Soliton spin excitations and their perturbation in a generalized inhomogeneous Heisenberg ferromagnet

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We investigate the nonlinear spin dynamics of a generalized inhomogeneous anisotropic Heisenberg ferromagnetic spin chain with bilinear and biquadratic exchange, octupole-dipole, and weak interactions in the semiclassical limit by using Glauber's coherent state method in combination with Holstein–Primakoff bosonic representation of spin operators. The dynamics is found to be governed by a generalized nonlinear Schrödinger equation in the continuum limit. We have identified several completely integrable spin models with soliton spin excitations for specific parametric choices. Finally, we carry out a multiple scale perturbation analysis to find the effect of discreteness and inhomogeneity on the soliton excitations in a more general case. The results show that the discreteness effect introduces symmetric fluctuations in the localized region of the soliton without altering its amplitude and velocity. On the other hand, the inhomogeneity introduces asymmetric fluctuations in the localized region with a decrease in the amplitude and velocity of the soliton with time. Further, the inhomogeneity reverses the velocity of the soliton after some time.

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I. INTRODUCTION

The study of nonlinear spin excitations in terms of solitary waves and solitons in the Heisenberg model of ferromagnets with different magnetic interactions in the classical and semiclassical limits under continuum approximation has attracted much interest in the past few years.[1–17] The bilinear nearest neighbor spin-spin interaction $(\mathbf{S}_i\cdot\mathbf{S}_{i+1})$ is fundamentally responsible for the parallel alignment of spins in ferromagnetic systems and its higher-order versions such as biquadratic exchange interaction[10] $(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2$ and octupole-dipole interaction,[16] which is a third-order interaction of the form $(\mathbf{S}_i\cdot\mathbf{S}_{i+1})(\mathbf{S}_{i+1}\cdot\mathbf{k})^2$, also play an important role. Even though classical spin systems admit an interesting class of nonlinear spin excitations such as solitons, for quantum spin systems, semiclassical treatment turns out to be a very suitable method for studying soliton excitations because of its consistency and validity.[16–21] Soliton spin excitations and solitary wave profiles have been identified in ferromagnets under semiclassical approximation in the long wavelength and low temperature limits, in which the spins are treated as bosonic operators under the Holstein–Primakoff (HP) approximation[22] in combination with Glauber's coherent state representation.[23] All of the recent semiclassical studies[16,17] in this direction pertain to spin systems with homogeneous spin-spin exchange interactions at different levels. The results of semiclassical as well as classical studies on homogeneous ferromagnets reveal that the dynamics is governed by the nonlinear Schrödinger (NLS) family of equations and, thus, the spin excitations are governed by solitons and perturbed solitons with small fluctuations.[15–17] However, in real systems, the bilinear exchange, biquadratic exchange, and octupole-dipole interactions may be site dependent and, hence, vary along the spin sites in the lattices. In other words, it is possible to have inhomogeneous magnetic systems in the continuum limit. The reasons for inhomogeneity or site dependence in exchange interactions in ordered magnetic systems may be one of the following: (i) the distance between neighboring atoms may vary along the magnetic lattice, (ii) the atomic wave function may vary from site to site, or (iii) there may be imperfections in the vicinity of a bond. Therefore, it is quite natural to understand the nonlinear spin dynamics of site-dependent or inhomogeneous Heisenberg ferromagnets. In this context, it should be mentioned that the dynamics of a Heisenberg spin chain with site-dependent bilinear and biquadratic exchange interactions in the classical continuum limit has been studied recently[15] and the study reveals that the dynamics is governed by an inhomogeneous generalized higher-order NLS equation. It is also shown that the above system exhibits an interesting phenomenon of soliton flipping, which leads to magnetization reversal in the ferromagnetic medium, due to inhomogeneity. Now, we consider a generalized anisotropic weak Heisenberg ferromagnetic spin system with varying bilinear, biquadratic, and octupole-dipole interactions and we study the nature of the nonlinear spin excitations and identify all possible integrable spin models so that the nonlinear spin excitations are governed by solitons. When anisotropy is present in the spin system, semiclassical treatment[16–21] makes the study of nonlinear dynamics easier. Hence, in the present paper, we study the nonlinear spin dynamics of the Heisenberg anisotropic weak ferromagnetic system with varying bilinear, biquadratic, and octupole-dipole interactions in the continuum limit under semiclassical approximation.

The paper is organized as follows. In Sec. II, we consider a Heisenberg model Hamiltonian for the generalized ferromagnet with different magnetic interactions and derive the dynamical equation in the continuum limit under semiclassical approximation. In Sec. III, the governing nonlinear dynamical equation in the continuum limit is then analyzed at different orders of the lattice parameter and integrable spin models have been identified. In Sec. IV, for the more general nonintegrable case, we carry out a multiple scale perturbation analysis to understand the effects of discreteness and inhomogeneity. The results are presented in Sec. V.

II. MODEL AND DYNAMICAL EQUATION

The Heisenberg spin Hamiltonian for an anisotropic weak ferromagnetic spin system with site-dependent bilinear, biquadratic, and octupole-dipole interactions (a generalized Heisenberg spin model) is written as

$$H' = -\sum_i \{J'_{1i}[b(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + cS_i^z S_{i+1}^z] + J'_{2i}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + J'_{3i}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{k})^2 + J'_4 \mathbf{D} \cdot (\mathbf{S}_i \times \mathbf{S}_{i+1}) - A'(S_i^z)^2 - A'_1(S_i^z)^4\}, \quad (1)$$

where J'_{1i}, J'_{2i}, and J'_{3i} represent site-dependent coefficients of bilinear, biquadratic, and octupole-dipole interactions, respectively, and the constants J'_4 and A' (and A'_1) correspond to weak and crystal field anisotropy interactions. $\mathbf{D}$ is the constant Dzyaloshinskii–Moriya (DM) vector, and the coefficients b and c introduce exchange anisotropy in the spin system. In Eq. (1), $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ represents the spin at the lattice site i and $\mathbf{S}_i \cdot \mathbf{S}_i = S(S+1)$. In order to express the spin Hamiltonian in dimensionless form, we write $\hat{\mathbf{S}}_i = \mathbf{S}_i/\hbar$ and introduce $\hat{S}_i^{\pm} = \hat{S}_i^x \pm i\hat{S}_i^y$. The $\mathbf{D}$ vectors are chosen to be $D^{\pm} = D^x \pm iD^y$ in analogy to the spin vectors. We also define $H = \frac{H'}{\hbar^2 S^2}$, $J_{1i} = J'_{1i}$, $J_{2i} = \hbar^2 S^2 J'_{2i}$, $J_{3i} = \hbar^2 S^2 J'_{3i}$, $J_{4i} = J'_{4i}$, $A = A'$, and $A_1 = \hbar^2 S^2 A'_1$. In view of this, the dimensionless form of Hamiltonian (1) is written as

$$H = -\frac{1}{2S^2}\sum_i \Bigg\{ (J_{10} + J_{11} f_i)[b(\hat{S}_i^+ \hat{S}_{i+1}^- + \hat{S}_i^- \hat{S}_{i+1}^+) + 2c\hat{S}_i^z \hat{S}_{i+1}^z] + \frac{1}{2S^2}(J_{20} + J_{21} g_i)[\hat{S}_i^+ \hat{S}_{i+1}^- \hat{S}_i^+ \hat{S}_{i+1}^- + \hat{S}_i^- \hat{S}_{i+1}^+ \hat{S}_i^- \hat{S}_{i+1}^+ + 2\hat{S}_i^- \hat{S}_{i+1}^+ \hat{S}_i^+ \hat{S}_{i+1}^- + 4\hat{S}_i^z \hat{S}_{i+1}^z \hat{S}_i^+ \hat{S}_{i+1}^- + 4\hat{S}_i^z \hat{S}_{i+1}^z \hat{S}_i^- \hat{S}_{i+1}^+ + 4\hat{S}_i^z \hat{S}_{i+1}^z \hat{S}_i^z \hat{S}_{i+1}^z] + \frac{1}{S^2}(J_{30} + J_{31} g_i)[\hat{S}_i^+ \hat{S}_{i+1}^- (\hat{S}_{i+1}^z)^2 + \hat{S}_i^- \hat{S}_{i+1}^+ (\hat{S}_{i+1}^z)^2 + 2\hat{S}_i^z (\hat{S}_{i+1}^z)^3] - iJ_4[D^-(\hat{S}_i^+ \hat{S}_{i+1}^z - \hat{S}_i^z \hat{S}_{i+1}^+) + D^+(\hat{S}_i^z \hat{S}_{i+1}^- - \hat{S}_i^- \hat{S}_{i+1}^z) + iD^z(\hat{S}_i^- \hat{S}_{i+1}^+ - \hat{S}_i^+ \hat{S}_{i+1}^-)] - 2A(\hat{S}_i^z)^2 - \frac{2A_1}{S^2}(\hat{S}_i^z)^4 \Bigg\}. \quad (2)$$

While writing Eq. (2), we have expressed the site-dependent coefficients J_{1i}, J_{2i}, and J_{3i} as $J_{1i} = J_{10} + J_{11} f_i$, $J_{2i} = J_{20} + J_{21} g_i$, and $J_{3i} = J_{30} + J_{31} h_i$, where f_i, g_i, and h_i are time-independent site-dependent functions related to the bilinear, biquadratic, and octupole-dipole interactions, respectively, and J_{10}, J_{11}, J_{20}, J_{21}, J_{30}, and J_{31} are constant coefficients. Different models of one-dimensional spin-1/2 ordered ferromagnetic chains with exchange interaction were proven to be exactly solvable with a complete description of their energy spectrum and eigenfunctions.[24,25] However, many real magnetic materials are characterized by higher values of spins, and solving them quantum mechanically is really a challenging task. An exact solution for spin chains with spin values greater than 1/2 does not exist. The excitation spectrum of spin systems with integer spins exhibit a gap.[26] On the other hand, advantageously, the higher values of spins reduce the quantum fluctuation and, hence, a semiclassical description of the models becomes meaningful in these cases. Since we have to bosonize the Hamiltonian in the semiclassical treatment, we need to express the Hamiltonian by using the Holstein–Primakoff representation[22] of spin operators, as follows:

$$\hat{S}_n^+ = (2S)^{1/2}\left[1 - \frac{a_n^\dagger a_n}{2S}\right]^{1/2} a_n, \quad (3a)$$

$$\hat{S}_n^- = (2S)^{1/2} a_n^\dagger \left[1 - \frac{a_n^\dagger a_n}{2S}\right]^{1/2}, \quad (3b)$$

$$\hat{S}_n^z = [S - a_n^\dagger a_n]. \quad (3c)$$

The bosonic operators a_n and $a_n^\dagger$ satisfy the following usual commutation relations:

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad (4a)$$

$$[a_m, a_n] = [a_m^\dagger, a_n^\dagger] = 0. \quad (4b)$$

In the low temperature limit, for large spins, the ground state expectation value of $a_n^\dagger a_n$ is small compared to $2S$ and, therefore, we use the semiclassical expansions for $\hat{S}_n^+$ and $\hat{S}_n^-$ in the following form:

$$\frac{\hat{S}_n^+}{S} = \sqrt{2}\left[1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - \frac{\epsilon^4}{32} a_n^\dagger a_n a_n^\dagger a_n - \frac{\epsilon^6}{128} a_n^\dagger a_n a_n^\dagger a_n a_n^\dagger a_n - O(\epsilon^8)\right]\epsilon a_n, \quad (5a)$$

$$\frac{\hat{S}_n^-}{S} = \sqrt{2}\epsilon a_n^\dagger \left[1 - \frac{\epsilon^2}{4} a_n^\dagger a_n - \frac{\epsilon^4}{32} a_n^\dagger a_n a_n^\dagger a_n - \frac{\epsilon^6}{128} a_n^\dagger a_n a_n^\dagger a_n a_n^\dagger a_n - O(\epsilon^8)\right], \quad (5b)$$

where $\epsilon = 1/\sqrt{S}$ is a small dimensionless parameter. By using Eqs. (5a) and (5b) in Eq. (2), we obtain a different Hamiltonian that is written as a power series in ϵ. As the Hamiltonian is very lengthy in form, in which it contains a large number of terms, it is separately presented in Appendix A [Eq. (A1)].

