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Entanglement entropy in the two-dimensional random transverse field Ising model

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The scaling behavior of the entanglement entropy in the two-dimensional random transverse field Ising model is numerically studied through the strong disordered renormalization group method. We find that the leading term of the entanglement entropy always linearly scales with the block size. However, besides this *area law* contribution, we find a subleading logarithmic correction at the quantum critical point. This correction is discussed from the point of view of an underlying percolation transition, both at finite and at zero temperature.

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The study of novel quantum phases and related quantum phase transitions is at the forefront of many recent developments in condensed matter physics. It heavily relies on the concept of entanglement entropy.

A state $|\Phi\rangle$ of a bipartite system $A\cup B$ is entangled if it cannot be accurately described in either subsystem A or B. A convenient measure of this entanglement is the entropy, $S_A=-\mathrm{Tr}\,\rho_A \log_2 \rho_A$, where $\rho_A=\mathrm{Tr}_B|\Phi\rangle\langle\Phi|$. Denote the linear dimensions of $A\cup B$ and A as M and L, respectively. An important question in quantum many-body systems is to study how $S_A(L)$ scales with L in the limit of $M\to\infty$ in different quantum phases. This question has been extensively investigated in one dimension.[1–7] There, it is now well understood that for noncritical systems, $S(L)$ saturates to a constant as $L\to\infty$, whereas in critical systems, a logarithmic modification stands out as the leading term: $S(L)\sim \ln L$, and its coefficient is associated with the central charge of the related (1+1) conformal field theory (CFT).[7] In higher dimensions, it is generally believed that an *area law* holds at least for noncritical systems: the entanglement entropy scales as the area of the boundary between subsystem A and B, $S(L)\sim L^{d-1}$. This has been confirmed by studies on bosonic harmonic lattice systems.[6,8] For critical systems, the area law is shown to be violated in free fermion systems with a finite Fermi surface.[9–11] However, it still holds for fermionic systems without a finite Fermi surface[10,11] and critical bosonic systems.[11]

In d-dimensional system where the area law holds, $S(L)\sim f_s L^{d-1}$. Here, f_s is a boundary free energy determined by the short-distance properties of the system and is hence not universal. It is thus interesting to wonder about subleading[12] terms in $S(L)$, where universal coefficients depending only on the model and the topological properties of the system may appear. For instance, it was recently found that in two-dimensional (2D) gapped systems, a subleading constant term in $S(L)$ is related to the topological order.[13] Also, for a class of $z=2$ conformal quantum critical systems in 2D, a universal logarithmic correction to the area law term has been found.[14] Clearly, the problem is not fully settled.

It is of course also possible to investigate entanglement in quantum disordered systems. In a series of studies in one-dimensional (1D) based on strong disorder renormalization group (SDRG) techniques,[17] it was found that for the class of 1D infinite randomness fixed points (IRFPs), a $\ln L$ term in $S(L)$ is also present.[3,15] Similar results were also discovered for 1D aperiodic systems.[16]

In this Rapid Communication, we report on our study of the 2D random transverse field Ising (RTFI) model, and the numerical calculation of the entanglement entropy using the SDRG technique. The model is defined on a 2D square lattice with linear dimension M and open boundary condition. The subsystem A is an $L\times L$ square region located in the center of the square lattice. The Hamiltonian reads

$$H=-\sum_{\langle i,j\rangle} J_{ij}S_i^zS_j^z-\sum_i h_iS_i^x. \tag{1}$$

The Ising coupling J_{ij} and the transverse field h_i take random values drawn from the following box shape distributions: $P(J)=\Theta(J)-\Theta(J-1)$, $P(h)=\frac{1}{h_0}[\Theta(h)-\Theta(h-h_0)]$.

