

Ballistic propagation of thermal excitations near a vortex in superfluid $^3\text{He-B}$

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Andreev scattering of thermal excitations is a powerful tool for studying quantized vortices and turbulence in superfluid $^3\text{He-B}$ at very low temperatures. We write Hamilton's equations for a quasiparticle in the presence of a vortex line, determine its trajectory, and find under which conditions it is Andreev reflected. To make contact with experiments, we generalize our results to the Onsager vortex gas and find values of the intervortex spacing in agreement with less rigorous estimates.

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I. MOTIVATION

Superfluid turbulence consists of a disordered tangle of quantized vortex filaments which move under the velocity field of each other.^{1,2} If the temperature T is sufficiently smaller than the critical temperature T_c , then the normal fluid can be neglected and the vortices do not experience any friction effect.³ The simplicity of the vortex structures (discrete vortex lines) and the absence of dissipation mechanisms, such as friction and viscosity, make superfluid turbulence a remarkable fluid system, particularly when compared to turbulence in ordinary fluids. Current experimental, theoretical, and numerical investigations attempt to determine the similarities and the dissimilarities between superfluid turbulence and ordinary turbulence. Questions which are currently addressed concern (i) the existence of a Kolmogorov energy cascade at length scales larger than the typical intervortex spacing,^{4,5} (ii) the existence of a Kelvin wave cascade at length scales smaller than the Kolmogorov length^{6–9} followed by (iii) acoustic emission at even shorter length scales,^{10,11} (iv) the possible existence of a bottleneck^{12,13} between the Kolmogorov cascade and the Kelvin wave cascade, (v) the nature of the fluctuations of the observed vortex line density^{14–17} and (vi) their decay,^{18,19} (vii) whether there are two forms of turbulence,²⁰ a structured one, which consists of many length scales (Kolmogorov turbulence), and an unstructured, more random one (Vinen turbulence), and (viii) the effects of rotation on turbulence.^{21–24} Most of these questions refer to the important limit $T/T_c \ll 1$, where fundamental distinctions between a perfect Euler fluid and a superfluid becomes apparent.²⁵

Superfluid turbulence experiments are currently performed in both ^4He (Refs. 14, 15, and 26–28) and in $^3\text{He-B}$.^{17,21,29,30} In the last few years, it has been recognized that, to make progress in answering the above questions, it is necessary to develop better measurement techniques which are suitable for turbulence in quantum fluids. In ^4He , the application of the classical particle image velocimetry (PIV) method^{31–33} was a breakthrough. In $^3\text{He-B}$, a nonclassical, powerful measurement technique which is suitable in the limit $T/T_c \ll 1$ is the Andreev scattering,²⁹ developed at the University of Lancaster.

This paper is concerned with the Andreev scattering. The plan of the paper is the following. In Sec. II, we shall de-

scribe the basic ideas behind the Andreev scattering and review what is a quantized vortex line. In Sec. III, we shall write down the governing equations of motion. In Sec. IV, we shall determine the ballistic trajectories of excitations in the vicinity of the velocity field of a vortex line, and, in Sec. V, we shall study the transport of heat by ballistic quasiparticles through a tangle of vortices. Section VI will apply our result to the current experiments. Finally, in Sec. VII, we shall draw the conclusions.

II. ANDREEV SCATTERING AND QUANTIZED VORTICES

The study of the motion of quasiparticle excitations in a superfluid was pioneered by Andreev.³⁴ Consider an excitation which moves in the direction of increasing excitation gap. The excitation propagates at constant energy and gradually reaches the minimum of the rising excitation spectrum, where its group velocity becomes zero. Thereafter, it retraces its path but as an excitation on the other side of the minimum. An incoming quasiparticle is thus reflected as a quasihole and an incoming quasihole is reflected as a quasiparticle. The effect is a consequence of the fact that the minimum of the energy spectrum of the excitation lies at nonzero momentum.

The case of p -wave triplet pairing appropriate to superfluid ^3He has been discussed by various authors studying the interaction of excitations with the boundaries,³⁵ motion of quasiparticles through the A - B phase boundary in ^3He ,³⁶ ballistic motion of quasi-particle in slow varying textural field of $^3\text{He-A}$,³⁷ scattering of ballistic quasiparticles in $^3\text{He-B}$ by a moving solid surface,^{38,39} and calculation of the friction force on quantized vortices.^{40,41} References 41 and 42 are concerned with Andreev reflection within the vortex core and therefore apply to the bound states. Our concern is the propagation of thermal excitations outside vortex cores.

