Plasma picture of the fractional quantum Hall effect with internal SU(K) symmetries

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We consider trial wave functions exhibiting SU(K) symmetry, which may be well suited to grasp the physics of the fractional quantum Hall effect with internal degrees of freedom. Systems of relevance may be either spin-unpolarized states ($K=2$), semiconductor bilayers ($K=2,4$), or graphene ($K=4$). We find that some introduced states are unstable, undergoing phase separation or phase transition. This allows us to strongly reduce the set of candidate wave functions eligible for a particular filling factor. The stability criteria are obtained with the help of Laughlin's plasma analogy, which we systematically generalize to the multicomponent SU(K) case. The validity of these criteria is corroborated by exact-diagonalization studies for SU(2) and SU(4). Furthermore, we study the pair-correlation functions of the ground state and elementary charged excitations within the multicomponent plasma picture.

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I. INTRODUCTION

Soon after the discovery of the fractional quantum Hall effect (FQHE),[1] Laughlin successfully described the underlying strongly correlated electron liquid with the help of a simple trial wave function.[2] The reasons for the success of this approach were twofold: first, the calculated energy of this state is lower than that of charge-density waves or Wigner crystals,[3] which are natural candidates for the ground state within the partially filled lowest Landau level (LL) due to the quenched kinetic energy.[2] The second reason for its success is the fact that Laughlin's wave function is in very sharp agreement with exact-diagonalization studies.[4]

A powerful tool in the understanding of Laughlin's wave function is the quantum-classical analogy, in which its probability is interpreted as the (classical) Boltzmann weight of a two-dimensional one-component plasma (2DOCP).[2] Most strikingly, this analogy shows that Laughlin's trial wave functions have no free parameter that may be optimized by any variational calculation.

Laughlin's original proposal was concerned only with a single species of fermions, namely, spin-polarized electrons. In spite of its success, this is at first sight a very crude assumption in view of the relatively weak (effective) Zeeman effect when compared to the leading energy scale set by the Coulomb interaction $e^2/\epsilon l_B$ in terms of the magnetic length $l_B=\sqrt{\hbar/eB}$. Indeed, the latter is almost 2 orders of magnitude larger than the bare spin splitting for typical magnetic fields of $B\sim 10$ T. In order to account for an internal SU(2) spin symmetry, Halperin proposed a generalized trial wave function,[5] which includes Laughlin's as a special case. The latter may indeed be viewed as a Halperin wave function with a spontaneous ferromagnetic spin ordering.[6]

Halperin's SU(2) wave functions have been a first step in the understanding of general multicomponent systems. In the case of bilayer quantum Hall systems, the same wave functions may be applied if one supposes a complete polarization of the physical spin and if one interprets the two layer indices as the two possible orientations of a *pseudospin*.[6,7] However, the hypothesis of a complete spin polarization is *a priori* as feebly justified in bilayer as in monolayer quantum Hall systems, again due to a relatively weak Zeeman effect. A more appropriate approach is, therefore, one that takes into account the internal SU(4) spin-pseudospin symmetry. Such approaches have indeed been proposed in the description of ferromagnetic states when the LL filling factor $\nu=n_{el}/n_B$, in terms of the electronic, n_{el}, and the flux, $n_B=eB/h$, densities are 1, 2, or 3, respectively.[8,9]

Another example of a multicomponent quantum Hall system is graphene, where the internal SU(4) symmetry is due to the physical spin accompanied by a twofold valley degeneracy.[10–12] In contrast to the above-mentioned bilayer quantum Hall systems, where the pseudospin symmetry is explicitly broken because of the difference between intra- and interlayer Coulomb interactions, the SU(4) symmetry is almost perfectly preserved in graphene from an interaction point of view—a possible (valley) symmetry breaking may be due to lattice effects, which are suppressed by the small parameter a/l_B, where $a=0.14$ nm is the distance between nearest-neighbor carbon atoms in graphene, as compared to $l_B=26\sqrt{B[\mathrm{T}]}$ nm.[13–17] In order to describe a possible, yet unobserved, FQHE in graphene, taking into account the appropriate form of the interaction potential,[13,18] exact-diagonalization studies have been performed in the framework of an internal SU(2) valley symmetry[19,20] as well as in a SU(4) composite-fermion approach.[21,22]

More recently, two of us have proposed a generalization of Halperin's wave functions to K components, i.e., systems with an internal SU(K) symmetry, in order to describe a possible FQHE in K-component systems, namely, graphene with $K=4$.[23] Here, we investigate the stability of these wave functions from two complementary perspectives—first, we derive the stability criteria within a generalized plasma picture. This analogy allows one to interpret the SU(K) Halperin wave functions in terms of K correlated 2DOCP and to describe their ground-state properties in a compact manner as well as the elementary excitations with fractional charge. Second, we corroborate the validity of the generalized plasma picture with the help of exact-diagonalization studies.

After a brief review of Laughlin's plasma analogy (Sec. II), we generalize the plasma picture to K-component sys-

tems in Sec. III. In Sec. IV, we derive the general stability criteria within the plasma analogy, on the basis of which we discuss the stability of specific SU(2) and SU(4) wave functions. We complete this paper with a discussion of ground-state properties, such as sum rules for the pair-correlation functions (Sec. V), and fractional charges of quasiparticle and/or quasihole excitations (Sec. VI).

II. LAUGHLIN'S PLASMA ANALOGY

In order to describe a correlated electron liquid to account for the FQHE, Laughlin proposed the N-particle trial wave function[2]

$$\Psi_m(\{z_k\}) = \prod_{k<l}^{N} (z_k - z_l)^m \exp\left(-\sum_k^N \frac{|z_k|^2}{4}\right), \tag{1}$$

where $z_k = x_k + iy_k$ denotes the position of the kth electron in the complex plane. Here and in the following, we set the magnetic length $l_B \equiv 1$ for notational convenience. The form of this trial wave function is solely determined by the analyticity condition for the lowest LL—i.e., all single-particle states are of the form $z^\ell \exp(-z^2/4)$, where ℓ is a positive integer—and by symmetry considerations. In order to have a translational and rotational invariant state and, thus, an incompressible state with no gapless Goldstone mode, the wave function may only depend on the relative distance $z_k - z_l$ of the kth and the lth particle. Furthermore, fermion statistics for electrons requires the exponent m to be an odd integer, which is the only variational parameter of Laughlin's wave function (1).

However, the parameter m turns out to be fixed by the electron density, or by the filling factor $\nu = 1/m$, as Laughlin showed with the help of a plasma analogy.[2] Indeed, one may interpret the modulus square of the wave function as the Boltzmann weight

$$|\Psi_m(\{z_k\})|^2 = e^{-\beta \mathcal{H}_N} \tag{2}$$

of a classical system, namely, a 2DOCP described by the classical Hamiltonian[2,24,25]

$$\mathcal{H}_N = -m \sum_{k<l} \ln|z_k - z_l| + \sum_k \frac{|z_k|^2}{4}, \tag{3}$$

where one has somewhat arbitrarily set the inverse "temperature" $\beta \equiv 2$. Note that the true temperature does not intervene in the analysis because the system is placed at $T=0$. The first term describes two-dimensional (2D) interacting particles of charge $\sqrt{m}$, whereas the second term may be interpreted as a homogeneous background of charge $-1/\sqrt{m}$ (jellium). The minimization of the classical Hamiltonian corresponds, via the relation (2), to a maximal quantum probability of the original quantum system of electrons within the lowest LL. The classical ground state of the Hamiltonian (3), however, is obtained when the plasma particles of charge $\sqrt{m}$ are fully neutralized by the background, i.e., when

$$m n_{el} = \frac{1}{2\pi} \Leftrightarrow m\nu = 1. \tag{4}$$

It is evident from the last equation that the variational parameter must be positive—otherwise, one would have to deal with unphysical negative densities. Note that from the wave function point of view, $m<0$ is not physical because it violates the analyticity condition for wave functions in the lowest LL. This point, which may seem obvious, is worth emphasizing and recalling in the following sections when the plasma picture is generalized to more components.