This model is known to have a quantum phase transition which is governed by an IRFP.[18,19] Here, the critical point is tuned by h_0. Starting from the original Hamiltonian [Eq. (1)], the SDRG finds the ground state by successively eliminating the highest energy degrees of freedom.[17,20] At each renormalization group (RG) step, we look for the largest term in the Hamiltonian; its coupling (or field) is defined as the energy scale Ω at this step. As illustrated in Fig. 1, there are two basic decimation procedures. If $\Omega=h_i$, the local spin is frozen in the eigenstate of S_i^x and is eliminated from the system. The renormalized coupling between its two neighboring spins at sites j and k is thus $J'_{jk}\approx\max(J_{jk},J_{ji}J_{ik}/h_i)$. If $\Omega=J_{ij}$, the two spins involved respond to the field uniformly and are combined into a new effective spin (a cluster). The local field at this effective spin is $h'_i=h_ih_j/J_{ij}$. Numerically, the RG is processed until only one cluster is left in the system. The ground state then consists of independent clusters each of which is frozen into a *GHZ* state: $|C(n)\rangle=\frac{1}{\sqrt{2}}(|\uparrow\rangle^{\otimes n}+|\downarrow\rangle^{\otimes n})$. Since each *GHZ* state will contribute either 1 (we

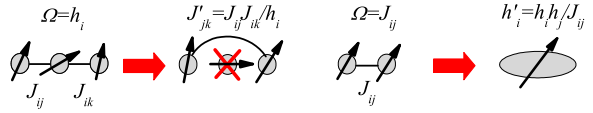

FIG. 1. (Color online) Basic RG transformations (see text for details). Left: energy scale is a field and right: energy scale is an Ising coupling.

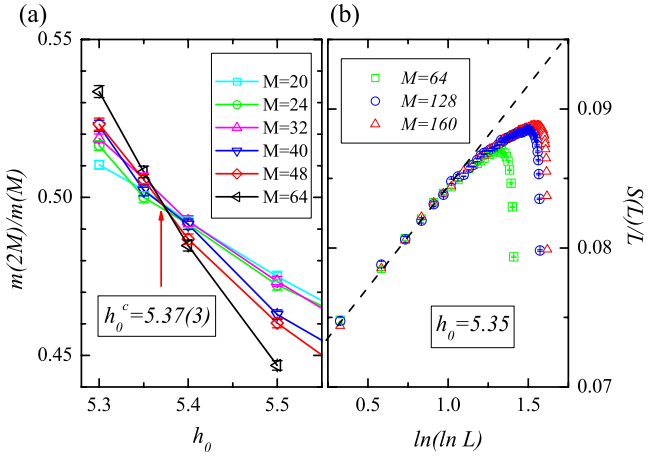

FIG. 2. (Color online) (a) Finite-size scaling of magnetization ratio given in Eq. (2). (b) Entropy per surface $S(L)/L$ vs $\ln \ln L$ at critical field $h_0^c=5.35$. The dashed line is a linear fit in $\ln \ln L$ scale. The deviation from $\ln \ln L$ is clearly seen in systems with $M \geqslant 128$ for $40 \lesssim L \lesssim 80$, where $S(L)/L$ scales almost independently of M.

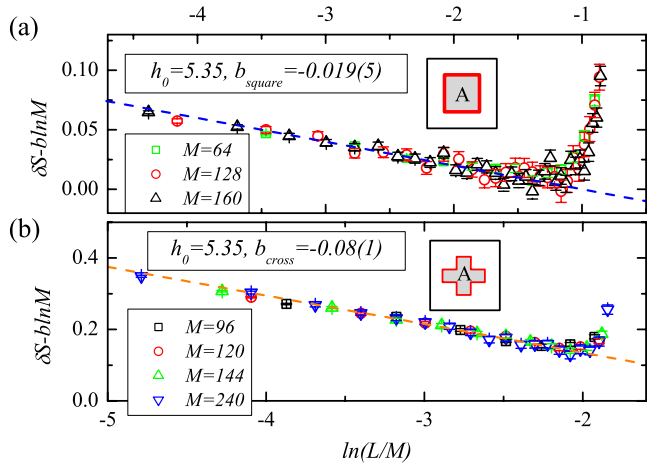

FIG. 3. (Color online) (a) Scaling plots of $\delta S(L)$ to reveal the subleading term of $S(L)$ at critical field $h_0^c=5.35$ for different geometries of subsystem A: a square [in (a)] and a cross shape [in (b)]. The dashed lines are linear fits in $\ln L$ scale with $b_{square}=-0.019$ and $b_{cross}=-0.08$, respectively.

take logarithms in base two) to the entropy if it consists of degrees of freedom in both subsystems A and B, or 0 otherwise, calculating the entanglement entropy between two subsystems is reduced to a pure *cluster counting* problem: $S(L)$ is proportional to the number of clusters $N(L)$ that cross the boundary between the two subsystems. Finally, $S(L)$ is averaged over 10^5-10^6 different disorder configurations to ensure a sufficient representation of rare realizations.