Collisions between the quasiparticles can cause some spreading of the incoming beam. However, the spreading can be made arbitrary small by lowering the density of the excitations, that is, to say, by lowering the temperature. At low enough temperatures, the mean free path exceeds the dimensions of the experimental cell, and we can consider undamped excitations moving along straight paths until they hit

a boundary or any potential barrier, particularly a barrier formed by a vortex. Andreev reflection of excitations thus gives the opportunity to probe flows in superfluid ^3He at ultralow temperatures. The most fruitful and promising application of Andreev scattering is thus superfluid turbulence in $^3\text{He-B}$ in the low temperature limit, that is, to say, for $T/T_c \leq 0.4$ K.^{17,29,43–45}

Superfluid $^3\text{He-B}$ is described by a macroscopic wave function, called the order parameter, with a well defined phase φ . The superfluid velocity \mathbf{v}_s is proportional to the gradient of the phase,

$$\mathbf{v}_s = \frac{\hbar}{2m} \nabla \varphi, \quad (2.1)$$

where m is the mass of one ^3He atom. Consequently, in contrast to classical fluids, superfluid motion is irrotational and vorticity exists only in the form of quantized vortices. Quantized vortices are line defects around which the phase φ changes by 2π . The superfluid order parameter is distorted within the relatively narrow core of the vortex, and the superfluid flows around the core with speed which is inversely proportional to the distance from the vortex core. Since both the real and the imaginary parts of the order parameter are zero on the axis of a vortex, vortex lines can be considered as topological defects. Vortices cannot terminate in the middle of the flow, so they are either closed loops or extend to the walls.

Superfluid turbulence consists of a tangle of quantized vortices. The complex flow field within the tangle acts as a potential barrier for quasiparticles, causing the Andreev reflection of a fraction of a beam of thermal excitations incident upon the tangle. The use of Andreev scattering as a visualization technique of ultralow temperature turbulence requires finding out exactly what happens to a single quasiparticle which moves in the velocity field of a vortex, which is what we set out to do.

III. EQUATIONS OF MOTION OF THERMAL EXCITATIONS

Our first aim is to formulate, in the (x, y) plane, the equations of motion of a single excitation moving in the velocity field of a single straight vortex which we assume to be fixed and aligned along the z axis. We are thus concerned with a two-dimensional problem only. The quantities (here and below, the numerical values of the quantities are taken at the 0 bar pressure⁴⁶) which are necessary to describe the motion of the excitation are the Fermi velocity, $v_F \approx 5.48 \times 10^3$ cm/s, the Fermi momentum, $p_F = m^* v_F \approx 8.28 \times 10^{-20}$ g cm/s, and the Fermi energy, $\epsilon_F = p_F^2 / (2m^*) \approx 2.27 \times 10^{-16}$ erg. The quantity

$$\epsilon_p = \frac{p^2}{2m^*} - \epsilon_F \quad (3.1)$$

is the “kinetic” energy of the excitation measured with respect to the Fermi energy ϵ_F , where $m^* \approx 3.01 \times m = 1.51 \times 10^{-23}$ g is the effective mass of the excitation, and \mathbf{p} the momentum, $p = |\mathbf{p}|$. Let Δ_0 be the magnitude of the superfluid

energy gap. Near the vortex axis, at radial distances r smaller than the zero-temperature coherence length $\xi_0 = \hbar v_F / \pi \Delta_0 \approx 0.8 \times 10^{-5}$ cm, the energy gap falls to zero and can be approximated by $\Delta(r) \approx \Delta_0 \tanh(r/\xi_0)$.^{47,48} Since we are mainly concerned with what happens for $r \gg \xi_0$, we neglect the spatial dependence of the energy gap and assume the constant value $\Delta_0 = 1.76 k_B T_c \approx 2.43 \times 10^{-19}$ erg.

Using polar coordinates (r, φ) in the (x, y) plane, the velocity field of a superfluid vortex set along the z axis is

$$\mathbf{v}_s = \frac{\kappa}{2\pi r} e_\varphi, \quad (3.2)$$

where

$$\kappa = \frac{h}{2m} = \frac{\pi \hbar}{m} = 0.662 \times 10^{-3} \text{ cm}^2/\text{s} \quad (3.3)$$

is the quantum of circulation and e_φ is the unit vector in the azimuthal direction on the (x, y) plane.

In the presence of the vortex, the energy of the excitation becomes

$$E = \sqrt{\epsilon_p^2 + \Delta_0^2} + \mathbf{p} \cdot \mathbf{v}_s. \quad (3.4)$$

