Valence bond solid order near impurities in two-dimensional quantum antiferromagnets

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Recent scanning tunneling microscopy (STM) experiments on underdoped cuprates have displayed modulations in the local electronic density of states, which are centered on a Cu-O-Cu bond [Kohsaka *et al.*, Science **315**, 1380 (2007)]. As a paradigm of the pinning of such bond-centered ordering in strongly correlated systems, we present the theory of valence bond solid (VBS) correlations near a single impurity in a square lattice antiferromagnet. The antiferromagnet is assumed to be in the vicinity of a quantum transition from a magnetically ordered Néel state to a spin-gap state with long-range VBS order. We identify two distinct classes of impurities: (i) local modulation in the exchange constants and (ii) a missing or additional spin, for which the impurity perturbation is represented by an uncompensated Berry phase. The "boundary" critical theory for these classes is developed. In the second class, we find a VBS pinwheel around the impurity, accompanied by a suppression in the VBS susceptibility. Implications for numerical studies of quantum antiferromagnets and for STM experiments on the cuprates are noted.

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I. INTRODUCTION

A number of recent scanning tunneling microscopy experiments have highlighted spatial modulations in the local density of states in the cuprate compounds, nucleated by external perturbations. In Ref. 1, the spatial modulation was observed in the normal state above T_c, presumably nucleated by impurities. In Refs. 2–4, the order was found in a halo around vortices, which were in turn pinned by impurities. Most recently, in Ref. 5, similar charge-ordering patterns were found to be ubiquitous in the underdoped cuprates at low temperatures, and it was established that the charge ordering was "bond centered" and had an anisotropic structure similar to a valence bond solid state.[6–9]

In the light of these observations, it is of general interest to study the appearance of varieties of charge order [including "valence bond solid" (VBS) order[8–10]] near impurities in strongly correlated systems. For superfluid states, such a theory has been presented in earlier work[11,12] and compared quantitatively with some of the above experiments. It was argued that the charge order was linked to quantum fluctuations of vortices/antivortices in the superfluid order. Consequently, the problem was mapped onto the pinning of the vortices by impurities and the quantum zero-point motion of vortices about the pinning site. In both zero and nonzero magnetic fields, enhanced charge order was found in the spatial region over which the vortex executed its zero-point motion.[11] This charge order was present even when the net vorticity was zero everywhere (as is the case in zero magnetic field): The vorticity canceled between the vortex and antivortex fluctuations, but the charge order did not.

This paper will present an extensive field-theoretic analysis of a paradigm of the problem of charge order near impurities in correlated systems. We will consider insulating $S=1/2$ antiferromagnets on the square lattice across a quantum phase transition from the magnetically ordered Néel state to a spin-gap (VBS) state.[13–16] By representing the $S=1/2$ spins as hard-core bosons, our results can be reinterpreted as applying to the superfluid-insulator transition of bosons at half-filling on the square lattice: The Néel state of the antiferromagnet is mapped onto the superfluid state of the bosons, while the VBS state is mapped onto a Bose insulator with a bond-centered charge order. The bond-centered charge correlations in the underdoped cuprates now appear to have two possible physical mechanisms ("disordered" antiferromagnet/superfluid), but it was argued in Ref. 17 that they represent the same underlying physics. Our results here will go beyond the earlier work[11,12] in two important respects:

(i) We will describe the critical singularities in the impurity-induced VBS/charge order at the quantum critical point.

(ii) We will consider a wider class of impurity perturbations.