The foregoing technique has been applied with success in the 1D case. The technical difference in 2D is first that the quantum critical point is not exactly known. To locate it accurately we study the scaling behavior of the average magnetization $m(M)$.[21] At the critical field h_0^c,

$$\left.\frac{m(2M)}{m(M)}\right|_{h_0=h_0^c} = 2^{-x_m} \tag{2}$$

is independent of M, where x_m is the anomalous dimension of the bulk magnetization. Our result is given in Fig. 2(a). The critical field is estimated to be $h_0^c=5.37\pm0.03$, with $x_m=1.01\pm0.05$, which is consistent with a previous RG study.[22] The entropy $S(L)$ is calculated for various values of h_0. For both critical and noncritical h_0, we find that the area law holds: $S(L)\sim aL$ in the leading term. The result of $S(L)/L$ for different system sizes at critical $h_0^c=5.35$ based on 10^6 realizations of random configurations is shown in Fig. 2(b). At critical the coefficient $a=0.089\pm0.002$, and is independent of system sizes and the number of realizations.

This conclusion is quite different from the one in a recent study,[23] where a very weak double-logarithmic modification of the area law in the same model was reported at the critical point. We find that for small systems the double-logarithmic fit is reasonable, but that for system size $M \geqslant 128$, $S(L)/L$ definitely increases slower than $\ln \ln L$ for $L \gtrsim 40$, strongly suggesting $S(L)\sim L$ in the limit of $L\to\infty$, without modification, and that the observation of Ref. 23 is biased by finite size effects. We also note that our results are largely independent of the distribution of couplings, confirming the idea of universal behavior for $S(L)$.

Having established the validity of the area law in this system, it is natural to investigate subleading terms. We thus consider $\delta S(L) \equiv 2S(L)-S(2L)$, in which the terms linear in L exactly cancel. We find in both disordered and ordered phases that $\delta S(L)$ saturates to a constant term, indicating $S(L)=aL+c$. Meanwhile, at critical, we find that $\delta S(L)$ scales linearly as $\ln L$, suggesting

$$S(L) = aL + b \ln L + c, \tag{3}$$

i.e., a logarithmic correction to the area law. The coefficient of this logarithmic correction is determined to be $b=-0.019\pm0.005$ through finite-size scaling in Fig. 3(a). It is interesting to note that a similar logarithmic correction was also found in Ref. 14 for a class of conformal critical models with dynamical exponent $z=2$. However, there is no reason to expect that the $\ln L$ term we find in the critical RTFI model has much to do with the one in Ref. 14. This can be substantiated by calculating the amplitudes of the logarithmic term for different geometries, which obey some precise relations in the case of Ref. 14. As an example, we considered a cross shape geometry, as shown in Fig. 3(b). In this case as well, we can resolve a $\ln L$ term in $S(L)$ in addition to the area law contribution, with the coefficient $b_{cross}=-0.08\pm0.01$. We can then calculate and compare the ratios in our model, where we obtain $b_{cross}/b_{square}\approx4$, and in the conformal quantum critical models, where $b_{cross}/b_{square}=3$ exactly. This implies that the $\ln L$ term in $S(L)$ in our model most probably has a different origin, an unsurprising conclusion since, for the IRFP, $z\to\infty$.