In writing Eq. (3.5), the spatial variation of the order parameter is not taken into account for the sake of simplicity. We also assume that the interaction term $\mathbf{p} \cdot \mathbf{v}_s$ varies on a spatial scale which is larger than ξ_0 and that the excitation can be considered a compact object of momentum $\mathbf{p} = \mathbf{p}(t)$, position $\mathbf{r} = \mathbf{r}(t)$, and energy $E = E(\mathbf{p}, \mathbf{r})$. This gives us the opportunity to use the method developed in Ref. 37, and consider Eq. (3.5) as an effective Hamiltonian, for which the equations of motion are

$$\frac{d\mathbf{r}}{dt} = \frac{\partial E(\mathbf{p}, \mathbf{r})}{\partial \mathbf{p}} = \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta_0^2} m^*} \mathbf{p} + \mathbf{v}_s, \quad (3.5)$$

$$\frac{d\mathbf{p}}{dt} = - \frac{\partial E(\mathbf{p}, \mathbf{r})}{\partial \mathbf{r}} = - \frac{\partial}{\partial \mathbf{r}} (\mathbf{p} \cdot \mathbf{v}_s). \quad (3.6)$$

Equations (3.6) and (3.7) have one immediate integral of motion, the energy

$$E(\mathbf{p}, \mathbf{r}) = E = \text{constant}. \quad (3.7)$$

Equation (3.6) represents the group velocity of the excitation in the velocity field of the vortex. Excitations such that $\epsilon_p > 0$ are called quasiparticles, and excitations such that $\epsilon_p < 0$ are called quasiholes. The right-hand side of Eq. (3.6) is thus the force acting on the excitation.

IV. PROPAGATION OF EXCITATION IN THE VELOCITY FIELD OF A VORTEX

We want to determine the trajectory of an excitation which moves in the two-dimensional velocity field of the vortex. It is convenient to rewrite the Hamiltonian, Eq. (3.5), and Hamilton’s equations (3.6) and (3.7) in polar coordinates (r, φ) . We notice that the system consisting of a single excitation and a single vortex has a second integral of motion:

the component of the angular momentum in the z direction, perpendicular to the plane of motion, (x, y) . Consequently, we can introduce two pairs of canonically conjugated variables, $(p_r; r)$ and $(J=p_\varphi r; \varphi)$, where p_r and p_φ are the radial and azimuthal components of \mathbf{p} and J is the angular momentum. Since J is constant, it is convenient to write it in the form $J=p_F\rho_0$, thereby defining the constant ρ_0 for a particular trajectory. Under special initial conditions, as we shall see, ρ_0 can be interpreted as the impact parameter.

Equations (3.4), (3.1), and (3.5) become

$$E = \sqrt{\epsilon_p^2 + \Delta_0^2} + p_F\rho_0 \frac{\kappa}{2\pi r^2}, \quad (4.1)$$

$$\epsilon_p = \frac{p_r^2}{2m^*} + \frac{(p_F\rho_0)^2}{2m^*r^2} - \epsilon_F, \quad (4.2)$$

$$\dot{r} = \frac{dr}{dt} = \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta_0^2}} \frac{p_r}{m^*}, \quad (4.3)$$

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta_0^2}} \frac{p_F\rho_0}{m^*r^2} + \frac{\kappa}{2\pi r^2}. \quad (4.4)$$

By setting $dE/dt=0$ and using Eq. (4.3), we find

$$\epsilon_p = \frac{d\epsilon_p}{dt} = p_F\rho_0 \frac{p_r}{m^*} \frac{\kappa}{\pi r^3}, \quad (4.5)$$

and from Eq. (4.2), we have

$$p_r = p_F \left(1 + \frac{\epsilon_p}{\epsilon_F} - \frac{\rho_0^2}{r^2} \right)^{1/2}. \quad (4.6)$$

Equations (4.1)–(4.6) form a closed set which allows us to determine the trajectory of the excitation.

It is apparent from Eq. (4.3) that a quasiparticle incident upon the vortex has $\epsilon_p > 0$ and $p_r < 0$, whereas a quasiparticle moving away from the vortex has $\epsilon_p > 0$ and $p_r > 0$. Vice versa, a quasihole incident upon the vortex is characterized by $\epsilon_p < 0$ and $p_r > 0$, whereas a quasihole moving away from the vortex has $\epsilon_p < 0$ and $p_r < 0$.

Later, we shall consider a quasiparticle which leaves a point of the wall of the cylindrical experimental cell; this quasiparticle is initially characterized by $r=R$ (where R is the radius of the cell), $p=p_F$, and $p_r < 0$. The axis of the vortex will still be at the center of the coordinate system. In such case, the quasiparticle with initial momentum directed along the x axis will feel the effective pairing potential $\Delta_{eff} \approx \Delta_0 - p_F y \kappa / (2\pi r)$ (Fig. 1).

It is obvious from Eq. (4.3) that unless ρ_0 is exactly zero ($J=0$), the radial velocity of the excitation will eventually vanish. This may happen either because $p_r=0$ (classical turning point) or because $\epsilon_p=0$ (Andreev turning point).