In the previous work,[11,12] an "impurity" was assumed to be a generic deformation of the underlying Hamiltonian, which broke its space group symmetry. For the Néel-VBS transition, such an impurity is realized, e.g., by the modulation in the magnitude of a particular exchange coupling (see Fig. 1). We will briefly discuss the critical singularities describing the *enhancement* of the VBS order near such an impurity in Sec. I A below; these results have a natural extension to the models of charge order near the superfluid-insulator transition discussed above. However, the primary focus of the present paper is on a distinct class of impurities, in which the valence bond structure of the non-magnetic ground state of the antiferromagnet is more strongly disrupted and a "Berry phase" contribution of an unpaired spin is the crucial impurity-induced perturbation.[18,19] Such impurities are realized by replacing the $S=1/2$ Cu spins in antiferromagnets by a nonmagnetic Zn ion or a $S=1$ Ni ion (see Fig. 2). For the superfluid-insulator transition, such an impurity is a site from which particles are excluded, and so a local "phase shift" is induced in the charge order of the insulating state (replacing a Cu atom by Zn or Ni is expected to have the desired "Cooper pair" exclusion effect[17]). Our main results will include a description of the *suppression* of VBS order near such Berry phase impurities. These results are summarized in Sec. I B below and described in the body of the paper. A simple sketch of how such an impurity disrupts the VBS order is shown in Fig. 2; this figure builds upon the dual theory of spinons in the VBS state developed by Levin and Senthil.[20] The bulk of this paper will describe how quantum fluctuations of the type sketched in Fig. 2 lead to a modification of the scaling dimension of the VBS order in the vicinity of the vacancy.

The remainder of the paper will be presented in the language of the Néel-VBS transition in quantum antiferromagnets. For this model, a field-theoretic description of the vicinity of the quantum critical point[15,16,21,22] is provided by the $\mathbb{CP}^{N-1}$ theory at $N=2$,

$$\mathcal{S} = \int d^2x d\tau \left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + \frac{g}{2}(|z_\alpha|^2)^2 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]. \tag{1.1}$$

Here, μ, ν, and λ are space-time indices, z_α, $\alpha=1,\ldots,N$ is a complex scalar, which is a SU(N) fundamental, and A_μ is a noncompact U(1) gauge field. The Néel order of the antiferromagnet is $n^a = z^\dagger T^a z$, where T^a is a SU(N) generator. The SU(N) symmetry is spontaneously broken in the Néel phase, $\langle n^a \rangle \neq 0$, which is realized for $s < s_c$, where s_c is the critical value of the tuning parameter s for the quantum phase transition. For $s > s_c$, the $\mathbb{CP}^{N-1}$ theory above describes a U(1) spin liquid state of the antiferromagnet, with gapped spinons z_α and a gapless U(1) photon. However, as has been argued at length elsewhere,[13,14] lattice effects not included in the continuum field theory [Eq. (1.1)] eventually render the U(1) spin liquid unstable to spinon confinement and fully gapped state with VBS order. The VBS order parameter, V, is an operator[13,16] that creates a *Dirac monopole* with total flux 2π in the U(1) gauge field A_μ. This paper will therefore be concerned with correlations of the monopole/VBS operator V under the field theory $\mathcal{S}$ after including the impurity perturbations described below. The bulk scaling dimension of the monopole operator at the $s=s_c$ critical point will make frequent appearances in our analysis, and so we define this as

$$\Delta^V = \dim[V(\vec{x},\tau)] \quad \text{in the theory } \mathcal{S} \text{ without an impurity}. \tag{1.2}$$

The following subsections will now describe the two classes of impurity perturbations to the theory $\mathcal{S}$ shown in Figs. 1 and 2, respectively.

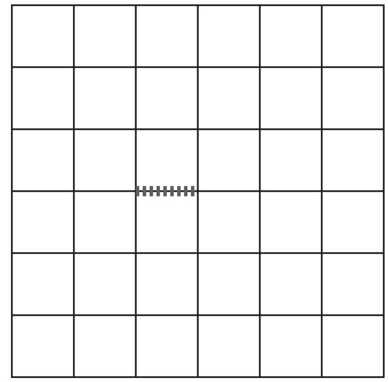

FIG. 1. A modulated exchange impurity, which is described in Sec. I A. The dashed line indicates a different value of the antiferromagnetic exchange constant. We expect VBS order to be enhanced near such an impurity because the modulated exchange will lock in a preferred orientation and offset of the VBS state.

