Supersymmetric field theory of local light diffusion in semi-infinite media

Chushun Tian
Institut für Theoretische Physik, Universität zu Köln, 50937 Köln, Germany
(Received 29 October 2007; published 26 February 2008)

A supersymmetric field theory of light diffusion in semi-infinite disordered media is presented. With the help of this technique we justify—at the perturbative level—the local light diffusion proposed by Tiggelen, Lagendijk, and Wiersma [Phys. Rev. Lett. **84**, 4333 (2000)], and show that the coherent backscattering line shape of medium bar displays a crossover from two-dimensional weak to quasi-one-dimensional strong localization.

DOI: 10.1103/PhysRevB.77.064205 PACS number(s): 42.25.Dd, 42.25.Hz

I. INTRODUCTION

The Anderson localization of light has been one of the most fascinating phenomena in condensed matter physics since the mid 1980s.[1,2] Like electron systems this phenomenon finds its origin in coherent multiple scattering which slows down diffusion of photons and eventually brings them to a stop. Parallel to studies of disordered conductors the subject in this field ranges from light localization near or far below the mobility edge in bulk (infinite) systems[1,3] (where low-energy photon motion enjoys the translational symmetry) to their detection such as transmission measurements in the slab geometry (e.g., Refs. 4–6).

A unique subject of localization in optical (and other classical wave) systems is the enhanced coherent backscattering (CBS) phenomenon.[7,8] In this subject the issue of semi-infinite geometry is heavily addressed because the CBS line shape is responsible for by optical paths near the vacuum-medium interface. Although it is well known that in the weak disorder region, i.e., $l \gg \lambda$ (l the mean free path and λ the wavelength) incident photons enjoy diffusion as in bulk media,[7,8] in the strong disorder region $l \lesssim \lambda$ the role played by the leakage at the interface has been of long term interest[9] and still remains in the central position of CBS studies, particularly to forecast or observe the CBS line shape.[10–13]

Pressingly, in the latter region strong localization emerges in the bulk and a new scale, namely, the localization length ξ appears. On the experimental side, there has been increasing evidence indicating that inside the boundary layer of thickness $\gtrsim \xi$ the photon leakage at the vacuum-medium interface strongly interplays with strong localization.[5,6,11,14] On the theoretical side, some time ago exact solution of semi-infinite one-dimensional disordered chains shed the light on the existence of so-called radiative localization states in the boundary layer,[15] which led to anomalous slowing decay of reflected (backscattered) incident light pulses.[15,16]

Recently, in an insightful theoretical work[12] it was realized that (in three-dimensional disordered media) inside the boundary layer the translational symmetry of low-energy (hydrodynamic) photon motion is strongly destroyed resulting in the so-called "local diffusion." Remarkably, constructive wave interference renders the static diffusion coefficient depending on the distance from the interface. Consequently, the local diffusion was found to lead to a rounded CBS line shape resembling that observed experimentally[11] and thus, might overcome the conceptual difficulty of an earlier theoretical proposal.[9] Surprisingly, the dynamic generalization of the local diffusion equation[17,18] provides an explanation of some key phenomena observed in a quasi-one-dimensional microwave experiment,[5] and well captures the anomalous slowing decay of reflected incident pluses in both quasi-one-dimensional[16,17,19] and three-dimensional disordered media.[18] Moreover, such a novel prediction—the position dependence of a diffusion coefficient—seems to have been confirmed by a very recent experiment.[14]

Despite this progress theoretical investigations on local diffusion are, however, restricted on the self-consistent diagrammatical method.[12,17,18] Thus, an intellectual challenge is to seek the genuine microscopic origin underlying this novel concept. This is, indeed, the purpose of this work. The last few decades have witnessed spectacular success of applications of supersymmetric field theory to various disordered systems in the absence of interactions. (Such a condition is perfectly satisfied by optical systems.)[20] Among them there are a few exact nonperturbative results for quasi-one-dimensional disordered wires such as density-density correlation function (in the infinite geometry).[21] and transmission statistics.[22] They allow one to obtain important insight on the strong localization. Most importantly, for periodic disordered media by using the supersymmetric field-theoretic method a local light diffusion equation, similar to that proposed in Ref. 12, recently has been derived at the microscopic level.[23] In view of these it is natural and inevitably necessary to proceed along the same line to explore the concept of local light diffusion and its effects for more general—fully disordered—media, which differ drastically from the former[23] in both physical and technical view.

The main results of this paper are as follows. (i) We present a field-theoretic proof showing that, contrary to the conjecture of Ref. 9, no scaling behavior exists inside a layer of thickness $\sim l$ extrapolating into the vacuum. (ii) We justify—at the perturbative level—the local diffusion equation proposed in Refs. 12 and 17. (iii) We analyze signatures of the static local diffusion in the CBS line shape. It should be stressed that in this paper the suspersymmetric field theory is treated perturbatively, and the nonperturbative treatise will be reported in the forthcoming paper.

The rest of this paper is organized as follows. In the next section we produce a nonlinear supersymmetric σ model in the context of optical systems. Most importantly, we derive the boundary constraint satisfied by the supersymmetric matrix field. The supersymmetric field theory is then applied to the two-dimensional medium bar. Section III is devoted to exploring states residing deeply inside the semi-infinite medium bar (namely, far away from the interface) by investigating renormalization effects of infinite medium bar. In Sec.

IV weak localization in the semi-infinite medium bar is studied, where the general dynamic local diffusion equation is justified. The static limit of the local diffusion equation is studied in Sec. V. In particular, the weak localization correction to the bare diffusion constant is explicitly calculated, and its signatures in the CBS line shape are analyzed. We conclude in Sec. VI and give some technical details in Appendixes A–E.

II. SUPERSYMMETRIC FIELD-THEORETIC FORMALISM

In this section a supersymmetric field-theoretic formalism is presented for light scattering in a semi-infinite disordered medium.

A. Nonlinear σ model

We first show that as interactionless electron systems low-energy photon motion in bulk disordered media is well described by the nonlinear σ model. The derivation is rather standard.[20] Here we only outline the scheme with an emphasis on the main difference, while we refer the reader to Ref. 20 for the details.

In the present work for simplicity the scalar wave will be considered. The wave propagation in a bulk disordered medium is described by the Helmholtz equation as follows:

$$\{\nabla^2 + \Omega^2[1+\epsilon(\mathbf{r})]\}E(\mathbf{r}) = j(\mathbf{r}), \tag{1}$$

where the field E has the radiation frequency Ω (velocity c set to be unity), and $j(\mathbf{r})$ is the source. Here the fluctuating dielectric field $\epsilon(\mathbf{r})$ has zero mean and is distributed according to the Gaussian δ-correlated law as follows:

$$\Omega^4\langle \epsilon(\mathbf{r})\epsilon(\mathbf{r}')\rangle = \Delta\,\delta(\mathbf{r}-\mathbf{r}'). \tag{2}$$

The Helmholtz equation resembles the Schrödinger equation with the Hamiltonian now read out as $\hat{H}=-\nabla^2-\Omega^2\epsilon(\mathbf{r})$. As usual we may introduce the retarded or advanced Green function $G^{R,A}_{\Omega^2}$ defined as

$$\{\Omega^2_{\pm} - \hat{H}\}G^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{3}$$

where $\Omega_{\pm}=\Omega \pm i0^+$. The electric field and the source are related via $E(\mathbf{r})=\int d\mathbf{r}'\, G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')j(\mathbf{r}')$. We may further introduce the diffusion $\mathcal{Y}^{\mathrm{D}}$ and the cooperon $\mathcal{Y}^{\mathrm{C}}$ propagator defined as

$$\mathcal{Y}^{\mathrm{D}}(\mathbf{r},\mathbf{r}';\omega) \equiv \overline{G^R_{(\Omega+\omega^+/2)^2}(\mathbf{r},\mathbf{r}')G^A_{(\Omega-\omega^+/2)^2}(\mathbf{r}',\mathbf{r})},$$

$$\mathcal{Y}^{\mathrm{C}}(\mathbf{r},\mathbf{r}';\omega) \equiv \overline{G^R_{(\Omega+\omega^+/2)^2}(\mathbf{r},\mathbf{r}')G^A_{(\Omega-\omega^+/2)^2}(\mathbf{r},\mathbf{r}')}, \tag{4}$$

with $\omega^+=\omega+i0^+$ and $\omega \ll \Omega$, where the overline stands for the average over random dielectric field. These two propagators describe elegantly the light propagation over large scales.

The propagators above are represented in terms of superintegrals. For this purpose we define a supervector field ψ as follows:

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad \psi^m = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi^{m*} \\ \chi^m \\ S^{m*} \\ S^m \end{pmatrix}, \quad m=1,2, \tag{5}$$

with S's (χ's) complex commuting (anticommunting) variables, where the superscript 1 (2) refers to retarded (advanced) Green function, and its charge conjugate $\bar{\psi}\equiv \psi^\dagger\Lambda$. Here Λ is an 8×8 supermatrix as follows:

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\mathrm{ar}} \otimes \mathbf{1}^{\mathrm{bf}} \otimes \mathbf{1}^{\mathrm{tr}}. \tag{6}$$

Hereafter supermatrices are defined on the retarded or advanced (ar), bosonic or fermonic (bf), and time-reversal (tr) sector. Then

$$\mathcal{Y}^{\mathrm{D}}(\mathbf{r},\mathbf{r}';\omega) = -4\int d[\psi]\psi^1_\alpha(\mathbf{r})\bar{\psi}^1_\alpha(\mathbf{r}')\psi^2_\beta(\mathbf{r}')\bar{\psi}^2_\beta(\mathbf{r})\overline{e^{-\mathfrak{L}[\psi,\bar{\psi}]}},$$

$$\mathcal{Y}^{\mathrm{C}}(\mathbf{r},\mathbf{r}';\omega) = -4\int d[\psi]\psi^1_\alpha(\mathbf{r})\bar{\psi}^1_\alpha(\mathbf{r}')\psi^2_\beta(\mathbf{r})\bar{\psi}^2_\beta(\mathbf{r}')\overline{e^{-\mathfrak{L}[\psi,\bar{\psi}]}}. \tag{7}$$

Here

$$\mathfrak{L} = i\int \bar{\psi}(\mathbf{r})[-\mathcal{H}_0 - \Omega^2\epsilon(\mathbf{r}) - \Omega\omega^+\Lambda]\psi(\mathbf{r})d\mathbf{r}, \tag{8}$$

with $\mathcal{H}_0=-\nabla^2-\Omega^2$, where the ω^2 term is omitted since $\omega \ll \Omega$. Performing the average we arrive at

$$\overline{e^{-\mathfrak{L}[\psi,\bar{\psi}]}} = e^{-i\int\bar{\psi}[-\mathcal{H}_0-\Omega\omega^+\Lambda]\psi d\mathbf{r}-(\Delta/2)\int(\bar{\psi}\psi)^2 d\mathbf{r}}. \tag{9}$$