Now the dynamics of spins can be expressed in terms of the Heisenberg equation of motion for the boson operators by substituting the Hamiltonian, as given in Eq. (A1), in the following equation of motion:

$$i\hbar\frac{\partial a_n}{\partial t}=[a_n,H]=F(a_n^\dagger,a_n,a_{n+1}^\dagger,a_{n+1}). \tag{6}$$

We are concerned with extended nonlinear excitations of spins induced by nonlinearity in the magnon system, in which a cluster of spins may undergo a large excursion as compared to the rest of the spins. The quantum state of such large amplitude collective modes may be represented by coherent states. Hence, we introduce Glauber's coherent-state representation[23] for the bosonic operators, $a_n^\dagger|u\rangle=u_n^*|u\rangle$, $a_n|u\rangle=u_n|u\rangle$, and $|u\rangle=\Pi_n|u_n\rangle$ with $\langle u|u\rangle=1$, where u_n is the coherent amplitude of the operator a_n for the system in the state $|u\rangle$. Now, we write down the equation of motion by using Eq. (6) for the average $\langle u|a_j|u\rangle$, the explicit form of which is given in Appendix B [Eq. (B1)]. Equation (B1) represents the dynamics of spins in a discrete generalized weak, inhomogeneous, and anisotropic Heisenberg ferromagnetic spin system. It is very difficult to solve Eq. (B1) in its present form because of its high nonlinearity and discreteness. Hence, we make a continuum approximation, which is valid in the low temperature and long wavelength limit by assuming that the lattice constant is very small compared to the length of the lattice. For this, we assume that the spins vary slowly over the distance of the lattice parameter α (lattice constant) and the coefficients f_j, g_j, and h_j vary slowly comparatively over a different distance scale β. Thus, we introduce the following series expansions for $u_{j\pm1}$, f_{j-1}, g_{j-1}, and h_{j-1} by assuming $u_j(t)$, f_j, g_j, and h_j, respectively, as $u(x,t)$, $f(x)$, $g(x)$, and $h(x)$, where x is a continuous variable:

$$u_{j\pm1}(t)=u(x,t)\pm\alpha u_x+\frac{\alpha^2}{2!}u_{xx}\pm\frac{\alpha^3}{3!}u_{xxx}+\frac{\alpha^4}{4!}u_{xxxx}+\cdots, \tag{7a}$$

$$\begin{pmatrix}f_{j-1}\\ g_{j-1}\\ h_{j-1}\end{pmatrix}=\begin{pmatrix}f(x)\\ g(x)\\ h(x)\end{pmatrix}-\beta\begin{pmatrix}f_x\\ g_x\\ h_x\end{pmatrix}+\frac{\beta^2}{2!}\begin{pmatrix}f_{xx}\\ g_{xx}\\ h_{xx}\end{pmatrix}-\cdots. \tag{7b}$$

In Eqs. (7a) and (7b), the suffix x on the right hand side represents a partial derivative with respect to x. In normal cases, α and β are expected to coincide with each other. Nevertheless, there are situations where $\alpha\neq\beta$. Therefore, by expanding the spins up to $O(\alpha^4)$ and the interaction coefficients up to $O(\beta^2)$ in the discrete equation of motion [Eq. (B1)], we obtain the following continuous equation of motion written up to $O(\epsilon^m\alpha^n\beta^l)$, where $m+n+l=6$, by using Eqs. (7a) and (7b):

$$\begin{aligned}
-iu_t={}&\epsilon^2\Big\{[2(b-c)(J_{10}+J_{11}f)-2(J_{30}+J_{31}h)+2A+4A_1]u-\alpha(2iD^z)u_x-\beta[(b-c)J_{11}f_x-2J_{31}h_x]u\\
&+\frac{\eta^2}{2}[(b-c)J_{11}f_{xx}-2J_{31}h_{xx}]u+\alpha^2[b(J_{10}+J_{11}f)+2(J_{20}+J_{21}g)+(J_{30}+J_{31}h)]u_{xx}+\alpha\beta[bJ_{11}f_x+2J_{21}g_x+J_{31}h_x]u_x\\
&-\frac{\alpha^2\beta}{2}[bJ_{11}f_x+2J_{21}g_x+J_{31}h_x]u_{xx}-\frac{\alpha^2\beta}{2}[bJ_{11}f_{xx}+2J_{21}g_{xx}+J_{31}h_{xx}]u_x+\alpha^3\left[\frac{-iD^z}{3}\right]u_{xxx}+\alpha^4\left[\frac{1}{12}b(J_{10}+J_{11}f)\right.\\
&\left.+\frac{1}{6}(J_{20}+J_{21}g)+\frac{1}{12}(J_{30}+J_{31}h)\right]u_{xxxx}+\frac{\beta\alpha^3}{6}[bJ_{11}f_x+2J_{21}g_x+J_{31}h_x]u_{xxx}+\frac{\alpha^2\beta^2}{4}[bJ_{11}f_{xx}+2J_{21}g_{xx}+J_{31}h_{xx}]u_{xx}\Big\}\\
&-i\epsilon^3\left\{\alpha[(\sqrt{2}D^-)uu_x+(\sqrt{2}D^+)u^*u_x]+\alpha^3\left[\left(\frac{D^-}{3\sqrt{2}}\right)uu_{xxx}+\left(\frac{D^+}{\sqrt{2}}\right)\left(\frac{1}{3}u^*u_{xxx}+u_x^*u_{xx}+u_{xx}^*u_x\right)\right]\right\}\\
&-\epsilon^4\Big\{[2(b-c)(J_{10}+J_{11}f)+2(J_{30}+J_{31}h)+2A+12A_1]|u|^2u-\alpha(2iD^z)|u|^2u_x-\beta[(b-c)J_{11}f_x-2J_{31}h_x]|u|^2u\\
&+\frac{\beta^2}{2}[(b-c)J_{11}f_{xx}-2J_{31}h_{xx}]|u|^2u+\beta\alpha\left[((b-c)J_{11}f_x+2J_{31}h_x)|u|^2u_x+\frac{1}{2}((b-2c)J_{11}f_x-2J_{21}g_x-J_{31}h_x)u_x^*u^2\right]\\
&+\alpha^2\Big[((b-c)(J_{10}+J_{11}f)+2(J_{30}+J_{31}h))|u|^2u_{xx}-\frac{1}{2}((2c-b)(J_{10}+J_{11}f)+2(J_{20}+J_{21}g)+(J_{30}+J_{31}h))u^2u_{xx}^*\\
&+\frac{1}{2}(b(J_{10}+J_{11}f)+2(J_{20}+J_{21}g)+5(J_{30}+J_{31}h))u^*u_x^2-((2c-b)(J_{10}+J_{11}f)+2(J_{20}+J_{21}g)+(J_{30}+J_{31}h))|u_x|^2u\Big]\Big\}\\
&+i\epsilon^5\alpha\left\{\left(\frac{D^-}{2\sqrt{2}}\right)|u|^2uu_x-\frac{D^+}{\sqrt{2}}\left(|u|^2uu_x+\frac{1}{2}u^3u_x^*-|u|^2uu_x^*-|u|^2u^*u_x\right)\right\}+\epsilon^6\{(12A_1)|u|^4u\}.
\end{aligned} \tag{8}$$

Equation (8) describes the spin dynamics of the generalized weak inhomogeneous anisotropic ferromagnetic systems in the continuum limit. As the various spin-spin interactions are chosen to be site dependent here, the coefficients of various terms in Eq. (8) are still functions of x (spatial dependent) and, hence, it is difficult to solve Eq. (8) in its present form. Therefore, before making any attempts to solve Eq. (8), we make the following assumptions, transformations, and rescalings to arrive at an equation that is closely related to integrable nonlinear evolution equations. We assume that the DM interaction takes place along the z direction so that $D^{\pm}=0$. As the three inhomogeneous functions f, g, and h and their derivatives appear in the same order in many of the coefficients in Eq. (8), we consider them to be linearly proportional to each other, so that $g=\frac{J_{11}}{J_{21}}(\frac{c}{2}-b)f$ and $h=\frac{J_{11}}{J_{31}}(b-c)f$. Further, we transform the terms proportional to u and u_x in Eq. (8) by making the dependent variable transformation $u\to ue^{i[2(b-c)J_{10}-2J_{30}+2A+4A_1]t}$ and the Galilean transformation $x\to x+2\alpha D^z t$, respectively. After a suitable rescaling and redefinition of parameters, Eq. (8) can be written as

$$iu_t+\left\{\beta(b-c)J_{11}\left(f_x-\frac{\beta}{2}f_{xx}\right)u+\alpha^2(bJ_{10}+2J_{20}+J_{30})u_{xx}-i\frac{\alpha^3D^z}{3}u_{xxx}+\frac{\alpha^4}{12}(bJ_{10}+2J_{20}+J_{30})u_{xxxx}\right\}$$
$$+\epsilon^2\Bigg\{2\left[J_{30}-(b-c)J_{10}-A-6A_1+\beta(b-c)J_{11}\left(f_x-\frac{\beta}{2}f_{xx}\right)\right]|u|^2u+\alpha(2iD^z)|u|^2u_x-\alpha\beta[3(b-c)J_{11}f_x|u|^2u_x$$
$$+(b-c)J_{11}f_xu_x^*u^2]-\alpha^2\Bigg[[(b-c)J_{10}+2J_{30}+3(b-c)J_{11}f]|u|^2u_{xx}-\frac{1}{2}[(2c-b)J_{10}+2J_{20}+J_{30}+2(b-c)J_{11}f]u^2u_{xx}$$
$$+\frac{1}{2}[bJ_{10}+2J_{20}+5J_{30}+4(b-c)J_{11}f]u^*u_x^2+[(2c-b)J_{10}+2J_{20}+J_{30}+2(b-c)J_{11}f]|u_x|^2u\Bigg]\Bigg\}+\epsilon^4\{12A_1|u|^4u\}=0. \quad (9)$$

The second term in Eq. (9), which is proportional to u, can be transformed when its coefficient is a constant. Hence, we assume $\frac{\beta^2}{2}(c-b)J_{11}c_0=c_1$, where

$$f_{xx}-\frac{2}{\beta}f_x=c_0. \quad (10)$$

Here, c_0 and c_1 are constants. The solution to Eq. (10) is given by

$$f=a_1x+a_0, \quad (11)$$

where $a_1=-\frac{\beta}{2}c_0$ and a_0 is an arbitrary constant. The above form of the linear function f assumes importance, because in the case of an inhomogeneous isotropic Heisenberg ferromagnetic spin chain, the system proved to be completely integrable when the inhomogeneity assumes the form of a linear function[12,13] in x. Also, in a different context, the generalized nonlinear Schrödinger equation with linear inhomogeneity proved to be integrable.[7] Now, Eq. (9) reduces to the following equation after making the transformation $u\to e^{ic_1t}u$:

$$iu_t+\alpha^2k_1u_{xx}-i\frac{\alpha^3}{3}D^zu_{xxx}+\frac{\alpha^4}{12}k_1u_{xxxx}$$
$$+\epsilon^2\Bigg\{2k_2|u|^2u+\alpha[2iD^z]|u|^2u_x-\alpha\beta\zeta_x[3|u|^2u_x+u_x^*u^2]$$
$$-\alpha^2\Bigg[p_1(x)|u|^2u_{xx}-p_2(x)\left(\frac{1}{2}u^2u_{xx}^*+|u_x|^2u\right)$$
$$+\frac{p_3(x)}{2}u^*u_x^2\Bigg]\Bigg\}+12\epsilon^4A_1|u|^4u=0, \quad (12a)$$

where

$$k_1=bJ_{10}+2J_{20}+J_{30}, \quad (12b)$$

$$k_2=J_{30}-(b-c)J_{10}-A-6A_1+c_1, \quad (12c)$$

$$\zeta(x)=(b-c)J_{11}f(x), \quad (12d)$$

$$p_1(x)=(b-c)J_{10}+2J_{30}+3\zeta(x), \quad (12e)$$

$$p_2(x)=(2c-b)J_{10}+2J_{20}+J_{30}+2\zeta(x), \quad (12f)$$

$$p_3(x)=bJ_{10}+2J_{20}+5J_{30}+4\zeta(x). \quad (12g)$$

Equation (12a) represents the spin dynamics of a weak inhomogeneous anisotropic Heisenberg ferromagnetic spin chain with biquadratic and octupole-dipole interactions under semiclassical approximation. It is interesting to note that Eq. (12a) contains several integrable spin models, which will be discussed in detail in the next section.

