Griffiths-McCoy singularities in random quantum spin chains: Exact results

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We consider random quantum (tight-binding, XX, and Ising) spin chains in the off-critical region and study their Griffiths-McCoy singularities. These are obtained from the density of states of the low-energy excitations, which is exactly calculated by the Dyson-Schmidt method. In large finite systems, the low-energy excitations are shown to follow the statistics of extremes and their distribution is given by the Fréchet form. The relation between the Dyson-Schmidt technique and the strong disorder renormalization group method is also discussed.

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I. INTRODUCTION

Quenched disorder has a profound effect on the low-energy, low-temperature, and long wavelength properties of quantum systems. The interplay between quantum fluctuations, correlations, and disorder fluctuations generally results in strong singularities in the thermodynamical quantities and in the (dynamical) correlation functions.[1,2] This type of effect takes place even outside the quantum critical region, e.g., in the quantum paramagnetic phase at zero temperature, $T=0$, where spatial correlations are short ranged.[2,3] The origin of this phenomenon, as pointed out by Griffiths,[4] is due to rare regions, in which strong bonds are accumulated by extreme fluctuations so that the system in these regions is locally in the thermodynamically unstable ferromagnetic phase. As a consequence, the excitation energy E in the rare regions is very small, the relaxation process is very slow, and the associated relaxation time $\tau \sim E^{-1}$ is divergent in the thermodynamic limit. If we consider a finite part of a sample with linear size ℓ, the characteristic time scale of the slowest relaxation process also stays finite and is asymptotically given by

$$\tau \sim \ell^z. \tag{1}$$

Here, $z=z(\delta)$ is the dynamical exponent, which is generally a continuous function of the quantum control parameter δ, which measures the distance from the quantum critical point.

According to a scaling theory,[5,6] the distribution of the low-energy excitations, $n(E,\ell)$, depends on the scaling combination, $E\ell^z$, and for a small but fixed E it is proportional to the volume ℓ^d since the probability of finding a rare region goes linearly with the volume. From this the asymptotic behavior of the distribution function in the thermodynamic limit reads as

$$n(E) \sim E^{d/z-1}. \tag{2}$$

Thermodynamical quantities which are obtained through an integration of the density of states are also singular. For example, the low-temperature behavior of the average linear susceptibility, $\chi(T)$, and that of the specific heat, $c_v(T)$, is expected to scale as[2,3]

$$\chi(T) \sim T^{-1+d/z}, \quad c_v(T) \sim T^{d/z}, \tag{3}$$

whereas the small-field H dependence of the zero-temperature magnetization is given by

$$m(H) \sim H^{d/z}. \tag{4}$$

One can see from Eq. (3) that the susceptibility is divergent at zero temperature for $z(\delta)>d$, which was first noticed by McCoy[7] in an exact calculation of the random transverse-field Ising chain (RTFIC).

Detailed results about Griffiths-McCoy singularities are partially obtained for one-dimensional systems by numerical investigations [e.g., free-fermionic techniques,[6,8,9] density matrix renormalization method,[10,11] quantum Monte Carlo (MC) simulations[12]] and by analytical calculations[11,13–15] based on the use of a strong disorder renormalization group (SDRG) method.[2] For higher dimensional systems, Griffiths-McCoy singularities are numerically studied, either by quantum MC simulations[16] or by numerical implementation of the SDRG method.[17]

Analytical and conjectured exact results about Griffiths-McCoy singularities are scarce and these are practically restricted to the RTFIC. An analytical solution of the SDRG equations is first obtained in the vicinity of the quantum critical point,[13] i.e., in the weakly disordered and weakly ordered Griffiths phases, where the dynamical exponent is shown to diverge as $z(\delta) \sim 1/|\delta|$. The solution is then extended to the complete Griffiths phase[11,15] and the calculated value of $z(\delta)$ is shown to agree with that obtained through a mapping to a random walk problem in a random environment.[18]