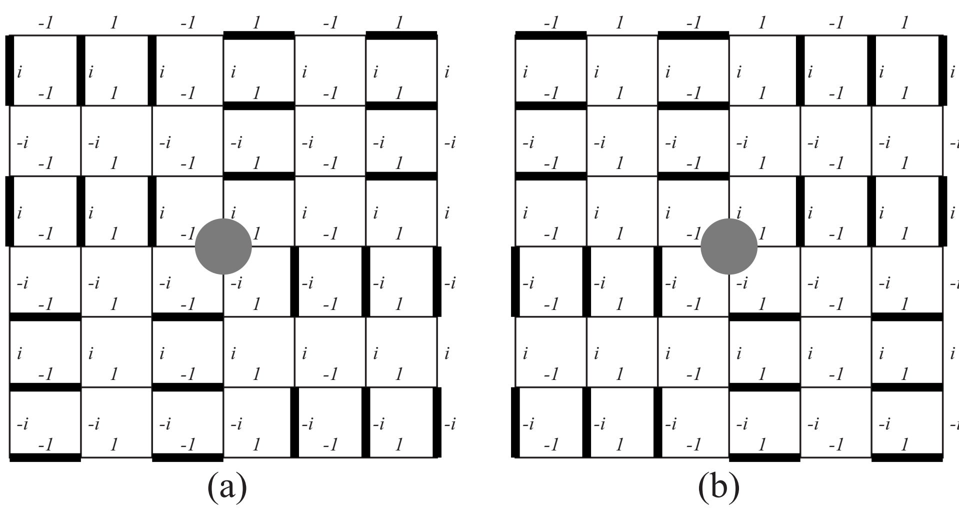

FIG. 2. A vacancy (the shaded circle) in a square lattice quantum antiferromagnet, which is described in Sec. I B and the remainder of the paper. The thick lines represent singlet bonds between the spins. The local value of the VBS order is measured by the phase factors on the singlet bonds. Moving anticlockwise from the right in both figures, we observe that VBS order cycles as $1 \to i \to -1 \to -i$ in (a) and as $i \to -1 \to -i \to 1$ in (b). Thus, *both* configurations are "vortices" in the VBS order, which we name "VBS pinwheels" (these VBS pinwheels are "dual" to the vortices in the superfluid/Néel order that are discussed in the beginning of the paper). Anti-pinwheels in the VBS order appear only around vacancies on the other sublattice; in other words, VBS pinwheels transform to VBS anti-pinwheels under translation by a single site (see Fig. 3 later).

A. Modulated exchange

A modulation in the magnitude of an exchange constant in the underlying antiferromagnet (see Fig. 1) breaks the lattice space group symmetry but preserves the spin rotation symmetry. Also, the number of spins on each sublattice is preserved, so no Berry phase term is expected. Consequently, we need to consider all local perturbations to $\mathcal{S}$ which preserve the required symmetries. The simplest allowed possibility is a local shift in the position of the critical point. For an impurity at the spatial origin, $x=0$, this would lead to a term

$$\tilde{s}\int d\tau |z_\alpha(\vec{x}=0,\tau)|^2. \qquad (1.3)$$

However, a simple computation[23] shows that $\tilde{s}$ is very likely an irrelevant perturbation at the bulk critical point. We have $\dim[\tilde{s}]=1-(D-1/\nu)$, where $D=3$ is the space-time dimension and ν is the correlation length exponent of $\mathcal{S}$. Because it is almost certainly the case that $\nu>1/2$, we conclude that $\tilde{s}$ is irrelevant. However, a more interesting perturbation is that considered in previous work[11,12] on the superfluid-insulator transition. In the present context, this perturbation follows from the fact that with a broken space group symmetry, a linear coupling to the monopole operator is permitted. So, we have the impurity action

$$\tilde{\mathcal{S}}_{\text{imp}} = \int d\tau [h^* V(\vec{x}=0,\tau) + \text{c.c.}], \qquad (1.4)$$

where h is a complex-valued constant whose value depends on the details of the modulated exchange near $x=0$. Now, the renormalization group (RG) flow of h follows from Eq. (1.2) to linear order

$$\frac{dh}{d\ell} = (1-\Delta^V)h + \mathcal{O}(h^2). \qquad (1.5)$$

The remainder of this subsection will analyze the correlations of the monopole/VBS operator $V(\vec{x},\tau)$ in the theory $\mathcal{S}+\tilde{\mathcal{S}}_{\text{imp}}$.