III. PLASMA PICTURE FOR SU(K) HALPERIN WAVE FUNCTIONS

Based on Halperin's idea to write down a Laughlin-type wave function for a two-component quantum Hall system, in order to take into account the spin degree of freedom,[5] two of us have proposed a SU(K) generalization for a K-component system,[23]

$$\Psi^{\mathrm{SU}(K)}_{m_1,\ldots,m_K;n_{ij}} = \Phi^L_{m_1,\ldots,m_K} \times \Phi^{\mathrm{inter}}_{n_{ij}} \times \exp\left(-\sum_{i=1}^{K}\sum_{k_i=1}^{N_i} \frac{|z^{(i)}_{k_i}|^2}{4}\right),$$

$$\Phi^L_{m_1,\ldots,m_K} = \prod_{i=1}^{K} \prod_{k_i<l_i}^{N_i} (z^{(i)}_{k_i} - z^{(i)}_{l_i})^{m_i},$$

$$\Phi^{\mathrm{inter}}_{n_{ij}} = \prod_{i<j}^{K} \prod_{k_i=1}^{N_i} \prod_{k_j=1}^{N_j} (z^{(i)}_{k_i} - z^{(j)}_{k_j})^{n_{ij}}. \tag{5}$$

There are K different types of electrons [denoted with superscript (i)] with inter- (n_{ij}) and intracomponent (m_i) quantum correlations, and $z^{(i)}_{k_i}$ is the complex position of the k_ith electron of type $i=1,\ldots,K$. The lowest-LL analyticity condition imposes that all exponents, m_i and n_{ij}, must be integers. Furthermore, m_i must be odd in the case of fermions. Apart from $K=2$, discussed by Halperin,[5] $K=4$ wave functions may be physically significant in the case of bilayer quantum Hall systems and graphene. In the former example, the internal degrees of freedom do not only contain the physical SU(2) spin $(\uparrow,\downarrow)$, but also a layer index, which may be mimicked by an additional SU(2) isospin $(+,-)$. There are, thus, four internal states, $1=(\uparrow,+)$, $2=(\uparrow,-)$, $3=(\downarrow,+)$, and $4=(\downarrow,-)$. In the case of graphene, an isospin $(+,-)$ must be introduced in order to account for the twofold valley degeneracy. Wave functions similar to those in Eq. (5) have been proposed by Qiu *et al.*[26] for multilayer quantum Hall systems, by Morf[27] as potential candidates for the FQHE hierarchy states, and by Yang *et al.*[21] in the study of a possible FQHE in graphene.

Again, the starting point of the plasma analogy is Eq. (2), and one associates the new Hamiltonian $\mathcal{H}_N$ with a physical system. In the case of $K=2$, this system has been interpreted as a generalized plasma, which consists of K different types of particles (each of which corresponds to a different electron type in the original quantum system) plus a neutralizing background.[6,28] With the identification (2), one obtains for the wave functions (5) the classical Hamiltonian

$$\mathcal{H}_N = -\sum_{i=1}^{K} m_i \sum_{k_i<l_i}^{N_i} \ln|z_{k_i}^{(i)} - z_{l_i}^{(i)}| - \sum_{i<j}^{K} n_{ij} \sum_{k_i,k_j} \ln|z_{k_i}^{(i)} - z_{k_j}^{(j)}| + \sum_{i=1}^{K} \sum_{k_i=1}^{N_i} \frac{|z_{k_j}^{(i)}|^2}{4}. \tag{6}$$

Here, the first term represents a sum over K 2D interaction terms for (i)-type particles of charge $\sqrt{m_i}$, whereas the second one takes into account interactions between particles of different types, (i) and (j). However, this generalized plasma does not satisfy the charge superposition principle[29] unless $n_{ij}=\sqrt{m_i m_j}$, which is a rather special case.

Instead of one single plasma of K types of particles, it seems, therefore, more appropriate to interpret this generalized plasma in terms of K different 2DOCPs (one for each type of electrons) with correlations between them. For this purpose, we introduce the continuum limit, in which the density for particles of type (i) (electrons or plasmatic particles) is $\rho_i(\mathbf{r})=\Sigma_{k_i}\delta(\mathbf{r}-\mathbf{r}_{k_i})$. In order to distinguish the resulting Hamiltonian from the original discrete one, we supress the N subscript in Eq. (3) and one obtains the energy functional

$$\mathcal{H}[\{\rho_i(\mathbf{r})\}] = -\int\int_{\Omega} d^2r d^2r' \begin{pmatrix} \rho_1(\mathbf{r}) \\ \vdots \\ \rho_K(\mathbf{r}) \end{pmatrix}^{\mathsf{T}} \frac{M_K}{2} \ln|\mathbf{r}-\mathbf{r}'| \begin{pmatrix} \rho_1(\mathbf{r}') \\ \vdots \\ \rho_K(\mathbf{r}') \end{pmatrix} + \int_{\Omega} d^2r \begin{pmatrix} \rho_1(\mathbf{r}) \\ \vdots \\ \rho_K(\mathbf{r}) \end{pmatrix}^{\mathsf{T}} \frac{|r|^2}{4} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{7}$$

Here, M_K is the symmetric exponent matrix, with $n_{ij}=n_{ji}$ and $n_{ii}\equiv m_i$,[23] and Ω is the surface occupied by the plasma. Similar to the one-component case, the configurations with maximal probability [Eq. (2)] are obtained by minimizing $\mathcal{H}$ with respect to all densities. The stationary points are found at

$$\left.\frac{\delta\mathcal{H}}{\delta\rho_i(\mathbf{r})}\right|_{\rho_{j,j\neq i}} = 0. \tag{8}$$

In order to have a minimum, the Hessian matrix

$$\frac{\delta^2\mathcal{H}}{\delta\rho_i(\mathbf{r})\,\delta\rho_j(\mathbf{r})} \propto M_K, \tag{9}$$

which is identical to the exponent matrix M_K up to a positive constant, needs to be positive, i.e., have positive eigenvalues. One may interpret Eq. (8) as the stationary point of a 2DOCP of (i)-type particles, whereas the positions of all other particles of type $(j\neq i)$ are fixed and constitute a quasistatic impurity potential felt by the (i)-type particles. For the 2DOCP of this type, the interactions between all other types of particles yield only an unimportant constant with respect to the ρ_i derivative. In this sense, one may indeed interpret the system as K correlated 2DOCPs rather than a single plasma of K different types of particles.

As for a single 2DOCP, Eq. (8) is satisfied when each of the K plasmas exhibits quasineutrality,[25] but now contributions from the impurities have to be taken into account,

$$m_i\rho_i(\mathbf{r}) + \sum_{j\neq i} n_{ij}\rho_j(\mathbf{r}) = \frac{1}{2\pi} \tag{10a}$$

$$\Leftrightarrow M_K \begin{pmatrix} \rho_i(\mathbf{r}) \\ \vdots \\ \rho_K(\mathbf{r}) \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{10b}$$

Here, $1/2\pi$ on the right-hand side of Eq. (10a) represents the neutralizing background, as for a single 2DOCP [Eq. (4)]. The second term in Eq. (10a) represents the contributions from type-$(j\neq i)$ particles due to intercomponent correlations. One notes that Eq. (10b) is the matrix generalization of Eq. (4). This result was previously derived by counting the zeros of the wave function (5).[23] Invertible matrices yield a unique solution with all densities being uniform, $\rho_i(\mathbf{r})=\rho_i$, as is the case of U(1) Laughlin's liquid. The case of noninvertible matrices will be discussed in the next section.

Unlike the U(1) case, fixing the total filling factor $\nu_T=\nu_1+\cdots+\nu_K$ does not uniquely determine the exponent matrix. It has been pointed out in Ref. 23 that several candidate wave functions may give rise to a FQHE at the same filling factor ν_T, especially in the case of larger internal symmetry groups. Moreover, even if one fixes all component filling factors $\nu_i=2\pi\rho_i$, the wave function is not unambiguous. As an example, we consider the SU(2) Halperin wave functions (m_1,m_2,n), with (i) $m_1=m_2=3$, $n=1$ [(331) wave function], and (ii) $m_1=m_2=1$, $n=3$ [(113) wave function]. Both wave functions describe a situation with $\nu_1=\nu_2=1/4$ and have been considered in the past within the study of a possible unpolarized $\nu_T=1/2$ state.[30] Indeed, an even-denominator quantum Hall state has been observed at $\nu=5/2$ and $\nu=7/2$ in the first excited LL.[31] However, it is strongly unlikely that this state is spin unpolarized[32–34] and that more sophisticated theories, in terms of a Pfaffian state, need to be invoked to account for a spin-polarized state.[35] As we will show below, stability conditions related to Eq. (9) allow one, in the case of unpolarized states, to discriminate between (mmn) and (nnm) wave functions for $m>n$.

IV. STABILITY

In order to obtain a stable state, the stationary point obtained from Eq. (8) must be minimum, i.e., the Hessian matrix $\propto M_K$ in Eq. (9) must be positive. Otherwise, the plasmas would neither be stable nor be in the state of lowest energy. Hence, a first stability condition imposes that all eigenvalues λ_i of M_K must be positive.