III. INTEGRABLE SPIN MODELS

Equations (12a)–(12g) involve at least three small parameters, namely, ϵ, α, and β, which at different orders of $(\epsilon^k\alpha^l\beta^m)$, $k,l,m=0,1,2,\ldots$, lead to different completely integrable soliton equations. In the following, we present the details of these integrable spin models as obtained from Eq. (12a).

A. $O(\epsilon^k\alpha^l\beta^m)$ with $k+l+m=2$

At this order, when $\alpha=\epsilon$, Eq. (12a) reduces to

$$iu_t+u_{xx}+2|u|^2u=0. \quad (13)$$

While writing Eq. (13), we have rescaled the variables u and t as $u\to(\frac{k_1}{k_2})^{1/2}u$ and $t\to\frac{1}{\epsilon^2k_1}t$. Equation (13) is the well known completely integrable cubic NLS equation, which admits N-soliton solutions. The N-soliton solutions of Eq. (13)

were found by using the inverse scattering transform (IST) method,[27] by writing the equation in a bilinear form,[28] or through Bäcklund transformation[29] or other methods. For instance, the one-soliton solution of Eq. (13) that is obtained by using the IST method is written as

$$u = \eta \operatorname{sech}\frac{\xi}{\eta}\exp[i\xi(x-2\xi t-\theta_0)+i(\eta^2+\xi^2)t], \quad (14)$$

where ξ and η are related to the real and imaginary parts of the spectral parameter and θ_0 is the phase constant.

B. $O(\epsilon^k\alpha^l\beta^m)$ with $k+l+m=3$

In this order, Eq. (12a)–(12g) becomes

$$iu_t+\alpha^2 k_1 u_{xx}-i\frac{\alpha^3}{3}D^z u_{xxx}+\epsilon^2[2k_2|u|^2u+\alpha(2iD^z)|u|^2u_x]=0. \quad (15)$$

Equation (15) contains at least three integrable models, two of which correspond to the case when $\alpha=\epsilon$ and the other one is obtained when $\alpha \ll \epsilon$.

1. Case (i)

For instance, when $\alpha=\epsilon$, after making the same rescalings for u and t as before and choosing the constant c_0 as $c_0 = \frac{2[cJ_{10}+2(J_{20}+J_{30})-(A+6A')]}{\beta^2(c-b)J_{11}}$, we get the following completely integrable Hirota equation, which admits N-soliton solutions:[30]

$$iu_t+u_{xx}+2|u|^2u-i\gamma_1(u_{xxx}+6|u|^2u_x)=0, \quad (16)$$

where $\gamma_1=\frac{\epsilon D^z}{3(bJ_{10}+2J_{20}+J_{30})}$. The one-soliton solution of Eq. (16) that is obtained through Hirota's bilinear method is written as

$$u=\frac{1}{2}\operatorname{sech}\left[\frac{1}{2}(\eta_1+\eta_1^*+\eta_1^{(0)})\right]\exp\left[\frac{1}{2}(\eta_1-\eta_1^*+\eta_1^{(0)})\right], \quad (17)$$

where $\eta_1=\kappa_1 x+i\kappa_1^2 t+\eta_1^{(0)}$ and κ_1 and $\eta_1^{(0)}$ are complex constants.

2. Case (ii)

In Eq. (15), by choosing the exchange anisotropy constant b as $b=-\frac{J_{20}+J_{30}}{J_{10}}$ and the constant c_0 as $c_0 = \frac{2[(c-b)J_{10}+J_{30}+(A+6A')]}{\beta^2(c-b)J_{11}}$ and by rescaling t as $t\to(-\frac{3}{\alpha^3 D^z})t$, we obtain

$$u_t+u_{xxx}-6u^2u_x=0, \quad (18)$$

where u is real. Equation (18) is the modified Korteweg–de Vries (MKdV) equation, which admits N-soliton solutions.[31] The one-soliton solution of the MKdV equation is written as[32]

$$u=2a\operatorname{sech} a\Omega, \quad (19)$$

where $\Omega=x-\lambda_1 t$ and a and λ_1 are real constants.

3. Case (iii)

When $\alpha \ll \epsilon$, the term proportional to α^3 is negligibly small compared to the other terms and, hence, Eq. (15) becomes

$$iu_t+\alpha^2 k_1 u_{xx}+2\epsilon^2 k_2|u|^2u+2i\epsilon^2\alpha(D^z)|u|^2u_x=0. \quad (20)$$

Now, by choosing the constant c_0 as in the previous case and by using the same rescaling for t as given after Eq. (13), Eq. (20) reduces to

$$iu_t+u_{xx}+i\gamma_2|u|^2u_x=0, \quad (21)$$

where $\gamma_2=\frac{2\epsilon^2 D^z}{\alpha(bJ_{10}+2J_{20}+J_{30})}$. Equation (21) is one form of the derivative NLS equation and is known as the Chen–Lee–Liu (CLL) equation,[33] which is completely integrable and admits N-soliton solutions. The one-soliton solution of Eq. (21) that is obtained by using Hirota's bilinear method is written as[34]

$$u=\frac{1}{2}\operatorname{sech}\left[\frac{1}{2}(\eta_2+\eta_2^*+P)\right]\exp\left[\frac{1}{2}(\eta_2-\eta_2^*-P)\right], \quad (22)$$

where $\exp P=1+\frac{i\gamma_2\kappa_2}{2(\kappa_2+\kappa_2^*)^2}$, $\eta_2=\kappa_2 x+i\kappa_2^2 t+P$, and κ_2 is a complex constant.

C. $O(\epsilon^k\alpha^l\beta^m)$ with $k+l+m=4$

At this order, Eq. (12a) contains five integrable models, four of which correspond to the cases $\beta \gg \epsilon$ and the other one is obtained when $\epsilon=\alpha=\beta$.

1. Case (i)

For instance, when $\beta \gg \epsilon$, if we put $\alpha=\epsilon$ and consider terms up to $O(\epsilon^3\beta)$, Eq. (12a) reduces to

$$iu_t+u_{xx}+2|u|^2u+i\gamma_3[u_{xxx}+6|u|^2u_x+3(|u|^2)_x u]=0, \quad (23)$$

where $\gamma_3=\frac{\epsilon D'}{3(bJ_{10}+2J_{20}+J_{30})}$ and $D'=iD^z$. While writing Eq. (23), we have transformed x, u, and t to $x\to -ix$, $u\to(\frac{-k_1}{k_2})^{1/2}u$, and $t\to -\frac{1}{\epsilon^2 k_1}t$, respectively, and have chosen the parameter D' and the constant c_0 to be $D'=3\beta(b-c)J_{11}a_1$ and $c_0 = \frac{2}{3}[\frac{2J_{30}-(4b-3c)J_{10}-2J_{20}-3(A+6A')}{\beta^2(c-b)J_{11}}]$, respectively. Equation (23) is the completely integrable Sasa–Satsuma equation, which is a higher-order nonlinear Schrödinger equation[35,36] that admits N-soliton solutions, and its one-soliton solution that is obtained through the IST method is given by[37]

$$u(x,t)=\frac{\eta_3 e^{iB}[2\cosh A+(\kappa_3-1)e^{-A}]}{\cosh(2A-\log|\kappa_3|)+|\kappa_3|}, \quad (24)$$

where $A=\eta_3\{x-[\xi_3-\gamma_3(\eta_3^2-3\xi_3^2)]t-x^{(0)}\}$, $B=\xi_3\{x+[\frac{\eta_3^2-\xi_3^2}{2\xi_3}+\gamma_3(\xi_3^2-3\eta_3^2)]t-x^{(1)}\}$, and $\kappa_3=1-\frac{i\eta_3}{[\xi_3-\frac{1}{(6\gamma_3)}]}$. The quantities ξ_3 and η_3 are real parameters, whereas $x^{(0)}$ and $x^{(1)}$ are real phase constants.

2. Case (ii)

In Eq. (12a), if we choose the values for the exchange anisotropy constant b and the constant c_0 to be equal to the value given before Eq. (18) and by rescaling x and choosing the parameter D' as in the previous case [case (i)], we obtain the complex modified KdV (CMKdV) equation given below,

$$u_t + \gamma_4[u_{xxx} + 6|u|^2 u_x + 3(|u|^2)_x u] = 0, \qquad (25)$$

where $\gamma_4 = \frac{\epsilon^3 D'}{3}$. Equation (25) is completely integrable and admits N-soliton solutions.[38,39] The one-soliton solution of the CMKdV equation obtained through Hirota's bilinear method is written as[38]

$$u(x,t) = \delta_0 \operatorname{sech}\left[\eta_4 + \frac{p_0}{2}\right], \qquad (26)$$

where $\eta_4 = \kappa_4 x - \gamma_4 \kappa_4^3 t + \eta_4^{(0)}$, $\delta_0 = \frac{1}{2\sqrt{2}\kappa_4}$, and $\exp(p_0) = \frac{1}{2\kappa_4^2}$. The quantities κ_4 and $\eta_4^{(0)}$ are complex constants.

3. Case (iii)

When $\beta \gg \epsilon$ and $\alpha = \epsilon^2$, by ignoring the effect of higher-order anisotropy (i.e., $A' = 0$), Eq. (12a) reduces to

$$iu_t + \epsilon^2(2k_2)|u|^2 u + \epsilon^4[k_1 u_{xx} + 2D'|u|^2 u_x - \beta\zeta_x(3|u|^2 u_x + u_x^* u^2)] = 0. \qquad (27)$$

By transforming x, u, and t as given after Eq. (23) and by choosing the parameter D' as $D' = \frac{5}{2}\beta(b-c)J_{11}a_1$, Eq. (27) reduces to

$$iu_t + u_{xx} + 2|u|^2 u + i\gamma_5(|u|^2 u)_x = 0, \qquad (28)$$

where $\gamma_5 = \frac{\epsilon^2[2D' - 3\beta(b-c)J_{11}a_1]}{2[J_{30} - (b-c)J_{10} - A + c_1]}$. Equation (28) is the completely integrable mixed derivative NLS (MDNLS) equation that admits N-soliton solutions. For instance, the one-soliton solution of Eq. (28) that is obtained using Hirota's bilinear method is given by[38]

$$u = \frac{1}{2}\cosh\left[\frac{1}{2}(\eta_5 + \eta_5^* + \alpha_1)\right]\operatorname{sech}^2\left[\frac{1}{2}(\eta_5 + \eta_5^* + \alpha_2)\right] \times \exp\left[\frac{1}{2}(\eta_5 + \alpha_1 - \alpha_2)\right], \qquad (29)$$

where $\exp(\alpha_1) = \frac{\delta' - 3i\gamma_5\omega_1^*}{(\omega_1 + \omega_1^*)^2}$, $\exp(\alpha_2) = \frac{\delta' + 3i\gamma_5\omega_1}{(\omega_1 + \omega_1^*)^2}$, $\delta' = 1$, $\eta_5 = \kappa_5 x + \omega_1 t + \eta_5^{(0)}$, $\omega_1^2 = -2i\kappa_5$, and κ_5 and $\eta_5^{(0)}$ are complex constants.