First, let us consider the likely possibility that $\Delta^V<1$. In this case, h is a relevant perturbation, and higher order corrections to Eq. (1.5) cannot be ignored. By analogy with results in the theory of boundary critical phenomena[24] and, in particular, with the theory of the "extraordinary" transition,[25–27] we conclude that a likely possibility is that the RG flow is to strong coupling, to a fixed point with $|h|=\infty$. In this, case some powerful statements on the correlations of $V(\vec{x},\tau)$ can be immediately made. It is useful to express the correlations in the vicinity of the impurity by an operator product expansion (OPE). In general, this expansion will have the structure

$$\lim_{|\vec{x}|\to 0} V(\vec{x},\tau) \sim |\vec{x}|^{\Delta^V_{\text{imp}}} V_{\text{imp}}(\tau), \qquad (1.6)$$

where V_{imp} is an operator localized on the impurity site and Δ^V_{imp} is the difference in scaling dimensions between V and V_{imp}. Specifically, Eq. (1.6) implies that

$$\Delta^V = -\Delta^V_{\text{imp}} + \dim[V_{\text{imp}}]. \qquad (1.7)$$

Now, at a $|h|=\infty$ fixed point, we expect that fluctuations of V near the impurity are strongly suppressed, and so it is a reasonable conclusion that V_{imp} is just the identity operator

$$V_{\text{imp}} = \mathbb{1}. \qquad (1.8)$$

Consequently, $\dim[V_{\text{imp}}]=0$, and we have our main result,

$$\Delta^V_{\text{imp}} = -\Delta^V. \qquad (1.9)$$

The combination of Eqs. (1.6) and (1.9) appears to be a promising route to measuring the scaling dimension of a monopole operator in numerical studies of quantum antiferromagnets.

To complete our analysis of modulated exchange, we also address the case with $\Delta^V>1$. In this situation, by Eq. (1.5), the perturbation h is irrelevant, and so we may compute the consequences of h by perturbation theory. Computing correlations to first order in h, we see that Eq. (1.6) is now replaced by

$$\lim_{|\vec{x}|\to 0} V(\vec{x},\tau) \sim h|\vec{x}|^{-2\Delta^V+1}. \qquad (1.10)$$

B. Missing spin

Next, we will consider the behavior of the monopole/VBS operator V near the missing spin impurity illustrated in Fig. 2. As discussed in some detail in Ref. 19, the dominant consequence of such an impurity is an exactly marginal perturbation to $\mathcal{S}$ given by

$$\mathcal{S}_{\text{imp}} = iQ\int d\tau A_\tau(\vec{x}=0,\tau), \qquad (1.11)$$

where Q is a "charge" characterizing the impurity. The value of Q does not flow under the RG, and so Q is a pure number which controls all universal characteristics of the impurity response. For an impurity of Fig. 2 with a single missing spin, $Q=\pm 1$. The remainder of this paper presents an analysis of the critical properties of the $\mathcal{S}+\mathcal{S}_{\text{imp}}$ defined in Eqs. (1.1) and (1.11).

The magnetic correlations of the theory $\mathcal{S}+\mathcal{S}_{\text{imp}}$ (and of a related theory[29]) have been computed in previous papers[19,28] which obtained the scaling dimensions of the Néel order parameter, n^a, and of the uniform magnetization density in the vicinity of the impurity. It was found that the impurity significantly enhanced the local magnetic susceptibilities. For the case of double-layer antiferromagnets, which have magnetic ordering transitions described by the Landau-Ginzburg-Wilson theory, such impurity magnetic correlations have also been computed by similar methods,[18,23,30] and have been found to be in excellent agreement with numerical studies.[31–33]

This paper will describe the "charge-order" correlations of the theory $\mathcal{S}+\mathcal{S}_{\text{imp}}$ by a computation of the OPE of the monopole/VBS operator $V(\vec{x},\tau)$ as $\vec{x}\to 0$. Our principal result is that the OPE is modified from the form in Eq. (1.6) to