There is indeed a limiting case for which one (or more) eigenvalue(s) is (are) zero. Because the potential is quadratic, the minimum point now becomes a line of minima, which correspond to different ground states. If at least one eigenvalue is zero, the exponent matrix M_K is no longer invertible and all densities may not be fully determined from Eq. (10b).

This point may be alternatively interpreted in terms of SU(K) ferromagnets—different states of equal energy may, e.g., occur at various combinations of two (or more) filling factors ν_i and ν_j although the sum $\nu_{ij}=\nu_i+\nu_j$ is fixed. In this case, one may introduce a pseudospin operator $S_{ij}^z=N(\nu_i$

$-\nu_j)/2$, which can possibly take all values in between $-N/2 \leq S_{ij}^z \leq N/2$. The simplest example of such a case is the Laughlin wave function with an internal spin degree of freedom [a Halperin (*mmm*) wave function], which is, for odd m, completely antisymmetric in its orbital part. For fermions, the spin wave function must, therefore, be completely symmetric and, thus, represent a SU(2) ferromagnet. If the total spin is oriented along the positive z direction, all electrons reside in the upper spin branch ($\nu_1=1/m$, $\nu_2=0$). In the absence of a Zeeman effect, this state has the same energy as the one with a total spin in the $-z$ direction ($\nu_1=0$, $\nu_2=1/m$), as well as any intermediate state with $\nu_1+\nu_2=1/m$. In the general SU(K) case, the ferromagnetic properties are determined by the rank r of the matrix M_K. Indeed, if $r<K$ and one introduces common fixed filling factors ν_{ij} for the relevant components, one may describe the resulting state by a SU(r) Halperin wave function with an invertible exponent matrix $\tilde{M}_r$ with additional pseudospin degrees of freedom for the components, the density of which remains undetermined.[23]

As mentioned earlier, some exponent matrices can lead to negative density solutions for Eq. (10b). A second class of stability conditions needs to be imposed in order to prevent this unphysical situation, which may occur even in the case of a positive matrix M_K.

In order to illustrate the two conditions, we discuss some specific examples for different K's. The case of $K=1$ has already been presented above.

A. The case $K=2$

We first study Halperin's wave function (m_1,m_2,n) for the SU(2) case, which is described by the exponent matrix

$$M_{K=2} = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix}.$$

Even if all exponents are positive, as required by the lowest-LL analyticity condition, the eigenvalues $\lambda_\pm$ and the filling factors $\nu_{1/2}$ are not necessarily so,

$$\lambda_\pm = \frac{m_1+m_2 \pm \sqrt{(m_1-m_2)^2+4n^2}}{2}, \tag{11a}$$

$$\nu_1 = \frac{m_2-n}{m_1m_2-n^2} \quad \nu_2 = \frac{m_1-n}{m_1m_2-n^2}. \tag{11b}$$

In order to obtain only positive eigenvalues (first stability condition), one needs to require

$$m_1m_2-n^2 \geq 0. \tag{12}$$

The case $m_1m_2=n^2$ corresponds to a situation of a noninvertible matrix of rank $r=1$. Because of Eq. (12), positive densities (filling factors) are found in Eq. (11b) only for

$$m_1 \geq n \quad \text{and} \quad m_2 \geq n, \tag{13}$$

which we, thus, need to impose as a second class of stability conditions. Furthermore, one notices from Eqs. (12) and (13) that the only states with a noninvertible exponent matrix of rank $r=1$ are the ferromagnetic Laughlin states (*mmm*) discussed above.

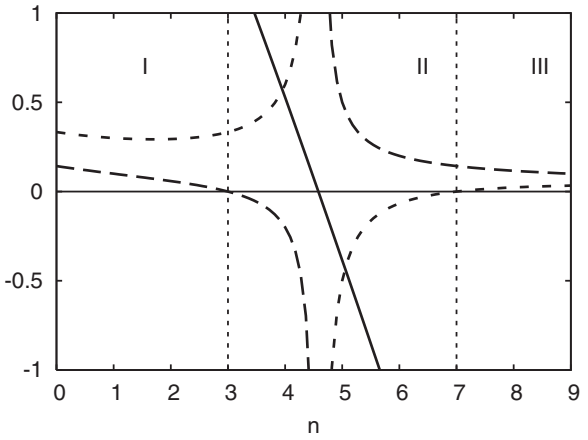

FIG. 1. Stability of the (73n) wave function. Both filling factors, ν_1 (long dashed line) and ν_2 (short dashed line), and the λ_- eigenvalue (solid line) of the (73n) wave function are plotted as a function of n. For $n \leq 3$ (part I), all quantities are positive and the corresponding state is stable. In part II, one of the filling factors is negative. Part III exhibits positive filling factors, but the state is still unstable because of the negative eigenvalue; the system eventually undergoes a phase separation.

The final stability criterion Eq. (13) for SU(2) wave functions has a compelling physical interpretation: intracomponent correlations must always be *stronger* than intercomponent correlations. Within the plasma picture, this may also be understood from Figs. 2(a) and 2(b). For illustration, we consider the (73n) wave function, where n is left as a variable, which we treat in a rather artificial manner as a continuous variable in the following discussion. It is evident that only integer values may be taken into account for physical candidate wave functions.

Figure 1 shows the plot of both component filling factors and the lower eigenvalue, λ_-. This graph can be split into three distinct parts I, II, and III.

Part I ($n \leq 3$). Each type-(1) particle carries a charge $\sqrt{7}$ and is affected by those from the type-(2) plasma through a charge coupling of $n/\sqrt{7} \leq \sqrt{7}$, which may be interpreted as constituting a quasistatic impurity distribution interacting with type-(1) particles. Alternatively, one may concentrate on type-(2) particles with charge $\sqrt{3}$, which see type-(1) particles as a distribution of charge $n/\sqrt{3} \leq \sqrt{3}$ impurities. Therefore, both types of particles are more strongly repelled by those of their own species than by particles of a different type. One, thus, obtains a stable homogeneous mixture of two plasmas, which is shown in Fig. 2(a).

Part II ($3<n<7$). Although type-(1) particles are still more strongly repelled by those of their own species than by type-(2) particles ($n/\sqrt{7} \leq \sqrt{7}$), this is not the case for type-(2) particles. Because their interspecies repulsion is now weaker than that which they experience from type-(1) particles ($n/\sqrt{3} > \sqrt{3}$), they prefer to gather— rather than be mixed —with type-(1) particles. This indicates a tendency to phase separate, and the effect manifests itself in an unphysical negative filling factor ν_1. The divergence of the filling factors at the artificial value of $n_c=\sqrt{21} \simeq 4.6$ in Fig. 1 is due

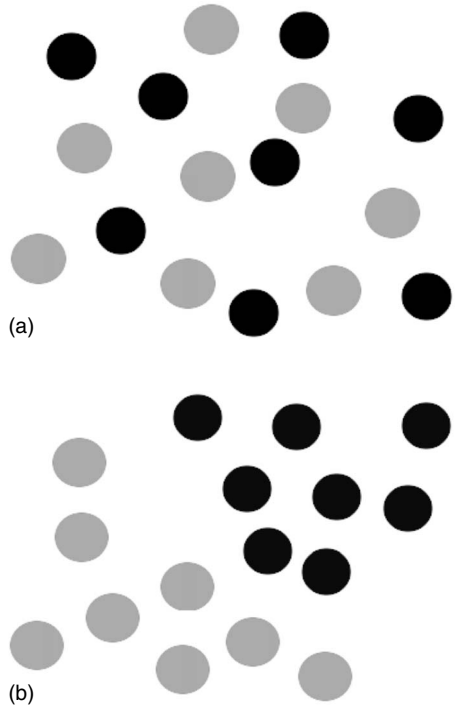

FIG. 2. Sktech of the plasma corresponding to the $(73n)$ wave function for (a) $n \leq 3$ and (b) $7 \leq n$. (a) Intracomponent repulsions are stronger than intercomponent repulsions $(n \leq m_1, m_2)$. Type-(1) particles (black) are therefore, on the average, surrounded by type-(2) particles (gray). This yields a stable state of two homogeneous interpenetrating plasmas. (b) If the intercomponent repulsion is stronger than the intracomponent repulsion, the plasmas have a tendency to phase separate to minimize the number of neighbors from different types.

to the vanishing eigenvalue λ_-. Above n_c, the original minimum of the energy functional (7) evolves into a saddle point and the first stability condition of non-negative eigenvalues is no longer satisfied. Indeed, one notices that the filling factors interchange their roles. Although the repulsion between type-(1) particles is stronger than that between type-(2) particles and one would, therefore, intuitively expect that $\nu_1 < \nu_2$, one finds $\nu_1 > \nu_2$ for $n > n_c$.