4. Case (iv)

However, if we choose $c_0 = \frac{2[(c-b)J_{10} + J_{30} + (A + 6A')]}{\beta^2(c-b)J_{11}}$ and $k_2 = 0$, the second term in Eq. (27) drops out. By carrying out the same rescaling as in the previous case, Eq. (27) reduces to

$$iu_t + u_{xx} + i\gamma_6(|u|^2 u)_x = 0, \qquad (30)$$

where $\gamma_6 = \frac{3\beta(b-c)J_1 a_1 - 2D'}{2(bJ_{10} + 2J_{20} + J_{30})}$. Equation (30) is known as the Kaup–Newell (KN) equation,[33,34,40] which is a completely integrable derivative NLS equation that admits N-soliton solutions. The one-soliton solution of Eq. (30) is given in the same form as Eq. (29) with $\delta' = 0$ instead of 1.

5. Case (v)

Finally, when $\alpha = \beta = \epsilon$, in the homogeneous limit (i.e., when f = constant), Eq. (12a) reduces to

$$iu_t + u_{xx} + 2|u|^2 u + \frac{\epsilon^2}{12}[u_{xxxx} + 8|u|^2 u_{xx} + 2u^2 u_{xx}^* + 6u^* u_x^2 + 4|u_x|^2 u + 6|u|^4 u] = 0. \qquad (31)$$

While writing Eq. (31), we have completely ignored the DM interaction and rescaled u and t as given after Eq. (13). Further, we have chosen c_0 and the parameters J_{30}, A_1, and A, respectively, to be $c_0 = \frac{(3b-7c)J_{10} - 2(4J_{20} + A + 6A')}{\beta^2(c-b)J_{11}}$, $J_{30} = \frac{1}{4}[(b-3c)J_{10} - 4J_{20}]$, $A_1 = \frac{3[(3b-5c)J_{10} - 4J_{20}]^2}{32[(5b-3c)J_{10} + 4J_{20}]}$, and $A = \frac{1}{2}[(3b-7c)J_{10} - 8J_{20}] - 6A_1 + c_1$. Equation (31) is the completely integrable fourth-order NLS equation that admits N-soliton solutions. For instance, the one-soliton solution[38] that is obtained through Hirota's bilinear method is written as

$$u = \frac{1}{2}\operatorname{sech}\left[\frac{1}{2}(\eta_6 + \eta_6^* + \mu_6)\right]\exp\left[\frac{1}{2}(\eta_6 - \eta_6^* - \mu_6)\right], \qquad (32)$$

where $\eta_6 = \kappa_6 x + i\epsilon\kappa_6^4 t + \eta_6^{(0)}$, $\exp(\mu_6) = -\frac{(\kappa_6 - \kappa_6^*)^2}{(\kappa_6 + \kappa_6^*)^4}$, and κ_6 and $\eta_6^{(0)}$ are complex constants. One can verify that all of the above integrable models can also be obtained by carrying out the Painlevé singularity structure analysis[41] on Eq. (12a).

In Table I, we present a class of Heisenberg spin models with different magnetic interactions, which are governed by different completely integrable soliton equations that are obtained in the semiclassical limit under a continuum approximation for different parameter values of magnetic energies at different orders. It is interesting to note that the nonlinear spin dynamics of a given spin model is governed by different completely integrable soliton equations for different parametric values at different orders. This implies that soliton spin excitations can be generated in a particular spin system in the continuum limit under a semiclassical approximation by suitably fixing the magnetic interaction parameters, which generates solitons of different forms through various competing magnetic interactions. Further, it may be noted that the soliton of the completely integrable cubic NLS equation represents the basis of spin excitations in the semiclassical continuum ferromagnetic spin chain.

The spin model that we have considered in the present paper includes all of the well known magnetic interactions of a ferromagnet and is governed by different known completely integrable soliton equations; thus, the model considered here acts as a generalized model. In the next section, we study the effect of discreteness and inhomogeneity in a more general case (other than the integrable cases) by carrying out a multiple scale perturbation analysis. While doing this, we perturb the cubic NLS soliton since the cubic NLS equation forms the basis of all the models.

IV. PERTURBATION OF SOLITON DUE TO DISCRETENESS AND INHOMOGENEITY

Having identified a class of integrable spin models exhibiting soliton spin excitations for a generalized Heisenberg spin system, we now investigate the nature of nonlinear spin excitations in a more general case (other than the integrable

TABLE I. A class of integrable spin models with the associated magnetic energy and the corresponding completely integrable soliton equations in the semiclassical limit.

Integrable spin models	Magnetic energy	Completely integrable soliton equations
	Homogeneous case	
$H=-\Sigma_i\{J_1(\mathbf{S}_i\cdot\mathbf{S}_{i+1})-A(S_i^z)^2\}$	Bilinear exchange +crystal field anisotropy	Cubic NLS
$H=-\Sigma_i\{J_1(\mathbf{S}_i\cdot\mathbf{S}_{i+1})-A(S_i^z)^2+J_4\mathbf{D}\cdot(\mathbf{S}_i\times\mathbf{S}_{i+1})\}$	Bilinear exchange +crystal field anisotropy +DM interaction	Cubic NLS, Hirota, CLL(DNLS-I), MKdV
$H=-\Sigma_i\{J_1[b(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+c(S_i^zS_{i+1}^z)]$ $-A(S_i^z)^2-A_1(S_i^z)^4\}$	Bilinear exchange (nonuniform) +crystal field anisotropies	Cubic NLS, Fourth-order NLS
$H=-\Sigma_i\{J_1(\mathbf{S}_i\cdot\mathbf{S}_{i+1})-A(S_i^z)^2-A_1(S_i^z)^4$ $+J_3(\mathbf{S}_i\times\mathbf{S}_{i+1})(\hat{k}\cdot\mathbf{S}_{i+1})^2\}$	Bilinear exchange +crystal field anisotropies +octupole-dipole interaction	Cubic NLS, Fourth-order NLS
$H=-\Sigma_i\{J_1[b(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+c(S_i^xS_{i+1}^x)]$ $-A(S_i^z)^2-A_1\ (S_i^z)^4)+J_2(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2\}$	Bilinear exchange (nonuniform) +crystal field anisotropies +biquadratic exchange	Cubic NLS, Fourth-order NLS
	Inhomogeneous case	
$H=-\Sigma_i\{J_1f_i(\mathbf{S}_i\cdot\mathbf{S}_{i+1})-A(S_i^z)^2$ $+J_4\mathbf{D}\cdot(\mathbf{S}_i\times\mathbf{S}_{i+1})\}$	Inhomogeneous bilinear exchange +crystal field anisotropies +DM interaction	Cubic NLS, MDNLS, KN(DNLS-II), Sasa–Satsuma, Complex MKdV

cases) by carrying out a multiple scale perturbation analysis[42] on Eq. (12a). The perturbed higher-order nonlinear Schrödinger equation [Eq. (12a)] [after setting $\epsilon=\alpha$ and choosing $D^z=0$, and rescaling u and t as given after Eq. (13)] is written as

$$iu_t+u_{xx}+2|u|^2u+\chi a_1(3|u|^2u_x+u_x^*u^2)+\lambda\Bigg[u_{xxxx}+P_1|u|^2u_{xx}+P_2\left(\frac{1}{2}u^2u_{xx}^*+|u_x|^2u\right)+P_3(u^*u_x^2+P_4|u|^4u)\Bigg]=0, \quad (33)$$

where $\chi=\frac{\alpha\beta(c-b)J_{11}}{k_2}$, $\lambda=\frac{\alpha^2}{12}$, $P_1=\frac{-12p_1}{k_2}$, $P_2=\frac{12p_2}{k_2}$, $P_3=\frac{-6p_3}{k_2}$, and $P_4=\frac{144A_1k_1}{k_2^2}$. Further, the coefficients χ and λ are assumed to be very small, i.e., χ, $\lambda\ll 1$. In Eq. (33), while the terms proportional to χ originate from the inhomogeneity of the spin system, the terms proportional to λ get contribution from the higher-order terms in the continuum approximation and also from the effect of discreteness of the spin system. It may be noted that the cubic NLS equation forms the basis of Eq. (33). Therefore, we study the effect of inhomogeneity and the effect of discreteness on the soliton spin excitations of the NLS model separately through a perturbation analysis. As the nonlinearity in the second case is more complicated, we first investigate the effect of discreteness in a homogeneous (f=constant) spin system.

A. Effect of discreteness

When $\chi=0$, Eq. (33) becomes

$$iu_t+u_{xx}+2|u|^2u+\lambda\Bigg[u_{xxxx}+w_1|u|^2u_{xx}+w_2\left(\frac{1}{2}u^2u_{xx}^*+|u_x|^2u\right)+w_3u^*u_x^2+w_4|u|^4u\Bigg]=0. \quad (34)$$

Here, w_1, w_2, w_3, and w_4 are constants that are obtained by treating f as a constant in P_1, P_2, P_3, and P_4, respectively. In Eq. (34), the terms proportional to λ are responsible for the discreteness effect and, hence, we treat these terms as a weak perturbation by taking λ as the perturbation parameter. When $\lambda=0$, Eq. (34) reduces to the completely integrable cubic NLS equation, for which the one-soliton solution given in Eq. (14) can be written as

$$u=\eta\ \mathrm{sech}\ \eta(\theta-\theta_0)e^{i\xi(\theta-\theta_0)+i(\sigma-\sigma_0)}, \quad (35)$$

where $\frac{\partial\theta}{\partial t}=-2\xi$, $\frac{\partial\theta}{\partial x}=1$, $\frac{\partial\sigma}{\partial t}=\eta^2+\xi^2$, $\frac{\partial\sigma}{\partial x}=0$, and η and ξ are related to the scattering parameter of the IST. We now carry out a multiple scale perturbation analysis on Eq. (34) following the perturbation method of Kodama and Ablowitz.[42] For this, first, we introduce a slow time variable $T=\lambda t$ and write the quantities η, ξ, θ_0, and σ_0 as functions of this time scale so that the envelope soliton solution of Eq. (34) is written as

$$u=\hat{u}(\theta,T;\lambda)\exp[i\xi(\theta-\theta_0)+i(\sigma-\sigma_0)]. \quad (36)$$

Under this assumption of quasistationarity, Eq. (34) can be written as

$$-\eta^2\hat{u}+\hat{u}_{\theta\theta}+2\hat{u}^2\hat{u}^*=\lambda F(\hat{u}), \quad (37a)$$

where

$$F(\hat{u})=\Bigg[(\xi_T(\theta-\theta_0)-\xi\theta_{0T}-\sigma_{0T}-\xi^4)\hat{u}-\hat{u}_{\theta\theta\theta\theta}+6\xi^2\hat{u}_{\theta\theta}-w_1|\hat{u}|^2\hat{u}_{\theta\theta}-\frac{w_2}{2}(\hat{u}^2\hat{u}_{\theta\theta}^*-\xi^2\hat{u}^2\hat{u}^*-2|\hat{u}_\theta|^2\hat{u})+w_3\hat{u}_\theta^2\hat{u}^*-(w_1-w_2+w_3)\xi^2|\hat{u}|^2\hat{u}+w_4|\hat{u}|^4\hat{u}\Bigg]+i[-\hat{u}_T-4\xi\hat{u}_{\theta\theta\theta}+4\xi^3\hat{u}_\theta-(2w_1-w_2+2w_3)\xi|\hat{u}|^2\hat{u}_\theta]. \quad (37b)$$

