(41 + 1408E^2 - 322\gamma^2 - 492\gamma^4) - 512a_B^2E\beta^{12}(3 + 208E^2 - 18\gamma^2 + 15\gamma^4) - 32a_B^6E^3\beta^8(79 + 1152E^2 + 802\gamma^2 + 172\gamma^4) + 1024\beta^{14}[48E^2 + (-1+\gamma^2)^2]\} - 64E^2\beta^8[-2048a_B^2E\beta^6 + 1024\beta^8 + 48a_B^4\beta^4(1 + 32E^2 + \gamma^2 - 12\gamma^4) - 16a_B^6E\beta^2(3 + 32E^2 + 24\gamma^2(2+\gamma^2) + a_B^8E^2\{64E^2 - 3[11 + 8\gamma^2(7+\gamma^2)]\})]\ln\left(\frac{a_B^2E}{2\beta^2}\right)\Big\}. \tag{36}$$

The dimensionless variables E and a_B in the right-hand side of Eq. (36) must be replaced by $E(\beta)\hbar\Omega/E_B^2$ and a_B/R, respectively.

The results obtained by using the hydrogenlike trial function are presented in Table I. We used more significant figures to stress the fact that the magnetoexciton energies, which are calculated by applying the BS formalism, are extremely close to those provided by the Schrödinger equation.

The magnetoexciton dispersion is determined by Coulomb interaction (30) and the BS term in Eq. (29). The contribution from the Coulomb interaction to the energy of the magnetoexciton (in E_B units) quadratically increases for

TABLE I. Variational calculations of the heavy-hole exciton ground-state energies with $\mathbf{Q}=0$ for various well widths L and weak magnetic fields B. The trial function (34) depends on the variational parameter β. The energy gap is $E_g=1.519$ eV. The electron and hole confinement energy levels E_{c0} and E_{v0} are calculated by assuming square-well potentials of finite depths. The E_{var} column represents the results from the variational calculations with the following Luttinger parameters: $\gamma_1=7.36$ and $\gamma_2=2.57$ (Ref. 14). The measured ground-state energies E_{exp} are reproduced from Ref. 15. The E_{Schr} column represents the ground-state energies calculated according to the Schrödinger equation with the Hamiltonian (2) and a variational parameter β_{Schr}.

L (nm)	B (T)	β	E_{c0} (meV)	E_{v0} (meV)	E_{var} (eV)	E_{exp} (eV)	E_{Schr} (eV)	β_{Schr}
4.03	0	0.786	100	26.9	1.6355	1.638	1.6355	0.786
4.03	2	0.810	100	26.9	1.6356	1.639	1.6357	0.810
4.03	4	0.869	100	26.9	1.6365	1.640	1.6367	0.867
4.32	0	0.776	93.5	24.3	1.6262	1.630	1.6262	0.776
4.32	2	0.802	93.5	24.3	1.6265	1.631	1.6266	0.802
4.32	4	0.861	93.5	24.3	1.6274	1.632	1.6275	0.859
7.2	0	0.702	51.0	11.0	1.5716	1.571	1.5716	0.702
7.2	2	0.734	51.0	11.0	1.5719	1.572	1.5720	0.734
7.2	4	0.803	51.0	11.0	1.5730	1.573	1.5731	0.800

small wave vectors $Qa_B \ll 1$ and can be written as $(Qa_B)^2\mu/M_C$. The hydrogenlike trial function provides the following expression for the in-plane exciton mass M_C:

$$\frac{\mu}{M_C} = 32\beta^3\left(\frac{R}{a_B}\right)^4 \int_0^\infty dx \frac{x^2 f\left(x\frac{L}{a_B}\right)}{(16\beta^2+x^2)^{3/2}}.$$

The contribution to the exciton dispersion due to the BS term can be analytically evaluated. We found that it also quadratically increases for small wave vectors, but for $B_\perp < 4$ T, this contribution is about one-tenth of $(Qa_B)^2\mu/M_C$. Thus, in a weak magnetic field, there is no measurable difference between the results calculated by the Schrödinger equation and those obtained by the more complicated BS formalism. For a weak perpendicular magnetic field and small wave vectors, the Coulomb interaction dominates, which means that a hydrogen type of ground state that is slightly modified by the magnetic field exists.