The quartic term is decoupled by the standard Hubbard-Stratonovich transformation. Introduce an 8×8 supermatrix field $Q(\mathbf{r})$ conjugate to $\frac{2}{\pi N(\Omega^2)}\psi(\mathbf{r})\otimes\bar{\psi}(\mathbf{r})$. Here $N(\Omega^2)$ is related to the photon density of states per unit volume $\nu(\Omega)$ by $\nu(\Omega)=2\Omega N(\Omega^2)$. Then,

$$\exp\left[-\frac{\Delta}{2}\int(\bar{\psi}\psi)^2 d\mathbf{r}\right] = \int \exp\left[-\pi\Delta N(\Omega^2)\int\left(\bar{\psi}Q\psi + \frac{\pi N(\Omega^2)}{4}Q^2\right)d\mathbf{r}\right]D[Q]. \tag{10}$$

Substituting it into Eqs. (7) and (9) and integrating out the ψ fields using the Wick theorem, we obtain

$$\mathcal{Y}^{\mathrm{D,C}}(\mathbf{r},\mathbf{r}';\omega) = \left[\frac{\pi N(\Omega^2)}{4}\right]^2\langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)Q(\mathbf{r})(1-\Lambda) \times(1\mp\tau_3)kQ(\mathbf{r}')]\rangle, \tag{11}$$

with

$$k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\mathrm{bf}} \otimes \mathbf{1}^{\mathrm{ar}} \otimes \mathbf{1}^{\mathrm{tr}}, \tag{12}$$

and τ_k the Pauli matrices defined on the time-reversal sector. In Eq. (11) the following average is introduced:

$$\langle P[Q]\rangle = \int D[Q]P[Q]e^{-F[Q]}, \tag{13}$$

where the action $F[Q]$ is

$$F[Q] = \int d\mathbf{r}\ \mathrm{str}\left\{\left(\frac{\pi N(\Omega^2)}{2}\right)^2 \Delta Q^2 - \frac{1}{2}\ln[-i\mathcal{H}_0 - i\Omega\omega^+\Lambda + \pi\Delta N(\Omega^2)Q(\mathbf{r})]\right\}. \tag{14}$$

Minimizing $F[Q]$ gives the saddle-point equation as follows:

$$Q = \frac{1}{\pi N(\Omega^2)}\{-i\mathcal{H}_0 - i\Omega\omega^+\Lambda + \pi\Delta N(\Omega^2)Q\}^{-1} \equiv \frac{1}{\pi N(\Omega^2)}\mathcal{G}_0. \tag{15}$$

In the limit $\Omega \gg \omega, \pi\Delta N(\Omega^2)/\Omega$, Eq. (15) gives the saddle point as $Q(\mathbf{r})=\Lambda$.

So far the derivation is exact. Fluctuations analysis around the saddle point may be performed for Eqs. (13) and (14). Yet, we could not proceed further and only give the results here; and instead refer the reader to Ref. 20 for all the details. First, after standard procedure the mean field approximation, namely, $Q(\mathbf{r})=\Lambda$ gives the averaged retarded or advanced Green function as

$$\overline{G^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')} = \langle \mathbf{r}|\{\Omega^2 + \nabla^2 \pm i\pi\Delta N(\Omega^2)\}^{-1}|\mathbf{r}'\rangle. \tag{16}$$

The imaginary part of the self-energy gives the elastic mean free path which is

$$l = \frac{\Omega}{\pi\Delta N(\Omega^2)}, \tag{17}$$

and has the Rayleigh form, i.e., $l \sim \Omega^{-(d+1)}$.

Then, with $\Omega l \gg 1$ taken into account the action is simplified to be $F[Q]=\int d\mathbf{r}\mathcal{L}[Q]$, where (from now on we set $\nu \equiv \nu(\Omega)$ to shorten the formula)

$$\mathcal{L}[Q] = \frac{\pi\nu}{8}\mathrm{str}[D_0(\partial Q)^2 + 2i\omega^+\Lambda Q], \tag{18}$$

with the bare diffusion constant $D_0=l/d$. Here $Q(\mathbf{r}) = T(\mathbf{r})\Lambda T^{-1}(\mathbf{r})$ describes Goldstone modes with $T(\mathbf{r})$ a matrix field taking the value in the coset space $U(2,2/4)/U(2/2)\times U(2/2)$ reflecting the orthogonal symmetry. An explicit parametrization of T will be given in the next section.

B. Q-field constraint at the vacuum-medium interface

The action $F[Q]$ obtained above is invariant under the translational symmetry, which is broken in the presence of the vacuum-medium interface. The broken translational symmetry may profoundly affect light propagation. Experience in mesoscopic physics shows that to take into account the vacuum-medium interface effect one may impose some appropriate boundary condition on the Q field in the field-theoretic formalism. However, this is a nontrivial task and in mesoscopic physics investigations so far have been restricted on interface structures of quasi-one-dimensional disordered wires and small quantum dots.[20,24–26] In this part we switch to optical systems and study the vacuum-medium coupling where the interface may be infinite and bear arbitrary geometry.

1. Vacuum-medium coupling action

Though the derivation below may be generalized to arbitrary dimension to simplify discussions we will focus on the two-dimensional case. Let us suppose an arbitrary curve C which divides the space $\mathbb{R}^2$ into two disconnected subspaces $\mathcal{V}_-$ and $\mathcal{V}_+$, i.e., $\mathbb{R}^2=\mathcal{V}_-\cup\mathcal{V}_+\cup C$ and $\mathcal{V}_-\cap\mathcal{V}_+=\emptyset$. We are interested in light propagation in some subspace, say $\mathcal{V}_+$ described by an effective Green function $\mathcal{G}^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')$, which is identical to $G^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')$ for $\mathbf{r},\mathbf{r}'\in\mathcal{V}_+$. To study such Green functions for $\mathbf{r},\mathbf{r}'\in\mathcal{V}_-$ we introduce auxiliary Green functions $g^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')$ satisfying

$$\{\Omega^2_\pm - \hat{H}\}g^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$
$$g^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}\ \mathrm{or}\ \mathbf{r}'\in C} = 0. \tag{19}$$

Then our starting point is the following theorem due to Zirnbauer[26] and refined by Efetov,[20] which was originally established for description of coupling between leads and mesoscopic devices. The theorem is stated as follows. (For the self-contained purpose the proof tailored to the present context is given in Appendix A.)

For $\mathbf{r},\mathbf{r}'\in\mathcal{V}_+$ the Green function $\mathcal{G}^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')$ solves

$$\{\Omega^2_\pm - \hat{H} \pm i\hat{B}\}\mathcal{G}^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$
$$\partial_{\mathbf{n}(\mathbf{r})}\mathcal{G}^{R,A}_{\Omega^2}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}\in C} = 0, \quad \mathbf{r}\in C, \tag{20}$$

where the normal unit vector $\mathbf{n}(\mathbf{r})$ at $\mathbf{r}$ points to $\mathcal{V}_+$. Here

$$(\hat{B}f)(\mathbf{r}) \equiv \int_C d\mathbf{r}'\ \mathrm{Im}[B(\mathbf{r},\mathbf{r}')]f(\mathbf{r}'),$$

$$B(\mathbf{r},\mathbf{r}') = \partial_{\mathbf{n}(\mathbf{r})}\partial_{\mathbf{n}(\mathbf{r}')}g^R_{\Omega^2}(\mathbf{r},\mathbf{r}'), \quad \text{for } \mathbf{r},\mathbf{r}'\in C. \tag{21}$$

The effective Hamiltonian for the retarded (advanced) Green function is $\hat{H}\mp i\hat{B}$. Remarkably, it is non-Hermitian due to the escape from $\mathcal{V}_+$ into $\mathcal{V}_-$ through C.

In the present case the vacuum-medium interface, namely, the curve C is a straight line. To proceed we choose the coordinate system $(r_\perp, z)$ with the z $(r_\perp)$ direction perpendicular (parallel) to the vacuum-medium interface. The vacuum fills the space $z<0$ where no dielectric scatterers are available. For technical reasons we assume that the dielectric scatterers located at $(r^i_\perp,0)$ $(r^1_\perp < r^2_\perp < \cdots)$ are uniformly (in the statistical sense) distributed with the distance between nearest scatterers $l_i=r^i_\perp - r^{i-1}_\perp$ order of l. C is located at $z=0^-$ and $\mathcal{V}_+$ $(\mathcal{V}_-)$ is set to be the medium (vacuum).

Taking into account the boundary condition specified in Eq. (19) we find that the Green function $g^R_{\Omega^2}$ is

$$g^R_{\Omega^2}(r_\perp,z,r'_\perp,z') = \frac{1}{\pi^2}\int dk_\perp \int_0^\infty dk \frac{e^{ik_\perp(r_\perp - r'_\perp)}\sin(kz)\sin(kz')}{\Omega^2 - k_\perp^2 - k^2 + i0^+}. \quad (22)$$

Upon the substitution of Eq. (22) into Eq. (21) the operator $\hat{B}$ is simplified to be

$$(\hat{B}f)(r_\perp) = \int_{|k_\perp|\le\Omega} \frac{dk_\perp}{2\pi}\int dr'_\perp \sqrt{\Omega^2 - k_\perp^2}e^{ik_\perp(r_\perp - r'_\perp)}f(r'_\perp). \quad (23)$$

Repeating the derivation of Sec. II A with the effective Hamiltonian $\hat{H}\mp i\hat{B}$ we arrive again at Eq. (14) except that the action is modified according to $F[Q]\to F[Q]+F_{\text{inter}}[Q]$ with

$$F_{\text{inter}}[Q] = -\frac{1}{2}\,\text{str}\ln\{1 - \hat{B}\Lambda\mathcal{G}_0\}. \quad (24)$$