In this paper, we use a direct and simple method to calculate exact values of the Griffiths-McCoy singularities in a class of random quantum spin chains. These models include the random tight-binding chain, the random antiferromagnetic XX chain, and the RTFIC. The low-energy excitations for each model have the same form: they are obtained from the eigenvalue problem of a symmetric tridiagonal matrix $\mathcal{M}$ [see Eq. (7)], with random (positive) entries. In the off-critical region of the spin chains there is an even-odd asymmetry: the matrix elements of $\mathcal{M}$ are taken from different distributions at even and odd bonds. We calculate the density of states, $n(E)$, in the center of the band by the Dyson-

Schmidt technique[19] by using the random walk idea of Eggarter and Riedinger.[20] In Ref. 20, $n(E)$ is calculated in the continuum approximation for an even-odd symmetric $\mathcal{M}$, which corresponds to the critical point of the random quantum spin chains. In the present paper, $\mathcal{M}$ has a general even-odd asymmetric form, which corresponds to a strongly disordered quantum Griffiths phase and for which the continuum approximation is no longer valid. Having the exact behavior of $n(E)$ at hand, we then calculate the singularities of the thermodynamic quantities (i.e., specific heat, susceptibility, and magnetization).

The structure of the paper is the following. Random quantum chain models studied in this paper are presented in Sec. II. In Sec. III, the density of states of the low energy excitations is calculated by the Dyson-Schmidt technique and the relation of this technique with the SDRG method is discussed. Thermodynamic singularities are calculated in Sec. IV and the results are discussed in Sec. V.

II. RANDOM QUANTUM CHAINS

A. Random tight-binding model

The first model we consider is a one-dimensional tight-binding model with off-diagonal disorder,[21] which is defined by the Hamiltonian

$$\mathcal{H} = \sum_i t_i(|i\rangle\langle i+1| + |i+1\rangle\langle i|), \tag{5}$$

with random hopping matrix elements t_i. The hopping matrix elements are generally taken from different distributions at even (t_e) and odd (t_o) sites so that a quantum control-parameter is defined as

$$\delta = \frac{[\ln t_o]_{\rm av} - [\ln t_e]_{\rm av}}{\mathrm{var}(\ln t_e) + \mathrm{var}(\ln t_o)}, \tag{6}$$

where $[\cdots]_{\rm av}$ stands for averaging over quenched disorder and $\mathrm{var}(x)$ stands for the variance of x. For $\delta>0$ ($\delta<0$), the model is asymmetric and the particles are preferentially at odd (even) bonds. The symmetric model with $\delta=0$ corresponds to a quantum critical point.

In the basis $|i\rangle$, the Hamiltonian is represented by a tridiagonal matrix,

$$\mathcal{M} = \begin{pmatrix} 0 & t_1 & & & \\ t_1 & 0 & t_2 & & \\ & t_2 & 0 & t_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{7}$$

and we are interested in its eigenvalue problem,

$$\mathcal{M}\vec{\alpha} = E\vec{\alpha}, \tag{8}$$

and the corresponding density of states $n(E)$ at the center of the band.

B. Random antiferromagnetic XX chain

The second model is the random antiferromagnetic XX chain that is defined by the Hamiltonian

$$\mathcal{H}_{\rm XX} = \sum_i J_i(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) \tag{9}$$

in terms of the spin-1/2 operators $S_i^{x,y}$ at site i. Here, the J_i exchange couplings are random variables which have different distributions at even (J_e) and odd (J_o) sites. Using the Jordan-Wigner transformation, i.e., $a_j^{\pm} = S_j^x \pm iS_j^y$ and $c_i^{\dagger} = a_i^+ \exp[\pi i \Sigma_j^{i-1} a_j^+ a_j^-]$, and $c_i = \exp[\pi i \Sigma_j^{i-1} a_j^+ a_j^-] a_i^-$, this Hamiltonian is expressed in terms of fermion creation ($c_i^{\dagger}$) and annihilation (c_i) operators as[22]

$$\mathcal{H}_{\rm XX} = \sum_i \frac{1}{2}(J_i c_i^{\dagger} c_{i+1} + \mathrm{H.c.}). \tag{10}$$