Next, we consider the case of a strong magnetic field. In this regime, one can apply the lowest Landau level (LLL) approximation, according to which one can ignore transitions between Landau levels and consider only the states on the lowest Landau level. In the LLL approximation, we choose the trial wave function $\psi_\beta(r)$ to be similar to the corresponding ground-state wave function of a charge particle in a magnetic field but which depend on a variational parameter β as follows:

$$\psi_\beta(r) = \frac{1}{\sqrt{2\pi}\beta}\exp\left(-\frac{r^2}{4\beta^2}\right). \tag{37}$$

Here, and in what follows, we use the exciton cyclotron energy $\hbar\Omega$ for unit energy and magnetic length R for unit length. The ground-state magnetoexciton energy will be calculated by minimizing the energy functional $E'(\beta)=(E-E_g-E_{0c}-E_{0v})/\hbar\Omega$ with respect to the variational parameter β as follows:

$$E' = \frac{1}{4}\left(\frac{1}{\beta^2}+\beta^2\right) + V_C(\beta) + V_{BS}(\beta,E') + V_C(\beta,\mathbf{Q}) + V_{BS}(\beta,E',\mathbf{Q}). \tag{38}$$

Note that (i) all terms in Eq. (38) are dimensionless (in a cyclotron energy $\hbar\Omega$ unit) and (ii) we have written the contributions from the Coulomb interaction and from the BS term (28a) as a sum of Q-independent terms, $V_C(\beta)$ and $V_{BS}(\beta,E')$, and Q-dependent terms, $V_C(\beta,\mathbf{Q})$ and $V_{BS}(\beta,E',\mathbf{Q})$. The Q-dependent terms will be used to obtain the magnetoexciton dispersion. The second and third terms in Eq. (38) are given by

$$V_C(\beta) = -\frac{E_b}{\hbar\Omega}\sqrt{\frac{2}{\pi}}\int_0^\infty dx f\left(x\frac{L}{R}\right)\exp\left(-\frac{x^2\beta^2}{2}\right), \tag{39}$$

$$V_{\mathrm{BS}}(\beta,E') = \frac{e^{-4E'\beta^2}\beta^2(-1+\gamma^2)}{64E'^2}\{e^{4E'\beta^2}[-56E'^2\beta^4\gamma^4 + 32E'^3\beta^6\gamma^4 + (-1+\gamma^2)^2 + 4E'\beta^2(-1-2\gamma^2 + 3\gamma^4)] - 32E'^2(-1+\beta^4\gamma^2\{-1+[3+4E'\beta^2 \times(-2+E'\beta^2)]\gamma^2\})Ei(4E'\beta^2)\}. \tag{40}$$

Here, $E_b=\sqrt{\pi/2}e^2/(\epsilon_\infty R)$ is the binding energy of the two-dimensional $(L=0,\ \beta=1)$ magnetoexciton, which is calculated according to the Schrödinger equation.

The energy of the magnetoexciton quadratically increases for small wave vectors $(QR \ll 1)$: $V_C(\beta,\mathbf{Q}) = [\mu/2M_C(L,B,\beta)](QR)^2$ and $V_{BS}(\beta,E',\mathbf{Q}) = [\mu/2M_{BS}(L,B,\beta)](QR)^2$. The in-plane mass $M_C(L,B,\beta)$ is due to the Coulomb interaction and does not depend on the electron or hole mass as follows:

TABLE II. Variational calculations of the heavy-hole exciton ground-state energies for various well widths L and strong magnetic fields B. Trial function (37) depends on the variational parameter β. The energy gap is $E_g=1.519$ eV for the $L=4.03$, 4.32, 7.2, and 7.49 nm wells, and $E_g=1.512$ eV for the $L=7.5$ nm. The electron and hole confinement energy levels E_{c0} and E_{v0} are calculated by assuming square-well potentials of finite depths. $V_C(\beta)$ is the average Coulomb energy. The E_{var} column represents the energies obtained by the variational method using the following Luttinger parameters: $\gamma_1=6.9$ and $\gamma_2=2.4$ (Ref. 16). The measured ground-state energies E_{exp} for the $L=4.03$, 4.32, 7.2, and 7.49 nm wells are reproduced from Ref. 15, and for the $L=7.5$ nm well from Ref. 17. The E_{Schr} column represents the ground-state energies calculated according to the Schrödinger equation with β_{Schr}. The M_C and M_{BS} are the masses calculated according to Eqs. (41) and (42).

L (nm)	B (T)	β	E_{c0} (meV)	E_{v0} (meV)	$V_C(\beta)$ (meV)	E_{var} (eV)	E_{exp} (eV)	E_{Schr} (eV)	β_{Schr}	M_C/m_0	M_{BS}/m_0
4.03	20	0.85	100	25.9	23.7	1.650	1.644	1.651	0.85	0.145	0.0025
4.03	18	0.84	100	25.9	22.9	1.648	1.643	1.649	0.83	0.127	0.0010
4.03	16	0.84	100	25.9	21.8	1.647	1.642	1.647	0.82	0.114	0.0002
4.32	20	0.84	93.5	23.4	23.8	1.641	1.636	1.642	0.85	0.147	0.0026
4.32	18	0.83	93.5	23.4	23.1	1.639	1.635	1.640	0.83	0.129	0.0011
4.32	16	0.83	93.5	23.4	22.0	1.638	1.634	1.638	0.82	0.116	0.0002
7.2	20	0.86	51.1	10.5	22.2	1.587	1.583	1.588	0.86	0.176	0.0044
7.2	18	0.84	51.1	10.5	21.7	1.585	1.582	1.586	0.85	0.159	0.0022
7.2	16	0.84	51.1	10.5	20.7	1.583	1.581	1.584	0.84	0.142	0.0007
7.49	20	0.86	48.3	9.84	22.1	1.584	1.580	1.584	0.86	0.178	0.0046
7.49	18	0.84	48.3	9.84	21.6	1.582	1.579	1.582	0.85	0.161	0.0024
7.49	16	0.84	48.3	9.84	20.6	1.580	1.578	1.580	0.84	0.144	0.0008
7.5	14.5	0.67	48.3	9.82	23.6	1.577	1.577	1.572	0.83	0.131	0.0302
7.5	12	0.64	48.3	9.82	22.8	1.575	1.573	1.570	0.60	0.049	0.0160
7.5	8.5	0.60	48.3	9.82	20.9	1.572	1.570	1.569	0.52	0.026	0.0071

$$\frac{M_{2D}}{M_C(L,B,\beta)} = \sqrt{\frac{2}{\pi}}\int_0^\infty dx f\left(x\frac{L}{R}\right)x^2 \exp\left(-\frac{x^2\beta^2}{2}\right), \quad (41)$$

where $M_{2\mathrm{D}}=2^{3/2}\epsilon_\infty\hbar^2/(\sqrt{\pi}e^2R)$. The second in-plane mass, M_{BS}, has its origin in the fact that the BS term depends on $\mathbf{Q}$ and, for $QR \ll 1$, M_{BS} is defined by the following:

$$\frac{\mu}{2M_{BS}(L,B,\beta)} = \frac{e^{-4E'\beta^2}(-1+\gamma^2)}{256E'^3}[e^{4E'\beta^2}\{256E'^5\beta^{12}\gamma^6 + 64E'^4\beta^{10}\gamma^4(5-17\gamma^2) - 3\beta^2(-1+\gamma^2)^3 - 2E'\beta^4(-1+\gamma^2)^2(1+12\gamma^2)$$
$$- 48E'^2\beta^6\gamma^2(2-7\gamma^2+5\gamma^4) + 16E'^3[-2+2\beta^4+\beta^8\gamma^2(4-53\gamma^2+74\gamma^4)]\} - 64E'^3\beta^2\{-\beta^2+16E'^3\beta^{12}\gamma^6$$
$$+ 4E'^2\beta^{10}\gamma^4(5-18\gamma^2) + \beta^6\gamma^2(-7+33\gamma^2-30\gamma^4) + 2E'[-1+\beta^4+\beta^8\gamma^2(2-29\gamma^2+45\gamma^4)]\}Ei(4E'\beta^2)], \quad (42)$$

where $Ei(x)=-\int_{-x}^{\infty}dt \exp(-t)/t$ is the exponential integral function (the principle value of the integral is taken).

Table II gives the results of our variational calculations. It can be seen that the BS equation provides similar results for the ground-state energies as does the Schrödinger equation. Since the BS mass is much smaller than the Coulomb mass, one can say that in strong magnetic fields, the exciton dispersion for small wave vectors ($QR \ll 1$) is determined by the BS term rather than the Coulomb interaction.

V. COUPLED QUANTUM WELLS IN STRONG MAGNETIC FIELDS

In this section, we consider exactly the same double well electron-hole system as in Ref. 18. The electron layer and hole layer have finite widths, denoted below by L_c and L_v, and they are separated by a distance D. As in Ref. 18, we assume that the electrons and holes are confined between two parallel, infinitely high potential barriers. This assumption greatly simplifies our numerical calculations of the magnetoexciton energy and the Coulomb mass, but by neglecting

TABLE III. Variational calculations of the magnetoexciton energies for various strong magnetic fields B, measured relative to the $E_g+E_{0c}+E_{0v}$ level. The trial function (37) depends on the variational parameter β. The $E_{\rm var}$ column contains the energies calculated by the variational method with the following parameters: $m_c=0.067m_0$, $m_v=0.18m_0$, $\epsilon_\infty=12.35$, $L_c=L_v=8$ nm, and $D=11.5$ nm. The $E_{\rm Schr}$ column represents the magnetoexciton energies calculated according to the Schrödinger equation with $\beta_{\rm Schr}$. M_C is the in-plane mass defined by Eq. (43). The $M_{\rm BS}$ is the mass calculated according to Eq. (42).

B (T)	β	$E_{\rm var}$ (meV)	$E_{\rm Schr}$ (meV)	$\beta_{\rm Schr}$	M_C/m_0	$M_{\rm BS}/m_0$
10	0.96	6.36	6.56	0.98	2.06	0.228
9	0.96	5.17	5.43	0.97	1.75	0.221
8	0.96	4.03	4.31	0.97	1.46	0.216
7	0.96	2.94	3.20	0.96	1.19	0.215
6	0.95	1.91	2.11	0.95	0.95	0.218
5	0.94	0.95	1.04	0.94	0.72	0.230
4	0.92	0.01	0.01	0.92	0.52	0.247

the existence of the finite confinement potentials, we cannot provide a more realistic value for this part of the exciton energy related to the exciton confinement along the z direction than the sum of the well-known terms $\hbar^2\pi^2/2m_{c,v}L_{c,v}^2$. Obviously, the more realistic model of a symmetric (or asymmetric) DQW with finite quantum-well widths[19] will cause minor corrections to our main conclusions, which are as follows: (i) the BS formalism provides a term that does not exist in the Schrödinger equation, and (ii) the term plays an important role in determining the magnetoexciton dispersion.