Part III $(7 \leq n)$. Above $n=7$, the intercomponent repulsion is stronger than that between particles of the same type. The phase separation between the two plasmas, which we have alluded to in the discussion of part II, is well pronounced. Due to this strong intercomponent repulsion, the interface between the two plasmas needs to be minimized, and this results in an inhomogeneous state of two spatially separated plasmas, as shown in Fig. 2(b). Should the partial densities not be fixed, i.e., particles could flip from state 1 to state 2, it is clear that the system would favor a distribution where only one type of particles would remain. The interface between the two plasmas disappears.

Figure 3 shows the stability graph for an (mmn) wave function (here with $n=3$). In this case, the eigenvalues (11a) and the component filling factors (11b) become

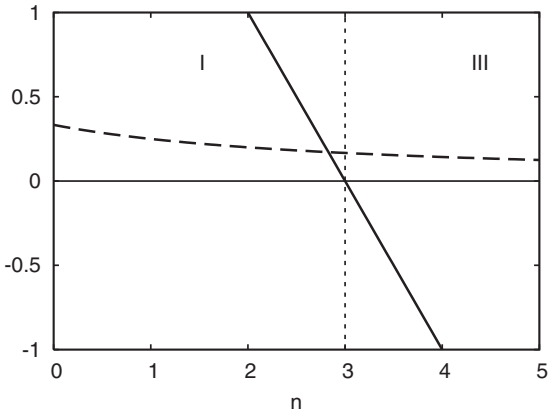

FIG. 3. Stability of the $(33n)$ wave function. Equal filling factors, ν_1 and ν_2 (long dashed line), and the λ_- eigenvalue (solid line) of the $(33n)$ wave function are plotted as a function of n. For $n \leq 3$ (part I), the state is stable as in Fig. 1. There is no corresponding part II, since the densities never vanish. In part III, the plasmas tend to phase separate.

$$\lambda_\pm = m \pm n \quad \text{and} \quad \nu_1 = \nu_2 = \frac{1}{m+n}, \tag{14}$$

respectively. Although the component filling factors remain positive for all choices of n, the eigenvalue λ_- becomes negative for $n > m$, where one would expect a phase separation between the two plasmas, as in the case of the $(73n)$ wave function [Fig. 2(a)]. The critical value $n=m$ corresponds to the Laughlin case with a SU(2) ferromagnetic spin wave function, as discussed above. For the case of $n < m$, both trial wave functions, (mmn) and (nnm), are valuable candidates for the description of a potential FQHE at $\nu_T = 2/(m+n)$ if only the symmetry considerations for trial wave functions in the lowest LL are taken into account. However, the plasma analogy clearly indicates that only one of the two wave functions, namely, (mmn), yields a stable physical state. At half-filling, the only SU(2) Halperin wave function which might yield a stable FQHE state is (331), whereas (113) corresponds to an unstable plasma, which is not evident from wave function calculations alone.[30]

B. The case of $K=4$

We now consider the generalized Halperin wave functions with an internal SU(4) symmetry. We restrict our studies to a particular subset of the latter, $(m_1 m_2 m_1 m_2, n_e n_+ n_-)$, the corresponding exponent matrices of which may be written as[23]

$$M_K = \begin{pmatrix} m_1 & n_e & n_+ & n_e \\ n_e & m_2 & n_e & n_- \\ n_+ & n_e & m_1 & n_e \\ n_e & n_- & n_e & m_2 \end{pmatrix}. \tag{15}$$

If applied to graphene, those correlation coefficients imply that one should treat all intervalley components (n_e) on the same footing and the intravalley ones separately $(n_+, n_-$, for + and − valleys, respectively). This is an even more natural assumption in the case of bilayer quantum Hall systems in

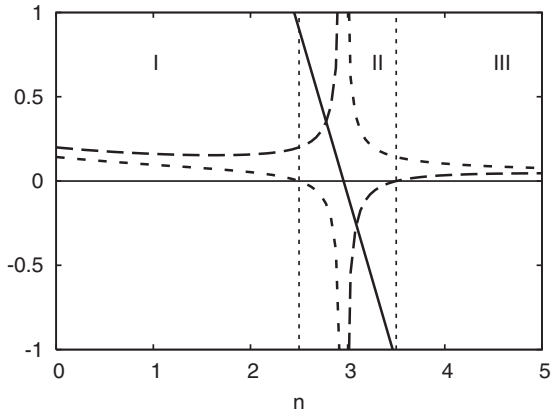

FIG. 4. Stability of the (3535,n22) wave functions. Filling factors ν_1 and ν_3 (long dashed line), ν_2 and ν_4 (short dashed line), and the third eigenvalue λ_3 (solid line) of the (3535,n22) generalized Halperin wave function are plotted as a function of n. For $n \leq 2$ (part I), all quantities are positive and the corresponding state is stable. Part II corresponds to unphysical negative densities with a negative eigenvalue (−0.08). For $n \geq 4$, because of the eigenvalue λ_3 being still negative, the plasmas (1)–(3) phase separate from (2)–(4).

semiconductor heterostructures, where interlayer correlations (described by the exponents n_e) are weaker than intralayer ones (n_+ and n_-, which couple the different spin orientations within the + and − layers, respectively). Moreover, some intracomponent correlations are fixed to the same value so that an explicit calculation of the eigenvalues λ and filling factors ν may be carried out. However, here we will only settle the conditions for all quantities to be positive (the first and second stability arguments), and these conditions are satisfied if

$$m_1 \geq n_+,$$

$$m_2 \geq n_-,$$

$$m_2 + n_- \geq 2n_e,$$

$$m_1 + n_+ \geq 2n_e. \tag{16}$$

The case where one of the first two inequalities, or one of the last two, turns into an equality corresponds to matrices of rank $r < 4$. For $r = 1$, Eq. (16) becomes an equation set and, once again, the only stable state is of Laughlin type ($mmmm$,mmm) with SU(4)-ferromagnetic ordering. The stability criteria [Eq. (16)] yield a slightly more complex interpretation: not only do intracomponent correlations have to be stronger than some intercomponent ones, but there are also conditions between intercomponent coefficients. This may be more easily understood within the (3535,n22) state, where n is left as a variable, similar to the SU(2) case discussed in Sec. IV A. Again, we have chosen this state purely for illustration reasons.

Figure 4 plots all filling factors and the relative signed eigenvalue λ_3. As for the case of $K=2$, the graph is split into parts I, II, and III.

Part I ($n \leq 2$). One notices that type-(1/3) and (2/4) particles act identically; therefore, one can virtually treat the problem as for the case $K=2$. Type-(1/3) and (2/4) plasmas are stable separately with respect to the previous section on SU(2) wave functions. Moreover, type-(1/3) particles carry a $\sqrt{3}$ charge and are affected by type-(2/4) quasistatic impurities of $n/\sqrt{3} \leq \sqrt{3}$ charge. Similarly, type-(2/4) particles with a $\sqrt{5}$ charge interact with type-(1/3) plasma through a $n/\sqrt{5} \leq \sqrt{5}$ charge. Hence, one plasma suffers weaker repulsion from the other and type-(1/3) and (2/4) plasmas will mix in order to form a stable homogeneous state.

Part II ($n=3$). The λ_3 eigenvalue is now negative (−0.08) as well as some filling factors. In the plasma picture, type-(1/3) particles are equally repelled by their own species and type-(2/4) quasistatic impurities, each carrying a $\sqrt{3}$ charge. On the contrary, the type-(2/4) plasma still experiences more "favorable" repulsion from type-(1/3) particles. One cannot conclude about the stability at this level and some care has to be taken of the inner composition of the type-(1/3) particles. Indeed, type-(1) particles carry a $\sqrt{3}$ charge and interact with type-(3) quasistatic impurities via a $2/\sqrt{3}$ charge, which is less than any other charge for this plasma. Similarly, type-(3) particles will be less repelled by type-(1) particles than by any other particles. Hence, type-(1/3) plasma will tend to phase separate, which is contradictory to the phase mixing tendency of type-(2/4) particles. As in the $K=2$ case, this yields unphysical negative densities for type-(1/3) particles.

Part III ($4 \leq n$). Above $n=4$, all filling factors are positive, but λ_3 becomes more and more negative. For $n=4$, the same argument as above can be developed: type-(1/3) plasma tends to separate from type-(2/4) plasma, whereas type-(2/4) plasma tends to mix with type-(1/3) plasma. Surprisingly, this is not related to a negative density, as in any other case previoulsy discussed. This may be due to the composite nature of type-(2/4) plasma. For $n \geq 5$, both plasmas are more severely repelled by each other such that the phase separation is complete.