Expanding the logarithm and substituting Eq. (23) into it we obtain

$$F_{\text{inter}}[Q] = -\frac{1}{2}\,\text{str}\left\{\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}[\hat{B}\Lambda\mathcal{G}_0]^n\right\}, \quad (25)$$

where the supertrace str includes the integration over $r_\perp$, and $\mathcal{G}_0(\mathbf{r},\mathbf{r}';Q)$ exponentially decays for $|\mathbf{r}-\mathbf{r}'|\gtrsim l$ according to Eq. (15). To calculate Eq. (25) we introduce, for arbitrary i, the auxiliary variable $x_i\equiv r^i_\perp - r_\perp, r_\perp\in[r^{i-1}_\perp, r^i_\perp]$. Following Ref. 20 in the layer $0\le z\le l$ the Green function $\mathcal{G}_0(\mathbf{r},\mathbf{r}';Q)\equiv\mathcal{G}_0(r^i_\perp - x_i, z, r^{i'}_\perp - x'_{i'}, z';Q)$ may be approximated by

$$\mathcal{G}_0(r^i_\perp - x_i,z,r^{i'}_\perp - x'_{i'},z';Q) = \frac{2\delta_{ii'}}{\pi}\sum_{N\ge1}\int_0^\infty dk \frac{\varphi^i_{\pi N/l_i}(x_i)\varphi^i_{\pi N/l_i}(x'_i)\cos(kz)\cos(kz')}{\Omega^2(1+\epsilon^i) - k_\perp^2 - k^2 + i\pi\Delta N(\Omega^2)Q_i}, \quad (26)$$

where the longitudinal wave function is determined by the boundary condition of Eq. (20). Here Q^i and $1+\epsilon^i$ stand for the Q and the dielectric field, respectively, in the regime: $[r^{i-1}_\perp, r^i_\perp]\times[0,l]$. They are considered to be a constant (matrix) since both Q and ϵ varies over the scale l. Moreover, the transverse component $\varphi^i_{k_\perp}$ is defined as

$$\varphi^i_{\pi N/l_i}(x_i) = \sqrt{\frac{2}{l_i}}\sin\frac{\pi N x_i}{l_i}. \quad (27)$$

Substituting Eq. (26) into Eq. (25), with the help of the following identity:

$$\int_0^{l_i}\int_0^{l_{i'}} \frac{dx_i dx'_{i'}}{2\pi}e^{ik_\perp(-x_i + x'_{i'} + r^i_\perp - r^{i'}_\perp)}\varphi^i_{\pi N/l_i}(x_i)\varphi^{i'}_{\pi N'/l_{i'}}(x'_{i'}) \approx \frac{\delta_{ii'}\delta_{NN'}}{2}\left[\delta\left(k_\perp - \frac{\pi N}{l_i}\right) + \delta\left(k_\perp + \frac{\pi N}{l_i}\right)\right], \quad (28)$$

we obtain

$$F_{\text{inter}}[Q] = -\frac{1}{2}\sum_i \sum_{0<k_\perp/\Omega\le1}\text{str}\ln\{1+\alpha^i_{k_\perp}\Lambda Q_i\}, \quad (29)$$

with

$$\alpha^i_{k_\perp} = \sqrt{\frac{\Omega^2 - k_\perp^2}{\Omega^2(1+\epsilon^i) - k_\perp^2}}. \quad (30)$$

Taking advantage of the large channel number $\Sigma_{0<k_\perp/\Omega\le1}\approx\Omega l_i/\pi$ we may simplify it to be (the details are given in Appendix B)

$$F_{\text{inter}}[Q] = -\frac{1}{4}\sum_i \frac{\Omega l_i}{\pi}T_0(i)\,\text{str}(\Lambda Q_i) \quad (31)$$

by assuming that $\alpha^i_{k_\perp}$ does not depend on $k_\perp$, i.e., $\alpha^i_{k_\perp}\equiv\alpha^i$. Here

$$T_0(i) = \frac{4\alpha^i}{(1+\alpha^i)^2}\le 1 \quad (32)$$

is the well known transmission coefficient of electromagnetic wave.[27] Passing to the continuum limit: $\Sigma_i l_i\to\int dr_\perp$ we rewrite the action $F_{\text{inter}}[Q]$ as

$$F_{\text{inter}}[Q] = -\frac{\Omega}{4\pi}\int dr_\perp T_0(r_\perp)\,\text{str}[\Lambda Q(r_\perp, z=0)]. \quad (33)$$

In the quasi-one-dimensional geometry the summation over i is suppressed, and the coupling action $F_{\text{inter}}[Q]$, namely, Eq. (31) recovers the one obtained previously.[24,25] For $d>2$ although to generalize the derivation above is straightforward, the coupling action may be obtained by the simple physical arguments below. Notice that the coefficient of Eq. (31) allows a simple physical explanation.[24] According to Eq. (16) the (single) photon Green function decays over the scale l. Suppose that the medium is partitioned into boxes of volume l^d, then the states (denoted as μ) in different boxes are uncorrelated. The box states neighboring to the interface may be translated into the vacuum state (denoted as a)—so-called lead channel in the terminology of mesoscopic physics. The coupling strength is $\propto\Sigma_\mu W_{a\mu}W_{\mu a}$ with $W_{a\mu}$ the scattering matrix element, which scales as l^{d-1}/A with A the interface area. Thus, although the total channel number is $\propto A\Omega^{d-1}$,the number of channels to which the interface box state is transmitted (denoted as N_d) is much smaller $N_d\sim A\Omega^{d-1}\times l^{d-1}/A=(\Omega l)^{d-1}$. More precisely, N_d may be found to be

$$N_d = \frac{l^{d-1}}{(2\pi)^{d-1}}\frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\int_0^\Omega dk_\perp k_\perp^{d-2}. \quad (34)$$

For $d=2$ this gives $N_2=\Omega l/\pi$, namely, the coefficient of Eq. (31). For arbitrary d with the replacement of $N_2\to N_d$ (and $l\to l_i$) in Eq. (31) the vacuum-medium action becomes

$$\tilde{F}_{\rm inter}[Q] = -\frac{\tilde{N}_d}{4}\sum_i \frac{(\Omega l_i)^{d-1}}{\pi} T_0(i)\,{\rm str}(\Lambda Q_i)$$
$$= -\frac{\tilde{N}_d \Omega^{d-1}}{4}\int d\mathbf{r}\,\delta_C T_0(\mathbf{r})\,{\rm str}[Q(\mathbf{r})\Lambda], \quad (35)$$

with

$$\tilde{N}_d = \frac{1}{(2\pi)^{d-1}} \frac{\pi^{(d-1)/2}}{\frac{d-1}{2}\Gamma\left(\frac{d-1}{2}\right)}. \quad (36)$$

In the last equality of Eq. (35) we again pass to the continuum limit, and the operator δ_C is defined as $\int d\mathbf{r}\,\delta_C f(\mathbf{r}) \equiv \int d\mathbf{r}_\perp f(\mathbf{r}_\perp, z=0)$.

2. Boundary condition

We then come to derive the boundary condition satisfied by Q. For this purpose we employ the so-called boundary Ward identity.[28] It states that an arbitrary local observable, say $P(\mathbf{r})$ (with $\mathbf{r}$ inside the medium), which is expressed in terms of the average of the functional $\mathcal{P}[Q(\mathbf{r})]$, namely,

$$P(\mathbf{r}) \equiv \int D[Q]\mathcal{P}[Q(\mathbf{r})] e^{-\int_{z\in\mathbb{R}^+} d\mathbf{r}\mathcal{L}[Q(\mathbf{r})] - \tilde{F}_{\rm inter}[Q]}, \quad (37)$$

must be invariant under an infinitesimal boundary rotation below.

$$Q \to e^{-R} Q e^{R} \approx Q - [R,Q],$$
$$R = \begin{pmatrix} 0 & \mathcal{R}(\mathbf{r}_\perp, z=0) \\ \bar{\mathcal{R}}(\mathbf{r}_\perp, z=0) & 0 \end{pmatrix}^{\rm ar} \otimes \mathbf{1}^{\rm bf} \otimes \mathbf{1}^{\rm tr}. \quad (38)$$

Notice that the boundary rotation alters neither $\mathcal{P}[Q(\mathbf{r})]$ nor $\mathcal{L}[Q(\mathbf{r})]$ for $\mathbf{r}$ inside the medium. The boundary Ward identity then demands $\delta\tilde{F}_{\rm inter} \equiv 0$, i.e.,

$$\delta\tilde{F}_{\rm inter} = \int d\mathbf{r}\,\delta_C\,{\rm str}\left\{R\left(\frac{\pi\nu D_0}{2} Q\partial_z Q + \frac{\tilde{N}_d\Omega^{d-1}}{4} T_0[Q,\Lambda]\right)\right\}$$
$$= 0. \quad (39)$$

As $\mathcal{R}, \bar{\mathcal{R}}$ are arbitrary this requires

$$(\tilde{l} Q\partial_z Q + T_0[Q,\Lambda])_\perp|_{z=0} = 0 \quad (40)$$

to be met, where the subscript $\perp$ stands for the off-diagonal component in the retarded or advanced sector (thereby anticommunting with Λ) and $\tilde{l} = 2\pi\nu D_0/(\tilde{N}_d\Omega^{d-1})$.

Equation (40) is the field-theoretic version of the radiative boundary condition.[12,29,30] As we will show in Sec. IV, it describes that the low-energy coherent dynamics penetrates into the vacuum with the extrapolation length

$$\zeta = \frac{\tilde{l}}{2T_0} = \frac{\pi\nu D_0}{\tilde{N}_d\Omega^{d-1}}\frac{1}{T_0}. \quad (41)$$

Notice that it is proportional to the inverse transmission coefficient in agreement with Ref. 29. For $d=3$ in the case of perfect transmission, i.e., $T_0=1$ the extrapolation length is $\zeta = \frac{2}{3} l$ in agreement with Ref. 12, and is closed to the one that was obtained by solving the Milne equation,[31] which gives $\zeta = 0.7l$. Traditionally the radiative boundary condition is imposed to diffusion equation to mimic the leakage at the interface[12,29,30,32] and is justified for one-dimensional discrete random walk.[32]

III. TWO-DIMENSIONAL RENORMALIZATION EFFECTS OF INFINITE MEDIUM BAR

In the rest of this paper we will apply the supersymmetric field-theoretic formalism to the semi-infinite two-dimensional medium bar (with the width $a \gg l$), where in the transverse (ρ) direction the photon motion is confined. The purpose of this section is twofold: On the physical side, we wish to explore how a finite width affects localization in the bulk, which differs in essentially from localization in an infinite bar. Accordingly, throughout this section the action reads out as $F[Q] = \int_{-\infty}^{\infty} dz \int_0^a d\rho \mathcal{L}[Q]$. On the technical side, by presenting some details we wish to address the difference of calculations between semi-infinite and infinite bar, which originates at the fact that in the former system the translational symmetry of low-energy modes, i.e., the Q field is broken.

Following the standard strategy we factorize the T field into the slow and fast mode in terms of $T = T_> T_<$, where rotations $T_>$ ($T_<$) involve spatial fluctuations on short (large) scales. Substituting it into the action we then obtain

$$F[Q] = \frac{\pi\nu}{8}\int_{-\infty}^{\infty} dz \int_0^a d\rho\,{\rm str}\{D_0((\partial Q_>)^2 + 4Q_> \partial Q_> \Phi$$
$$+ [\Phi, Q_>]^2) + 2i\omega^+ Q_> T_<^{-1}\Lambda T_<\}, \quad (42)$$

where $Q_> = T_> \Lambda T_>^{-1}$ and $\Phi = T_<^{-1}\partial T_<$. Integrating out $Q_>$ results in an effective action of $Q_<$.

