PHYSICAL REVIEW B **77**, 060502(R) (2008)

Kolmogorov and Kelvin-wave cascades of superfluid turbulence at $T=0$: What lies between

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(Received 1 March 2007; revised manuscript received 19 December 2007; published 8 February 2008)

As long as vorticity quantization remains irrelevant for long-wave physics, superfluid turbulence supports a regime macroscopically identical to the Kolmogorov cascade of a normal liquid. At high enough wave numbers, the energy flux in wavelength space is carried by individual Kelvin-wave cascades on separate vortex lines. We analyze the transformation of the Kolmogorov cascade into the Kelvin-wave cascade, revealing a chain of three distinct intermediate cascades supported by local-induction motion of the vortex lines and distinguished by specific reconnection mechanisms. The most prominent qualitative feature predicted is unavoidable production of vortex rings of a characteristic size.

DOI: 10.1103/PhysRevB.77.060502 PACS number(s): 67.25.dk, 03.75.Kk, 47.37.+q

Nowadays, superfluid turbulence[1,2]—a structured or nonstructured tangle of quantized vortex lines—has been attracting much attention,[3] stimulated, in particular, by advances in experimental techniques allowing studies of different turbulent regimes in diverse superfluid systems, such as ^{4}He,[2,4] ^{3}He-B,[5,6] and Bose-Einstein condensates of ultacold atoms.[2,7] In superfluids at $T=0$, vorticity can only exist in the form of topological defects—vortex lines of microscopic thickness, around which the circulation of velocity is equal to the liquid-specific quantum κ. Speaking generally, the dynamical mechanisms governing superfluid turbulence are fundamentally different from those of classical turbulence (see, e.g., the recent review in Ref. 3 and references therein).

A wave of interest in the dynamics of superfluid turbulence came with the experiment by Maurer and Tabeling,[8] who observed that superfluid turbulence in ^{4}He formed by counterrotating disks is indistinguishable from classical turbulence at large length scales, in particular exhibiting the classical Kolmogorov cascade. Shortly, the same effect was found in superfluid turbulence generated by a towed grid.[9] In the experiments,[8,9] the fraction of normal component is considerable, making analysis of vortex tangle dynamics and structure significantly complicated.[10,11] (Considerations regarding possible energy spectra in this case are presented in Ref. 12.) However, the similarity between classical and superfluid turbulence exists even at practically zero temperature, which was first observed in numerical simulations[13–15] and, just recently, for the first time confirmed by measurements in ^{3}He-B.[6]

By the nature of a cascade regime, implying that the kinetic times get progressively shorter down the hierarchy of length scales, the instantaneous structure of turbulence follows the evolution at the largest length scales (typically of the order of the system size), where the energy flux (per unit mass) ε is formed. At very low temperatures, due to the absence of frictional dissipation, the flux ε must be carried down to scales significantly smaller than the (related to ε) typical separation between the vortex lines l_0. At small enough length scales, the energy flux is carried by pure Kelvin-wave cascades on separate vortex lines,[16,17] the cutoff being due to sound radiation.[10,18,19]

The fact that superfluid turbulence at large compared to l_0 length scales may be consistent with the classical Kolmogorov law is not surprising (a formal argument is mentioned below). It is well known[1] that macroscopic velocity profile of a rapidly rotated superfluid mimics solid-body rotation, which is accomplished by formation of a dense array of vortex lines aligned along the rotation axis. By the same mechanism, by "stirring" a superfluid one can produce vorticity in the *course-grained* (up to length scales larger than l_0) superfluid velocity field, indistinguishable from that of a normal fluid, the underlying vortex tangle being organized in polarized "bundles" of vortex lines. What turns out to be a puzzle,[20] however, is what the vortex tangle looks like when one zooms in down to scales of the order of the interline separation l_0, where the vorticity is essentially discrete.

In this Rapid Communication, we analyze the structure of turbulence at all length scales, tracing the transformation of the *classical* regime, described by the Kolmogorov law at large length scales, into the *quantized* regime, in which the discreteness of vortex lines is important, in the fundamental case of zero temperature. The analysis relies on the large parameter

$$\Lambda = \ln(l_0/a_0) \gg 1, \tag{1}$$

where a_0 is the vortex core radius. In realistic ^{4}He experiments, $\Lambda \sim 15$. Attention to the problem of linking the two regimes was drawn recently by L'vov *et al.*,[20] who realized that it is impossible to directly cross over from the Kolmogorov regime to the pure Kelvin-wave cascade and put forward the idea of a bottleneck with specific dynamical implications. The Achilles' heel of the treatment of Ref. 20 is taking it for granted,[10,11] that the coarse-grained macroscopic description of quantized vorticity remains valid down to the scale of l_0.