Now, we assume for $\hat{u}$ the Poincaré type asymptotic expansion, $\hat{u}(\theta,T;\lambda)=\Sigma_{m=0}^{\infty}\lambda^m\hat{u}_m(\theta,T)$. We restrict ourselves to calculations of order (λ) such that $\hat{u}(\theta,T;\lambda)=\hat{u}_0(\theta,T)+\lambda\hat{u}_1(\theta,T)$, where $\hat{u}_0=\eta\ \mathrm{sech}\ \eta(\theta-\theta_0)$. We further substitute $\hat{u}_1=\phi_1+i\psi_1$ (ϕ_1 and ψ_1 are real) and rewrite Eqs. (37a) and (37b) as a set of equations in real variables:

$$L_1\phi_1 \equiv -\eta^2\phi_1 + \phi_{1\theta\theta} + 6\hat{u}_0^2\phi_1 = \mathrm{Re}\, F_1(\hat{u}_0), \quad (38a)$$

$$L_2\psi_1 \equiv -\eta^2\psi_1 + \psi_{1\theta\theta} + 2\hat{u}_0^2\psi_1 = \mathrm{Im}\, F_1(\hat{u}_0), \quad (38b)$$

where L_1 and L_2 are self-adjoint operators, and Re $F_1(\hat{u}_0)$ and Im $F_1(\hat{u}_0)$ are the real and imaginary parts of $F_1(\hat{u}_0)$ as follows:

$$\begin{aligned}\mathrm{Re}\, F_1(\hat{u}_0) = {}& [\xi_T(\theta-\theta_0) - \xi\theta_{0T} - \sigma_{0T} - \xi^4]\hat{u}_0 - \hat{u}_{0\theta\theta\theta\theta} \\ &+ 6\xi^2\hat{u}_{0\theta\theta} - w_1|u|^2\hat{u}_{0\theta\theta} \\ &- w_2\left(\frac{1}{2}\hat{u}_0^2\hat{u}_{0\theta\theta}^* - \frac{1}{2}\xi^2\hat{u}_0^2\hat{u}_0^* - |\hat{u}_{0\theta}|^2\hat{u}_0\right) + w_3\hat{u}_{0\theta}^2\hat{u}_0^* \\ &+ (w_1 - w_2 + w_3)\xi^2|\hat{u}_0|^2\hat{u}_0 + w_4|\hat{u}_0|^4\hat{u}_0, \quad (39a)\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\, F_1(\hat{u}_0) = {}& -\hat{u}_{0T} - 4\xi\hat{u}_{0\theta\theta\theta} + 4\xi^3\hat{u}_{0\theta} - (2w_1 - w_2 \\ &+ 2w_3)\xi|\hat{u}_0|^2\hat{u}_{0\theta}. \quad (39b)\end{aligned}$$

As $\hat{u}_{0\theta}$ and $\hat{u}_0$ are solutions of the homogeneous parts of Eqs. (38a) and (38b), respectively, we have the following secularity conditions:

$$\int_{-\infty}^{\infty} \hat{u}_{0\theta}\, \mathrm{Re}\, F_1(\hat{u}_0)d\theta = 0, \quad (40a)$$

$$\int_{-\infty}^{\infty} \hat{u}_0\, \mathrm{Im}\, F_1(\hat{u}_0)d\theta = 0. \quad (40b)$$

By substituting the values of $\hat{u}_{0\theta}$, $\hat{u}_0$, Re $F_1(\hat{u}_0)$, and Im $F_1(\hat{u}_0)$ into Eqs. (40a) and (40b) and after evaluating the integrals, we obtain $\xi_T = \eta_T = 0$, which imply that the discreteness effect does not alter the velocity and amplitude of the soliton during propagation.

The perturbed soliton solution can now be constructed by solving Eqs. (38a) and (38b). The homogeneous part of Eq. (38a) admits the following two particular solutions:

$$\phi_{11} = \mathrm{sech}\ \eta(\theta-\theta_0)\tanh(\theta-\theta_0), \quad (41a)$$

$$\begin{aligned}\phi_{12} = -\frac{1}{\eta}\Bigg[&\mathrm{sech}\ \eta(\theta-\theta_0) - \frac{3}{2}\eta(\theta-\theta_0)\mathrm{sech}\ \eta(\theta-\theta_0) \\ &\times \tanh\ \eta(\theta-\theta_0) - \frac{1}{2}\tanh\ \eta(\theta-\theta_0)\sinh\ \eta(\theta-\theta_0)\Bigg]. \quad (41b)\end{aligned}$$

The general solution is now found using the formula

$$\begin{aligned}\phi_1 = {}& W_1\phi_{11} + W_2\phi_{12} - \phi_{11}\int \phi_{12}\, \mathrm{Re}\, F_1(\hat{u}_0)d\theta \\ &+ \phi_{12}\int \phi_{11}\, \mathrm{Re}\, F_1(\hat{u}_0)d\theta, \quad (42)\end{aligned}$$

where W_1 and W_2 are arbitrary constants. By substituting the values of ϕ_{11}, ϕ_{12}, and Re F_1 into Eq. (42) and by evaluating the integrals, we obtain ϕ_1 as

$$\begin{aligned}\phi_1 = {}& \frac{1}{192\eta}\{12[11\eta^4 - 3(\xi^4 + \xi\theta_{0T} + \sigma_{0T})] - [2(\eta^2 - 9\xi^2)(w_1 + w_3) + 9w_2(\eta^2 + 3\xi^2) + 8w_4\eta^2]\eta^2 + 4\{12(\eta^4 - \xi^4 - \xi\theta_{0T} - \sigma_{0T}) \\ &+ [(2w_1 + 3w_2 + 2w_3)(\eta^2 - 3\xi^2) + 4w_4\eta^2]\eta^2\}\cosh 2\eta(\theta-\theta_0) - \{12(7\eta^4 + \xi^4 + \xi\theta_{0T} + \sigma_{0T}) - [2w_1(5\eta^2 - 3\xi^2) + 3w_2(\eta^2 \\ &+ 3\xi^2) - 2w_3(\eta^2 - 3\xi^2) + 8w_4\eta^2]\eta^2\}\cosh 4\eta(\theta-\theta_0) + [12\eta(\eta^4 + \xi^4 - 6\eta^2\xi^2 + \xi\theta_{0T} + \sigma_{0T})(\theta-\theta_0)][2\sinh 2\eta(\theta-\theta_0) \\ &+ \sinh 4\eta(\theta-\theta_0)] + 192W_1\eta\cosh^3\ \eta(\theta-\theta_0)\sinh\ \eta(\theta-\theta_0) + 24W_2[12\eta(\theta-\theta_0)\sinh(\theta-\theta_0) - 9\cosh\ \eta(\theta-\theta_0) \\ &+ \cosh 3\eta(\theta-\theta_0)]\cosh^3\ \eta(\theta-\theta_0)\}\mathrm{sech}^5\ \eta(\theta-\theta_0). \quad (43)\end{aligned}$$

In Eq. (43), the last term is a secular term that makes the solution unbounded and, hence, it is removed by choosing the arbitrary constant $W_2=0$. Further, we use the boundary conditions $\phi_1|_{\theta=\theta_0=0}=0$ and $\phi_{1\theta}|_{\theta=\theta_0=0}=0$, and obtain $W_1=0$ and $(\xi\theta_{0T}+\sigma_{0T})=\frac{1}{12}[12(\eta^4-\xi^4)+\eta^2[2w_1(\eta^2-3\xi^2)+3w_2(\eta^2+3\xi^2)+w_3(\eta^2-3\xi^2)+4w_4\eta^2]]$. By using the above in Eq. (43), the real part of the perturbed soliton is found to be

$$\begin{aligned}\phi_1 = {}& \frac{1}{96}\eta^2\{3[-3(2w_1 - 3w_2 + 2w_3 + 24)\xi^2 + (2w_1 + 3w_2 + w_3 + 4w_4 + 24)\eta^2](\theta-\theta_0)\cosh\ \eta(\theta-\theta_0) \\ &+ [-3(2w_1 - 3w_2 + 2w_3 + 24)\xi^2 + (2w_1 + 3w_2 + w_3 + 4w_4 + 24)\eta^2](\theta-\theta_0)\cosh 3\eta(\theta-\theta_0) \\ &+ 2[(8w_1 - w_3 + 4w_4 - 96) + (8w_1 - 3w_3 + 4w_4 - 96)\cosh 2\eta(\theta-\theta_0)]\eta\sinh\ \eta(\theta-\theta_0)\}\mathrm{sech}^4\ \eta(\theta-\theta_0)\tanh\ \eta(\theta-\theta_0). \quad (44)\end{aligned}$$

Similarly, the homogeneous part of Eq. (38b) admits the following two particular solutions:

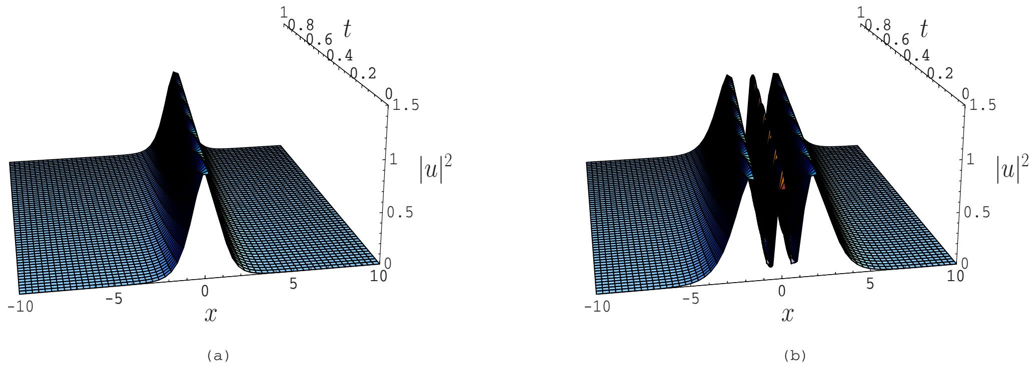

FIG. 1. (Color online) A plot of (a) $|u|^2$ of the unperturbed soliton [Eq. (35)] and (b) $|u|^2$ of the perturbed soliton [Eq. (36)] for the parametric choices $\lambda=0.5$, $\eta=\xi=1.0$, $w_1=1.0$, $w_2=0.5$, and $w_3=w_4=0.75$.

$$\psi_{11}=\text{sech }\eta(\theta-\theta_0), \tag{45a}$$

$$\psi_{12}=\frac{1}{2\eta}[\eta(\theta-\theta_0)\text{sech }\eta(\theta-\theta_0)+\sinh\,\eta(\theta-\theta_0)]. \tag{45b}$$