The low-energy states of the model contain one fermion, which can be written in the form $|\psi\rangle = \Sigma_i \alpha_i c_i^{\dagger}|0\rangle$, where $|0\rangle$ denotes the fermionic vacuum. Energies in this one fermion subspace are obtained by the solution of the eigenvalue problem of $\mathcal{M}$ in Eq. (7) with the correspondence

$$t_i = J_i/2. \tag{11}$$

Then, the quantum control parameter of the model is just given by δ in Eq. (6). In the asymmetric model with $\delta>0$ ($\delta<0$), there is an enforced dimerization, and the system is in the random dimer phase[23] with preference of odd (even) bonds. On the other hand, at the quantum critical point with $\delta=0$ the system is in the so-called random singlet phase.[24]

C. Random transverse-field Ising chain

Our third and final model is the RTFIC, which is a prototypical model of random quantum systems having an order-disorder transition.[13] This system is defined by the Hamiltonian

$$\mathcal{H}_I = -\frac{1}{2}\sum_i \lambda_i \sigma_i^x \sigma_{i+1}^x - \frac{1}{2}\sum_{i=1}^{L} h_i \sigma_i^z \tag{12}$$

in terms of the Pauli-matrices $\sigma_i^{x,z}$ at site i, and the λ_i couplings and the h_i transverse fields are random numbers.

As for the XX chain, $\mathcal{H}_I$ is expressed in terms of fermion operators as[25]

$$\mathcal{H}_I = -\sum_i h_i\left(c_i^{\dagger}c_i - \frac{1}{2}\right) - \frac{1}{2}\sum_i \lambda_i (c_i^{\dagger} - c_i)(c_{i+1}^{\dagger} + c_{i+1}), \tag{13}$$

which is then diagonalized through a canonical transformation. Now, the low-energy excitations contain one free fermion, the possible energy of which is given by the positive eigenvalues of the following symmetric matrix:[9,26]

$$\mathcal{T} = \begin{pmatrix} 0 & h_1 & & & \\ h_1 & 0 & \lambda_1 & & \\ & \lambda_1 & 0 & h_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{14}$$

This is equivalent to $\mathcal{M}$ in Eq. (7) with the correspondences

$$t_{2i-1} = h_i, \quad t_{2i} = \lambda_i. \tag{15}$$

Using this relation together with Eq. (6), the control parameter of the RTFIC is given by the difference in the average log fields and the average log couplings. For $\delta > 0$ ($\delta < 0$), the system is in the paramagnetic (ferromagnetic) phase, and $\delta = 0$ represents the quantum critical point.

We can thus conclude that the low-energy properties of all the three models are related to the eigenvalue problem of $\mathcal{M}$ in Eq. (7). In the next section, we calculate the density of states of matrix $\mathcal{M}$ around $E=0$ by the Dyson-Schmidt method.

III. DENSITY OF STATES AT THE CENTER OF THE SPECTRUM

Here, in Secs. III A and III B, we recapitulate the basic ingredients of the Dyson-Schmidt method and present the solution in the continuum approximation. Our findings, which are obtained in the strongly disordered regimes, are presented in Secs. III C and III D.

A. Random walk method

In order to calculate the density of states of $\mathcal{M}$, we introduce a new vector $\vec{\Delta}$ with the components $\Delta_i = \alpha_{i-1} t_{i-1} / \alpha_i$, which satisfy the equation $\Delta_{i+1} = t_i^2/(E - \Delta_i)$. The basic ingredient of the Dyson-Schmidt method[19] is the *node counting theorem* of one-dimensional Hamiltonians, which states that the integrated density of states, $N(E) = \int_{-\infty}^{E} n(E')\mathrm{d}E'$, is given by the fraction of positive terms in the sequence of Δ_i. At the center of the band, $E=0$, the components of $\vec{\Delta}$ have alternating signs. Thus, here, the "sign variables" $s_i \equiv \mathrm{sgn}[\Delta_i(-1)^i]$ have a fully ordered state, $\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots$ and $N(0) = 1/2$. For nonzero E, the iterated equations for Δ_i are the following:[20]