C. Comparison with exact-diagonalization studies

The stability discussed so far is only related to the particular form of the wave functions and it somehow has to be linked to the true ground state of the quantum system with N interacting electrons. We, therefore, investigate the stability of generalized Halperin wave functions within exact-diagonalization studies. The system is mapped onto a sphere, in the center of which a magnetic monopole is fixed to ensure a magnetic field orthogonal to the surface. This magnetic monopole creates $2S$ flux quanta threading the surface of the sphere. At a particular filling factor ν_T, the relation between the number of particles and that of flux quanta is

$$2S = N/\nu_T - \delta,$$

where

$$\nu_T = \sum_{i,j} M_{ij}^{-1}, \tag{17}$$

and the shift

$$\delta = \frac{1}{\nu_T} \sum_{i,j} M_{ij}^{-1} m_i \tag{18}$$

is due to the finite-size geometry and depends on the particular wave function considered.

All calculations are performed within the lowest LL with the help of Haldane's pseudopotentials,[36] V_{ij}^l, which determine the interaction between two electrons of types i and j with a relative angular momentum l. Halperin's wave functions (m_i, n_{ij}) represent the exact ground state for a model interaction, such that

$$V_{ii}^l = \begin{cases} 1 & \text{for } l < m_i \\ 0 & \text{for } l \geq m_i, \end{cases}$$

$$V_{ij}^l = \begin{cases} 1 & \text{for } l < n_{ij} \\ 0 & \text{for } l \geq n_{ij}. \end{cases} \tag{19}$$

One of the simplest nontrivial cases occurs when all intra-(inter-)component correlations are the same, i.e., $m_i = m$ and $n_{ij} = n$. These states are fully unpolarized and the corresponding filling factor and shift are

$$\nu_T^{\{m,n\}} = \frac{K}{m + (K-1)n} \quad \text{and} \quad \delta_{\{m,n\}} = m, \tag{20}$$

respectively. We have already shown in Secs. IV A and IV B that, within the plasma picture, $m > n$ yields a stable state and $m < n$ an unstable state, whereas $m = n$ represents a (stable) Laughlin state with SU(K) ferromagnetic order. This stability criterion may also be obtained directly from the interaction model corresponding to the $(m_i = m, n_{ij} = n)$-Halperin state: whenever this state is unstable, other zero-energy states with respect to the model interaction appear in the fully polarized sectors. These zero-energy states are quasihole excitations of the $(m_i = n_{ij} = m)$-Laughlin state, the model interaction of which matches that of the Halperin state in that sector. This is a direct consequence of $\nu_T^{\{m,n\}} < 1/m$ for unstable states, as may be seen from Eq. (20) with $n > m$. Thus, any Zeeman-type perturbation or any extra pseudopotentials beyond the model interaction dramatically change the polarization, as suggested by the plasma picture. For more generic Halperin wave functions, similar conclusions can be drawn when one of the m_i is lower than $1/\nu_T$.

When the polarization is regarded as fixed, the phase separation is clearly observed through the study of pair-correlation functions, which are discussed in Sec. V B. As mentioned before, unstable systems will exhibit instability through unphysical correlation functions.[37]

In addition to these general stability arguments, we investigate via exact diagonalization the $K = 2$, (331) and (113), wave functions studied by MacDonald *et al.*[30] The first state is realized for the particular model $V_{\uparrow\uparrow}^1 = V_{\downarrow\downarrow}^1 = V_{\uparrow\downarrow}^0 = 1$, all other potentials being zero, and $2S = 2N - 3$. The second one is related to $V_{\uparrow\downarrow}^0 = V_{\uparrow\downarrow}^1 = V_{\uparrow\downarrow}^2 = 1$ and $2S = 2N - 1$. For $N = 6$ electrons and 9 and 11 flux quanta, exact-diagonalization calculations yield an energy gap of 0.8 for (331) as compared to 0.01 for (113). Hence, the (113) state has an energy gap, almost 2 orders of magnitude smaller than the characteristic energy, which is set to 1. One may, therefore, expect that the (113) state is much less stable than the (331) state, as indicated by the plasma analogy and the above-mentioned argument. This is indeed the case as may be seen when other pseudopotentials are chosen as nonzero in a perturbative manner. We choose, for this investigation, to vary $V_A^3 \equiv V_{\uparrow\uparrow}^3 = V_{\downarrow\downarrow}^3$ and $V_E^1 \equiv V_{\uparrow\downarrow}^1$ continuously from zero to 1, for the (331) state, and $V_A^1 \equiv V_{\uparrow\uparrow}^1 = V_{\downarrow\downarrow}^1$ and $V_E^3 \equiv V_{\uparrow\downarrow}^3$ for the (113) state. Our exact-diagonalization results show that the unpolarized state described by the above-mentioned wave functions (there is an equal number of spin $\uparrow$ and $\downarrow$) is conserved only for the (331) case. Moreover, there is indeed an instability of the (113) state such that even a small perturbation in the pseudopotentials ($V_A^3 = 0.1$, for example) completely polarizes the state, which is in agreement with the general polarization argument given above.

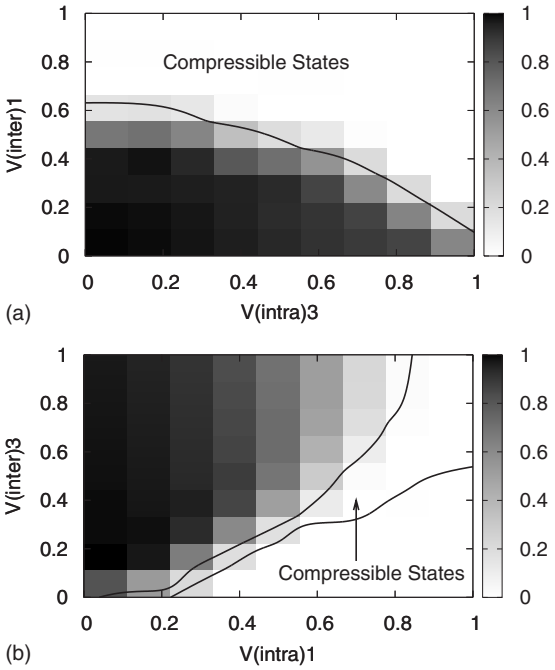

FIG. 5. Phase diagram for (a) 331 and (b) 113 ground states. Exact diagonalization is performed with $N = 6$ electrons and (a) $2S = 9$ (b) $2S = 11$ flux quanta. An unpolarized state is assumed. Pseudopotentials $V_{\uparrow\uparrow} = V_{\downarrow\downarrow}$ are denoted V(intra), whereas $V_{\uparrow\downarrow}$ is V(inter). The overlap between Halperin's wave functions (exact ground state of the unperturbed model) and the exact ground state is plotted in gray scale. We also indicate the separation between incompressible and compressible states (i.e., whether the ground state is in the $L = 0$ sector or not) by a black line.

We now focus on the unpolarized sector although the system is no longer in its ground state in the (113) case. The unpolarized sector may be physically relevant for a system with constrained polarizations, such as those for a bilayer configuration with equal densities in both layers. Figure 5 presents phase diagrams with compressible and incompressible states for (331) and (113) varying states. The overlap between the exact ground state and the Halperin wave functions is represented by gradual shading. For the (331) case, it can be observed that there is a finite energy gap around the values $V_A^3 = V_E^1 = 0$ of the exact model, and the overlap be-

tween the true ground state and the trial wave function remains large whenever the pseudopotentials V_A^3 and V_E^1 remain in this area. A further increase in the potentials leads to a gap collapse at relatively large values of the pseudopotentials. Hence, this state is stable even in the case of more realistic interactions beyond the model situation.

In the case of the (113) wave function, the gap collapses rapidly even at small values of $V_{\uparrow\downarrow}^3$ (≈ 0.1). A rather subtle perturbation would, therefore, completely change the system since a zero-energy gap is incompatible with any FQHE. Outside the funnel-shaped area of compressible states, cf. Fig. 5(b), the overlap can be either quite small (≈ 0, at larger values of V_A^1 and V_E^3) or quite large (≈ 1, in the vicinity of the model situation). This indicates that the incompressible states are described by states with a different symmetry. However, even though the system appears stable up to relatively high $V_{\uparrow\downarrow}^3$, the gap remains small (0.01–0.1). Moreover, we emphasize that, in the case of an unfixed polarization, the ground state is no longer in the unpolarized sector.

Note that the phase diagram in Fig. 5(b) is not generic for all unstable wave functions, but may be attributed to the pathologic model interaction of (113). General considerations on stability should only treat the polarization and pair-correlation function arguments.