Now, the general solution of Eq. (38b) is written in the form

$$\psi_1=W_3\psi_{11}+W_4\psi_{12}-\psi_{11}\int\psi_{12}\,\text{Im }F_1(\hat{u}_0)d\theta + \psi_{12}\int\psi_{11}\,\text{Im }F_1(\hat{u}_0)d\theta, \tag{46}$$

where W_3 and W_4 are arbitrary constants. By substituting the values of ψ_{11}, ψ_{12}, and Im F_1 in Eq. (46) and by evaluating the integrals, we obtain the form of ψ_1. While evaluating ψ_1, the secular term that makes the solution unbounded is removed by choosing the arbitrary constant $W_4=0$. By using the boundary conditions $\psi_1|_{\theta=\theta_0=0}=0$ and $\psi_{1\theta}|_{\theta=\theta_0=0}=0$, we obtain $W_3=0$ and $\theta_{0T}=\frac{\xi}{2}[\eta^2(2w_1-w_2+2w_3-16)-8\xi^2]$. The final form of the general solution ψ_1 is obtained as

$$\psi_1=\frac{1}{4}(2w_1-w_2+2w_3-24)\eta^2\xi[\eta(\theta-\theta_0)\cosh\,\eta(\theta-\theta_0) - \sinh\,\eta(\theta-\theta_0)]\text{sech}^2\,\eta(\theta-\theta_0). \tag{47}$$

By using Eqs. (44) and (47), the first-order perturbed part of the soliton $\hat{u}_1=\phi_1+i\psi_1$ is obtained as

$$\begin{aligned}\hat{u}_1=&\frac{1}{96}\eta^2\,\text{sech}^4\,\eta(\theta-\theta_0)\{3[-3(2w_1-3w_2+2w_3+24)\xi^2+(2w_1+3w_2+w_3+4w_4+24)\eta^2](\theta-\theta_0)\cosh\,\eta(\theta-\theta_0)\\&+[-3(2w_1-3w_2+2w_3+24)\xi^2+(2w_1+3w_2+w_3+4w_4+24)\eta^2](\theta-\theta_0)\cosh\,3\eta(\theta-\theta_0)\\&+2[8w_1-w_3+4w_4-96+(8w_1-3w_3+4w_4-96)\cosh\,2\eta(\theta-\theta_0)]\eta\sinh\,\eta(\theta-\theta_0)\}\tanh\,\eta(\theta-\theta_0)\\&+i\left[\frac{1}{4}(2w_1-w_2+2w_3-24)\eta^2\xi[\eta(\theta-\theta_0)\cosh\,\eta(\theta-\theta_0)-\sinh\,\eta(\theta-\theta_0)]\text{sech}^2\,\eta(\theta-\theta_0)\right].\end{aligned} \tag{48}$$

Knowing $\hat{u}_1$, the general first-order perturbed solution $u=(\hat{u}_0+\lambda\hat{u}_1)e^{i\xi(\theta-\theta_0)+i(\sigma-\sigma_0)}$ can be written down by using Eq. (36), from which the spin components can be constructed. For a better understanding of the effect of the first-order perturbation that is due to discreteness on the NLS soliton, we have plotted the perturbed one-soliton by choosing the parameter values to be $\lambda=0.5$, $\eta=\xi=1.0$, $w_1=1.0$, $w_2=0.5$, and $w_3=0.75$ [see Fig. 1(b)]. The above parameter values correspond to the following values of the original coefficients that appear in the Hamiltonian, namely, $b=1$, $c=0.25$, $J_{20}=J_{30}=\frac{1}{32}J_{10}$, $A=6.69J_{10}$, and $A_1=0.453J_{10}$. However, for comparison, we have also plotted the unperturbed one-soliton of the cubic NLS equation in Fig. 1(a) by choosing the same parameter values. By comparing Figs. 1(a) and 1(b), we observe that the overall robust nature of the soliton is not affected by the perturbation. In other words, the discreteness does not introduce any significant shape changing character in the soliton. However, symmetric fluctuations are noticed in the localized region of the soliton with the tails of the soliton intact. In order to accommodate these fluctuations, the width of the soliton has increased a little [see Fig. 1(b)], thereby increasing the number of participating spins for the formation of localized excitations. As already found from secularity conditions, the velocity and amplitude of the soliton do not change.

B. Effect of inhomogeneity

Now, in order to understand the effect of inhomogeneity, we consider Eq. (33) by setting $\lambda=0$,

$$iu_t + u_{xx} + 2|u|^2u + \chi a_1[f_x(3|u|^2u_x + u_x^* u^2)] = 0. \qquad (49)$$

In Eq. (49), $\chi=\frac{\alpha\beta(c-b)J_{11}}{k_2}$ and we treat χ in this case as a small perturbation parameter. In order to carry out the multiple scale perturbation analysis on Eq. (49), we introduce another slow variable $T'=\chi t$ and write the quantities η, ξ, θ_0, and σ_0 that appear in the one-soliton solution given in Eq. (35) as functions of this time scale so that the envelope soliton solution of Eq. (49) is written as given in Eq. (36). We carry out a perturbation analysis on Eq. (49) by following the same procedure as in the previous case and obtain Eqs. (38a), (38b), (39a), and (39b) with Re $F_1(\hat{u})$ and Im $F_1(\hat{u})$ as given by

$$\text{Re } F_1(\hat{u}_0) = [\xi_{T'}(\theta-\theta_0) - \xi\theta_{0T'} - \sigma_{0T'} - \xi^4]\hat{u}_0 - 3a_1[|\hat{u}_0|^2\hat{u}_{0\theta} - \hat{u}_{0\theta}^*\hat{u}_0^2], \qquad (50a)$$

$$\text{Im } F_1(\hat{u}_0) = -\hat{u}_{0T'} - 2\xi a_1|\hat{u}_0|^2\hat{u}_0. \qquad (50b)$$

By using the above expressions for Re $F_1(\hat{u}_0)$ and Im $F_1(\hat{u}_0)$ and the values of $\hat{u}_0$ and $\hat{u}_{0\theta}$ found earlier in the secularity conditions given in Eq. (40a) and (40b), we obtain the following time evolution equations for the amplitude (η) and velocity (ξ) of the soliton:

$$\eta_{T'} = -\frac{4}{3}a_1\eta^3, \qquad (51a)$$

$$\xi_{T'} = -\frac{16}{15}a_1\eta^4. \qquad (51b)$$

By solving Eqs. (51a) and (51b), we explicitly obtain the amplitude and velocity of the soliton as

$$\eta = \frac{\eta_0}{\left(1+\frac{8}{3}a_1\eta_0^2T'\right)^{1/2}}, \qquad (52a)$$

$$\xi = \xi_0 - \frac{2\eta_0^2}{5\left(1+\frac{3}{8a_1\eta_0^2T'}\right)}, \qquad (52b)$$

where η_0 and ξ_0 are the initial amplitude and velocity of the soliton, respectively. From Eqs. (52a) and (52b), we observe that (i) when there is inhomogeneity present in the spin lattice, the amplitude and velocity of the soliton decrease with time (however, when there is no inhomogeneity, the amplitude and velocity of the soliton remain the same during propagation); (ii) the amplitude and velocity of the soliton decrease when the amount of inhomogeneity (a_1) increases; and, finally, (iii) the soliton velocity decreases with an increase in the initial amplitude of the soliton. From Eq. (52b), we further calculate the time taken for the soliton to stop $T'(\xi=0)$ and obtain

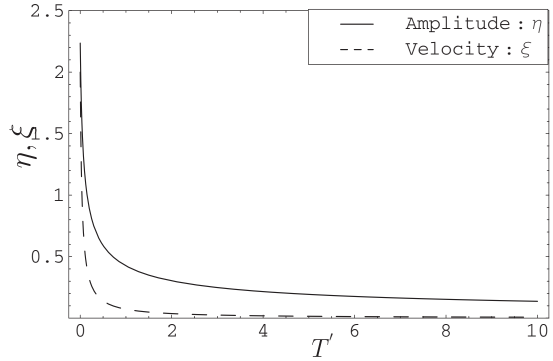

FIG. 2. Variation in amplitude (η) and velocity (ξ) of the soliton [Eqs. (52a) and (52b)] for the case $\xi_0=\frac{2}{5}\eta_0^2$ by choosing $a_1=2.0$, $\xi_0=2.0$, and $\eta_0=2.236$.

$$T'(\xi=0) = \frac{3}{8a_1\eta_0^2\left(\frac{2\eta_0^2}{5\xi_0}-1\right)}. \qquad (53)$$

Equation (53) indicates that when the inhomogeneity is large, the soliton comes to rest within a short period. However, when the initial velocity is related to the initial amplitude according to $\xi_0=\frac{2}{5}\eta_0^2$, the soliton never stops and continues to propagate. When the initial velocity ξ_0 is less than $\frac{2}{5}\eta_0^2$, the velocity of the soliton after some time becomes negative and, therefore, it travels in the opposite direction. This is because the inhomogeneity gives rise to an effective potential (attractive or repulsive) to the soliton such that the soliton is either captured or reflected by the inhomogeneity. However, the values in the range $\xi_0>\frac{2}{5}\eta_0^2$ are not allowed, which is obvious from Eq. (53). In Fig. 2, we have plotted the time evolution of the amplitude (η) and velocity (ξ) of the soliton for the case $\xi_0=\frac{2}{5}\eta_0^2$ by choosing $a_1=2.0$, $\xi_0=2.0$, and, hence, $\eta_0=2.236$. We observe that both the amplitude and velocity of the soliton decrease with time and the soliton amplitude and velocity asymptotically approach zero, which is in conformity with the above discussion in relation to Eqs. (52a) and (52b). From Eq. (52b), as ξ becomes negative when the initial velocity $\xi_0<\frac{2}{5}\eta_0^2$, we observe that the soliton is reflected by the potential offered by the inhomogeneity and the amount of inhomogeneity determines the time taken for the soliton to reverse its direction. In addition, when the inhomogeneity is large, the corresponding potential becomes large and, hence, the soliton reverses its direction quickly. In Figs. 3(a)–3(d), the time evolution of the velocity and amplitude of the soliton has been plotted for the case $\xi_0<\frac{2}{5}\eta_0^2$ for different values of a_1 [(a) $a_1=0.2$, (b) $a_1=0.5$, (c) $a_1=2.0$, and (d) $a_1=20.0$] by choosing $\eta_0=2.236$ and $\xi_0=1.5$. We observe that the amplitude of the soliton decreases and approaches zero asymptotically. However, the soliton is reversed after some time without affecting the amplitude and asymptotically attains a uniform speed. By comparing Figs. 3(a)–3(d), we also notice that as the inhomogeneity (a_1) increases, the amplitude of the soliton quickly decreases and the soliton also quickly reverses. This adds to the fact that, in one dimension, noninteracting excitations are strongly localized by a disorder.

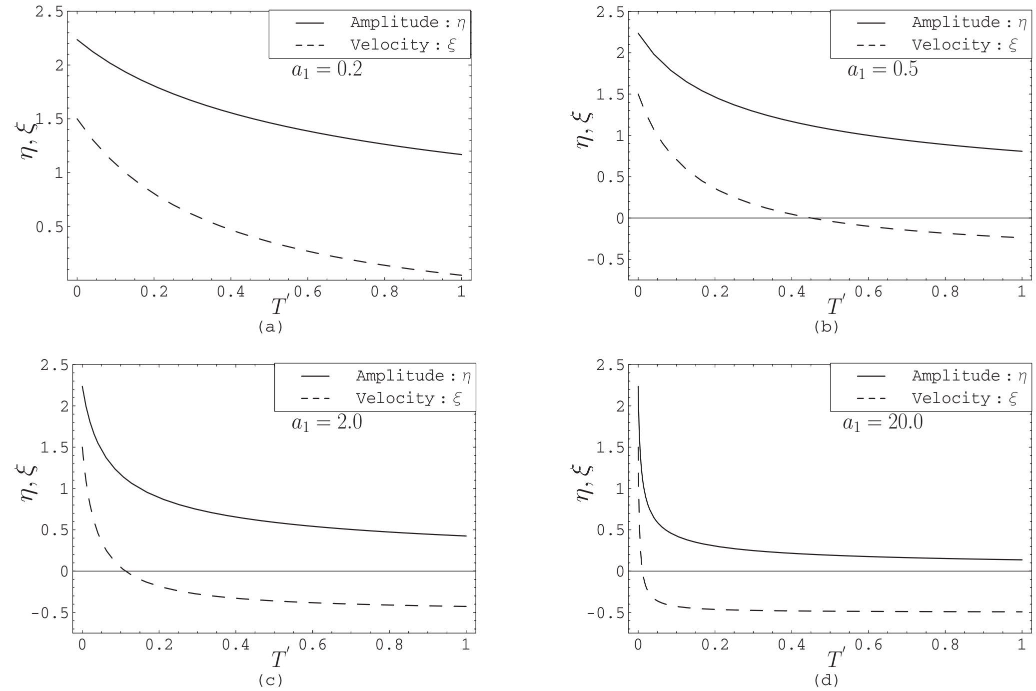

FIG. 3. Variation in amplitude (η) and velocity (ξ) of the soliton [Eqs. (52a) and (52b)] when $\xi_0 < \frac{2}{5}\eta_0^2$ for different values of the inhomogeneity: (a) a_1=0.2, (b) a_1=0.5, (c) a_1=2.0, and (d) a_1=20.0 by choosing ξ_0=1.5 and η_0=2.236.