It can be seen from Eqs. (4.1) and (4.6) that the classical turning point is reached first when

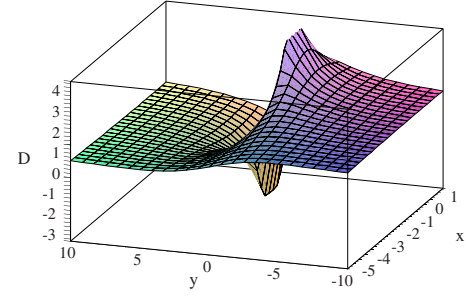


FIG. 1. (Color online) The dimensionless effective potential $D = \Delta_{eff}(r/\xi_0)/\Delta_0$ seen by quasiparticles with momentum parallel to the x axis and moving from $x=-\infty$. The dimensionless coordinates x and y are in units of ξ_0 .

$$E > \Delta_0 + p_F \frac{\kappa}{2\pi\rho_0} \approx \Delta_0 \left(1 + \frac{3\pi\xi_0}{2\rho_0} \right) \quad (4.7)$$

(here and in the equations below, the numerical factor of 3 is introduced by the ratio between the effective mass of quasiparticle and the bare mass of a ^3He atom: $m^*/m \approx 3$) in which case a quasiparticle with this energy follows a trajectory which is of the “normal” type: The quasiparticle retains its “particle” nature and moves past the vortex, across the experimental cell to the wall on the opposite side. On the contrary, a quasiparticle with energy E such that

$$\Delta_0 < E < \Delta_0 + p_F \frac{\kappa}{2\pi\rho_0} \approx \Delta_0 \left(1 + \frac{3\pi\xi_0}{2\rho_0} \right) \quad (4.8)$$

reaches the Andreev turning point first, undergoes Andreev reflection, and returns to a point near its starting point after changing its nature and becoming a quasihole.

Of these two cases, our concern is the case of Andreev reflection. We first determine the locus of Andreev turning points, defined by the minimum radial distance from the vortex core:

$$r_{min} = \left(\frac{\kappa}{2\pi} \frac{p_F\rho_0}{(E - \Delta_0)} \right)^{1/2} = \left(\frac{3\pi\xi_0\rho_0}{2} \frac{\Delta_0}{(E - \Delta_0)} \right)^{1/2}. \quad (4.9)$$

Consider a quasiparticle which has reached $r=r_{min}$. At this point, the radial velocity \dot{r} vanishes, but the excitation does not stop. It still has a nonzero azimuthal velocity, $r\dot{\varphi}$. Thereafter, the excitation propagates as a quasihole (characterized by a negative value of ϵ_p).

In order to calculate the trajectory of a reflected quasiparticle, it is convenient to simplify the governing equations of motion using the fact that at the ultralow temperatures which interest us, $T \ll T_c$, most quasiparticles have energies $\epsilon_p \ll \Delta_0$. We can then make the following approximation:

$$\sqrt{\epsilon_p^2 + \Delta_0^2} \approx \Delta_0 + \frac{\epsilon_p^2}{2\Delta_0} = \Delta_0 + \frac{(p^2 - p_F^2)^2}{8m^{*2}\Delta_0} \approx \Delta_0 + \frac{(p - p_F)^2}{2\Delta_0 v_F^2}. \quad (4.10)$$

This spectrum is similar to Landau’s spectrum of excitations in superfluid He II near the roton minimum ($p=p_0$), $E \approx \Delta_0 + (p-p_0)^2/(2m_r)$ (where m_r is the effective roton mass),

which was used to calculate the mutual friction force,⁴⁹ note that in Eq. (4.10), the role of the roton mass is played by the ratio Δ_0/v_F^2 .

Using Eqs. (4.10), (4.3), and (4.4), and the smallness of the ratios ϵ_p/p_F and Δ_0/ϵ_F , we obtain

$$dt = \frac{m^* \Delta_0}{\epsilon_p p_r} dr = - \frac{m^*}{p_F} \frac{r_{min}}{(3\pi\xi_0\rho_0)^{1/2}} \frac{r^2 dr}{(r^2 - r_{min}^2)^{1/2} (r^2 - \rho_0^2)^{1/2}}, \quad (4.11)$$

$$d\varphi = - \left(\frac{\rho_0}{r^2(1 - \rho_0^2/r^2)^{1/2}} \pm \frac{3br_{min}}{2(3\pi\xi_0\rho_0)^{1/2}(r^2 - r_{min}^2)^{1/2}(r^2 - \rho_0^2)^{1/2}} \right) dr, \quad (4.12)$$

where $b = \hbar/p_F$; the sign plus is used for quasiparticles and the sign minus for quasiholes.