As in the case of $K=2$, we compare the plasma picture with exact-diagonalization results for $K=4$. For the special subset of matrices previously discussed, it has been checked numerically that unstable states are related to the existence of partially polarized zero-energy states with a lower number of flux quanta. As in the case of $K=2$, those states will be favored when any Zeeman-type perturbation is introduced. The phase transition predicted within the plasma picture is recovered. We checked this criterion for several particular wave functions, such as (3333, 233) and (3333, 311), and it appears that wave functions with unstable corresponding plasma do polarize, partially or completely, which is in agreement with the classical stability discussed in Sec. IV B. For a given fixed polarization, phase separation is likely to be observed, as in the $K=2$ case.

V. GROUND-STATE PROPERTIES

Although the plasma picture is a powerful tool for the study of intrinsic properties of Laughlin and generalized Halperin wave functions, as shown in the previous section, it gives in itself no indication of the physical state chosen by the true interaction Hamiltonian. The Hamiltonian (6) is obtained from a formal mapping of the wave functions to a corresponding plasma model, but it is not related to the original Hamiltonian of interacting particles in the lowest or partially filled higher LL. Indeed, incompressible quantum liquids, which display the FQHE, are not found at all possible filling factors for which one may write down a trial wave function. For example, in the lowest LL, a Wigner crystal is energetically favorable at $\nu < 1/6.5$ (Ref. 38); in the first excited LL, a succession of FQHE states and Wigner and bubble crystals[39] gives rise to a re-entrant integral quantum Hall effect[40]; and in even higher LLs, stripe phases[41,42] yield a highly anisotropic longitudinal transport.[43] Which of these competing phases is indeed chosen depends on the precise form of the true interaction of electrons in a fixed partially filled LL.

It is, however, possible to express the energy of Laughlin's wave function in terms of the three-dimensional Coulomb interaction potential $\mathcal{V}(\mathbf{r})=e^2/\epsilon r$ and the pair-correlation function

$$g(r) \propto \int d^2z_3 \ldots . d^2z_N |\Psi_m(z_1=0, z_2=r; z_3, \ldots, z_N)|^2,$$

apart from a normalization constant

$$E = \int d^2r \mathcal{V}(\mathbf{r})[g(\mathbf{r}) - 1]. \tag{21}$$

The plasma picture may be of use here because the pair-correlation functions for electrons and for plasmatic particles are the same, as may be seen from Eq. (2). The pair-correlation function may be expanded as[28]

$$g(z) = 1 - e^{-|z|^2/2} + \sum_{n=1}^{\infty}{}' \frac{2}{n!}\left(\frac{|z|^2}{4}\right)^n c_n e^{-|z|^2/4}, \tag{22}$$

where the prime indicates a summation only over odd n (due to Fermi statistics) and the expansion parameters c_n vanish in the large-n limit. These expansion parameters are constrained in several respects. First, the short-range behavior $g \overset{|z|\to 0}{\to} |z|^{2m}$ implies that $c_n=-1$ for $n<m$. Second, particular properties of the logarithmic potential in the plasma picture can be used to derive sum rules, which act as further constraints.[24,44–46]

A. Sum rules for SU(K) pair-correlation functions

For wave functions with SU(K) symmetry, there are $K(K+1)/2$ pair-correlation functions if all densities are well defined, i.e., if M_K is invertible. The correlation function between type-(i) and type-(j) electrons is denoted by $g_{ij}(z)$, and one may generalize the expression (22) to the case of K-component wave functions,

$$g_{ij}(z) = 1 - e^{-|z|^2/2} + \sum_{n=1}^{\infty}{}' \frac{2}{n!}\left(\frac{|z|^2}{4}\right)^n c_n^{(ij)} e^{-|z|^2/4}, \tag{23}$$

where the expansion coefficients $c_n^{(ij)}$ vanish for large n. Here, the prime indicates a summation over odd n only for the intraspecies functions g_{ii}. Indeed, Fermi statistics is no more relevant when considering distinguishable electrons of types i and $j \neq i$. As for the Laughlin ($K=1$) case, the short-range behavior $g_{ij}(z) \overset{|z|\to 0}{\to} |z|^{2n_{ij}}$ implies that $c_n^{(ij)}=-1$ for $n < n_{ij}$.

Further sum rules may be derived within the picture of K correlated plasmas introduced in the previous section. In order to derive those in the simplest manner, we decouple the different plasmas with the help of an orthogonal transformation on the densities,

$$\begin{pmatrix} \rho_1'(\mathbf{r}) \\ \vdots \\ \rho_K'(\mathbf{r}) \end{pmatrix} = P \begin{pmatrix} \rho_1(\mathbf{r}) \\ \vdots \\ \rho_K(\mathbf{r}) \end{pmatrix}, \tag{24}$$

which diagonalizes the exponent matrix, $M_K = P^{\mathsf{T}} D P$, in terms of the orthogonal matrix P. The diagonalized Hamiltonian thus reads

$$\mathcal{H}[\{\rho_i'(\mathbf{r})\}] = \sum_{i=1}^{K} \mathcal{H}^{(i)}[\rho_i'(\mathbf{r})],$$

$$\mathcal{H}^{(i)}[\rho_i'(\mathbf{r})] = -\int\int_{\Omega} d^2r d^2r' \rho_i'(\mathbf{r}) \frac{\lambda_i}{2} \ln|\mathbf{r}-\mathbf{r}'| \rho_i'(\mathbf{r}') + \int_{\Omega} d^2r \rho_i'(\mathbf{r}) \frac{\alpha_i |r|^2}{4}, \tag{25}$$

where λ_i is the ith eigenvalue of M_K and $\alpha_i = \Sigma_j [P]_{ij}$. The Hamiltonian $\mathcal{H}[\{\rho_i'(\mathbf{r})\}]$ is a sum of K independent Hamiltonians $\mathcal{H}^{(i)}[\{\rho_i'(\mathbf{r})\}]$, each of which corresponds to a single 2DOCP. The correlation functions for these K plasmas must therefore obey the usual sum rules for 2DOCPs as follows:[24,44,46]

$$\mathcal{M}_0 = -1, \tag{26a}$$

$$\mathcal{M}_1 = -\frac{4}{2\pi\beta\lambda_i}, \tag{26b}$$

$$\mathcal{M}_2 = -\frac{64}{(2\pi\beta\lambda_i)^2}\left(1 - \frac{\beta\lambda_i}{4}\right), \tag{26c}$$

$$\mathcal{M}_3 = -6\frac{(\beta\lambda_i - 6)(8 - 3\beta\lambda_i)}{(\pi\beta\lambda_i)^3}, \tag{26d}$$

for the different moments

$$\mathcal{M}_m \equiv (\rho_i')^{m+1} \int d^2r r^{2m} [g_{ii}'(\mathbf{r}) - 1].$$

Here, primes indicate quantities in the diagonal basis. Equation (26a) is due to the charge neutrality of the system, Eq. (26b) reflects its perfect-screening property, and Eq. (26c) is a compressibility sum rule. The third moment [Eq. (26d)] has no apparent physical interpretation. Because the plasmas are decoupled in the diagonal basis, there are no correlations between different plasmas, i.e., $g_{ij}'(\mathbf{r}) = 1$ for $j \neq i$.

The pair-correlation functions $g_{ij}(\mathbf{r})$ in the original basis may be obtained from $g_{ij}'(\mathbf{r})$ with the help of the inverse orthogonal transformation. It is useful to start from the definition of the structure factor in reciprocal space, which is related to the pair-correlation function by Fourier transformation,

$$S(\mathbf{k}) - 1 = \rho \int d^2r e^{i\mathbf{k}\cdot\mathbf{r}} [g(\mathbf{r}) - 1]$$

for the simplest $K=1$ case. It may also be expressed in terms of density operators,

$$\rho\Omega S(\mathbf{k}) = \langle \rho(\mathbf{k})\rho(-\mathbf{k})\rangle - |\langle \rho(\mathbf{k})\rangle|^2, \tag{27}$$

where the quantities in brackets are averages with respect to the probability density function. In the case of $K \neq 1$, the structure factor has a matrix form,

$$\Omega\sqrt{\rho_i\rho_j} S_{ij}(\mathbf{k}) = \langle \rho_i(\mathbf{k})\rho_j(-\mathbf{k})\rangle - |\langle \rho_i(\mathbf{k})\rangle||\langle \rho_j(\mathbf{k})\rangle|,$$

and the associated pair-correlation functions $g_{ij}(\mathbf{r})$ may be obtained from those in the diagonal basis with the help of $S_{ij}(\mathbf{r}) = \delta(\mathbf{r})\delta_{ij} + (\rho_i\rho_j)^{1/2}[g_{ij}(\mathbf{r}) - 1]$,

$$\begin{pmatrix} \rho_1^2[g_{11}(\mathbf{r})-1] & \cdots & \rho_1\rho_K[g_{1K}(\mathbf{r})-1] \\ \vdots & \ddots & \vdots \\ \rho_K\rho_1[g_{K1}(\mathbf{r})-1] & \cdots & \rho_K^2[g_{KK}(\mathbf{r})-1] \end{pmatrix} = \delta(\mathbf{r}) \times \left[-\begin{pmatrix} \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_K \end{pmatrix} + P^{\mathsf{T}} \begin{pmatrix} \rho_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_K' \end{pmatrix} P \right] + P^{\mathsf{T}} \begin{pmatrix} \rho_1'^2[g_{11}'(\mathbf{r})-1] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho_K'^2[g_{KK}'(\mathbf{r})-1] \end{pmatrix} P. \tag{28}$$