To better understand the $\ln L$ term in $S(L)$ at the IRFP in the 2D RTFI model, we notice that there is a striking difference between the model in 2D and in 1D. In 2D, for any $h_0<h_0^c$, there is a finite-temperature phase transition at $T_c(h_0)$ (Ref. 18): the IRFP in 2D can then be considered as an extension of this finite-temperature transition right down

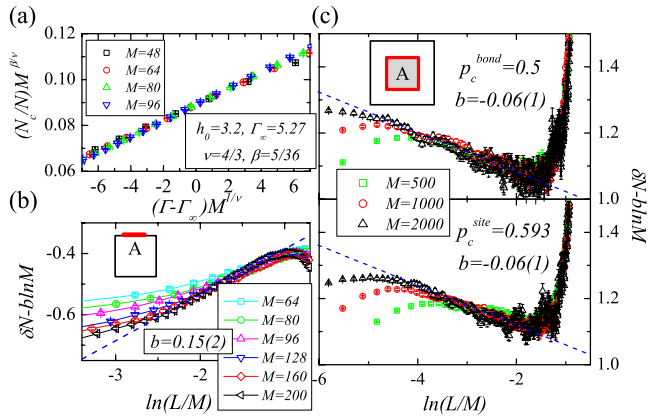

FIG. 4. (Color online) (a) Scaling of the largest active cluster size N_c during RG shows a signature of classical percolation at a finite energy scale. (b) Scaling of δN at percolation threshold $\Gamma_\infty=5.27$ for $h_0=3.2$; A is a line interval on the boundary. (c) Scaling of δN at bond (upper) and site (lower) classical percolation thresholds; A takes the geometry of a square. In (b) and (c), the dashed lines are linear fits in $\ln L$ scale with corresponding slope b marked on the plot.

to $T=0$. Through the SDRG, the transition to a ferromagnetically ordered phase can be mapped to a percolation transition in 2D:[19] the magnetic transition corresponds to the development of an infinite percolating spin cluster during RG. It is widely expected that this percolation process at the IRFP (which occurs at energy scale $\Omega_\infty=0$) is different from the one at finite temperature since at $h_0=h_0^c$, the critical behavior is controlled by quantum fluctuations. This leads one to think of the IRFP as a type of "quantum percolation," with fractal dimension $d_f=2-x_m\approx 1.0$. For $h_0<h_0^c$ meanwhile, the percolation takes place at finite energy scale $\Omega_\infty\sim T_c$ in the RG and is expected to be in the universality class of conventional classical percolation.[19] We have confirmed this picture at finite energy scale by studying the scaling of the largest active cluster size during RG. Some numerical results at $h_0=3.2$ are presented in Fig. 4(a). The percolation threshold is at $\Gamma_\infty=\ln(\Omega_0/\Omega_\infty)=5.27\pm0.02$, where the extracted exponents β and ν take the values of classical percolation indeed.

The number of clusters crossing the boundary between two subsystems can be investigated at these percolation transitions as well (even though it does not correspond to an entanglement entropy except when $h_0=h_0^c$). It is easy to see then that a $\ln L$ correction to the area law is expected and related to conformal invariance, even though it depends on more complicated parameters than the central charge and the topology (in contrast to the example in Ref. 14).

To see this, consider the "baby" case where the subsystem A takes the geometry of a line interval of length L on the boundary of the lattice, so the boundary between the two subsystems is A itself. Now, $N(L)$ is simply equal to the number of clusters touching A. Its scaling can be studied by using CFT techniques. For this, consider first the problem on the upper half complex plane, with A on the real axis. Use the well known expansion of the partition function of the critical Q-state Potts model in terms of clusters, or equivalently, dense loops,[24]

$$Z=\sum_P \sqrt{Q}^{N_P}, \tag{4}$$

where every loop gets the same weight $\sqrt{Q}$. Now, introduce a boundary condition changing operator[25] (BCCO) ϕ_y such that the two point function of ϕ is defined through a sum similar to the one for Z, but loops touching the boundary between the two insertions ζ_1 and ζ_2 get a different weight y instead of $\sqrt{Q}$,

$$\langle\phi_y(\zeta_1)\phi_y(\zeta_2)\rangle=\frac{1}{Z}\sum\sqrt{Q}^{N_P}\left(\frac{y}{\sqrt{Q}}\right)^{N_P^b(\zeta_1,\zeta_2)}, \tag{5}$$