$$\Delta_{2i} = f_{2i-2}\left(\frac{t_{2i-1}}{t_{2i-2}}\right)^2 \Delta_{2i-2},$$
$$f_{2i-2} = \frac{1 - E/\Delta_{2i-2}}{1 + (E\Delta_{2i-2} - E^2)/t_{2i-2}^2}, \tag{16}$$

which lead to different iteration behaviors for small positive E for various limiting values of Δ_{2i}. These are summarized as

$$\Delta_{2i+1}/\Delta_{2i} < 0 \quad \text{if} \quad \Delta_{2i} < E, \tag{17a}$$

$$f_{2i} = 1 \quad \text{if} \quad E \gg \Delta_{2i} \gg \tilde{t}^2/E, \tag{17b}$$

$$\Delta_{2i+2}/\Delta_{2i} < 1 \quad \text{if} \quad \Delta_{2i} \approx \tilde{t}^2/E, \tag{17c}$$

where $\tilde{t}$ denotes the typical (average) value of the matrix-element. According to Eq. (17b), we can identify an interval, $[E, \tilde{t}^2/E]$, in which the "signs" stay ordered, say, $s_i = \uparrow$. There is a finite upper boundary value at $\Delta_{\max} = \tilde{t}^2/E$, where the iterated sequence is reflected but s_i stays $\uparrow$ [see Eq. (17c)]. As the sequence arrives at the lower boundary value, $\Delta_{\min} = E$, the "spins" their change signs [see Eq. (17a)] and the iteration process starts again, however, in a new domain with $s_i = \downarrow$. Consequently, for a small $E > 0$, the sign variables have a fragmented domain structure $\cdots\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\cdots$ and therefore the fraction of positive terms in the sequence of Δ_i is somewhat larger than 1/2 due to extra positive terms appearing at the domain walls. If the typical (average) size of a domain is denoted by $\tilde{\ell}$, then the density of states is asymptotically given by

$$N(E) - N(0) = \frac{1}{2\tilde{\ell}}. \tag{18}$$

We can thus summarize that to obtain the density of states at the center of the spectrum it is enough to follow the evaluation of the sequence Δ_i within one typical domain and calculate its size $\tilde{\ell}$. Within this domain, we formally set $f_{2i-2} = 1$ in Eq. (16) and set (i) a reflecting boundary at $\Delta_{\max}$ and (ii) an absorbing boundary at $\Delta_{\min}$. If we introduce the logarithmic variable $\ln \Delta_{2i} = u_{2i}$, we obtain a random walk (directed polymer) problem,

$$u_{2i} = 2(\ln t_{2i-1} - \ln t_{2i-2}) + u_{2i-2}, \tag{19}$$

with reflecting ($u = u_{\max}$) and absorbing ($u = u_{\min}$) boundary conditions. In this language, the walker (polymer) starts at $u_0 = u_{\max}$ and its mean first-passage time (length) at the position $u_{\min}$ is just $\tilde{\ell}$; thus, $u_{\min} = u_{\tilde{\ell}}$.

B. Analysis in terms of the diffusion equation

In order to set the length scales in the random walk problem, we use a continuum approximation in which Eq. (19) is transformed into a diffusion equation, as follows:

$$\frac{\partial P(u,\ell)}{\partial \ell} = D\frac{\partial^2 P(u,\ell)}{\partial u^2} - v\frac{\partial P(u,\ell)}{\partial u}. \tag{20}$$