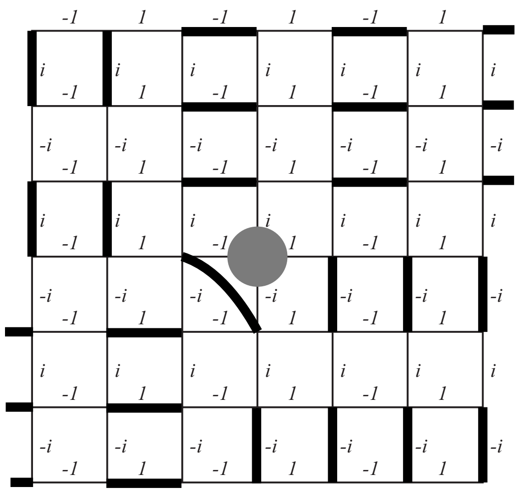

FIG. 3. A VBS *anti*-pinwheel in the presence of an impurity with a charge Q opposite in sign to that required by Eq. (1.12). This configuration has a higher energy cost than the VBS pinwheel configurations in Fig. 2.

$$\lim_{|\vec{x}|\to 0} V(\vec{x},\tau) \sim |\vec{x}|^{\Delta^V_{\rm imp}} e^{-iQ\theta} V_{\rm imp}(\tau), \qquad (1.12)$$

where θ is the azimuthal angle of $\vec{x}$. There are two important changes from Eq. (1.6). The first is that $V_{\rm imp}$ is no longer a trivial unit operator, but a fluctuating impurity degree of freedom with a nontrivial scaling dimension. The second is the presence of the $e^{-iQ\theta}$ factor, which indicates a Q-fold winding in the phase of the VBS order parameter around the impurity. The sketches in Fig. 2 give a simple physical interpretation of this winding in terms of the valence bond configurations of the underlying antiferromagnet: we will call this vortex-like winding in the VBS order a "VBS pinwheel." Also, as we discussed earlier,[19] the sign of Q is determined by the sublattice location of the missing spin. Thus, the result of Eq. (1.12) indicates that VBS pinwheels will occur preferentially around impurities on one sublattice, while VBS anti-pinwheels will occur around impurities on the other sublattice. This same result is also obtained from the intuitive microscopic pictures in Fig. 2. Also, we show in Fig. 3 an illustration of an anti-pinwheel in the presence of an impurity on the disfavored sublattice. The same sublattice bond indicates that this configuration has a higher energy.

Apart from establishing the form of Eq. (1.12), we will also describe computations of the exponent $\Delta^V_{\rm imp}$. There are general reasons for expecting that $\Delta^V_{\rm imp}>0$, and this will be the case in the explicit result we obtain. This positive value of $\Delta^V_{\rm imp}$ characterizes the suppression of VBS order near the impurity and should be contrasted with the negative value in Eq. (1.9) for the impurity in Fig. 1.

Our analysis will begin in Sec. II by a large N analysis of the theory $\mathcal{S}+\mathcal{S}_{\rm imp}$ with a full SU(N) spin symmetry. We will establish Eq. (1.12) in this limit. We will also find that the $N=\infty$ limit (at fixed Q) of the exponent $\Delta^V_{\rm imp}$ vanishes, but we will not evaluate the subleading correction in the $1/N$ expansion here.

The remainder of the paper will explore another approach to estimating $\Delta^V_{\rm imp}$. This relies[15,16] on examining the "easy-plane" limit of the $\mathbb{CP}^{N-1}$ model, in which the global SU(N) spin symmetry is reduced to U$(1)^{N-1}$. With this simplification to an Abelian global symmetry, an explicit duality transformation of the theory becomes possible. In the dual theory, the monopole/VBS operator V has a local expression in terms of the dual fields, and so this facilitates the analysis of the impurity critical property. We will begin the dual analysis in Sec. III by considering the simplest $N=1$ case:[34] This model describes the onset of VBS order in a $S=1/2$ quantum antiferromagnet in the presence of a staggered magnetic field[16] and is the simplest setting in which several technical issues can be described. We then extend the analysis to general N in Sec. IV. The exponent $\Delta^V_{\rm imp}$ will be estimated in these sections by a self-consistent theory of Gaussian fluctuations about a mean-field state; in the physically interesting case of $N=2$ and $Q=1$, which describes both the easy-plane antiferromagnet and the boson superfluid/insulator transition, we obtain the estimate