We show that the locally induced motion of the vortex lines changes the dynamical picture already at the scale

$$r_0 \sim \Lambda^{1/2} l_0, \tag{2}$$

with the interline separation related to the energy flux by

$$l_0 \sim (\Lambda \kappa^3/\varepsilon)^{1/4}. \tag{3}$$

In the range of wavelength $r_0 > \lambda > \lambda_*$,

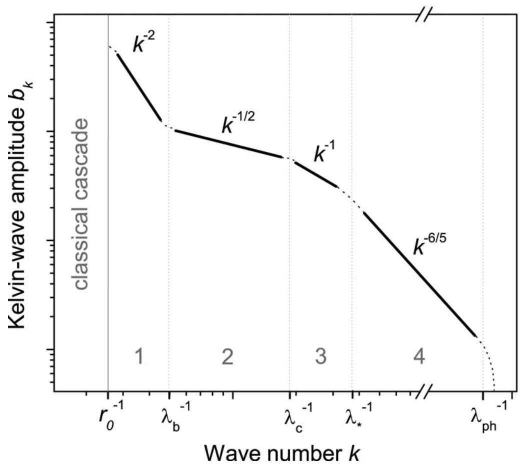

FIG. 1. Spectrum of Kelvin waves in the quantized regime. The inertial range consists of a chain of cascades driven by different mechanisms: (1) reconnections of vortex-line bundles, (2) reconnections between nearest-neighbor vortex lines in a bundle, (3) self-reconnections on single vortex lines, and (4) nonlinear dynamics of single vortex lines without reconnections. Regimes (3) and (4) are familiar in the context of nonstructured vortex tangle decay (Refs. 16 and 21).

$$\lambda_* = l_0/\Lambda^{1/2}, \tag{4}$$

there takes place a chain of three cascade regimes, in which the energy flux ε is carried by locally induced motion combined with vortex-line reconnections. The three regimes are distinguished by their specific types of reconnections: (i) reconnections of vortex-line bundles, (ii) reconnections between nearest-neighbor vortex lines, and (iii) self-reconnections on single vortex lines—the mechanism introduced earlier by one of us[21] in the context of the decay of nonstructured superfluid turbulence. The existence of regime (iii) means an unavoidable production of vortex rings of typical size λ_* at a rate immediately following from (3) by conservation of energy. Namely, $\sim\kappa\Lambda^{1/2}/l_0^2$ rings are emitted per unit time in the characteristic volume of l_0^3. Note that this rate is $\sim\Lambda^{3/2}$ times smaller than the rate of vortex ring production characteristic of nonstructured superfluid turbulence.[16,21]

For realistic values of Λ, a sharp distinction between the three subregimes is likely to be lost, although characteristic features of strong turbulence, such as generation of a spectrum of vortex rings by mechanism (iii), might manifest themselves. At the wavelength scale λ_*, self-reconnections cease and the weak-turbulent regime sets in with a purely nonlinear Kelvin-wave cascade.[16] This regime covers a significant part of the inertial range until eventually at the scale[19]

$$\lambda_{ph} = [\Lambda^{27}(\kappa/c)^{25} l_0^6]^{1/31} \tag{5}$$

(c is the sound velocity) the cascade is cut off due to the radiation of sound by Kelvin waves. With $\kappa/c \sim a_0$ we have $\lambda_{ph} \lll \lambda_*$.

The structure of turbulence is summarized in Fig. 1. We emphasize that the notion of energy spectrum $E(k)$, where $E(k)dk$ gives the energy per unit mass associated with variations of the fluid velocity over length scales $\sim k^{-1}$ in the interval dk, is practically meaningful only in the classical regime. In the quantized regime, the relevant degrees of freedom are waves on vortex lines, while even a perfectly straight vortex line has a nontrivial spectrum. On the experimental side, a recently introduced vortex-line visualization technique in ^{4}He (Ref. 4) could provide the most direct probe for the quantized regime.