Here, $P(u,\ell)$ is the probability distribution of the walk, $D = 2[\mathrm{var}(\ln t_e) + \mathrm{var}(\ln t_o)]$ is the diffusion coefficient, and $v = 2([\ln t_o]_{\mathrm{av}} - [\ln t_e]_{\mathrm{av}})$ is the drift velocity. The typical size of the transverse fluctuations of the walk is given by $\tilde{u} = D/v = \delta^{-1}$, whereas the average distance between two reflections ξ follows from the relation $\tilde{u} \sim \sqrt{D\xi}$. Thus, we obtain for the correlation length the following:

$$\xi \sim D^{-1}\delta^{-2}, \tag{21}$$

which agrees with the result of SDRG calculations.[13] The continuum approximation and thus the use of the diffusion equation is justified if the correlation length is much larger than the lattice spacing. This condition is satisfied if we are either at the critical point $\delta = 0$ or in the weakly disordered Griffiths phase with $|\delta| \ll 1$.

1. Critical point

At the critical point, both the correlation length ξ and the typical size of transverse fluctuations $\tilde{u}$ are divergent and they are related to the length scale $\tilde{\ell}$ as $\xi \sim \tilde{\ell}$ and $\tilde{u} \sim \sqrt{D\tilde{\ell}}$. Absorption of the walker in this case is due to *typical fluc-*

tuations when $\tilde{u}$ grows to the order of the width of the strip: $\tilde{u} \sim \Delta u = u_{\max} - u_{\min} = \ln(\tilde{t}^2/E^2)$. From this follows

$$\tilde{\ell} \sim \frac{1}{D} \ln(\tilde{t}^2/E^2)^2, \tag{22}$$

so that

$$N(E) - N(0) \sim D[\ln(\tilde{t}^2/E^2)]^{-2}. \tag{23}$$

This is the classical result derived by Eggarter and Riedinger.[20]

2. Weakly disordered Griffiths phase

In the weakly disordered Griffiths phase with $1 \gg \delta > 0$, the walker is drifted toward the reflecting boundary and both the correlation length and the typical size of the transverse fluctuations are finite, but much larger than the lattice spacing. Thus, the continuum approximation is valid. In this case $\tilde{u}$ is much smaller than the width of the strip and its absorption takes place with a very small probability, $p(\Delta u) \propto \exp(-\frac{\Delta u}{\tilde{u}})$. Thus, it is a rare region effect and is due to *extreme fluctuations*. Before having such large fluctuations, the walker is reflected several times and the typical number of independent excursions is given by $\tilde{\ell}/\xi$, the value of which follows from extreme-value statistics:[27] $p(\Delta u)\tilde{\ell}/\xi = O(1)$. From this, we have

$$\tilde{\ell} \sim \xi \exp\left(\frac{v}{D} \ln(\tilde{t}^2/E^2)\right) \sim \left(\frac{\tilde{t}}{E}\right)^{1/z}, \tag{24}$$

with

$$\frac{1}{z} = \frac{2v}{D} = 2\frac{[\ln t_o]_{\rm av} - [\ln t_e]_{\rm av}}{[\mathrm{var}(\ln t_e) + \mathrm{var}(\ln t_o)]} = 2\delta. \tag{25}$$

Here, z is just the dynamical exponent defined in Eq. (1). In the weakly disordered Griffiths phase at the center of the band, there is a power-law singularity of the density of states,

$$N(E) - N(0) \sim \left(\frac{\tilde{t}}{E}\right)^{-1/z}, \tag{26}$$

which is equivalent to the form in Eq. (2). This result for the random antiferromagnetic XX chain was presented in Ref. 28.

C. Analysis in the strongly disordered Griffiths phase

In the strongly disordered Griffiths phase, the correlation length is in the order of the lattice spacings and the continuum approximation is not valid. In this case, we use discrete variables and denote the (nonlogarithmic) position of the walker at the jth step of the kth independent excursion, which starts at $r(k)$, as $\Delta_{2j}^{(k)}$. Thus, $r(k)/k = \xi$ for large k and the normalized position is given by

$$\rho_{2j}^{(k)} \equiv \frac{\Delta_{2j}^{(k)}}{\Delta_{\max}} = \prod_{j'=1+r(k)}^{j+r(k)} \left(\frac{t_{2j'-1}}{t_{2j'-2}}\right)^2. \tag{27}$$