A. Parametrization of fast modes

To work out the strategy outlined above we set $T_> = 1 + iW_>$ with $W_>$ parametrized by

$$W_> = \begin{pmatrix} 0 & B_> \\ \bar{B}_> & 0 \end{pmatrix}^{\rm ar}. \quad (43)$$

Since photons are confined the current vanishes at $\rho = 0, a$, i.e., $\partial_\rho W_>(\mathbf{r})|_{\rho=0,a} = 0$. We may thus introduce the Fourier transformation

$$W_>(\mathbf{r}) = \frac{2}{a}\int_{|k|\ge k_0} \frac{dk}{2\pi}\sum_{n\ge n_0} W_{k,n\pi/a} e^{ikz}\cos\frac{n\pi\rho}{a}, \quad (44)$$

where k_0 and $\pi n_0/a$ are the ultraviolet cutoffs of the longitudinal and transverse wave number, respectively. In Eq. (43) the matrix $B_>$ has the structure as

$$B_> = \begin{pmatrix} a & i\sigma \\ \eta & ib \end{pmatrix}^{\rm bf}, \quad (45)$$

with

$$a=\begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix}^{\rm tr}, \quad b=\begin{pmatrix} b_1 & b_2 \\ b_2^* & b_1^* \end{pmatrix}^{\rm tr},$$

$$\sigma=\begin{pmatrix} \sigma_1 & \sigma_2 \\ -\sigma_2^* & -\sigma_1^* \end{pmatrix}^{\rm tr}, \quad \eta=\begin{pmatrix} \eta_1 & \eta_2 \\ \eta_2^* & \eta_1^* \end{pmatrix}^{\rm tr}, \tag{46}$$

where a's, b's (σ's, η's) are complex bosonic (Grassmann) numbers, and the charge conjugation transformation of a matrix M is defined as

$$\bar{M}=C_0M^TC_0^T, \quad C_0=\begin{pmatrix} -i\tau_2 & 0 \\ 0 & \tau_1 \end{pmatrix}^{\rm bf}. \tag{47}$$

Straightforward calculations justify the useful identity: $\overline{M_1M_2}=\bar{M}_2\bar{M}_1$.

Importantly, $W_>$ satisfies the following relation:

$$W_>=KW_>^\dagger K, \quad K=\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}^{\rm ar}, \tag{48}$$

which, as shown in Appendix C, enforces the invariance of $W_>$ under the charge conjugation, i.e., $\bar{W}_>=W_>$.

B. One-loop renormalization

Now we study the one-loop renormalization. In doing so we expand $Q_>$ to quadratic order in $W_>$. Consequently, the action separates into three contributions as follows:

$$F=F_S+F_F+F_{SF}, \tag{49}$$

where the slow mode action is

$$F_S=\frac{\pi\nu}{8}\int_{-\infty}^{\infty}dz\int_0^a d\rho\,{\rm str}[D_0(\partial Q_<)^2+2i\omega^+\Lambda Q_<], \tag{50}$$

with $Q_<=T_<\Lambda T_<^{-1}$, the fast mode action is

$$F_F=\frac{\pi\nu D_0}{2}\int_{-\infty}^{\infty}dz\int_0^a d\rho\,{\rm str}(\partial W_>)^2, \tag{51}$$

and the slow-fast mode coupling is described by the action

$$F_{SF}=\pi\nu\int_{-\infty}^{\infty}dz\int_0^a d\rho\,{\rm str}\Big\{D_0([W_>,\partial W_>]\Phi-(\Phi\Lambda W_>)^2 - (\Phi\Lambda)^2W_>^2-i\,\partial W_>\Phi-i(\Phi\Lambda)^2W_>) - \frac{i\omega^+}{2}(iW_>\Lambda T_<^{-1}\Lambda T_<+W_>^2\Lambda T_<^{-1}\Lambda T_<)\Big\}. \tag{52}$$

In order to calculate the general average with respect to the fast mode action: $\langle\cdots\rangle_F\equiv\int D[W_>](\cdots)e^{-F_F}$ we employ the contraction rule below.

$$2\pi\nu\langle W_>(\mathbf{r})\bar{M}W_>(\mathbf{r}')\rangle_F=-\mathcal{D}_F(\mathbf{r},\mathbf{r}')M, \quad M=\begin{pmatrix} 0 & M^{12} \\ M^{21} & 0 \end{pmatrix}^{\rm ar},$$

$$4\pi\nu\langle W_>(\mathbf{r})NW_>(\mathbf{r}')\rangle_F=\mathcal{D}_F(\mathbf{r},\mathbf{r}')({\rm str}\,N-\Lambda\,{\rm str}\,\Lambda N), \quad N=\begin{pmatrix} N^{11} & 0 \\ 0 & N^{22} \end{pmatrix}^{\rm ar},$$

$$2\pi\nu\langle{\rm str}[W_>(\mathbf{r})M_1]{\rm str}[W_>(\mathbf{r}')M_2]\rangle_F = \mathcal{D}_F(\mathbf{r},\mathbf{r}'){\rm str}[(M_1+\bar{M}_1)M_2], \quad M_{1,2}=\begin{pmatrix} 0 & M_{1,2}^{12} \\ M_{1,2}^{21} & 0 \end{pmatrix}^{\rm ar}, \tag{53}$$

with the fast mode propagator

$$\mathcal{D}_F(\mathbf{r},\mathbf{r}')=\frac{2}{a}\int_{|k|\ge k_0}\frac{dk}{2\pi}\sum_{n\ge n_0}\frac{\cos\frac{n\pi(\rho-\rho')}{a}e^{ik(z-z')}}{D_0\left[\left(\frac{n\pi}{a}\right)^2+k^2\right]}, \tag{54}$$

and the Wick theorem. Notice that we assume that the fast mode propagator does not depend on the transverse center-of-mass coordinate, i.e., $(\rho+\rho')/2$ and the self-average over this variable has been performed.

Performing the average of F_{SF} we obtain an effective action $F_{\rm eff}[Q_<]=F_S+\langle F_{SF}\rangle_F$, where

$$\langle F_{SF}\rangle_F=-\frac{I_0}{8}\int_{-\infty}^{\infty}dz\int_0^a d\rho\,{\rm str}(\partial Q_<)^2, \tag{55}$$

with

$$I_0=\frac{2}{a}\int_{|k|\ge k_0}\frac{dk}{2\pi}\sum_{n\ge n_0}\frac{1}{\left(\frac{n\pi}{a}\right)^2+k^2}. \tag{56}$$

Equations (55) and (56) show that the one-loop renormalization results in the weak localization correction to the bare diffusion constant D_0 as follows:

$$F_{\rm eff}[Q_<]=\frac{\pi\nu}{8}\int_{-\infty}^{\infty}dz\int_0^a d\rho\,{\rm str}\{[D_0+\delta D^{(2)}](\partial Q_<)^2 + 2i\omega^+\Lambda Q_<\}, \tag{57}$$

with the two-dimensional weak localization correction $\delta D^{(2)}=-I_0/(\pi\nu D_0)$.

C. Dimensional crossover of effective action

In the high-frequency region, i.e., $\omega\gg D_0/a^2$, the condition $k_0a/\pi,n_0\sim\sqrt{\omega a^2/D_0}\gg1$ is met. The photon motion is thus two-dimensional described by the action, Eq. (57). Furthermore, since $\Omega l\gg1$ the two-dimensional weak localization correction $\delta D^{(2)}$ is much smaller than D_0, the photon motion thereby is diffusive.

In the low-frequency region, i.e., $\omega\lesssim D_0/a^2$ one may further enforce $k_0=0$ and $n_0=1$ and thereby obtain a quasi-1D effective action of $\check{Q}\equiv Q_<(z)$, which is homogeneous in the transverse direction.

$$F_{\rm eff}[\check{Q}]=\int_{-\infty}^{\infty}dz\mathcal{L}_{\rm eff}[\check{Q}],$$

$$\mathcal{L}_{\rm eff}[\breve{Q}] = \frac{\pi\nu a}{8}\,{\rm str}[D_{\rm eff}(\partial_z\breve{Q})^2 + 2i\omega^+\Lambda\breve{Q}], \tag{58}$$

where the renormalized diffusion constant is

$$D_{\rm eff} = D_0\left(1 - \frac{1}{\pi\nu D_0}\sum_{n\geq 1}\frac{1}{n\pi}\right). \tag{59}$$

Note that in the above the second term suffers logarithmic divergence which, as usual, may be regularized by introducing the upper cutoff N, which is an order of $\sim a/(\pi l)$. As a result,

$$D_{\rm eff} \approx D_0\left(1 - \frac{1}{\pi^2\nu D_0}\ln\frac{a}{\pi l}\right). \tag{60}$$

Thus, in the low-frequency region $\omega \lesssim D_0/a^2$ the system is quasi-one-dimensional provided that the bar width satisfies $a \ll l e^{\pi^2\nu D_0}$. For wider bar the system displays two-dimensional strong localization, which is beyond the scope of the present perturbative analysis.

IV. WEAK LOCALIZATION IN SEMI-INFINITE TRANSPARENT MEDIUM BAR

The discussions of Sec. III break down in the semi-infinite geometry due to the absence of the translational symmetry. In this section and the next we turn to study the vacuum-medium interface effect on wave interference.

A. Simplified boundary condition

In order to explore the physics implied by the boundary constraint Eq. (40) let us parametrize T in the same way as Eqs. (43), (45), and (46). (To distinct notations from those of Sec. III we eliminate all the subscript $>$.) With the substitution of the parametrization and keeping Eq. (40) up to the first order in W we obtain

$$\begin{pmatrix} 0 & (\tilde{l}\partial_z - 2T_0)B \\ (\tilde{l}\partial_z - 2T_0)\bar{B} & 0 \end{pmatrix}^{\rm ar} = 0, \tag{61}$$

implying $B,\bar{B} \sim e^{z/\zeta}, z<0$ with $\zeta = \tilde{l}/(2T_0)$. Hence the low-energy Goldenstone modes penetrate into the vacuum of a depth ζ then exponentially decays. That is, the optical paths underlying coherent multiple scattering do not cross the line located at $z=-\zeta$.

From now on we assume that the interface is almost transparent, namely, T_0 closed to 1. In this case $\zeta = \frac{2}{3}l$. Since the mean free path l is much smaller than any other macroscopic scale we may safely assume that the crossing line where W vanishes coincides with C. Consequently, the boundary constraint Eq. (40) is simplified as

$$Q|_{z=0} = \Lambda. \tag{62}$$

It is the action $F[Q] = \int_0^\infty dz \int_0^a d\rho \mathcal{L}[Q]$ with the Q field subject to this boundary constraint that we will use in the rest of this paper.