Next, we construct the perturbed soliton solution by solving Eqs. (38a) and (38b), where Re $F_1(\hat{u}_0)$ and Im $F_1(\hat{u}_0)$ are now as given in Eqs. (50a) and (50b). The general solutions ϕ_1 and ψ_1 are determined from Eqs. (42) and (46), where the arbitrary constants W_1, W_2, W_3, and W_4 are now replaced by B_1, B_2, B_3, and B_4. Following the same procedure, we obtain the general solution ϕ_1 as

$$\phi_1 = \Bigg\{\left[-\frac{1}{2\eta}(\xi\theta_{0T}+\sigma_{0T}) - \frac{8}{15}a_1\eta^3(\theta-\theta_0) - \frac{9B_2}{8\eta}\right] + \Bigg[B_1 + \frac{3}{2}B_2(\theta-\theta_0) + \frac{1}{2}(\xi\theta_{0T}+\sigma_{0T})(\theta-\theta_0) + \frac{2}{15}a_1\eta^2[1+2\eta^2(\theta-\theta_0)^2] + \frac{4}{5}a_1\eta^2\log[\cosh\,\eta(\theta-\theta_0)]\Bigg]\tanh\,\eta(\theta-\theta_0) + \frac{B_2}{8\eta}\cosh 3\eta(\theta-\theta_0)\text{sech}\,\eta(\theta-\theta_0)\Bigg\}\text{sech}\,\eta(\theta-\theta_0). \tag{54}$$

In Eq. (54), the last term that is secular is removed by choosing the arbitrary constant B_2=0. By using the same boundary conditions for ϕ_1 as in the previous case, we get $B_1 = \frac{1}{2}a_1\eta^2$ and $(\xi\theta_{0T'}+\sigma_{0T'})$=0. By using these values, we obtain

$$\phi_1 = \frac{4}{15}a_1\eta^2[\{\eta^2(\theta-\theta_0)^2 + 3\log[\cosh\,\eta(\theta-\theta_0)] + 2\}\sinh\,\eta(\theta-\theta_0) - 2\eta(\theta-\theta_0) \times\cosh\,\eta(\theta-\theta_0)]\text{sech}^2\,\eta(\theta-\theta_0). \tag{55}$$

In a similar way, we construct the general solution ψ_1 and remove the secular term by choosing the arbitrary constant B_4=0. The boundary conditions on ψ_1 yield $B_3 = \frac{a_1\eta}{3}$ and θ'_{0T}=0. The final form of the solution ψ_1 after using the above conditions is written as

$$\psi_1 = \frac{1}{3}a_1\eta\{\eta^2(\theta-\theta)^2 - \log[\cosh\,\eta(\theta-\theta_0)]\}\text{sech}\,\eta(\theta-\theta_0). \tag{56}$$

By using Eqs. (55) and (56), the first-order perturbed part of the solution, $\hat{u}_1 = \phi_1 + i\psi_1$, is obtained. By knowing the first-order perturbed part of the solution $\hat{u}_1$, the perturbed solution u [Eq. (36)] can be obtained. In Fig. 4, we have plotted the square of the absolute value of u (i.e., $|u|^2$) by choosing χ =0.5, η_0=2.236, ξ_0=1.5, and a_1=1.0. From Figs. 4(a) and 4(b), we find that, although the overall nature of the soliton is not disturbed due to inhomogeneity, asymmetric fluctuations are introduced in the localized region of the soliton, in which the width of the soliton is increased by a small amount.

By comparing the perturbed solitons due to the effect of discreteness [Fig. 1(b)] and inhomogeneity [Fig. 4(b)] with the unperturbed soliton [Figs. 1(a) and 4(a)], we observe that, in both the cases, the width of the soliton is increased by the perturbation in order to accommodate the fluctuations in the localized region. We notice further that in spite of the fluctuations present, the symmetry about the center of the soliton is maintained in the case of the soliton perturbed due to discreteness. However, the symmetry is lost in the case of the soliton perturbed due to inhomogeneity.

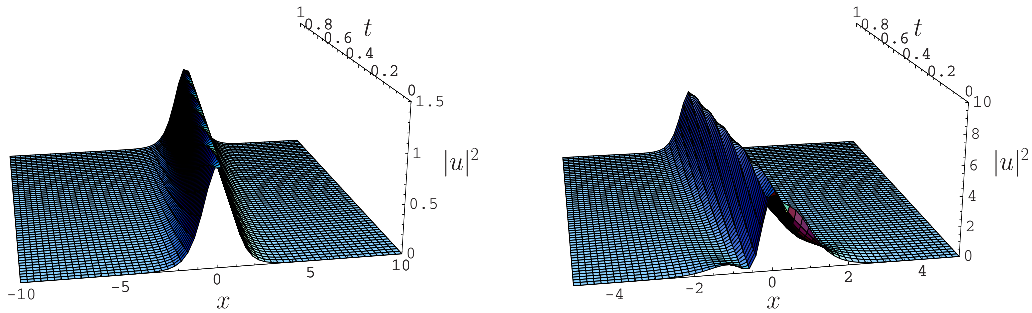

FIG. 4. (Color online) A plot of (a) $|u|^2$ of the unperturbed soliton [Eq. (35)] and (b) $|u|^2$ of the soliton perturbed due to linear inhomogeneity [Eq. (36)] for the parametric choices $\chi=0.5$, $\eta_0=2.236$, $\xi_0=1.5$, and $a_1=1.0$.

V. CONCLUSION

In this paper, the nonlinear spin excitations in the form of solitons in a generalized anisotropic weak Heisenberg ferromagnetic spin system with varying bilinear, biquadratic, and octupole-dipole interactions are investigated in the semiclassical limit under a continuum approximation, following HP expansion, in combination with Glauber's coherent-state method. The equation of motion governing the nonlinear spin dynamics of the above system is found to be governed by a generalized higher-order NLS equation, which is not completely integrable in general. However, for specific parametric choices, at different orders of continuum approximation, the dynamical equation reduces to the cubic NLS equation, the derivative NLS equation (Chen–Lee–Liu equation and Kaup–Newell equation), the modified MKdV equation, the MDNLS equation, the Hirota equation, the Sasa–Satsuma equation, the CMKdV equation, and the fourth-order NLS equation, which are completely integrable and possess N-soliton solutions. Thus, the model considered in the present paper acts as a generalized spin model since it includes the well known magnetic interactions of a ferromagnet and is governed by different known completely integrable soliton equations. Interestingly, from Table I, we observe that the nonlinear spin dynamics of a given spin model is governed by different completely integrable soliton equations for different parametric values at different orders. Thus, soliton spin excitations are generated in a particular spin system in the continuum limit under a semiclassical approximation by suitably fixing the magnetic interaction parameters and, thus, solitons of different forms are generated through various competing magnetic interactions. The effects of discreteness and inhomogeneity on the soliton are separately studied by carrying out a multiple scale perturbation analysis on the more general higher-order NLS equation. The results of the perturbation analysis show that the discreteness effect introduces symmetric fluctuations in the localized region (Fig. 1) of the soliton without altering the velocity and amplitude of the soliton. On the other hand, the amplitude and velocity of the soliton decrease with an increase in the amount of inhomogeneity (a_1) as time passes by. Also, the soliton velocity decreases with an increase in the initial amplitude of the soliton. We further observe that the presence of inhomogeneity makes the soliton stop and reverse its direction after some time. However, when the initial velocity (ξ_0) and the initial amplitude (η_0) are related according to the relation $\xi_0=\frac{2}{5}\eta_0^2$, the soliton never stops and continues to propagate. When the initial velocity ξ_0 is less than $\frac{2}{5}\eta_0^2$, the velocity of the soliton becomes negative after some time and, hence, it travels in the opposite direction. This is because the soliton is reflected by the potential offered by the inhomogeneity. When the inhomogeneity increases, the corresponding potential also becomes large and, hence, the amplitude of the soliton quickly decreases and the soliton also reverses its direction quickly [Figs. 3(a)–3(d)]. We further notice that the amplitude of the soliton approaches zero and the velocity of the soliton reaches a steady state value asymptotically. Moreover, asymmetric fluctuations occur in the localized region of the soliton due to the presence of inhomogeneity. When the soliton that is perturbed due to discreteness and inhomogeneity effects is compared with the unperturbed soliton, we observe that, in both cases, the width of the soliton is increased in order to accommodate either the symmetric or asymmetric fluctuations in the localized region.

ACKNOWLEDGMENTS

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APPENDIX A

The Hamiltonian under the Holstein–Primakoff representation is given by

$$H=-\sum_i\Bigg\{(J_{10}+J_{11}f_i+J_{30}+J_{31}h_i-A-A')-i\epsilon\left[\frac{J_4D^-}{\sqrt{2}}(a_i-a_{i+1})+\frac{J_4D^+}{\sqrt{2}}(a_{i+1}^\dagger-a_i^\dagger)\right]+\epsilon^2[(J_{10}+J_{11}f_i)(ba_ia_{i+1}^\dagger+ba_i^\dagger a_{i+1}$$
$$-ca_i^\dagger a_i-ca_{i+1}^\dagger a_{i+1})+2(J_{20}+J_{21}g_i)(a_ia_{i+1}^\dagger+a_i^\dagger a_{i+1}-a_i^\dagger a_i-a_{i+1}^\dagger a_{i+1})+(J_{30}+J_{31}h_i)(a_ia_{i+1}^\dagger+a_i^\dagger a_{i+1}-a_i^\dagger a_i-3a_{i+1}^\dagger a_{i+1})$$