The vortex dynamics at zero temperature is essentially described by the Kelvin-Helmholtz theorem, which states that a vortex-line element moves with local fluid velocity. Mathematically, this is reflected in the Biot-Savart equation[1]

$$\dot{\mathbf{s}} = \mathbf{v}(\mathbf{s}), \quad \mathbf{v}(\mathbf{r}) = \frac{\kappa}{4\pi}\int (\mathbf{s}_0 - \mathbf{r}) \times d\mathbf{s}_0/|\mathbf{s}_0 - \mathbf{r}|^3. \tag{6}$$

Here $\mathbf{v}(\mathbf{r})$ is the superfluid velocity field, $\mathbf{s}$ is the time-evolving radius vector of the vortex-line element, the overdot denotes differentiation with respect to time, the vector $\mathbf{s}_0$ has the same physical meaning as $\mathbf{s}$, understood as an integration variable, and integration is along all vortex lines. The Biot-Savart equation (6) rewritten in classical terms of vorticity $\mathbf{w} = \text{curl}\, \mathbf{v}$ in momentum space, $\mathbf{w}_\mathbf{k} = \int \mathbf{w}(\mathbf{r})\exp[-i\mathbf{k}\cdot\mathbf{r}]d^3r = \kappa\int \exp[-i\mathbf{k}\cdot\mathbf{s}]d\mathbf{s}$, is *identical* to the vorticity equation for a normal ideal incompressible fluid:

$$\frac{\partial \mathbf{w}_\mathbf{k}}{\partial t} = \mathbf{k} \times \int \frac{d^3q}{(2\pi)^3} q^{-2}[\mathbf{w}_{\mathbf{k}-\mathbf{q}} \times (\mathbf{w}_\mathbf{q} \times \mathbf{q})]. \tag{7}$$

If the energy is concentrated at the momentum scale $k_{en} \ll r_0^{-1}$, then, automatically, the superfluid turbulence must be equivalent to the classical ideal-incompressible-fluid turbulence in the inertial range $k_{en} \ll k \ll r_0^{-1}$, provided the decay scenario is local in momentum space, so that the quantized nature of vorticity is irrelevant for long-wavelength behavior. These are precisely the conditions under which the Kolmogorov cascade is obtained in classical fluids as well.

For our purposes, it is instructive to formally decompose the integral (6) into the self-induced part $\mathbf{v}^{SI}(\mathbf{s})$, for which the integration is restricted to the vortex line containing the element $\mathbf{s}$, and the remaining contribution induced by all the other lines $\mathbf{v}^{I}(\mathbf{s})$,

$$\mathbf{v}(\mathbf{s}) = \mathbf{v}^{SI}(\mathbf{s}) + \mathbf{v}^{I}(\mathbf{s}). \tag{8}$$

Since the velocities $\mathbf{v}^{SI}$ and $\mathbf{v}^{I}$ define the right-hand side of Eq. (6), competition between them is crucial for the problem.

The leading contribution to $\mathbf{v}^{SI}$ is given by the local induction approximation (LIA), which reduces the integral over the vortex line to its local differential characteristics,

$$\mathbf{v}^{SI}(\mathbf{s}) = \Lambda_R \frac{\kappa}{4\pi}\mathbf{s}' \times \mathbf{s}'', \quad \Lambda_R = \ln(R/a_0), \tag{9}$$

where the prime denotes differentiation with respect to the arclength and R is the typical curvature radius. The necessary condition of applicability of the LIA is $\Lambda_R \gg 1$. Taking into account that Λ_R is a very weak function of R, we shall treat it as a constant of typical value $\Lambda_R \sim \Lambda$. Note, however, that despite the fact that condition (1) is typically well satisfied, using the LIA is not always appropriate. Being an integrable model, the LIA does not capture the reconnection-free (purely nonlinear) kinetics of Kelvin waves.[16]

To determine the crossover scale r_0, consider the structure of the vortex tangle in the classical regime. By the definition of r_0, at length scales $r \gg r_0$ turbulence mimics classical vorticity by taking on the form of a dense coherently moving array of vortex lines bent at a curvature radius of order r. The velocity field of this configuration obeys the Kolmogorov law

$$v_r \sim (\varepsilon r)^{1/3}, \quad r \gg r_0. \tag{10}$$

Here and below the subscript r means typical variation of a field over distance $\sim r$. On the other hand, the value of v_r is fixed by the quantization of the velocity circulation around a contour of radius r—namely, $v_r r \sim \kappa n_r r^2$, where n_r is the areal density of vortex lines responsible for vorticity at the scale r. Note that scale invariance requires that on top of vorticity at the scale r there be a fine structure of vortex bundles of smaller sizes, so that, mathematically, $n_r r^2$ is the difference between large numbers of vortex lines crossing the area of the contour r in opposite directions. The quantity n_r is related to the flux by

$$n_r \sim \left[\frac{\varepsilon}{\kappa^3 r^2}\right]^{1/3}, \quad r \gg r_0. \tag{11}$$