B. Bare diffusive propagator

Expanding Q in terms of W gives

$$F[Q] = F_2[W] + F_4[W] + \cdots, \tag{63}$$

where the Gaussian action

$$F_2[W] = \frac{\pi\nu}{2}\int_0^\infty dz \int_0^a d\rho\,{\rm str}[D_0(\partial W)^2 - i\omega W^2], \tag{64}$$

and

$$F_4[W] = \frac{\pi\nu}{2}\int_0^\infty dz \int_0^a d\rho\{-2D_0\,{\rm str}[(\partial W)^2 W^2] + i\omega\,{\rm str}\,W^4\}. \tag{65}$$

From Eq. (64) immediately we obtain the same contraction rules as Eq. (53) except making the replacement

$$\mathcal{D}_F(\mathbf{r},\mathbf{r}') \to \mathcal{D}(\mathbf{r},\mathbf{r}';\omega), \tag{66}$$

where the propagator solves the diffusion equation as follows:

$$(-D_0\partial^2 - i\omega)\mathcal{D}(\mathbf{r},\mathbf{r}';\omega) = \delta(\mathbf{r}-\mathbf{r}'),$$

$$\mathcal{D}|_{\mathbf{r}\ \text{or}\ \mathbf{r}'\in C} = 0. \tag{67}$$

The boundary condition above is inherent from the constraint Eq. (62) which imposes $W(\mathbf{r})|_{\mathbf{r}\in C}=0$.

Keeping the prefactor of Eq. (11) up to quadratic term we obtain the leading cooperon propagator as follows:

$$\mathcal{Y}^{\rm C}_{(0)}(\mathbf{r},\mathbf{r}';\omega) = \left[\frac{\pi N(\Omega^2)}{2}\right]^2 \langle {\rm str}\{k(1+\Lambda)(1-\tau_3)W(\mathbf{r}) \times k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')\}\rangle_{F_2} = \frac{2\pi\nu}{\Omega^2}\mathcal{D}(\mathbf{r},\mathbf{r}';\omega). \tag{68}$$

It is easy to see that the propagator above preserves the symmetry $\mathcal{Y}^{\rm C}_{(0)}(\mathbf{r},\mathbf{r}';\omega) = \mathcal{Y}^{\rm C}_{(0)}(\mathbf{r}',\mathbf{r};\omega)$ inherent from Eq. (11). Equation (68) is traditionally obtained by summing up all the ladder diagrams and imposing an appropriate boundary condition.[7]

C. Weak localization correction

We proceed to calculate the one-loop correction to the bare propagator $\mathcal{Y}^{\rm C}_{(0)}$. For this purpose we keep the W expansion up to the quartic terms for both the prefactor and the action which, after straightforward calculations, gives the cooperon as $\mathcal{Y}^{\rm C} \approx \mathcal{Y}^{\rm C}_{(0)} + \delta\mathcal{Y}^{\rm C}$ with

$$\delta\mathcal{Y}^{\rm C}(\mathbf{r},\mathbf{r}';\omega) = -\left[\frac{\pi N(\Omega^2)}{2}\right]^2 \langle {\rm str}\{k(1+\Lambda)(1-\tau_3)W^3(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')\}$$

$$+ \mathrm{str}\{k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W^3(\mathbf{r}')\} - \mathrm{str}\{k(1+\Lambda)(1-\tau_3)W^2(\mathbf{r})k(1-\Lambda)(1+\tau_3)W^2(\mathbf{r}')\} + \mathrm{str}\{k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')\}F_4[W]\rangle_{F_2}. \quad (69)$$

First, it is easy to show that the third term in the right-hand side of Eq. (69) vanishes. Second, as shown in Appendix D the first two terms partly cancel the last term. Eventually Eq. (69) is reduced into

$$\delta\mathcal{Y}^C(\mathbf{r},\mathbf{r}';\omega) = -\left[\frac{\pi N(\Omega^2)}{2}\right]^2(\pi\nu D_0)\int_0^\infty dz_1\int_0^a d\rho_1\langle \mathrm{str}\{k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')\} \times \mathrm{str}[\partial^2 \overbrace{W(\mathbf{r}_1)W(\mathbf{r}_1)W}(\mathbf{r}_1)W(\mathbf{r}_1) + (\partial W(\mathbf{r}_1)W(\mathbf{r}_1))^2 + (\partial W(\mathbf{r}_1))^2W^2(\mathbf{r}_1)]\rangle_{F_2}, \quad (70)$$

where the overbrace fixes the contraction and the derivative acts only on the nearest W. We remark that $\delta\mathcal{Y}^C$ vanishes when $\mathcal{Y}^C_{(0)}$ is spatially homogeneous, which is a reflection of the flux conservation law or Ward identity at the one-loop level. Notice that $\delta\mathcal{Y}^C$ vanishes if either $\mathbf{r}$ or $\mathbf{r}'$ belongs to the interface C. Such property is inherent from Eq. (11), which vanishes upon sending either $Q(\mathbf{r})$ or $Q(\mathbf{r}')$ to Λ. Using the contraction rules and integral by parts we further reduce Eq. (70) into

$$\delta\mathcal{Y}^C(\mathbf{r},\mathbf{r}';\omega) = \frac{2D_0}{\Omega^2}\int_0^\infty dz_1\int_0^a d\rho_1\mathcal{D}(\mathbf{r}_1,\mathbf{r}_1;\omega) \times \partial_{\mathbf{r}_1}\mathcal{D}(\mathbf{r},\mathbf{r}_1;\omega)\partial_{\mathbf{r}_1}\mathcal{D}(\mathbf{r}',\mathbf{r}_1;\omega) \quad (71)$$

after tedious but straightforward calculations. Notice that $\delta\mathcal{Y}^C$ preserves the symmetry: $\delta\mathcal{Y}^C(\mathbf{r},\mathbf{r}';\omega) = \delta\mathcal{Y}^C(\mathbf{r}',\mathbf{r};\omega)$.

D. Local diffusion equation

Equation (71) justifies that $\mathcal{Y}^C = \mathcal{Y}^C_{(0)} + \delta\mathcal{Y}^C$ solves the following local diffusion equation:

$$\{-\partial D(\mathbf{r};\omega)\,\partial - i\omega\}\mathcal{Y}^C(\mathbf{r},\mathbf{r}';\omega) = \delta(\mathbf{r}-\mathbf{r}'),$$

$$\mathcal{Y}^C|_{\mathbf{r}\in C} = 0 \quad (72)$$

at the one-loop level. Here $D(\mathbf{r};\omega) = D_0 + \delta D(\mathbf{r};\omega)$ with the weak localization correction

$$\delta D(\mathbf{r};\omega) = -\frac{D_0}{\pi\nu}\mathcal{D}(\mathbf{r},\mathbf{r};\omega). \quad (73)$$

The local diffusion equation differs from the traditional one in that the diffusion coefficient is position dependent. It may amount to incompletely developed constructive interference between two counterpropagating optical paths—which leads to the weak localization—near the boundary. Indeed, although deep inside the medium $\delta D(\mathbf{r};\omega)$ saturates recovering the bulk weak localization, at the interface it vanishes, i.e.,

$$\delta D(\mathbf{r};\omega)|_{z=0} = 0. \quad (74)$$

Importantly, this is contrary to the theoretical proposal of Ref. 9, which claims that wave interference democratically renormalizes the diffusion constant appearing in both the diffusion equation in the bulk and the radiative boundary condition at the interface.

Here several remarks are in order: (i) Higher order loop corrections preserve Eq. (72). They affect the local diffusion equation by introducing higher order weak localization corrections which are also position dependent. This peculiar property reflects the photon number conservation law and is protected by Ward identity. (ii) In the presence of internal reflection, namely, $T_0(\mathbf{r})$ (far) below 1, (i) is no longer applicable because the simplification, namely, Eq. (62) breaks down due to large extrapolation length. In fact, Ref. 12 falls into this case. (iii) The concept of local diffusion originally introduced in Ref. 12 at the static limit, i.e., $\omega \to 0$ together with its dynamic generalization[17,18] is now justified at the perturbative level.

V. STATIC LIMIT OF LOCAL DIFFUSION EQUATION

In this section we study the static limit: $\omega \to 0$ (for this reason below we suppress the argument ω in all the formulas.) of the local diffusion equation, namely, Eq. (72) for a bar with the width satisfying $l \ll a \ll le^{\pi^2\nu D_0}$. In particular, we will explicitly calculate the weak localization correction Eq. (73), and study its effects on the coherent backscattering phenomenon.

A. Quasi-1D massive local diffusion equation

In the static limit the weak localization correction Eq. (73) becomes self-averaged over the center-of-mass ρ and thereby is ρ independent. That is,

$$\delta D(z) = -\frac{D_0}{\pi\nu a}\int_0^a d\rho\mathcal{D}(z,\rho,z,\rho). \quad (75)$$

Substituting Eq. (75) into Eq. (72) we find that $\mathcal{Y}^C(\mathbf{r}',\mathbf{r})$ depends on $\rho-\rho'$, but not on the center-of-mass $(\rho+\rho')/2$. Therefore, we may introduce the Fourier transform as follows:

$$\mathcal{Y}^{\mathrm{C}}(z,z',\rho-\rho') \equiv \frac{1}{2}\mathcal{Y}_0^{\mathrm{C}}(z,z') + \sum_{n\geq 1}\mathcal{Y}_{\pi n/a}^{\mathrm{C}}(z,z')\cos\frac{\pi n(\rho-\rho')}{a}, \tag{76}$$

and insert it into Eq. (72) to obtain $\left(q_\perp \equiv \frac{n\pi}{a}\right)$

$$\{-\partial_z D(z)\partial_z + D(z)q_\perp^2 - i0^+\}\mathcal{Y}_{q_\perp}^{\mathrm{C}}(z,z') = \delta(z-z'),$$

$$\mathcal{Y}_{q_\perp}^{\mathrm{C}}(z=0,z') = 0, \tag{77}$$

where 0^+ is an infinitesimal positive constant. Equation (77) may be considered to be a quasi-one-dimensional local diffusion equation with a mass $D(z)q_\perp^2$.