The condition of absorption is formulated as

$$\min_k \min_{1<j<\Delta r(k)} \rho_{2j}^{(k)} = \frac{\Delta_{\min}}{\Delta_{\max}} = \frac{\tilde{t}^2}{E^2}, \tag{28}$$

where $\Delta r(k) = r(k+1) - r(k)$, which can be replaced by $\Delta r = \infty$. Keeping in mind that $\rho_{2j}^{(k)}$ is typically much larger than its minimum value, we can estimate the order of magnitude of the minimum as

$$\min_k \min_{1<j<\infty} \rho_{2j}^{(k)} \propto \min_k \left[y^{(k)} \equiv \sum_j (\rho_{2j}^{(k)})^{-1} \right]^{-1}. \tag{29}$$

Here, $y^{(k)}$ is a Kesten variable[29] for any k, the distribution function of which displays a singularity for large arguments,

$$p(y) \sim_{y\to\infty} y^{-(1+\mu)}, \tag{30}$$

where the exponent μ is given by the positive root of the equation

$$\left[\left(\frac{t_o^2}{t_e^2}\right)^{\mu} \right]_{\rm av} = 1. \tag{31}$$

(For a pedagogical introduction to the theory of Kesten variables see Appendix C of Ref. 2.) In this way, the typical number of excursions $\tilde{\ell}/\xi$ follows from extreme-value statistics,[27] $\tilde{\ell}/\xi \int_{y_{\max}}^{\infty} p(y)dy = 1$, and we obtain

$$\tilde{\ell} \sim \xi \left(\frac{t}{E}\right)^{2\mu}. \tag{32}$$

Comparing Eq. (32) to Eq. (24), we see that the dynamical exponent in the strongly disordered Griffiths phase is given by

$$\frac{1}{z} = 2\mu, \tag{33}$$

which, in the limit $\delta \ll 1$, gives back the result obtained in the weakly disordered Griffiths phase[11,15] in Eq. (25). Then, with the correspondence in Eq. (33), the density of states at the center of the band is given in Eq. (26).

D. Relation with the strong disorder renormalization group method

The density of states in the center of the spectrum of $\mathcal{M}$ can also be analyzed by the SDRG method,[2] and here we outline this procedure. The first step in this study is to arrange the matrix-elements t_i in descending order and use the largest one, $\Omega = \max_i\{t_i\}$, to set the energy scale in the system. Let us denote the largest term by t_j, which connects sites j and $j+1$ and eliminate the two equations in the eigenvalue problem that contains t_j. In the second-order perturbational method, which is correct up to $O[(t_{j-1}/t_j)^2]$ and $O[(t_{j+1}/t_j)^2]$, we have

$$t' \approx \frac{t_{j-1}t_{j+1}}{t_j} \tag{34}$$

for the effective matrix-element t' between the remaining sites, $j-1$ and $j+2$. This new term has a length, $m' = m_{j-1} + m_j + m_{j+1} = 3$, where the original matrix elements have unit lengths.

TABLE I. Analogous quantities in the random walk (RW) and in the SDRG methods.

Method	Independent unit	Length scale	Energy scale
RW	Excursion	Size of the excursion	$\min_j \rho_{2j}^{(k)}$
SDRG	Cluster	Size of the cluster	$t_o'(k)$

In the following steps, we repeat the decimation transformation, during which the energy scale is reduced, the lengths are increased, and the distribution functions of the matrix-elements, $R_e(t_e,\Omega)$ and $R_o(t_o,\Omega)$, approach their fixed-point form. This type of RG equations have been analytically solved both at the critical point[13] and in the Griffiths phase.[11,15] Here, we summarize the known results for the Griffiths phase with $\delta>0$.