The underlying dynamics of a single vortex line in the bundle is governed by v_r^{I} and v_r^{SI}. While by definition $v_r^I \sim v_r$, which is given by Eq. (10), the self-induced part is determined by the curvature radius r of the vortex line according to Eq. (9),

$$v_r^{\mathrm{SI}} \sim \Lambda \frac{\kappa}{r}. \tag{12}$$

At length scales where $v_r^{\mathrm{I}} \gg v_r^{\mathrm{SI}}$, the vortex lines in the bundle move coherently with the same velocity $\sim v_r^{\mathrm{I}}$. However, at the scale $r_0 \sim (\Lambda^3 \kappa^3/\varepsilon)^{1/4}$, the self-induced motion of the vortex line becomes comparable to the collective motion, $v_r^{\mathrm{SI}} \sim v_r^{\mathrm{I}}$. At this scale, individual vortex lines start to behave independently of each other and thus r_0 gives the lower cutoff of the inertial region of the Kolmogorov spectrum (10).

Since r_0 is the size of the smallest classical eddies, the areal density of vortex lines at this scale is given by the typical interline separation $n_{r_0} \sim 1/l_0^2$. With Eq. (11), we arrive at (2) and (3).

At the scale r_0, turbulence consists of randomly oriented vortex-line bundles of size r_0, left by the classical regime. The typical number of vortices in the bundle is given by $n_{r_0} r_0^2 \sim \Lambda$. The length r_0 plays the role of a correlation radius in the sense that the relative orientation of two vortex lines becomes uncorrelated only if they are a distance $\gtrsim r_0$ apart. On the other hand, the crossover to the quantized regime means that each line starts moving according to its geometric shape, as prescribed by Eq. (9). Therefore, reconnections, at least between separate bundles, are inevitable and, as we show below, capable of sustaining the flux ε.

Reconnections play the leading role at $r_0 \gtrsim \lambda \gtrsim \lambda_*$. Although this region is relatively narrow as compared to the whole Kelvin-wave inertial range, it is significantly large in absolute units. Before going into the details of the reconnection-assisted regimes, we describe the remaining and dominant region of the cascade. As was shown by the authors,[16] at a sufficiently small wavelength, a strongly turbulent cascade of Kelvin waves is replaced by a purely nonlinear cascade, in which the reconnections are exponentially suppressed. The spectrum of Kelvin-wave amplitudes b_k, $k \sim \lambda^{-1}$, in the nonlinear cascade has the form

$$b_k = (\Theta/\kappa^3 \rho)^{1/10} k^{-6/5}, \tag{13}$$

where Θ is the flux of energy per unit vortex-line length supported by the nonlinear cascade. The value of λ_* can be determined by matching the energy flux ε with $\Theta/\rho l_0^2$, where $b_k \sim k^{-1} \sim \lambda_*$. With Eq. (3), we then obtain Eq. (4).

At $T=0$ Kelvin waves decay emitting phonons.[10] For Kelvin waves of wave number $\sim k$, the power of sound emission per unit line length is given by[19]

$$\Pi_k \sim \Lambda^6 \kappa^8 \rho b_k^4 k^{11}/c^5. \tag{14}$$

This dissipation mechanism is negligibly weak all the way down to wavelengths of order $\lambda_{\mathbf{ph}}$, given by Eq. (5), where $\Pi_k/\rho l_0^2$ becomes comparable to ε. The scale $\lambda_{\mathbf{ph}} \lll \lambda_*$ gives the lower dissipative cutoff of the Kelvin wave cascade.

Now we focus on the strongly turbulent regimes at $r_0 \gtrsim k^{-1} \gtrsim \lambda_*$. The key quantity here is the energy transferred to a lower scale after one reconnection of vortex lines at the scale k^{-1}, which, following Ref. 21, can be written as

$$\epsilon_k \sim f(\gamma) \Lambda \rho \kappa^2 k^{-1}. \tag{15}$$

Here, $f(\gamma)$ is a dimensionless function of the angle γ at which the vortex lines cross ($\gamma=0$ corresponds to parallel lines). Its asymptotic form is

$$f(\gamma) \sim \gamma^2, \quad \gamma \ll 1. \tag{16}$$

Although at the scale r_0 there is already no coupling between vortex lines to stabilize the bundles, they should still move coherently—the geometry of neighboring lines at this scale is essentially the same over distances $\sim r_0$—until whole bundles cross each other. It is possible, however, that vortex lines within the bundle reconnect. One can show that such processes cannot lead to any significant redistribution of energy and thus to a deformation of the bundle at the scale r_0 because they happen at small angles so that the energy (15) is too small. Indeed, the dimensional upper bound on the rate at which two lines at distance $l \ll r_0$ can cross each other is, from Eq. (9), $\Lambda\kappa/r_0 l$, while the actual value should be much smaller due to the strong correlations between line geometries. Taking into account that the number of lines in the bundle is $(r_0/l_0)^2$ and that $\gamma \sim l/r_0$, the contribution to the energy flux from these processes is bounded by $(l/r_0)\varepsilon$.