$$- iJ_4D^z(a_i^\dagger a_{i+1} - a_i a_{i+1}^\dagger) + 2(A + 2A')a_i^\dagger a_i] + i\epsilon^3/4\Bigg[\frac{J_4D^-}{\sqrt{2}}(4a_i a_{i+1}^\dagger a_{i+1} + a_i^\dagger a_i a_i - 4a_i^\dagger a_i a_{i+1} - a_{i+1}^\dagger a_{i+1} a_{i+1})$$
$$+ \frac{J_4D^+}{\sqrt{2}}(4a_i^\dagger a_i a_{i+1}^\dagger + a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - 4a_i^\dagger a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i^\dagger a_i)\Bigg] - \frac{\epsilon^4}{4}[(J_{10} + J_{11}f_i)(ba_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} + ba_i^\dagger a_i a_i a_{i+1}^\dagger$$
$$+ ba_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + ba_i^\dagger a_i^\dagger a_i a_{i+1} - 4ca_i^\dagger a_i a_{i+1}^\dagger a_{i+1}) - 2(J_{20} + J_{21}g_i)(2a_i a_{i+1}^\dagger a_i a_{i+1}^\dagger + 2a_i^\dagger a_{i+1} a_i^\dagger a_{i+1} + 12a_i^\dagger a_{i+1}^\dagger a_i a_{i+1}$$
$$- 5a_i^\dagger a_{i+1}^\dagger a_i a_i - 5a_{i+1}^\dagger a_{i+1}^\dagger a_i a_{i+1} - 5a_i^\dagger a_i^\dagger a_i a_{i+1} - 5a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + 2a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} + 2a_i^\dagger a_i a_i^\dagger a_i)$$
$$+ (J_{30} + J_{31}h_i)(9a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} + a_i^\dagger a_i a_i a_{i+1}^\dagger + 9a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + a_i^\dagger a_i^\dagger a_i a_{i+1} - 12a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} - 12a_i^\dagger a_i a_{i+1}^\dagger a_{i+1})$$
$$- iJ_4D^z(a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} + a_i^\dagger a_i^\dagger a_i a_{i+1} - a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i a_i a_{i+1}^\dagger) + 4(A + 6A')a_i^\dagger a_i a_i^\dagger a_i] + i\frac{\epsilon^5}{32}\Bigg[\frac{J_4D^-}{\sqrt{2}}(a_i^\dagger a_i a_i^\dagger a_i a_i$$
$$- 8a_i^\dagger a_i a_i a_{i+1}^\dagger a_{i+1} - a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} a_{i+1} + 8a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} a_{i+1}) + \frac{J_4D^+}{\sqrt{2}}(a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} - 8a_i^\dagger a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i^\dagger a_i a_i^\dagger a_i$$
$$+ 8a_i^\dagger a_i^\dagger a_i a_{i+1}^\dagger a_{i+1})\Bigg] + \frac{\epsilon^6}{32}[(J_{10} + J_{11}f_i)[b(2a_i^\dagger a_i a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i a_i^\dagger a_i a_i a_{i+1}^\dagger + 2a_i^\dagger a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} a_{i+1}$$
$$- a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} a_{i+1} - a_i^\dagger a_i^\dagger a_i a_i^\dagger a_i a_{i+1})] - 2(J_{20} + J_{21}g_i)(8a_i^\dagger a_{i+1}^\dagger a_{i+1}^\dagger a_i a_i a_i + 8a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1}^\dagger a_i a_i a_{i+1} + 8a_i^\dagger a_i^\dagger a_i^\dagger a_i a_{i+1} a_{i+1}$$
$$+ 8a_i^\dagger a_{i+1}^\dagger a_i^\dagger a_{i+1} a_{i+1} a_{i+1} + 48a_i^\dagger a_i^\dagger a_{i+1}^\dagger a_i a_i a_{i+1} + 48a_i^\dagger a_{i+1}^\dagger a_{i+1}^\dagger a_i a_{i+1} a_{i+1} - 50a_i^\dagger a_{i+1}^\dagger a_{i+1}^\dagger a_i a_i a_{i+1} - 7a_i^\dagger a_i^\dagger a_{i+1}^\dagger a_i a_i a_i$$
$$- 7a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_i a_{i+1} a_{i+1} - 50a_i^\dagger a_i^\dagger a_{i+1}^\dagger a_i a_{i+1} a_{i+1} - 7a_i^\dagger a_i^\dagger a_i^\dagger a_i a_i a_{i+1} - 7a_i^\dagger a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1} a_{i+1}) + (J_{30} + J_{31}h_i)$$
$$\times(47a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} + 18a_i^\dagger a_i a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i a_i^\dagger a_i a_i a_{i+1}^\dagger + 47a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} a_{i+1} + 18a_i^\dagger a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} a_{i+1}$$
$$- a_i^\dagger a_i^\dagger a_i a_i^\dagger a_i a_{i+1} - 32a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} - 96a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1}) + iJ_4D^z(a_i^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} a_{i+1} - 2a_i^\dagger a_i^\dagger a_i a_{i+1}^\dagger a_{i+1} a_{i+1}$$
$$+ a_i^\dagger a_i^\dagger a_i a_i^\dagger a_i a_{i+1} - a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} a_{i+1}^\dagger a_{i+1} + 2a_i^\dagger a_i a_i a_{i+1}^\dagger a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i a_i^\dagger a_i a_i a_{i+1}^\dagger) + 128A'(a_i^\dagger a_i a_i^\dagger a_i a_i^\dagger a_i)]\Bigg\}. \tag{A1}$$

APPENDIX B

By substituting the Hamiltonian given in Eq. (A1) in Eq. (6) and by using Glauber's coherent-state representation for bosonic operators given after Eq. (6), we obtain the following discrete equation of motion:

$$- i\frac{du_j}{dt} = \epsilon^2\{(J_{10} + J_{11}f_{j-1})[bu_{j-1} - cu_j] + (J_{10} + J_{11}f_j)[bu_{j+1} - cu_j] + 2(J_{20} + J_{21}g_{j-1})[u_{j-1} - u_j] + 2(J_{20} + J_{21}g_j)[u_{j+1} - u_j]$$
$$+ (J_{30} + J_{31}h_{j-1})[u_{j-1} - 3u_j] + (J_{30} + J_{31}h_j)[u_{j+1} - u_j] - iJ_4D^z[u_{j+1} - u_{j-1}] + 2(A + 2A')u_j\} + i\epsilon^3\Bigg\{\frac{J_4D^-}{\sqrt{2}}(u_{j-1}$$
$$- u_{j+1})u_j + \frac{J_4D^+}{\sqrt{2}}[(u_{j+1}^* - u_{j-1}^*)u_j + |u_{j-1}|^2 - |u_{j+1}|^2]\Bigg\} - \frac{\epsilon^4}{4}\{(J_{10} + J_{11}f_{j-1})[2b|u_j|^2u_{j-1} + b|u_{j-1}|^2u_{j-1} + bu_j^2u_{j-1}^*$$
$$- 4c|u_{j-1}|^2u_j] + (J_{10} + J_{11}f_j)[bu_j^2u_{j+1}^* + b|u_{j+1}|^2u_{j+1} + 2b|u_j|^2u_{j+1} - 4c|u_{j+1}|^2u_j] - 2(J_{20} + J_{21}g_{j-1})[4u_{j-1}^2u_j^*$$
$$+ 12|u_{j-1}|^2u_j - 10|u_j|^2u_{j-1} - 5u_{j-1}^*u_j^2 + 4|u_j|^2u_j] - 2(J_{20} + J_{21}g_j)[4u_{j+1}^2u_j^* + 12|u_{j+1}|^2u_j - 5u_{j+1}^*u_j^2 - 5|u_{j-1}|^2u_{j-1}$$
$$- 10|u_j|^2u_{j+1} - 5|u_{j+1}|^2u_{j+1} + 4|u_j|^2u_j] + (J_{30} + J_{31}h_{j-1})[18|u_j|^2u_{j-1} - |u_{j-1}|^2u_{j-1} + 9u_j^2u_{j-1}^* - 24|u_j|^2u_j - 12|u_{j-1}|^2u_j]$$
$$+ (J_{30} + J_{31}h_j)[u_j^2u_{j+1}^* + 9|u_{j+1}|^2u_{j+1} + 2|u_j|^2u_{j+1} - 12|u_{j+1}|^2u_j] - iJ_4D^z[|u_{j+1}|^2u_{j+1} + u_{j-1}^*u_j^2 + 2(u_{j+1} - u_{j-1})|u_j|^2$$
$$- u_j^2u_{j+1}^* - |u_{j-1}|^2u_{j-1}] + 8(A + 6A')|u_j|^2u_j\} - i\frac{\epsilon^5}{32}\Bigg\{\frac{8J_4D^-}{\sqrt{2}}[(|u_{j+1}|^2 - |u_{j-1}|^2)u_j^2 + (|u_{j-1}|^2u_{j-1} - |u_{j+1}|^2u_{j+1})u_j]$$
$$+ \frac{8J_4D^+}{\sqrt{2}}[|u_{j+1}|^2u_{j+1}^*u_j + 2|u_{j-1}|^2|u_j|^2 - 2|u_{j+1}|^2|u_j|^2 - |u_{j-1}|^2u_{j-1}^*u_j]\Bigg\}$$
$$+ \frac{\epsilon^6}{32}\{(J_{10} + J_{11}f_{j-1})[4b|u_{j-1}|^2|uj|^2u_{j-1} - 3b|u_j|^4u_{j-1} - b|u_{j-1}|^4u_{j-1} + 2b(|u_{j-1}|^2 - |u_j|^2)u_{j-1}^*u_j^2] + (J_{10} + J_{11}f_j)$$
$$\times[2b|u_{j+1}|^2u_{j+1}^*u_j^2 - 2b|u_j|^2u_j^2u_{j+1}^* + 4b|u_{j+1}|^2|u_j|^2u_{j+1} - b|u_{j+1}|^4u_{j+1} - 3b|u_j|^4u_{j+1}] - 2(J_{20} + J_{21}g_{j-1})[16|u_{j-1}|^2u_j^*u_{j-1}^2$$

$$+ 24|u_j|^2u_j^*u_{j-1}^2 + 8u_{j-1}^{*2}u_j^3 + 48|u_{j-1}|^4u_j + 96|u_j|^2|u_{j-1}|^2u_j - 100|u_{j-1}|^2|u_j|^2u_{j-1} - 7|u_{j-1}|^4u_j - 21|u_j|^4u_{j-1}$$
$$- 50|u_{j-1}|^2u_{j-1}^*u_j^2 - 14|u_j|^2u_j^2u_{j-1}^*] - 2(J_{20} + J_{21}g_j)[8u_{j+1}^{*2}u_j^3 + 24|u_j|^2u_{j+1}^2u_j^* + 16|u_{j+1}|^2u_{j+1}^2u_j^* + 96|u_{j+1}|^2|u_j|^2u_j$$
$$+ 48|u_{j+1}|^4u_j - 50|u_{j+1}|^2u_j^2u_{j+1}^* - 14|u_j|^2u_j^2u_{j+1}^* - 100|u_{j+1}|^2|u_j|^2u_{j+1} - 21|u_j|^4u_{j+1} - 7|u_{j+1}|^4u_{j+1}] + (J_{30} + J_{31}h_{j-1})$$
$$\times[141|u_j|^4u_{j-1} + 36|u_{j-1}|^2|u_j|^2u_{j-1} + 94|u_j|^2u_j^2u_{j-1}^* - |u_{j-1}|^4u_{j-1} + 18|u_{j-1}|^2u_j^2u_{j-1}^* - 96|u_j|^4u_j - 192|u_{j-1}|^2|u_j|^2u_j]$$
$$+ (J_{30} + J_{31}h_j)[18|u_{j+1}|^2u_j^2u_{j+1}^* - 2|u_j|^2u_j^2u_{j+1}^* + 47|u_{j+1}|^4u_{j+1} + 36|u_{j+1}|^2|u_j|^2u_{j+1} - 3|u_j|^4u_{j+1} - 96|u_{j+1}|^4u_j]$$
$$+ iJ_4D^z[|u_{j+1}|^4u_{j+1} + 2|u_j|^2u_j^2u_{j-1}^* - 4|u_{j+1}|^2|u_j|^2u_{j+1} - 2|u_{j-1}|^2u_j^2u_{j-1}^* + 3(u_{j+1} - u_{j-1})|u_j|^4 + 2|u_{j+1}|^2u_j^2u_{j+1}^*$$
$$+ 4|u_{j-1}|^2|u_j|^2u_{j-1} - 2|u_j|^2u_j^2u_{j+1}^* - |u_{j-1}|^4u_{j-1}] + 384A'|u_j|^4u_j\}. \quad \text{(B1)}$$

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