In the starting steps of the RG, both t_e and t_o terms are decimated, but the transformation in later steps become asymmetric. As the typical lengths are growing beyond $m' \sim \xi$, almost exclusively, the t_o' terms are decimated and the t_e' terms become very small, such that at the fixed point, $\Omega \to \Omega^*=0$, we have $t_e'/t_o' \to 0$. As a consequence, the energy of the low-energy excitations is simply $E \simeq t_o'$. At the fixed point, the distribution of t_o is given by[11,15]

$$R_o(t_o,\Omega) = \frac{2\mu}{\Omega}\left(\frac{\Omega}{t_o}\right)^{1-2\mu}, \tag{35}$$

where μ is defined in Eq. (31). This is just equivalent to the distribution of the excitation energies in Eq. (2), with the dynamical exponent defined in Eq. (33).

Now, to make a correspondence with the random walk method, the starting RG steps which lead to an effective $t_e'(k)[\gg t_o'(k)]$ of length $m'(k) \sim \xi$ are equivalent to an excursion (between two reflections) of the walk of size $\Delta r(k) \sim \xi$ and the minimal value of $\rho_{2j}^{(k)}$ for this excursion is just the renormalized value of $t_o'(k)$. The analogous quantities in the two approaches are collected in Table I.

IV. THERMODYNAMIC SINGULARITIES

Here, we consider the random tight-binding model with half filling, as well as the random antiferromagnetic XX chain and the RTFIC, and we note that all these models are expressed in terms of free fermions. The common form of the Hamiltonians is given by

$$\mathcal{H}_F = \sum_q E_q(\eta_q^+ \eta_q - 1/2), \tag{36}$$

where E_q denotes the qth eigenvalue of $\mathcal{M}$ and η_q^+ (η_q) are fermion creation (annihilation) operators. The ground-state energy per site of this system is given by

$$\mathcal{E} = -\frac{1}{2L}\sum_q E_q = -\frac{1}{2}\int_{E_{\min}}^{E_{\max}} n(E) E dE, \tag{37}$$

and the free energy per site is given by

$$\mathcal{F} = -\frac{T}{L}\sum_q \ln\left[2\ \cosh\left(\frac{E_q}{2T}\right)\right] = -T\left\{\ln 2 + \int_{E_{\min}}^{E_{\max}} n(E)\ln\left[\cosh\left(\frac{E}{2T}\right)\right]dE\right\}, \tag{38}$$

where L is the length of the chain. From the free energy, we obtain the internal energy,

$$\mathcal{E}(T) = -\frac{1}{2}\int_{E_{\min}}^{E_{\max}} n(E) E \tanh\left(\frac{E}{2T}\right)dE, \tag{39}$$

and the specific heat,

$$c_v(T) = \int_{E_{\min}}^{E_{\max}} n(E)\left(\frac{E}{2T}\right)^2 \cosh^{-2}\left(\frac{E}{2T}\right)dE. \tag{40}$$

Now, using the form of the density of states at the center of the band, we obtain the following for the low temperature behavior:

$$c_v(T) \propto \mathcal{A} T^{1/z}\int_{-\infty}^{\infty} \varepsilon^{1/z+1} \cosh^{-2}\varepsilon d\varepsilon, \tag{41}$$

which is in agreement with the scaling result in Eq. (3). Note that the prefactor in Eq. (41), $\mathcal{A}$, is proportional to $\xi^{-1}z^{-1}$, which means that in the weakly disordered Griffiths phase, we have $\mathcal{A} \sim \delta^3$, $\delta \ll 1$, which is in agreement with the SDRG result.[13]

Next, we consider the random antiferromagnetic XX chain for which in the Hamiltonian in Eq. (9) we introduce a homogeneous ordering field, $H\Sigma_i S_i^z$. This term with fermionic variables assumes the form $H/2\Sigma_i(c_i^\dagger c_i - 1/2)$. Thus, the eigenvalue matrix $\mathcal{M}$ also contains diagonal elements $\mathcal{M}_{i,i} = H/4, \forall i$, and the eigenvalues are shifted by $E \to E + H/4$. The magnetization is obtained through the following differentiation:

$$m(H,T) = -\frac{\partial \mathcal{F}}{\partial H} \sim \int_{-H/4}^{H/4} n(E)\tanh\left(\frac{E}{2T}\right)dE, \tag{42}$$

where we have used the fact that the spectrum of $\mathcal{M}$ in Eq. (7) is symmetric to $E=0$. At zero temperature, $m(H,0)$ is singular for small H,

$$m(H,0) \sim N(H/4) - N(-H/4) \sim H^{1/z}, \tag{43}$$

as in Eq. (4). However, in evaluating the integral in Eq. (42) for small H and T, with $H/T = O(1)$ we obtain the following for the low-temperature susceptibility:

$$\chi(T) \sim T^{1/z-1}, \tag{44}$$

which corresponds to the scaling result in Eq. (3).

V. DISCUSSION

In this paper, we have studied Griffiths-McCoy singularities in random quantum (tight-binding, XX, and Ising) spin chains, which can be represented in terms of free fermions. The main step of our investigation is the calculation of the density of states of the low energy excitations, in which ex-

TABLE II. Relation between the length-scales in different regimes of the quantum control parameter.

Critical point	$\delta=0$	$\tilde{\ell}\sim\xi\gg a$
Weakly disordered Griffiths	$\lvert\delta\rvert\ll 1$	$\tilde{\ell}\gg\xi\gg a$
Strongly disordered Griffiths	$\lvert\delta\rvert=O(1)$	$\tilde{\ell}\gg\xi\sim a$

citations are eigenvalues of a symmetric tridiagonal matrix with random entries with, however, an odd-even asymmetry. This latter problem is solved exactly by the Dyson-Schmidt technique[19,20] for any value of the quantum control parameter δ. Previous studies of this problem are restricted to the quantum critical point,[20] $\delta=0$, and to the weakly disordered Griffiths phase,[28] $\delta\ll 1$.

As we described in Sec. III B, in this problem there are three length scales: the mean-first passage length $\tilde{\ell}$, the correlation length ξ, and the lattice spacing a. In the different regimes of the quantum control parameter, their relative magnitudes are summarized in Table II.

In a finite system, there is still another length scale given by the size of the system L, and the mean-first passage length cannot exceed this value: $\tilde{\ell}\sim L$. Consequently, the lowest excitation energy is limited to $E_1\sim L^{-z}$. In this case, one is interested in the distribution of the scaling combination E_1L^z, which, in the random walk method in Sec. III C, is obtained from the statistics of extremes. Here, we recall that E_1 is just the minimum value of a set of L/ξ independent random numbers, each having the same parent distribution in a power-law form [see Eq. (30)]. Consequently, the distribution of $\epsilon_1=\alpha E_1L^z$ in the large L limit follows the Fréchet distribution,[27]

$$\tilde{P}_1(\epsilon_1)=\frac{1}{z}\epsilon_1^{1/z-1}\exp(-\epsilon_1^{1/z}), \tag{45}$$

where α is a nonuniversal constant which depends on the amplitude of the tail in Eq. (30). Here, one can go on and consider the second eigenvalue E_2 or, more generally, the qth smallest eigenvalue E_q. These are all obtained from the theory of extreme value statistics of independent and identically distributed (IID) random numbers and their distribution is given by the generalized Fréchet distribution [see Ref. 27]. In this way, we have shown that the distribution of the lowest energy levels of these strongly correlated physical systems is described in a form that holds for IID random numbers. This scenario, which is exactly shown here for the specific models, is expected to hold generally for all such random quantum systems, even in higher dimensions, for which the low-energy behavior is controlled by the so-called strong disorder fixed point in the SDRG framework.[30]

The dynamical exponent z, which is calculated exactly in this paper, is found to be a continuous function of the control parameter δ. Using the SDRG approach, the same result is obtained.[11,15] Thus, our present study gives further credit to the conjecture that the SDRG method asymptotically provides exact results even far outside the critical point, as far as dynamical quantities are considered. This latter statement is expected to hold for all systems with a strong disorder fixed point.

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