B. Dimensional crossover of weak localization

The weak localization correction Eq. (75) then becomes

$$\delta D(z) = -\frac{D_0}{\pi\nu a}\left[\mathcal{D}_0(z,z) + 2\sum_{n\geq 1}\mathcal{D}_{n\pi/a}(z,z)\right]. \tag{78}$$

Here $\mathcal{D}_{q_\perp}(z,z')$ satisfies

$$\{D_0[-\partial_z^2 + q_\perp^2] - i0^+\}\mathcal{D}_{q_\perp}(z,z') = \delta(z-z'),$$

$$\mathcal{D}_{q_\perp}(z=0,z') = 0. \tag{79}$$

It is solved by (with the introduction of $\tilde{q}_\perp = \sqrt{q_\perp^2 - i0^+}$)

$$\frac{\mathcal{D}_{q_\perp}(z,z')}{\pi\nu a} = \begin{cases} \dfrac{1}{2\xi}\dfrac{e^{\tilde{q}_\perp z'} - e^{-\tilde{q}_\perp z'}}{\tilde{q}_\perp}e^{-\tilde{q}_\perp z}, & z > z', \\ \dfrac{1}{2\xi}\dfrac{e^{\tilde{q}_\perp z} - e^{-\tilde{q}_\perp z}}{\tilde{q}_\perp}e^{-\tilde{q}_\perp z'}, & z < z', \end{cases} \tag{80}$$

where $\xi = \pi\nu a D_0$. Substituting it into Eq. (78) gives

$$\frac{\delta D(z)}{D_0} = -\frac{z}{\xi} - \frac{a}{\xi}\sum_{n\geq 1}\frac{1 - e^{-2n\pi z/a}}{n\pi}, \tag{81}$$

where the first term is the quasi-one-dimensional contribution, and the second term is the two-dimensional contribution with $n\pi/a$ standing for the transverse hydrodynamic wave number.

As expected at $z=0$ the weak localization correction δD vanishes. Away from the interface, i.e., $l \lesssim z \ll a$ it may be approximated by

$$\frac{\delta D(z)}{D_0} = -\frac{z}{\xi} - \frac{a}{\pi\xi}\ln\frac{z}{l}, \tag{82}$$

as shown in Appendix E. This suggests that in this region (even in the static limit) the two-dimensional low-energy motion dominates the weak localization.

The two-dimensional contribution saturates at $z \sim a$ as follows:

$$\frac{\delta D(z)}{D_0} = -\frac{z}{\xi} - \frac{a}{\xi}\sum_{n\geq 1}\frac{1}{n\pi}, \tag{83}$$

where the second term is none but the bulk weak localization correction [see Eq. (59)] renormalizing the bare diffusion constant D_0. With this taken into account Eq. (83) may be rewritten as

$$\frac{\delta D_{1\mathrm{D}}(z)}{D_{\mathrm{eff}}} = -\frac{z}{\xi_{1\mathrm{D}}}, \tag{84}$$

where $\delta D_{1\mathrm{D}}(z)$ stands for the quasi-one-dimensional weak localization correction, and $\xi_{1\mathrm{D}} = \pi\nu a D_{\mathrm{eff}}$ is the exact localization length.[20,21] Equation (84) agrees with the leading $z/\xi_{1\mathrm{D}}$ expansion of the local diffusion coefficient given in Ref. 12. It thereby justifies that for $z \gtrsim a$ the medium bar displays the quasi-one-dimensional (interface) weak localization. Indeed, at early times $t \lesssim D_0/a^2$ incident photons explore a region of size a^2 neighboring to the interface, approaching a uniform distribution in the transverse direction. At later times they diffuse as in a quasi-one-dimensional medium. Technically, starting from the quasi-one-dimensional σ model by performing the one-loop calculation one finds Eq. (84). Importantly, from Eqs. (82) and (84) we find that within the boundary layer $z \lesssim \xi$ weak *rather than strong* localization occurs even in the static limit.

C. CBS line shape: Crossover from 2D weak to quasi-1D strong localization

In this part we turn to investigate effects of local diffusion on the CBS line shape. We will consider a medium illuminated by the light of frequency Ω parallel to the bar, and calculate the angular resolution of the backscattered light intensity $\alpha(\theta)$ near the inverse incident direction. Since the bar is wide enough so that $l \ll a(\ll l e^{\pi^2\nu D_0})$ a large parametric region $\lambda/a \lesssim \theta \lesssim \lambda/l$ is opened. Below we pay particular attention to the line shape at $0 \lesssim \theta \lesssim \lambda/l$. (Notice that the line shape is symmetric with respect to $\theta=0$.)

It is well known that the backscattered light intensity may be decomposed into the background α_0 and the coherent part $\alpha_c(\theta)$ according to

$$\alpha(\theta) = \alpha_0 + \alpha_c(\theta). \tag{85}$$

Here

$$\alpha_0 = \int\int d\mathbf{r}d\mathbf{r}'e^{-(z+z'/l)}\mathcal{Y}^{\mathrm{D}}(\mathbf{r},\mathbf{r}'),$$

$$\alpha_c(\theta) = \int\int d\mathbf{r}d\mathbf{r}'e^{-(z+z'/l)}\cos[\mathbf{q}_\perp\cdot(\mathbf{r}-\mathbf{r}')]\mathcal{Y}^{\mathrm{C}}(\mathbf{r},\mathbf{r}'), \tag{86}$$

where $q_\perp = 2\Omega\sin(\theta/2) \approx 2\pi\theta/\lambda$ because of $\theta \ll 1$, and the overall normalization factor is omitted. First of all, it is easy to show that $\alpha_0 = \alpha_c(0)$ and therefore only the coherent part $\alpha_c(\theta)$, which determines the line shape, will be studied below. Inserting the Fourier transform, namely, Eq. (76) into $\alpha_c(\theta)$, we arrive at

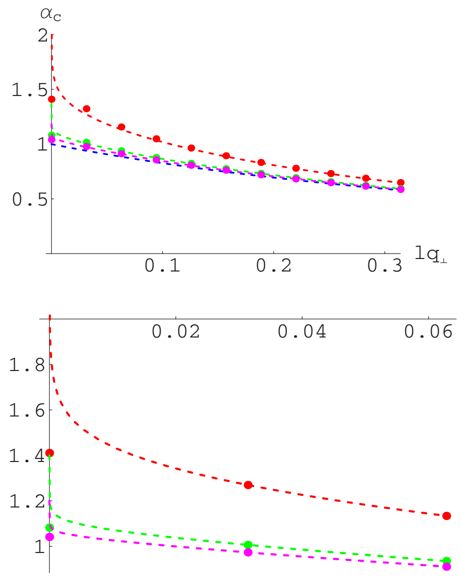

FIG. 1. (Color online) Coherent backscattering intensity (solid circle) in unit of l^3/D_0 versus $q_\perp = n\pi/a$ for the medium bar with $a/l=100$. Upper panel: The logarithmic enhancement–dashed line—is cut off at $q_\perp=0$. From top to bottom the parameter l/λ is 2 (red), 10 (green), 20 (purple), and ∞ (blue). Lower panel: The magnification of the upper panel for $lq_\perp$ closed to 0 (the blue curve is not plotted).

$$\alpha_c(\theta) = \int_0^\infty \int_0^\infty dz dz' e^{-(z+z'/l)} \mathcal{Y}^C_{q_\perp}(z,z') \approx l^2 \mathcal{Y}^C_{q_\perp}(l,l), \tag{87}$$

where the propagator $\mathcal{Y}^C_{q_\perp}(z,z')$ solves Eq. (77).

1. Signatures of 2D weak localization

The interfering optical paths penetrate into the medium of a depth $\sim q_\perp^{-1}$. Let us first study the CBS line shape in the region $\pi/a \ll q_\perp \lesssim l^{-1}$. Because of the condition $l \lesssim q_\perp^{-1} \ll a$ the CBS line shape is mainly responsible for photons which diffuse around the interface, i.e., $l \lesssim z \ll a$ and thereby undergo two-dimensional weak localization. Indeed, the first term of Eq. (81) is much smaller due to the condition $|q_\perp a|/\pi \gg 1$. Setting the ultraviolet cutoff $N \sim a/(\pi l)$ [as Eq. (60)] we may approximate Eq. (81) by

$$\frac{\delta D(z)}{D_0} \approx -\frac{a}{\xi} \sum_{aq_\perp/\pi}^{a/(\pi l)} \frac{1-e^{-2n\pi z/a}}{n\pi} \approx \frac{a}{\pi\xi} \ln|q_\perp l|. \tag{88}$$

With the substitution of such weak localization correction into Eq. (77) we find

$$\alpha_c(\theta) = \frac{l^3}{D_0}(1-2lq_\perp)\left[1 - \frac{a}{\pi\xi}\ln|q_\perp l|\right], \quad \pi/a \ll q_\perp \lesssim l^{-1}. \tag{89}$$

According to Eq. (89) the conventional triangular peak described by the factor $1-2lq_\perp$ is enhanced by a logarithmic factor (dashed line in Fig. 1). Equation (88) indicates that in the region $\pi/a \ll q_\perp \lesssim l^{-1}$ the local diffusion is of minor importance. It is the two-dimensional bulk weak localization that is responsible for such a logarithmic enhancement. Thus in such a region the scaling theory, still, is applicable.

2. Signatures of quasi-1D strong localization

At the exact backscattering direction, i.e., $\theta=0$ interfering optical paths penetrate into the medium bulk $z \gg \xi$ where quasi-one-dimensional (bulk) strong localization states are formed. In contrast to $\pi/a \ll q_\perp \lesssim l^{-1}$ for $q_\perp=0$ the local diffusion plays crucial roles and thus strongly affects the CBS line shape. Indeed, despite the nonperturbative nature of strong localization the backscattering light intensity at $q_\perp=0$ may be easily found provided that in the region $z \gg \xi$ the local diffusion equation is still valid.[12] For $q_\perp=0$ from Eq. (77) one may find

$$\mathcal{Y}^C_{q_\perp=0}(l,l) = \int_0^l dz \frac{1}{D(z)} \approx \frac{l}{D_{\text{eff}}}, \tag{90}$$

where in the second equality the substitution of Eq. (83) is made. This immediately gives

$$\alpha_c(0) = \frac{l^3}{D_{\text{eff}}}. \tag{91}$$

Equation (91) shows that although due to two-dimensional bulk weak localization the CBS line shape develops a logarithmic singularity at $\pi/a \lesssim q_\perp \lesssim l^{-1}$, the singularity is cut off at $q_\perp=0$ (Fig. 1) where quasi-one-dimensional strong localization occurs in the bulk.

It is in order to remark that formally there is a region $q_\perp \xi \lesssim 1$ where the rounding due to the local diffusion occurs [yet, the detailed rounding form depends on $D(z)$ in the region $z \gtrsim \xi$, to find which is beyond the present perturbative treatise], however, it is unobservable because the finite bar width renders $\xi^{-1} \ll \pi/a$. Finally, we anticipate that the predicted CBS line shape is qualitatively correct for $l \gtrsim \lambda$, though the analytical result here is obtained for $l \gg \lambda$.

VI. CONCLUSIONS

For light propagation in fully disordered media a supersymmetric field theory is presented. The supersymmetric σ model described by Eq. (18) may be applied to bulk (infinite) media for studies of optical localization transition. In this direction it may serve as an alternative to the replica field theory.[1] However, the supersymmetric field-theoretic formalism turns out to be far more powerful as propagation of incident light in semi-infinite media[23] concerned, which is the subject of this paper and is closely related to the coherent backscattering phenomenon.