$$\Delta^V_{\rm imp} \approx 0.57, \quad N=2, \quad Q=1. \qquad (1.13)$$

Our analysis of the easy-plane theory in Sec. IV also exhibits certain features, which we do not expect to be shared by the case with a global SU(N) symmetry: For $Q/N=1/2$, we find VBS-vortex solutions in which the $e^{-iQ\theta}$ factor in Eq. (1.12) is replaced by $e^{-i\ell\theta}$, with the integer $-Q\leqslant \ell \leqslant Q$. In the self-consistent theory we present here, all the values of ℓ are degenerate, but we expect that these degeneracies are partially lifted in the full easy-plane theory. These issues are discussed further in Sec. IV and in a forthcoming paper.

II. 1/N EXPANSION OF THE $\mathbb{CP}^{N-1}$ THEORY IN THE PRESENCE OF MONOPOLES

The insertion of one monopole into the partition function of the $\mathbb{CP}^{N-1}$ model in the disordered phase has been originally considered in Ref. 21. The $1/N$ expansion proceeds by replacing the quartic self-interactions in Eq. (1.1) by a fixed-length constraint on the spinons; so, we consider the action

$$\mathcal{S} = \int d^2x d\tau \left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + i\lambda\left(|z_\alpha|^2 - \frac{1}{g}\right)\right], \qquad (2.1)$$

where λ is a fluctuating Lagrange multiplier field. The procedure for generating the $1/N$ expansion is now simple. One first integrates over the z fields obtaining an effective action for A_μ and λ. However, instead of expanding this effective action around the trivial classical vacuum $A_\mu=0$, one expands around the monopole (instanton) solution, A^i_μ, with

$$F^i_\mu = 2\pi q \frac{(x-x_0)_\mu}{4\pi|x-x_0|^3}, \qquad (2.2)$$

where $F_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$, q is the monopole charge, and x_0 is the monopole position. In practice, integrating out the z fields in the background of spatially varying monopole fields is quite complicated (even more so due to the appearance of UV and IR divergences), so that only the leading term in the $1/N$ expansion has been computed in the past (that is, fluctuations of A_μ about the monopole solution have not been taken into account). At this order, one finds

$$\langle V^q(x)\rangle \sim \left(\frac{m}{\Lambda}\right)^{2N\rho_q}, \tag{2.3}$$

where $V^q(x)$ is the monopole operator of charge q, m is the mass gap of the theory, Λ is the ultraviolet cutoff, and ρ_q is a collection of universal numbers (depending only on the charge of the monopole), which have been computed in Ref. 21. Thus, the dimension of operator $V^q(x)$, $\dim[V^q]=2N\rho_q$.

If finding the expectation value of a monopole operator (and its scaling dimension) was very complicated, finding correlators of $V(x)$ with Wilson loops at $N=\infty$ turns out to be exceedingly simple. Indeed, we notice that at leading order in $1/N$, it is sufficient to simply replace A_μ in the Wilson loop by its monopole value,

$$\frac{\langle V^q(x)\exp(-iQ\int_{\mathcal{C}} A_\mu dx_\mu)\rangle}{\langle V^q\rangle} \to \exp\left(-iQ\int_{\mathcal{C}} A^i_\mu dx_\mu\right) = \exp\left(-iQ\int_{\mathcal{S}} F^i_\mu dS_\mu\right), \tag{2.4}$$

provided that we take the charge Q to be $\mathcal{O}(1)$ in N [otherwise, if $Q\sim\mathcal{O}(N)$, the Wilson line will change the background monopole field and the problem becomes intractable]. Here, $\mathcal{C}$ is some closed contour and $\mathcal{S}$ is any surface such that $\partial\mathcal{S}=\mathcal{C}$. Thus, all we have to do is find the flux of our monopole through the Wilson loop that we are considering. Fluctuations of A_μ about the monopole field [Eq. (2.2)] will contribute at $\mathcal{O}(1/N)$ to the correlator [Eq. (2.4)]. Likewise, if we denote the Wilson loop operator by $W(\mathcal{C})$, then in the absence of the monopole field $\langle W(\mathcal{C})\rangle \sim 1+\mathcal{O}(1/N)$ (saturated by fluctuations of A_μ around the trivial vacuum). So,

$$\frac{\langle V^q(x)W(\mathcal{C})\rangle}{\langle W(\mathcal{C})\rangle} = \langle V^q\rangle \exp\left(-iQ\int_{\mathcal{S}} F^i_\mu dS_\mu\right), \tag{2.5}$$

and at leading order in $1/N$, the external charge only changes the phase of the expectation value of the monopole operator but not its magnitude.