The basic features of the CQW magnetoexcitons are the same as those of the SQW magnetoexcitons. However, because of the separation between the electron and hole layers, the Coulomb energy and the Coulomb in-plane mass quantitatively differ from those of the SQW magnetoexciton. In other words, in strong magnetic fields, Eq. (38) holds, but the Coulomb interaction and the corresponding in-plane mass are defined as follows:

$$V_C(\beta) = -\frac{E_b}{\hbar\Omega}\sqrt{\frac{2}{\pi}}\int_0^\infty dx e^{-x^2\beta^2/2}F\left(x,\frac{L_c}{R},\frac{L_v}{R},\frac{D}{R}\right),$$

$$\frac{M_{2D}}{M_C(L,B,\beta)} = \sqrt{\frac{2}{\pi}}\int_0^\infty dx x^2 e^{-x^2\beta^2/2}F\left(x,\frac{L_c}{R},\frac{L_v}{R},\frac{D}{R}\right). \tag{43}$$

In CQW's, the structure factor is

$$F(x,\xi_c,\xi_v,d) = \frac{16\pi^4(1-e^{-\xi_c x})(1-e^{-\xi_v x})e^{-dx}}{\xi_c\xi_v x^2(4\pi^2+\xi_c^2x^2)(4\pi^2+\xi_v^2x^2)}.$$

Table III gives the result of our numerical calculation of the magnetoexciton energy relative to the $E_g+E_{0c}+E_{0v}$ level. We used the same parameters as in Refs. 5 and 18. It can be seen that the BS equation provides slightly different results for the binding energy than the Schrödinger equation.

The main difference between the BS and the Schrödinger equation is in their predictions about the in-plane magnetoexciton mass in a strong magnetic field. Unfortunately, optical experimental studies can provide information about the exciton dispersion only for $Q\leq Q_{\rm ph}$, where $\hbar Q_{\rm ph}$ is the photon momentum. Other studies, such as the photoluminescence measurement experiments that can measure the exciton-mass dependence of the recombination time or experimental data related to the polariton effects, can provide information about the magnetoexciton dispersion. Many of these experimental techniques[20] are used to measure the magnetoexciton dispersion in the presence of an in-plane magnetic field. As we mentioned above, the measurable differences between the magnetoexciton dispersions, as predicted by the BS formalism and by the Schrödinger equation, are to be expected in strong perpendicular magnetic fields. To the best of our knowledge, there is only one paper[18] wherein the exciton dispersion in $GaAs/Ga_{0.67}Al_{0.33}As$ CQW's in a weak perpendicular magnetic field was measured. There is good agreement between the mass M_C and the measured mass in a weak magnetic field. By referring to the conclusion that the BS term in a weak magnetic field has a very small contribution to the dispersion as compared to the contribution due to the Coulomb interaction, one can say that there exists good agreement between the BS formalism and the measurements.

Next, we discuss the fact that M_C increases by about four times if we increase the magnetic field from 4 to 10 T. If the magnetoexciton dispersion in strong magnetic fields ($B>5$ T) is determined mainly by the BS term, then the magnetoexciton mass should not increase so dramatically and, therefore, additional experimental points are needed to prove or disprove the conclusions drawn by applying the BS formalism.

VI. SUMMARY

In summary, we applied the BS formalism to the quantum-well excitons in an in-plane magnetic field and in a perpendicular magnetic field. We found that in the case of an in-plane magnetic field, the BS equation and the Schrödinger equation both provide exactly the same results. In contrast, in a perpendicular magnetic field, the BS equation contains an extra term (BS term). We applied a variational procedure to obtain the effect of the BS term on the magnetoexciton ground-state energy and magnetoexciton mass. We used a simple hydrogenlike trial wave function in a weak magnetic field and figured out that in a weak perpendicular magnetic field, the results obtained by the BS formalism are very close to the results calculated by means of the Schrödinger equation. In a strong magnetic field, we used a trial function similar to the wave function of a charged particle in a magnetic field. We calculated that in a strong magnetic field, the ground-state energy is very close to that obtained by means of the Schrödinger equation, but the magnetoexciton dispersion is determined by the BS term rather than by the electron-hole Coulomb term in the Schrödinger equation.

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