From Eqs. (4.11) and (4.12), we obtain the Andreev return time τ of the excitation (the time it takes to travel from the radial distance R to the Andreev reflection point and back) and the Andreev reflection angle $\Delta\varphi$:

$$\tau = 2 \frac{r_{min}}{v_F} \frac{1}{(3\pi\xi_0\rho_0)^{1/2}} \int_{r_{min}}^R \frac{r^2 dr}{(r^2 - r_{min}^2)^{1/2} (r^2 - \rho_0^2)^{1/2}}, \quad (4.13)$$

$$\Delta\varphi = 3 \frac{br_{min}}{(3\pi\xi_0\rho_0)^{1/2}} \int_{r_{min}}^R \frac{dr}{(r^2 - r_{min}^2)^{1/2} (r^2 - \rho_0^2)^{1/2}}. \quad (4.14)$$

The evaluation of these elliptic integrals is shown in the Appendix. We obtain

$$\tau = 2 \frac{r_{min}}{v_F} \frac{1}{(3\pi\xi_0\rho_0)^{1/2}} \left(\frac{(R^2 - r_{min}^2)^{1/2} (R^2 - \rho_0^2)^{1/2}}{R} + \frac{\pi}{4} \left(\frac{\rho_0}{r_{min}} \right)^{1/2} r_{min} \right), \quad (4.15)$$

which becomes, assuming $R \gg r$ and $R \gg \rho_0$,

$$\tau \approx 2 \frac{Rr_{min}}{v_F(3\pi\xi_0\rho_0)^{1/2}} = \frac{R}{v_F} \left(\frac{2\Delta_0}{E - \Delta_0} \right)^{1/2} \approx \frac{R}{v_F} \frac{2\Delta_0}{\epsilon_p}. \quad (4.16)$$

We conclude that the Andreev return time is longer if the excitation's energy is lower.

Similarly, assuming $\rho_0/R \ll 1$ and $r_{min}/R \ll 1$, the Andreev reflection angle is

$$\Delta\varphi \approx \frac{\pi b}{(3\pi\xi_0\rho_0)^{1/2}}. \quad (4.17)$$

To apply these results, we assume that the initial momentum of the quasiparticle is directed along the x axis and that the angular momentum $J = -py_0 = p_F\rho_0$. From Eq. (3.1), it follows that the momentum $p = p_F(1 + 2m^*\epsilon_p/p_F)^{1/2}$ and, in the

ultralow temperature limit, $(p - p_F)/p_F \ll 10^{-4}$. For y_0 , we have $y_0 = \rho_0(1 + 2m^*\epsilon_p/p_F)^{-1/2} \approx \rho_0$. In this case, ρ_0 becomes the impact parameter (Fig. 2), and Eq. (4.17) shows that quasiparticles with smaller impact parameter (hence smaller angular momentum) are Andreev reflected by smaller angles.

As it is seen from Eq. (4.9), the Andreev radius depends strongly on the initial energy of the excitation:

$$r_{min} = (3\pi\xi_0\rho_0)^{1/2} \frac{\Delta_0}{\epsilon_p}. \quad (4.18)$$

The same arguments apply to the critical value ρ_{0c} defined as a maximum value of ρ_0 which causes the Andreev reflection of quasiparticles with the given initial energy ϵ_p . To calculate ρ_{0c} , we assume that at the starting point of the trajectory, the quasiparticle has coordinates (R, φ_0) , where $\varphi_0 = \arcsin(y_0/R) \approx -\arcsin(\rho_0/R)$; the coordinates of the Andreev reflection point in this case should be $(r_{min}, -\pi/2)$. Thus, the difference between the reflection angle and the starting angle is

$$\Delta\varphi = -\frac{\pi}{2} + \arcsin\left(\frac{\rho_{0c}}{R}\right). \quad (4.19)$$

This difference can also be calculated from Eq. (4.12) where the second term (of the order of \hbar/p_F) in the integrand can be neglected. We obtain

$$\Delta\varphi = -\arcsin\left(\frac{\rho_{0c}}{r_{min}}\right) + \arcsin\left(\frac{\rho_{0c}}{R}\right). \quad (4.20)$$

By comparing Eqs. (4.19) and (4.20), we find

$$\rho_{0c} \approx 3\pi\xi_0 \left(\frac{\Delta_0}{\epsilon_p} \right)^2. \quad (4.21)$$

In the typical low temperatures experiments, $k_B T/\Delta_0 \approx 0.1$, and, for quasiparticles with initial energy $\epsilon_p \approx k_B T$, we find $r_{min} \approx 10(3\pi\xi_0\rho_0)^{1/2}$ and $\rho_{0c} \sim 10^3 \xi_0$, while the same quantities for the quasiparticles with $\epsilon_p \approx (\Delta_0 k_B T)^{1/2}$ are $r_{min} \sim 3(3\pi\xi_0\rho_0)^{1/2}$ and $\rho_{0c} \sim 10^2 \xi_0$.