Differing from infinite medium in the presence of the vacuum-medium interface C (the interface may bear arbitrary geometry but must be smooth over the scale of the mean free path), the supermatrix field (locally) satisfies the radiative boundary condition Eq. (40). Accordingly, the bare diffusion constant acquires a position-dependent wave interference correction, namely,

$$D(\mathbf{r};\omega) = D_0 + \delta D(\mathbf{r};\omega), \tag{92}$$

which roots in the incomplete constructive interference (weak localization) near the interface. Thus, we justify the (static) local diffusion equation, originally proposed in Ref. 12, as well as its dynamic (i.e., $\omega \neq 0$) generalization.[17] Most importantly, for (almost) transparent interface, i.e., $T_0(\mathbf{r}) \approx 1, \mathbf{r} \in C$, the weak localization correction $\delta D(\mathbf{r};\omega)$ vanishes at the interface. This immediately shows that the radiative boundary condition is protected against wave interference effects, and constitutes an explicit proof that no scaling hypothesis might exist in the extrapolation layer of thickness $\sim l$. Therefore, the present work supports the criticism of Refs. 10 and 12 on the earlier theoretical proposal.[9]

In the present work the static limit of the wave interference (weak localization) correction, namely, $\delta D(\mathbf{r};\omega\to 0)$ is explicitly calculated for the two-dimensional semi-infinite medium with a finite width a (the bar geometry), where $\delta D(\mathbf{r};\omega\to 0)$ solely depends on the distance from the interface z. For $l \ll a \ll l e^{\pi^2 \nu D_0}$ a dimensional crossover of the wave interference correction $\delta D(z) \equiv \delta D(\mathbf{r};\omega\to 0)$ is found. Indeed, $\delta D(z)$ displays two-dimensional weak localization at $z \ll a$ with a logarithmic dependence on z, while displays one-dimensional weak localization at $a \ll z (\ll \xi)$. Furthermore, for the latter region it is not difficult to generalize Eq. (84) to higher order loop corrections, which reads out as

$$\frac{\delta D_{1\mathrm{D}}(z)}{D_{\mathrm{eff}}} = -\frac{z}{\xi_{1\mathrm{D}}} + \sum_{n=2}^{\infty} c_n \left(\frac{z}{\xi_{1\mathrm{D}}}\right)^n. \tag{93}$$

This—at the perturbative level—formally confirms the result of Ref. 12 for one-dimensional geometry. Notice that the unimportant numerical expansion coefficients c_n may vary depending on the strict or quasi-one-dimensional geometry.

For wider medium bar such that $a \gg l e^{\pi^2 \nu D_0}$ (e.g., infinite medium plane) Eq. (82), in fact, is still applicable except that the one-dimensional contribution vanishes. That is,

$$\frac{\delta D(z)}{D_0} = -\frac{1}{\pi^2 \nu D_0} \ln \frac{z}{l}. \tag{94}$$

This suggests that in the medium there exists a boundary layer of thickness $\sim l e^{\pi^2 \nu D_0}$ outside which two-dimensional strong localization occurs. Surprisingly, inside the layer the diffusion coefficient logarithmically depends on the distance from the interface. How to reproduce this logarithmic dependence by the self-consistent diagrammatical method[12] is unclear.

Finally, it should be stressed that the present field-theoretic justification of local diffusion is perturbative. Therefore, the validity of such a concept in the nonperturbative strong localization region remains an important question. This problem is far beyond the scope of this paper and will be addressed in a forthcoming paper, especially the issue how Eq. (93) is extended to the nonperturbative region $z \gtrsim \xi$. It is well known that in Faraday-active medium the one-loop weak localization may be strongly suppressed.[33] Therefore, to take into account such medium within the present field-theoretic formalism remains another important problem. It is also interesting to generalize the present field-theoretic formalism to include medium gain.[34] These issues are left for future work.

Note added in proof. Recently, Cherroret and Skipetrov presented a diagrammatic derivation of Eqs. (72) and (73).[35]

ACKNOWLEDGMENTS

I am deeply grateful to A. Altland, T. Micklitz for useful conversations, and especially S. E. Skipetrov and M. R. Zirnbauer for several fruitful discussions. This work was supported by Transregio SFB 12 of the Deutsche Forschungsgemeinschaft.

APPENDIX A: PROOF OF EQ. (20)

Assuming that $\mathbf{r}, \mathbf{r}'' \in \mathcal{V}_-$ and $\mathbf{r}' \in \mathcal{V}_+$, from the Helmholtz equation (1) we obtain

$$G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')\{\nabla^2 + \Omega_+^2[1+\epsilon(\mathbf{r})]\}[g^A_{\Omega^2}(\mathbf{r},\mathbf{r}'')]^* = \delta(\mathbf{r}-\mathbf{r}'')G^R_{\Omega^2}(\mathbf{r},\mathbf{r}'), \tag{A1}$$

and

$$[g^A_{\Omega^2}(\mathbf{r},\mathbf{r}'')]^*\{\nabla^2 + \Omega_+^2[1+\epsilon(\mathbf{r})]\}G^R_{\Omega^2}(\mathbf{r},\mathbf{r}') = 0. \tag{A2}$$

Subtracting Eq. (A2) from Eq. (A1) gives

$$\nabla\cdot\{[\nabla g^{A*}_{\Omega^2}(\mathbf{r},\mathbf{r}'')]G^R_{\Omega^2}(\mathbf{r},\mathbf{r}') - [g^{A*}_{\Omega^2}(\mathbf{r},\mathbf{r}'')]\nabla G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}'')G^R_{\Omega^2}(\mathbf{r},\mathbf{r}'), \tag{A3}$$

where ∇ acts only on $\mathbf{r}$. Noticing that $g^A_{\Omega^2}(\mathbf{r},\mathbf{r}'')|_{\mathbf{r}\in C}=0$ and $[g^A_{\Omega^2}(\mathbf{r},\mathbf{r}'')]^* = g^R_{\Omega^2}(\mathbf{r}'',\mathbf{r})$, with $\mathbf{r}\in\mathcal{V}_-$ integrated out we find

$$G^R_{\Omega^2}(\mathbf{r}'',\mathbf{r}') = \int_C d\mathbf{r}\,\partial_{\mathbf{n}(\mathbf{r})}\{g^R_{\Omega^2}(\mathbf{r}'',\mathbf{r})\}G^R_{\Omega^2}(\mathbf{r},\mathbf{r}'). \tag{A4}$$

Here $\partial_{\mathbf{n}(\mathbf{r})}$ stands for the normal derivative at $\mathbf{r}$ with $\mathbf{n}(\mathbf{r})$ pointing to $\mathcal{V}_+$. Taking the derivative we obtain

$$\partial_{\mathbf{n}(\mathbf{r}'')}G^R_{\Omega^2}(\mathbf{r}'',\mathbf{r}') = \int_C d\mathbf{r} B(\mathbf{r}'',\mathbf{r})G^R_{\Omega^2}(\mathbf{r},\mathbf{r}'), \tag{A5}$$

with $B(\mathbf{r}'',\mathbf{r})$ following the definition of Eq. (21).

Now suppose that $\mathbf{r}$ is shuffled to C from the $\mathcal{V}_-$ side. Let us integrate out Eq. (3) over the line element along an infinitesimal piece of a curve passing from $\mathcal{V}_-$ to $\mathcal{V}_+$. In doing so we obtain

$$\partial_{\mathbf{n}(\mathbf{r})}\, G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}\in C^+} - \partial_{\mathbf{n}(\mathbf{r})}\, G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}\in C^-} = 0, \tag{A6}$$

where C^+ (C^-) stands for the curve infinitesimally closed to C from the $\mathcal{V}_+$ ($\mathcal{V}_-$) side. Taking Eq. (A5) into account we may rewrite Eq. (3) as

$$\{\Omega_+^2 - \hat{H}\}G^R_{\Omega^2}(\mathbf{r},\mathbf{r}') - \int_C d\mathbf{r}'' B(\mathbf{r},\mathbf{r}'')G^R_{\Omega^2}(\mathbf{r}'',\mathbf{r}') = 0, \quad \mathbf{r}\in C, \tag{A7}$$

which is supplemented by the boundary condition $\partial_{\mathbf{n}(\mathbf{r})}G^R_{\Omega^2}(\mathbf{r},\mathbf{r}')|_{\mathbf{r}\in C}=0$.

$B(\mathbf{r},\mathbf{r}')$ $(\mathbf{r},\mathbf{r}'\in C)$ consists of the real and imaginary part. The former is small and may be absorbed into Ω_+^2 renormalizing $\epsilon(\mathbf{r})$. It is thus ignored. In contrast, the latter is important determining the analytical structure. Taking this into account we prove Eq. (20) for the retarded (and similarly for the advanced) Green function.

APPENDIX B: SIMPLIFICATION OF THE COUPLING ACTION $F_{\text{inter}}[Q]$

For the moment let us suppress the indices i and $k_\perp$, and decompose Q according to

$$Q = Q_\perp + Q_\parallel,$$

$$[Q_\perp,\Lambda]=0, \quad \{Q_\parallel,\Lambda\}=0. \tag{B1}$$

Taking such decomposition into account we obtain

$$\text{str}\ln(1+\alpha\Lambda Q) = \text{str}\ln(1+\alpha\Lambda Q_\parallel) + \text{str}\ln\left(1+\frac{1}{1+\alpha\Lambda Q_\parallel}\alpha\Lambda Q_\perp\right). \tag{B2}$$

Upon Taylor expanding the second logarithm only the even order terms contribute. Thus, Eq. (B2) may be rewritten as

$$\text{str}\ln(1+\alpha\Lambda Q) = \text{str}\ln(1+\alpha\Lambda Q_\parallel) + \frac{1}{2}\,\text{str}\ln\left(1+\frac{1}{1+\alpha\Lambda Q_\parallel}\alpha\Lambda Q_\perp\right)$$

$$\frac{1}{2}\,\text{str}\ln\left(1-\frac{1}{1+\alpha\Lambda Q_\parallel}\alpha\Lambda Q_\perp\right)$$

$$=\frac{1}{2}\,\text{str}\ln\{(1+\alpha\Lambda Q_\parallel)^2+(\alpha Q_\perp)^2\}$$

$$=\frac{1}{2}\,\text{str}\ln(2+2\alpha^2+4\alpha\Lambda Q_\parallel)$$

$$=\frac{1}{2}\,\text{str}\ln(2-T_0+T_0\Lambda Q_\parallel), \tag{B3}$$

where in deriving the third equality we use the identity $Q_\parallel^2+Q_\perp^2=1$, and in deriving the last two equalities we use the identity $\text{str}\ln\mathbf{1}=0$.