where $N_P^b(\zeta_1,\zeta_2)$ gives the number of loops touching the boundary between ζ_1 and ζ_2 located on the real axis. We expect the two point function to have the following scaling form:

$$\langle\phi_y(\zeta_1)\phi_y(\zeta_2)\rangle\sim e^{-f(y)|\zeta_1-\zeta_2|}|\zeta_1-\zeta_2|^{-2h(y)}, \tag{6}$$

where $f(y)$ is the boundary free energy induced by the modified weight on the boundary, and the exponent $h(y)$ is the anomalous dimension of the BCCO. Now, differentiate the two point function of BCCOs with respect to the weight y, then take the limit $y=\sqrt{Q}$. This leads to

$$N_P^b(L)=aL+b\ln L, \tag{7}$$

where

$$b=-2\sqrt{Q}\left.\frac{\partial h(y)}{\partial y}\right|_{y=\sqrt{Q}}, \tag{8}$$

and $L=|\zeta_1-\zeta_2|$.

With the exact expression of $h(y)$,[26] we obtain

$$b=\frac{1}{2\pi p}\sqrt{Q(4-Q)}. \tag{9}$$

For percolation ($Q=1$), $b_{perc}=\frac{\sqrt{3}}{4\pi}\approx0.1378$ reproduces an early result by Cardy.[27] However, in Eq. (9), we generalize Cardy's result to general Q, and it is interesting to see that the $\ln L$ term vanishes at $Q=0$ and $Q=4$. It is also remarkable to see that b is related to the derivative of the anomalous dimension of BCCO, but not the central charge. This $\ln L$ term is also observed in our RG calculation at finite Γ_∞. In Fig. 4(b), we show the scaling of $\delta N\equiv 2N(L)-N(2L)$ at $\Gamma_\infty=5.27$ for $h_0=3.2$. b is estimated to be 0.15 ± 0.02, which is in agreement with the analytical result. This further confirms that the universality class at finite Γ_∞ is classical percolation. Interestingly, we find numerically for this case that $|b|<0.01$ at the quantum critical point, which is consistent again with the idea of a different universality class when $\Gamma_\infty\to\infty$.

Going back to the original problem where the subsystem A takes the geometry of a $L\times L$ square, we have not derived a similar analytical result for ordinary percolation. However, $N(L)$ can of course be calculated numerically. To get better scaling for large systems, we turn to a direct study of percolation. In Fig. 4(c), δN data at the percolation threshold are shown. As well expected, $\delta N\sim\ln L$ is resolved for both bond and site percolation, and the coefficient of the $\ln L$ term takes

the same value $b=-0.06\pm0.01$, which is in agreement with the idea that this term is universal. Note that we observe a negative value of b for subsystem A a square, just as in the case of the RTFI model. This is opposite to the sign of b in classical percolation when A is an interval.

The observation of a $\ln L$ term in $N(L)$ in percolation makes the presence of a similar term at the IRFP most likely: there will in fact always be such a term at energy scale Ω_∞. When this scale is finite, the coefficient b takes the value of classical percolation, $b=-0.06\pm0.01$. However, when $\Omega_\infty\to0$, i.e., at the quantum critical point, quantum fluctuations become dominant, leading to a quantum percolation belonging to a different universality class. A different b value, $b=-0.019\pm0.005$, reflecting this difference is then observed.

In summary, we have calculated the entanglement entropy of a 2D RTFI model by using a numerical SDRG method. In contrast to what is claimed in a recent paper, we find that the leading term of the entropy follows the *area law* and depends linearly on the block size L in both critical and noncritical phases. However, a $\ln L$ correction to the area law is discovered at criticality. While the presence of this correction may not have been expected from the entanglement point of view, it is very natural once the problem is geometrically reformulated. Indeed, the problem of counting clusters touching a boundary in 2D classical percolation is easily argued to give rise to sublogarithmic corrections, while the entanglement entropy in the RTFI model at criticality can be reformulated as a similar problem but in a different, "quantum percolation" universality class.

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