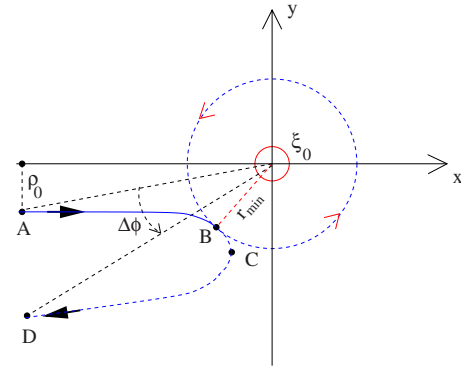


FIG. 2. (Color online) Schematic trajectory of the quasiparticle which starts at position A is Andreev reflected by the vortex (at the origin) at position B (where it becomes a quasi-hole), and then traces its way back with a small Andreev angle $\Delta\varphi$ (not to scale).

V. HEAT TRANSPORT THROUGH THE VELOCITY FIELD OF A VORTEX

In the experimental studies of superfluid turbulence in $^3\text{He-B}$ at the ultralow temperatures, the vortex tangle is studied by detecting the fraction of quasiparticles which are Andreev reflected by the vortices and measuring the heat which is transported by the quasiparticles. Using the results of previous sections, it is straightforward to calculate the fraction of energy (or heat) transmitted across the velocity field of a vortex. Once we know this fraction, we shall generalize it to a system of many vortices.

In Sec. IV, it was explained that the quasiparticles characterized by the particular impact parameter ρ_0 are Andreev reflected by a vortex if their energies satisfy the condition $\Delta_0 \leq E \leq \Delta_0(1 + 3\pi\xi_0/2\rho_0)$. If this condition is not satisfied, the quasiparticles pass freely across the vortex velocity field. If the system which consists of the vortex and quasiparticles is in thermal equilibrium, there is no preferred direction around the vortex. Incident and transmitted fluxes at one side of the vortex are canceled by the fluxes in the opposite direction, and no net flow of energy exists when the temperature everywhere around the vortex has the same value.

A net flux of quasiparticles and of energy results only if there is some small temperature difference, $\delta T \ll T$, between the two sides. If this is the case, the heat carried by the incident quasiparticles is described by the expression

$$\delta Q_{inc} = \int_{\Delta_0}^{\infty} N(E)v_g(E)E \frac{\partial f(E)}{\partial T} \delta T dE, \quad (5.1)$$

where

$$N(E) = N_F \frac{E}{(E^2 - \Delta^2)^{1/2}}. \quad (5.2)$$

Here,

$$N_F = \frac{mp_F}{\pi^2 \hbar^3} \quad (5.3)$$

is the density of states at the Fermi energy with corresponding Fermi momentum p_F . The group velocity of a Bogolubov quasiparticle v_g is given by the expression

$$v_g = \frac{\epsilon_p}{E} v_F = \frac{(E^2 - \Delta^2)^{1/2}}{E} v_F, \quad (5.4)$$

and $f(E)$ is the Fermi distribution function, which, at ultralow temperatures, is transformed into the Boltzmann distribution and describes the mean occupation number of a state with energy E :

$$f(E) = e^{-E/k_B T}. \quad (5.5)$$

The thermal flux of quasiparticles incident on the vortex velocity field per unit length per unit time is obtained with the help of Eqs. (5.2), (5.4), and (5.5); one has

$$\delta Q_{inc} = N_F v_F \frac{\delta T}{k_B T^2} \int_{\Delta}^{\infty} E^2 e^{-E/k_B T} dE \approx N_F v_F \Delta_0^2 \frac{\delta T}{T} e^{-\Delta_0/k_B T}. \quad (5.6)$$

If there is a plane current of quasiparticles with transverse cross section R_0 , then the total heat current incident on the vortex per unit time will be

$$Q_{inc} = 2R_0 \delta Q_{inc} = 2R_0 N_F v_F \Delta_0^2 \frac{\delta T}{T} e^{-\Delta_0/k_B T}. \quad (5.7)$$

We assume that, in the (x, y) plane orthogonal to the straight vortex line, the polarity of the vortex located at $(0, 0)$ is positive and consider quasiparticles incoming in the positive x direction. As discussed earlier, in this case, the upper half plane will be absolutely transparent for quasiparticles so that the heat transferred by quasiparticles through this half plane will meet no resistance. The lower half plane of vortex flow field will reflect a fraction of quasiparticles and induce some thermal resistance. A quasiparticle with the impact parameter ρ_0 is transmitted through the vortex velocity field if it carries the energy $E > \Delta_0(1 + \frac{3}{2}\pi\xi_0/\rho_0)$, in which case the heat which is transferred per unit time by such a quasiparticle can be calculated as

$$\begin{aligned} \delta Q(\rho) &= \int_{\Delta_0(1 + \frac{3\pi\xi_0}{2\rho_0})}^{\infty} N(E)v_g(E)E \frac{\partial f(E)}{\partial T} \delta T dE \\ &\approx Q_{inc} \frac{1}{2R_0} \left(1 + \frac{3\pi\xi_0}{\rho_0}\right) e^{-(\Delta_0/k_B T)(3\pi\xi_0/2\rho_0)}. \end{aligned} \quad (5.8)$$

Notice that estimating the ratio ξ_0/ρ_0 , we kept only the linear term.