Restoring the index i and substituting Eq. (B3) into Eq. (29) we obtain

$$F_{\text{inter}}[Q] = -\frac{\Omega l}{4\pi}\sum_i \text{str}\ln[2-T_0(i)+T_0(i)\Lambda Q_{i\parallel}]$$
$$\approx -\frac{\Omega l}{4\pi}\sum_i T_0(i)\text{str}(\Lambda Q_{i\parallel}) = -\frac{\Omega l}{4\pi}\sum_i T_0(i)\text{str}(\Lambda Q_i), \tag{B4}$$

where in the second line we take advantage of strong coupling, i.e., $\Omega l \gg 1$, and in the third equality we use the identity $\text{str}(\Lambda Q_\perp)=0$.

APPENDIX C: THE CHARGE-CONJUGATION SYMMETRY OF W

The charge-conjugation symmetry is irrespective of fast-slow mode separation and therefore we ignore the subscript $>(<)$. Substituting the parametrization of W into Eq. (48) we obtain

$$\begin{pmatrix} 0 & B \\ \bar{B} & 0 \end{pmatrix}^{\text{ar}} = \begin{pmatrix} 0 & \bar{B}^\dagger k \\ kB^\dagger & 0 \end{pmatrix}^{\text{ar}}, \tag{C1}$$

giving $kB^\dagger=\bar{B}$ and $\bar{B}^\dagger k=B$. The first relation may be rewritten as

$$B^* k = C_0 B C_0^T. \tag{C2}$$

Substituting the second relation into Eq. (C2) we obtain

$$B^* k = C_0 \bar{B}^{*T} k C_0^T, \tag{C3}$$

giving $Bk=C_0\bar{B}^T kC_0^T$. Noticing the relation $C_0kC_0^T=k$ one finds

$$B = C_0\bar{B}^T C_0^T = \bar{\bar{B}}, \tag{C4}$$

and thus justifies the charge-conjugation symmetry of W.

APPENDIX D: PRESERVATION OF WARD IDENTITY

Using integral by parts we transform Eq. (65) into (noticing that $\partial_\rho W(\mathbf{r})|_{\rho=0 \text{ or } a}=0$)

$$F_4[W] = \frac{\pi\nu}{2}\int_0^\infty dz \int_0^a d\rho\{2D_0\,\text{str}[\partial^2 WW^3+(\partial WW)^2 + (\partial W)^2W^2] + i\omega\,\text{str}\,W^4\}. \tag{D1}$$

Using the contraction rule we obtain

$$\pi\nu D_0\int_0^\infty dz_1\int_0^a d\rho_1\langle\text{str}[k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')]\text{str}[\partial^2 W(\mathbf{r}_1)W^3(\mathbf{r}_1)]\rangle_{F_2}$$
$$= I_a + \pi\nu D_0\int_0^\infty dz_1\int_0^a d\rho_1\langle\text{str}[k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')]\text{str}[\partial^2 \overbrace{W(\mathbf{r}_1)W(\mathbf{r}_1)W}(\mathbf{r}_1)W(\mathbf{r}_1)]\rangle_{F_2}, \tag{D2}$$

where the overbrace fixes the contraction and

$$I_a = \pi\nu D_0 \int_0^\infty dz_1 \int_0^a d\rho_1 \{\langle \mathrm{str}\, k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)\overbrace{W(\mathbf{r}')\mathrm{str}\, \partial^2 W}(\mathbf{r}_1)W^3(\mathbf{r}_1)\rangle_{F_2}$$
$$+ \langle \mathrm{str}\, k(1+\Lambda)(1-\tau_3)\overbrace{W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')\mathrm{str}\, \partial^2 W}(\mathbf{r}_1)W^3(\mathbf{r}_1)\rangle_{F_2}\}$$
$$= D_0 \int_0^\infty dz_1 \int_0^a d\rho_1 \{\partial^2_{\mathbf{r}_1}\mathcal{D}(\mathbf{r}',\mathbf{r}_1;\omega)\langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W^3(\mathbf{r}_1)]\rangle_{F_2}$$
$$+ \partial^2_{\mathbf{r}_1}\mathcal{D}(\mathbf{r},\mathbf{r}_1;\omega)\langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)W^3(\mathbf{r}_1)k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')]\rangle_{F_2}\}. \quad \text{(D3)}$$

Likewise, we also obtain

$$I_b \equiv \frac{i\pi\nu\omega}{2}\int_0^\infty dz_1 \int_0^a d\rho_1 \langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)\times(1+\tau_3)W(\mathbf{r}')]\mathrm{str}[W^4(\mathbf{r}_1)]\rangle_{F_2}$$
$$= i\omega\int_0^\infty dz_1 \int_0^a d\rho_1 \{\mathcal{D}(\mathbf{r}',\mathbf{r}_1;\omega)\langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)W(\mathbf{r})k(1-\Lambda)(1+\tau_3)W^3(\mathbf{r}_1)]\rangle_{F_2} + \mathcal{D}(\mathbf{r},\mathbf{r}_1;\omega)\times\langle \mathrm{str}[k(1+\Lambda)(1-\tau_3)W^3(\mathbf{r}_1)k(1-\Lambda)(1+\tau_3)W(\mathbf{r}')]\rangle_{F_2}\}. \quad \text{(D4)}$$

Noticing Eq. (67) we find that $I_a + I_b$ exactly cancels the first two terms of Eq. (69).

APPENDIX E: DERIVATION OF EQ. (82)

Let us introduce the function $f(z)=\Sigma_{n\geq 1}[1-e^{-2n\pi z/a}]/(n\pi)$. Taking its derivative we obtain

$$f'(z) = \frac{2}{a}\sum_{n\geq 1} e^{-2n\pi z/a} = \frac{2}{a}\frac{e^{-2\pi z/a}}{1-e^{-2\pi z/a}}. \quad \text{(E1)}$$

On the other hand, the low-energy diffusion occurs on the scale $\sim l$. Over this scale the interface where $f(z)$ vanishes is smeared. Therefore, without loss of any physics we may reformulate the boundary condition as $f(l)=0$. Taking it into account and integrating out Eq. (E1) we obtain $\pi f(z)=\ln\{(1-e^{-2\pi z/a})/(1-e^{-2\pi l/a})\}$ for $z \gtrsim l$, which gives $f(z) \approx \pi^{-1}\ln(z/l)$ for $l \lesssim z \ll a$ justifying Eq. (82).

[1] S. John and M. J. Stephen, Phys. Rev. B **28**, 6358 (1983); S. John, Phys. Rev. Lett. **53**, 2169 (1984).
[2] P. W. Anderson, Philos. Mag. B **52**, 505 (1985).
[3] J. Kroha, C. M. Soukoulis, and P. Wölfle, Phys. Rev. B **47**, 11093 (1993).
[4] A. A. Chabanov and A. Z. Genack, Phys. Rev. Lett. **87**, 153901 (2001).
[5] A. A. Chabanov, Z. Q. Zhang, and A. Z. Genack, Phys. Rev. Lett. **90**, 203903 (2003).
[6] M. Störzer, P. Gross, C. M. Aegerter, and G. Maret, Phys. Rev. Lett. **96**, 063904 (2006).
[7] A. A. Golubentsev, Zh. Eksp. Teor. Fiz. **86**, 47 (1984) [Sov. Phys. JETP **59**, 26 (1984)]; M. P. Van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985); P. E. Wolf and G. Maret, *ibid.* **55**, 2696 (1985).
[8] M. C. W. van Rosum and Th. M. Niuwenhuizen, Rev. Mod. Phys. **71**, 313 (1999).
[9] R. Berkovits and M. Kaveh, Phys. Rev. B **36**, 9322 (1987).
[10] I. Edrei and M. J. Stephen, Phys. Rev. B **42**, 110 (1990).
[11] D. S. Wiersma, P. Bartolini, A. Lagendijk, and R. Righini, Nature (London) **390**, 671 (1997); **398**, 207 (1999); F. Scheffold, R. Lenke, R. Tweer, and G. Maret, *ibid.* **398**, 206 (1999).
[12] B. A. van Tiggelen, A. Lagendijk, and D. S. Wiersma, Phys. Rev. Lett. **84**, 4333 (2000).
[13] X. D. Zhang and Z. Q. Zhang, Phys. Rev. B **65**, 155208 (2002).
[14] Z. Q. Zhang, A. A. Chabanov, S. K. Cheung, C. H. Wong, and A. Z. Genack, arXiv:0710.3155 (unpublished).
[15] V. Ya. Chernyak, K. I. Grigoshin, E. I. Ogievetsky, and V. M. Agranovich, Solid State Commun. **84**, 209 (1992).
[16] B. White, P. Sheng, Z. Q. Zhang, and G. Papanicolaou, Phys. Rev. Lett. **59**, 1918 (1987).
[17] S. E. Skipetrov and B. A. van Tiggelen, Phys. Rev. Lett. **92**, 113901 (2004).
[18] S. E. Skipetrov and B. A. van Tiggelen, Phys. Rev. Lett. **96**, 043902 (2006).
[19] M. Titov and C. W. J. Beenakker, Phys. Rev. Lett. **85**, 3388 (2000).
[20] K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge, Cambridge, UK, 1997).
[21] K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **85**, 764 (1983) [Sov. Phys. JETP **58**, 444 (1983)].
[22] A. Lamacraft, B. D. Simons, and M. R. Zirnbauer, Phys. Rev. B **70**, 075412 (2004).
[23] C. Tian, Pis'ma Zh. Eksp. Teor. Fiz. **86**, 651 (2007) [JETP Lett. **86**, 566 (2007)].
[24] S. Iida, H. A. Weidenmüller, and J. A. Zuk, Ann. Phys. **200**, 219 (1990).
[25] A. D. Mirlin, A. Müller-Groeling, and M. R. Zirnbauer, Ann. Phys. **236**, 325 (1994).
[26] M. R. Zirnbauer, Nucl. Phys. A **560**, 95 (1993).

[27] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Butterworth-Heinemann, Oxford, 1984).
[28] A. Altland, B. D. Simons, and D. Taras-Semchuk, Adv. Phys. **49**, 3 (2000).
[29] V. M. Agranovich and V. E. Kravtsov, Phys. Rev. B **43**, 13691 (1991).
[30] A. Lagendijk, B. Vreeker, and P. de Vries, Phys. Lett. A **136**, 81 (1989).
[31] B. Davison and J. B. Sykes, *Neutron Transport Theory* (Oxford, New York, 1957).
[32] N. G. van Kampen and I. Oppenheim, J. Math. Phys. **13**, 842 (1972).
[33] For example, see F. C. MacKintosh and S. John, Phys. Rev. B **37**, 1884 (1988).
[34] R. Frank, A. Lubatsch, and J. Kroha, Phys. Rev. B **73**, 245107 (2006).
[35] N. Cherroret and S. E. Skipetrov, arXiv:0711.2410 (unpublished).