In principle, we are interested in finding the correlator of the monopole operator [which we place at a point $x=(r\cos\theta, r\sin\theta, 0)$] and a straight, temporal Wilson line of charge Q (which we place at the origin). However, to regularize possible IR divergences, let us also place a charge $-Q$ on the positive x axis far away from the origin. As usual, we may connect the two oppositely directed Wilson lines in the far past and far future. Then, according to Eq. (2.4), we have to compute the magnetic flux due to the monopole field [Eq. (2.2)] through the $y=0$, $x>0$ half-plane,

$$\int \vec{F}\cdot d\vec{S} = -\frac{q}{2}\int_{-\infty}^{\infty} d\tau \int_0^\infty dx \frac{r\sin\theta}{[(x-r\cos\theta)^2+r^2\sin^2\theta+\tau^2]^{3/2}} \tag{2.6}$$

$$= -q\int_0^\infty dx \frac{r\sin\theta}{(x-r\cos\theta)^2+r^2\sin^2\theta} \tag{2.7}$$

$$= -q(\pi-\theta). \tag{2.8}$$

We see that the flux through the Wilson loop changes by $2\pi q$ as the monopole crosses the surface of the loop. However, the expectation value

$$\langle V^q(x)\rangle_{\text{imp}} = \frac{\langle V^q(x)W(\mathcal{C})\rangle}{\langle W(\mathcal{C})\rangle} = \langle V^q\rangle e^{iQq(\pi-\theta)} \tag{2.9}$$

remains single valued, as by Dirac's condition Q is an integer. [In what follows, we shall also often discuss Wilson loops with noninteger charge Q, which in the presence of monopole operators are defined by specifying a surface $\mathcal{S}$, $W(\mathcal{S})=e^{-iQ\int_{\mathcal{S}}F_\mu dS_\mu}$. The correlation functions then explicitly depend on the choice of the surface, as can be seen from Eq. (2.9).] Thus, we see that the phase of the monopole operator winds by $-2\pi Q$ as we move it in a full circle around the Wilson line; i.e., an external charge creates a vortex of the monopole field, consistent with the OPE in Eq. (1.12). We expect that once we go beyond the leading order in N, this vortex will also get a nontrivial spatial profile,

$$\langle V^q(x)\rangle_{\text{imp}} = \langle V^q\rangle f(m|\vec{x}|)e^{-iQq\theta}e^{i\chi}. \tag{2.10}$$

Here, $f(r)$ is the vortex profile function and $e^{i\chi}$ is some overall phase (discussed below). We expect that far away from the external charge, the monopole field tends to its vacuum expectation value so that $f(\infty)=1$. Moreover, by continuity, we expect the monopole field to vanish at the origin, $f(0)=0$. To the order to which we were working, $f(r)=1$, which implies that the impurity exponent $\Delta^V_{\text{imp}}\sim\mathcal{O}(1/N)$.

Notice that the result [Eq. (2.9)] is sensitive to the angular position of the distant charge relative to the one at the origin (we introduced the variable θ as the angle between the plane of the Wilson loop and the monopole operator). This is not unexpected. The monopole field is the order parameter for the flux symmetry, which is spontaneously broken in the disordered phase. As we rotate the distant charge, the overall phase $e^{i\chi}$ of the expectation value of the monopole operator changes—that is, we explore different states in our vacuum manifold.