The total amount of energy transferred through the vortex by quasiparticles originated within the interval $-R_0 \leq y \leq R_0$ is

$$Q_{tr} = \frac{Q_{inc}}{2} \left[1 + \frac{1}{R_0} \int_0^{R_0} \left(1 + \frac{3\pi\xi_0}{\rho_0}\right) e^{-(\Delta_0/k_B T)(3\pi\xi_0/2\rho_0)} d\rho_0 \right]. \quad (5.9)$$

The integral in Eq. (5.9) can be estimated as

$$\approx R_0 e^{-(\Delta_0/k_B T)(3\pi\xi_0/2\rho_0)}. \quad (5.10)$$

Thus, the fraction of heat which is transferred through the velocity field of the vortex is

$$\delta f_{tr} = \frac{1}{2} (1 + e^{-(\Delta_0/k_B T)(3\pi\xi_0/2\rho_0)}). \quad (5.11)$$

In experiments at ultralow temperatures we have $\Delta_0/k_B T \sim 10$, so that the cross section of the thermal flux is $R_0 \sim 10\xi_0$ and approximately 52% of the total heat is transferred through the vortex. If the heat current has the cross section $\sim 10^2\xi_0$, the fraction of the transferred heat is approximately 0.82. Therefore, the reflection of the heat flux takes place only in the close vicinity of the vortex core.

VI. ANDREEV REFLECTION IN A VORTEX GAS

To apply our result to experiments, we consider for simplicity a system of random parallel-antiparallel vortices [i.e., a system of vortex points in the (x, y) plane; such a system is known as the Onsager vortex gas]. This vortex system is penetrated by a quasiparticle current created by a temperature difference δT . It is convenient to introduce the effective radius R_0 of each vortex as the half of the mean intervortex distance, i.e., $R_0 = \ell/2$. We divide the vortex configuration in parallel layers of width ℓ each perpendicular to the quasiparticle current. Clearly, the transmittability of each layer is equal to the transmittability of a vortex within a region of radius $\ell/2$. Thus, the fraction of heat transmitted by each layer is

$$\delta f_{tr} = \frac{1}{2} (1 + e^{-(\Delta_0/k_B T)(3\pi\xi_0/\ell)}). \quad (6.1)$$

If we assume now that vortices are well separated and that their velocity fields do not overlap significantly, we obtain the conditions

$$\xi_0 \ll \ell, \quad \frac{\Delta_0}{k_B T} \frac{3\pi\xi_0}{\ell} \ll 1. \quad (6.2)$$

Equation (6.1) becomes

$$\begin{aligned} \delta f_{tr} &\approx \frac{1}{2} \left(1 + 1 - \frac{\Delta_0}{k_B T} \frac{3\pi\xi_0}{\ell} \right) = 1 - \frac{\Delta_0}{k_B T} \frac{3\pi\xi_0}{2\ell} \\ &\approx e^{-(\Delta_0/k_B T)(3\pi\xi_0/\ell)}. \end{aligned} \quad (6.3)$$

Driven by the temperature difference, the heat flux Q_0 reduces, after penetrating the first layer, to $Q_1 = Q_0 \delta f_{tr}$; after penetrating the second layer, it becomes $Q_2 = Q_1 \delta f_{tr}$. Hence, after penetrating the last n th layer, we obtain $Q_n = Q_{n-1} \delta f_{tr}$. Thus we have

$$Q_n = Q_{n-1} \delta f_{tr} = \dots = Q_0 \delta f_{tr}^n. \quad (6.4)$$

We conclude that the fraction of heat which is transferred through the system of vortices is

$$f_{tr} = (\delta f_{tr})^n. \quad (6.5)$$

If the total vorticity is confined within a region of size S , the number of layers, n can be estimated as $n \approx S/\ell$. From Eq. (6.5), we obtain

$$f_{tr} = e^{-(\Delta_0/k_B T)(3\pi\xi_0 S/2\ell^2)}. \quad (6.6)$$

Finally, we obtain the intervortex distance

$$\ell = \left(-\frac{\Delta_0}{k_B T} \frac{3\pi\xi_0 S}{2 \ln f_{tr}} \right)^{1/2}. \quad (6.7)$$

The quantities S (the size of the vortex system) and f_{tr} (the fraction of reflected quasiparticles) in Eq. (6.7) can be observed experimentally. From the available description of one experiment,²⁹ the maximum transmitted fraction of quasiparticle current is $f_{tr} \approx 0.75$ and the spatial extension of the vorticity is $S \sim 2 \times 10^{-1}$ cm. Since the zero-temperature coherence length is $\xi_0 \approx 0.8 \times 10^{-5}$ cm, we conclude that in the case where $\Delta_0/k_B T \sim 10$, the average intervortex distance is