Thus, the sum rules (26a)–(26d) for the diagonal basis immediately yield those for the K correlated 2DOCP due to Eq. (28). For instance, the zeroth- and first-moment rules are

$$\int d^2r \rho_i [g_{ij}(\mathbf{r}) - 1] = -\delta_{ij}, \tag{29a}$$

$$\int d^2r r^2 \rho_i\rho_j [g_{ij}(\mathbf{r}) - 1] = -\frac{4}{2\pi\beta}[M_K^{-1}]_{ij}, \tag{29b}$$

respectively. Equations (29a) and (29b) are generalizations of results which have been previously obtained for $K=2$.[28,37] The second- and third-moment rules may be derived in the same manner.

B. Exact-diagonalization results for pair-correlation functions

The pair-correlation functions of the ground state may also be obtained from exact-diagonalization results. We first focus on the SU(2) case by considering the same (331) and (113) wave functions discussed above and which were studied by MacDonald *et al.*[30] As mentioned in Sec. IV C, the first state is realized for the particular model $V_{\uparrow\uparrow}^1 = V_{\downarrow\downarrow}^1 = V_{\uparrow\downarrow}^0 = 1$, all other potentials being zero, and $2S = 2N - 3$. The second one is related to $V_{\uparrow\downarrow}^0 = V_{\uparrow\downarrow}^1 = V_{\uparrow\downarrow}^2 = 1$ and $2S = 2N - 1$. Figure 6(a) shows the ground-state pair-correlation functions for the (331) state with $N=10$ electrons and $2S=17$ flux quanta, and the results for the (113) case ($N=8$ and $2S=15$) are displayed in Fig. 6(b). The correlation function of a pair of electrons (i,j) is the relative density of type-j electrons when a type-i electron is fixed at the origin, $\rho_j(\mathbf{r}) = \rho_i g_{ij}(\mathbf{r})$. With this definition in mind, one can infer that the (331) state is rather well behaved. Correlation functions vanish near the origin as a result of repulsive interactions and Fermi statistics. The density peaks at finite distances indicate different layers of elec-

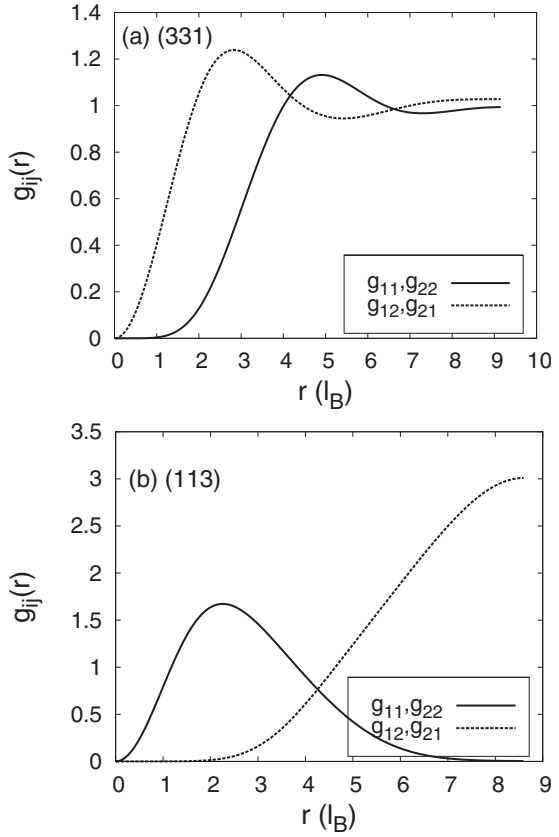

FIG. 6. Pair-correlation functions related to (a) (331) states for $N=10$ particles and (b) (113) states for $N=8$ particles, plotted as a function of distance (in units of magnetic length). Intracomponent pairs ($g_{11}=g_{22}$) are plotted as a solid line and intercomponent pairs ($g_{12}=g_{21}$) as dashed lines. In the (331) graph, all functions go to 1 at infinity, which is typical of uniform density. In contrast, the (113) intracomponent function vanishes at large distances, thus endowing a particle aggregate.

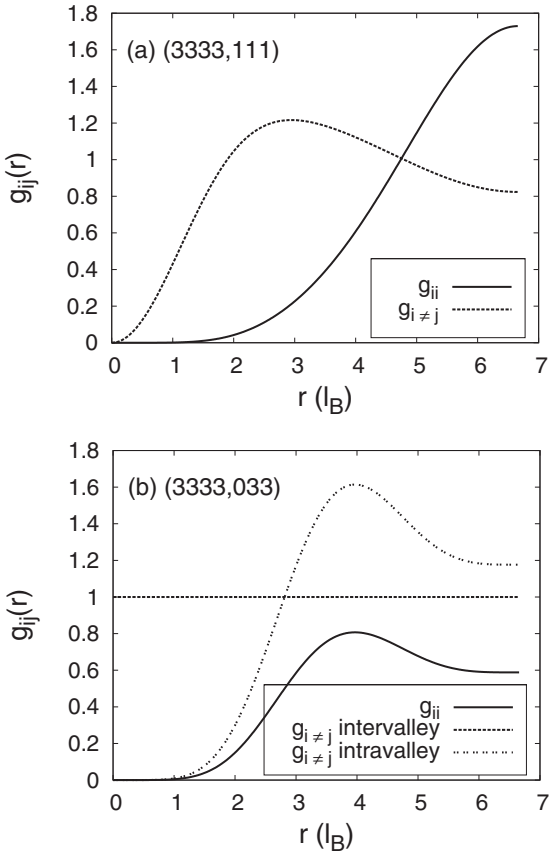

FIG. 7. Pair-correlation functions related to (a) (3333,111) and (b) (3333,033) states for $N=8$ particles, plotted as a function of distance (in units of magnetic length). Intracomponent pairs (g_{ii}) are plotted as a solid line. In the (3333,111) graph, the dashed line corresponds to the intercomponent pairs ($g_{i\neq j}$, all being equal). In the (3333,033) graph, the dotted line shows the correlation function for electrons with opposite spin within any of each layer or valley. The dashed line corresponds to intercomponent pairs between two different valleys or layers, which are trivially constant in this noncorrelated valley or layer example.

trons (on average) with a regular alternation of particles of types (1) and (2). The large-distance limit of type-(1) particles is related to a uniform density. Hence, the system discussed is composed of a uniform and homogeneous mixture of electrons of both types. Unlike this first case, the (113) state displays a vanishing intracomponent function ($g_{11}=g_{22}$) at large distances. Electrons of both types thus tend to aggregate on a finite-size location. Moreover, the intercomponent function ($g_{12}=g_{21}$) is maximum only at infinity, which implies that the different types of electrons tend to separate spatially. This corroborates the plasma picture of phase-separated particles.

We now turn to the SU(4) case by considering the two stable generalized Halperin wave functions studied in Ref. 23, namely, (3333,111) and (3333,033). The correlation functions are computed for $N=8$ electrons and $2S=9$ flux quanta (Fig. 7). While the partial filling factors of the (3333,111) state are fixed and all equal to 1/6, only the filling factor per layer is fixed for the (3333,033) ferromagnetic state. For a better comparison, we have set the partial filling factors to be the same as those of (3333,111). Thus, in both cases, there are only two electrons per species, leading to a prominent finite-size effect ($N=12$ is out of computational reach). The (3333,111) state is the straightforward SU(4) generalization of (331) and is therefore similar to its SU(2) counterpart (bear in mind the small system size we are considering). The (3333,033) state consists of two independent spin insensitive $\nu=1/3$-Laughlin states in each layer. Both intralayer and intracomponent correlation functions are therefore identical to their Laughlin counterpart (up to normalization factors). The interlayer pairs are trivially constant in this noncorrelated layer (3333,033) state.