Crossing of the bundles results in reconnections between their vortex lines, and Kelvin waves with somewhat smaller wavelength λ are generated. The coherence of the initial bundles implies that the waves on different vortex lines must be generated coherently. Thus, at the scale $\lambda \lesssim r_0$, vortex lines should be also organized in bundles of length λ that are bent with the amplitude of the Kelvin waves b_k, $k \sim \lambda^{-1}$, while the correlation radius for vortex-line configurations in the transversal direction is $\sim b_k$. Then reconnections between the bundles at the scale λ transport the energy to a lower scale. The cascade of bundles should repeat itself self-similarly in a range of wavelength $l_0 \ll k^{-1}$, $b_k < r_0$, in which the notion of bundles is meaningful. The spectrum of Kelvin waves b_k in this regime can be obtained from the condition $\tilde{\varepsilon}_k \equiv \varepsilon$, where $\tilde{\varepsilon}_k$ is the energy flux per unit mass transported by the reconnections at the scale k^{-1} given by

$$\tilde{\varepsilon}_k \sim (k/\rho b_k^2) N_k \epsilon_k \tau_k^{-1}. \quad (17)$$

Here, we take into account that the correlation volume is b_k^2/k, $N_k \sim (b_k/l_0)^2$ is the number of vortex lines in the bundle, and $\tau_k^{-1} \sim \Lambda \kappa k^2$ is the rate at which the bundles cross. Physically, b_k determines the typical crossing angle $\gamma \sim b_k k$, thereby controlling the energy lost in one reconnection. Thus, the spectrum of Kelvin waves in the bundles is

$$b_k \sim r_0^{-1} k^{-2}. \quad (18)$$

At the wavelength $\sim \lambda_b = \Lambda^{1/4} l_0$, the amplitudes become of the order of the interline separation $b_k \sim l_0$ and the cascade of bundles is cut off. At this scale, $b_k k \ll 1$, so that the mechanism of self-reconnections is strongly suppressed. On the other hand, the kinetic times of the purely nonlinear regime are too long to carry the flux ε.[16] We thus conclude that at $\lambda_c \lesssim \lambda \lesssim \lambda_b$ the cascade is supported by nearest-neighbor reconnections, the amplitudes b_k being defined by the condition of constant energy flux per unit length and the crossover scale λ_c being associated with the condition $b_{k\sim 1/\lambda_c} \sim \lambda_c$, meaning that at $\lambda \lesssim \lambda_c$ the self-crossing regime takes over. The observation crucial for understanding the particular mechanism of the cascade and thus finding b_k is that each nearest-neighbor reconnection (happening at the rate $\propto \Lambda / \lambda_b^2$ per each line element of the length $\sim \lambda_b$) performs a sort of *parallel processing* of the cascade for *each* of the wavelength scales in the range $[\lambda_c, \lambda_b]$. For the given wavelength scale $\lambda \sim 1/k$, the energy transferred by a single collision is $\propto \Lambda (b_k k)^2 \lambda$, and with the above estimate of the collision rate per length, λ_b, this readily yields the estimate $b_k \sim l_0 (\lambda_b k)^{-1/2}$ and, correspondingly, $\lambda_c \sim l_0 / \Lambda^{1/4}$.

In the range $\lambda_c \gg k^{-1} \gg \lambda_*$, the cascade is driven by self-reconnections of vortex lines giving the spectrum $b_k \sim k^{-1}$.[21] This regime is replaced by the purely nonlinear regime in the vicinity of $k^{-1} \sim \lambda_*$ (the actual transition region may be rather wide[16]).

To conclude, the transformation of a classical-fluid Kolmogorov cascade of superfluid turbulence into the pure Kelvin-wave cascade requires three intermediate stages associated with locally induced motion and reconnections of vortex lines as illustrated in Fig. 1.

We thank Victor L'vov and Sergei Nazarenko for drawing our attention to their work and are grateful to Nikolay Prokof'ev for fruitful discussions.

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