$\ell \sim 1.62 \times 10^{-2}$ cm, which is in good agreement with existing estimates.⁴³

VII. CONCLUSIONS

In conclusion, starting from Hamilton's equations, we have calculated the trajectories of quasiparticles which move in the velocity field of a quantized vortex in $^3\text{He-B}$ and determined the Andreev reflection point. Generalizing the result to a disordered system of many vortices, we have determined the precise location of turning point and showed how to recover the typical intervortex spacing in the turbulent $^3\text{He-B}$. Our result is in good agreement with less rigorous estimates.

Future work will investigate Andreev reflection of quasiparticles by a system of moving vortices. We shall also study how the Andreev reflection technique can be used to visualize vortex structures (e.g., coherent bundles of vortices) and determine turbulent fluctuations and turbulence statistics.

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APPENDIX

The Andreev return time τ and the Andreev reflection angle $\Delta\varphi$ are defined by formulae (4.13) and (4.14), where $0 < \rho_0 < r_{min} < r < R$. To evaluate these formulas we use the following integrals:

$$I_1 = \int_{r_{min}}^R \frac{r^2 dr}{(r^2 - r_{min}^2)^{1/2} (r^2 - \rho_0^2)^{1/2}} = \frac{\sqrt{R^2 - r_{min}^2}}{\sqrt{R^2 - \rho_0^2}} + r_{min} G, \quad (A1)$$

where

$$\begin{aligned} G = & K\left(\frac{\rho_0}{r_{min}}\right) - F\left(\arcsin \frac{r_{min}}{R}, \frac{\rho_0}{r_{min}}\right) - E\left(\frac{\pi}{2}, \frac{\rho_0}{r_{min}}\right) \\ & + E\left(\arcsin \frac{r_{min}}{R}, \frac{\rho_0}{r_{min}}\right), \end{aligned} \quad (A2)$$

and

$$\begin{aligned} I_2 = & \int_{r_{min}}^R \frac{dr}{(r^2 - r_{min}^2)^{1/2} (r^2 - \rho_0^2)^{1/2}} = \frac{1}{r_{min}} \left[K\left(\frac{\rho_0}{r_{min}}\right) \right. \\ & \left. - F\left(\arcsin \frac{r_{min}}{R}, \frac{\rho_0}{r_{min}}\right) \right], \end{aligned} \quad (A3)$$

where K , F , and E are elliptic integrals defined as

$$F(k, \theta) = \int_0^\theta \frac{d\varphi}{\sqrt{1 - k^2 \sin^2(\varphi)}}, \quad (A4)$$

$$K(k) = F\left(\frac{\pi}{2}, k\right), \quad (\text{A5})$$

$$E(k, \theta) = \int_0^\theta \sqrt{1 - k^2 \sin^2(\varphi)} d\varphi, \quad (\text{A6})$$

with $\theta = \arcsin(r_{\min}/R)$ and $k = \rho_0/R$.

For $k^2 < 1$, the elliptic integrals (A4)–(A6) are represented by the series

$$F(k, \theta) = \frac{2\theta}{\pi} K(k) - \sin \theta \cos \theta \frac{k^2}{4} + \dots, \quad (\text{A7})$$

$$K(k) = \frac{\pi}{2} + \frac{\pi^2}{8} k^2 + \dots, \quad (\text{A8})$$

$$E(k, \theta) = \frac{2\theta}{\pi} E(k) + \sin \theta \cos \theta \frac{k^2}{4} + \dots, \quad (\text{A9})$$

$$E\left(k, \frac{\pi}{2}\right) = E(k) = \frac{\pi}{2} - \frac{\pi}{8} k^2 + \dots, \quad (\text{A10})$$

using which we obtain

$$\tau \approx \frac{2r_{\min}}{v_F(3\pi\xi_0\rho_0)^{1/2}} \left[\frac{\sqrt{R^2 - r_{\min}^2} \sqrt{R^2 - \rho_0^2}}{R} + \frac{\pi}{4} \left(\frac{\rho_0}{r_{\min}} \right)^2 r_{\min} \right]. \quad (\text{A11})$$

Assuming $R \gg r_{\min}$, $R \gg \rho_0$, and $\rho_0 < r_{\min}$, we have

$$\tau \approx \frac{2R}{v_F} \frac{r_{\min}}{(3\pi\xi_0\rho_0)^{1/2}}. \quad (\text{A12})$$

Similarly,

$$\Delta\varphi \approx \pi \frac{b}{(3\pi\xi_0\rho_0)^{1/2}} \ll 1. \quad (\text{A13})$$

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