Furthermore, we have confirmed numerically the validity of the sum rules (29a) and (29b) for the SU(2) and SU(4) states discussed above. Note that the (3333, 033) state may be described alternatively by a SU(2) wave function, where the two components correspond to the two layer indices, regardless of the spin orientation. The relevant sum rules are therefore those of the SU(2) case, in which the exponent matrix M_K in Eq. (29a) and (29b) is invertible.

VI. CHARGED EXCITATIONS

In this section, we present our study of the charged excitations of SU(K) Halperin wave functions with the help of

TABLE I. Charged excitations of Halperin's wave functions.

m_1	m_2	n	ν_T	$e_1^{*(1)}$	$e_2^{*(1)}$	$e_1^{*(2)}$	$e_2^{*(2)}$
3	3	0	2/3	1/3	0	0	1/3
3	3	1	1/2	3/8	−1/8	−1/8	3/8
1	1	3	1/2	−1/8	3/8	3/8	−1/8
3	3	2	2/5	3/5	−2/5	−2/5	3/5
2	2	3	2/5	−2/5	3/5	3/5	−2/5
3	3	3	1/3		1/3		

the plasma analogy. For $K=1$, quasihole excitations of the Laughlin wave function may be written as

$$\prod_{l=1}^{N}(z_l - z_0)\Psi_m(\{z_k\}). \tag{30}$$

They consist of adding a zero density, or a magnetic flux, in the electron liquid at the position z_0.

Within the plasma picture, the extra Jastrow term adds a new potential term to the Hamiltonian (3),

$$\mathcal{H}_N^* = \mathcal{H}_N - \sum_{l=1}^{N} \ln|z_l - z_0|, \tag{31}$$

which thus describes a 2DOCP with a fixed impurity at z_0 and charge $1/\sqrt{m}$. Because of the plasma's perfect-screening ability,[25] the particles are rearranged so that they screen the effect of the impurity in its vicinity. This requires $1/m$ particles in the plasma picture such that, in the true electron liquid, the real charge e^* of the excitation must be

$$e^* = \frac{1}{m} \tag{32}$$

in units of the electron charge. Hence, one electron can screen m excitations.

We now investigate SU(K) Halperin wave functions for which there exist different types of excitation. The quasihole wave function

$$\prod_{k_i=1}^{N_i}(z_{k_i}^{(i)} - z_0) \times \Psi_{m_1,\ldots,m_K;n_{ij}}^{\mathrm{SU}(K)} \tag{33}$$

creates here an excitation of (i)-type electrons at the position z_0, i.e., adds a magnetic flux in the (i)th component of the electron liquid. The modified Hamiltonian in the plasma picture contains a new potential of an impurity that only affects particles of type (i),

$$\mathcal{H}_N^{(i)*} = \mathcal{H}_N - \sum_{k_i=1}^{N_i} \ln|z_{k_i}^{(i)} - z_0|.$$

Each of the K correlated 2DOCP exhibits a perfect-screening ability, i.e., the plasma of type (i) screens the impurity totally,

$$m_i e_i^{*(i)} + \sum_{j\neq i} n_{ij} e_j^{*(i)} = 1, \tag{34}$$

whereas plasmas of type (j), with $j \neq i$, screen a zero impurity

$$m_j e_j^{*(i)} + \sum_{k} n_{jk} e_k^{*(i)} = 0. \tag{35}$$

Here, $e_j^{*(i)}$ is the quasiparticle charge, in units of the electron charge, carried by electrons of type (j) in the electron liquid for excitations in the (i) component. Equations (34) and (35) were previously derived for the $K=2$ case,[6] and one may write them in a concise matrix form as

$$\sum_{k} n_{jk} e_k^{*(i)} = \delta_{ij} \Leftrightarrow e_j^{*(i)} = (M_K^{-1})_{ji}. \tag{36}$$

The last equation is valid only if M_K is invertible. Indeed, if this is not the case, some component densities remain unfixed, as described in the previous section, and one could only consider the excitation of groups of particles with a definite density. For instance, the (mmm)-Halperin wave function has a noninvertible exponent matrix; the densities ρ_1 and ρ_2 may fluctuate although their sum remains fixed, $\nu_T = 1/m$. Physical excitation must therefore not distinguish between the two components.

For the $K=4$ case, the (3333,111) wave function[23] exhibits four excitation types, each of which carries a 1/6 charge, whereas for the (3333,033) wave function, only joined (1)–(3) and (2)–(4) excitations are considered, each carrying a 1/3 charge.

Table I shows examples of charged excitations for SU(2) wave functions. The first example describes two independent Laughlin states and it is consistent with the assumption that a type-(1) excitation should only affect type-(1) particles. The four next examples are related to the "(mmn) and (nnm)" problem. One should note that there are inconsistencies for (113) and (223) states. Indeed, when an extra flux quantum is added to the type-(1) component, the number of flux quanta increases by 1 and the electron density remains the same. Therefore, the ν_1 filling factor should decrease. However, the total charge is conserved, so the sign of the quasihole charge should be the same as that of the electron in order to compensate for this "electronic" lack. This is not the case for (113) and (223) states which must thus be considered as unphysical, in addition to the conclusions drawn in the previous sections. The last example in Table I is simply a

TABLE II. Charged excitations of SU(4) generalized Halperin wave functions.

$m_1m_2m_1m_2,n_en_+n_-$	ν_T	$e^{*(1)}$	$e^{*(2)}$	$e^{*(3)}$	$e^{*(4)}$
3333,111	2/3	1/6	1/6	1/6	1/6
3555,222	2/5	1/5	1/15	1/15	1/15
3535,222	8/19	3/19	1/19	3/19	1/19
5555,222	4/11	1/11	1/11	1/11	1/11

Laughlin wave function split into two arbitrary sets. There is only one common excitation as for the usual U(1) case.

Similarly, Table II shows some charged excited states for SU(4) wave functions. All examples are associated with invertible matrices M_K. No proof is given here, but we conjecture that unstable states yield some inconsistencies concerning the charged excitations as for the $K=2$ case.

VII. CONCLUSIONS

In conclusion, we have investigated the stability of Halperin wave functions for K-component quantum Hall systems, with a particular emphasis on the cases of $K=2$ and 4. The associated SU(2) and SU(4) internal symmetries happen to be the physically most relevant if one considers, e.g., bilayer quantum Hall systems and graphene in a strong magnetic field. The $K=4$ case occurs when the Zeeman effect is relatively small with respect to the leading interaction energy scales. In order to derive the stability criteria, we have generalized, in a systematic manner, Laughlin's plasma analogy to multicomponent systems. The validity of the criteria is corroborated with the help of exact-diagonalization studies.

As for the conventional one-component quantum Hall system, the quantum-classical analogy yields a compelling physical interpretation of the trial wave functions in terms of K correlated 2DOCP. Besides the stability of the trial wave functions, it allows one to understand relevant ground-state properties, such as the associated pair-correlation functions, and fractionally charged quasiparticle excitations.

Whether the discussed trial wave functions correctly describe the true ground state in physically relevant multicomponent systems, such as bilayer quantum Hall systems or graphene, depends on the precise form of the interaction potential. The plasma analogy with its rather artificial interaction may not give insight here, and variational or exact-diagonalization studies need to be performed to determine the correct ground state for a physical interaction potential. It has indeed been shown that in the case of Coulomb interaction, a possible FQHE state at $\nu=2/3$ is not described by a generalized SU(4) Halperin wave function.[22,23] More complicated trial wave functions, such as composite-fermion[22] or even more exotic states, may describe FQHE states at this and possibly other filling factors in a more appropriate manner. However, in the U(1) one-component quantum Hall system, the inevitable starting point in the understanding of the FQHE is Laughlin's wave function[2]; other wave functions may be viewed as sophisticated generalizations of it. In the same manner, the study of SU(K) Halperin wave functions and the plasma analogy yield important physical insight into multicomponent quantum Hall systems, and one may conjecture that they play a similar basic role for possible generalizations as Laughlin's in the U(1) case.

Furthermore, it has been shown in the SU(2) case that not all possible, though stable from our analysis, Halperin wave functions are valid candidates from a symmetry point of view. Indeed, most of the (m,m,n) wave functions are not eigenstates of the total spin operator, the Casimir operator of SU(2), as they should for spin-independent interaction Hamiltonians.[47] The (331) wave function discussed here is, for example, not an eigenstate of the total spin. However, this problem may be cured by attaching the permanent of the matrix $(z_i^{(1)}-z_j^{(2)})^{-1}$ to the (331) wave function.[47,48] The situation is more complicated in the SU(4) case, where there are more Casimir (spin-pseudospin) operators, and the wave functions should be eigenstates of these operators. More detailed theoretical investigations are required to settle the question of whether some SU(4) wave functions may be corrected in a similar manner.

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