If we were instead considering a correlation function of a string of monopole operators $\Pi_i V^{q_i}(x_i)$ such that the overall combination is invariant under the flux symmetry (that is, $\Sigma_i q_i=0$), we expect the dependence on the angular position of the distant charge to drop out. We can check this in the limit $m|x_i-x_j|\gg 1$, $m|\vec{x}_i|\gg 1$, assuming clustering,

$$\left\langle \prod_i V^{q_i}(x_i)\right\rangle_{\text{imp}} \to \prod_i \langle V^{q_i}(x_i)\rangle_{\text{imp}} \to \prod_i \langle V^q_i\rangle e^{-iQq_i\theta_i}, \tag{2.11}$$

which is invariant under $\theta_i\to\theta_i+\chi$. Alternatively, in the same limit of far separated monopoles and at $N=\infty$, the classical magnetic field will just be a linear superposition of magnetic fields due to each monopole. Thus, the flux Φ through the Wilson loop will be given by $\Phi=-\Sigma_i q_i(\pi-\theta_i)$

$=\Sigma_i q_i \theta_i$, and using the equivalent of Eq. (2.5) for a string of monopole operators, we arrive at the same expression [Eq. (2.11)].

We expect the general form [Eq. (2.10)] to be preserved at any finite order in $1/N$. Nevertheless, in the flux-broken phase of the theory, there are also nonperturbative effects that should be taken into consideration. Indeed, the $U(1)_\Phi$ vortex nucleated by the external charge is global and will thus have a logarithmically divergent energy. Put into a more conventional language, the external charge creates a Coulomb potential, which is logarithmic in two dimensions, $V(r) \approx -\frac{e^2 Q}{2\pi} \log(mr)$ for $mr \gg 1$. The effective coupling constant e^2 can be calculated in the $1/N$ expansion to be $e^2 \sim \frac{1}{N} m$. Thus, it will be energetically favorable for the external charge to bind a dynamical spinon (we concentrate on the case $Q=1$ here for simplicity). This process can be analyzed by means of a nonrelativistic Schrodinger equation.[42] One finds a bound state of size $r_b \sim N^{1/2} m^{-1}$. We expect that for $r \gg r_b$, the external charge will be screened by the dynamical spinon. On the other hand, for $r \ll r_b$, this logarithmic confinement should generally have little effect on the physics. However, there is one notable exception. The expectation value of the monopole operator V^q [Eq. (2.9)] will be drastically altered on all distance scales by the screening. Indeed, if we assume that screening takes place, $\langle V^q(\vec{x})\rangle$ has to tend to its vacuum expectation value for $|\vec{x}| \gg r_b$ and should experience no phase winding. We do not expect the winding number to change abruptly as we decrease $|\vec{x}|$, so we will not see a phase winding of $\langle V^q(\vec{x})\rangle$ on short distances $|\vec{x}| \ll r_b$ as well.

A toy model for the disappearance of winding when screening effects are taken into account can be constructed as follows. We can use the charge $-Q$ that we previously put far away from the origin to represent the dynamical spinon that gets bound to the external charge. We first freeze the location of this spinon at some position $\vec{x}'$ away from the origin and compute the resulting expectation value of $V^q(\vec{x})$ using Eq. (2.5). We then average the resulting $\langle V^q(\vec{x})\rangle$ over the spinon positions x' with the probability distribution $|\psi(\vec{x}')|^2$, where $\psi(\vec{x})$ is the spinon wave function. Since this wave function will be azimuthally symmetric, one immediately learns that upon averaging over the angular position of the spinon, $\langle V^q(\vec{x})\rangle$ loses its finite winding number and will, in fact, carry a constant phase for all $\vec{x}$. This same averaging will also lead to an additional supression $\langle V^q(\vec{x})\rangle \sim |\vec{x}|$ as $\vec{x} \to 0$ (recall that at $N=\infty$ there was no supression of the vortex profile for $x \to 0$ before screening effects were taken into account). The origin of this supression is easy to see—for an external charge located infinitely far away, the averaging over the azimuthal position of the charge is identical to the averaging of the phase χ in Eq. (2.10), producing a zero result for $\langle V^q\rangle$.

Do the above findings invalidate the OPE [Eq. (1.12)]? The answer is no. The above discussion simply implies that $\langle V_{\mathrm{imp}}\rangle=0$ and, thus, the expectation value $\langle V(\vec{x})\rangle$ for $\vec{x} \to 0$ is controlled by higher order terms in the OPE (namely, by the impurity operator with angular momentum zero). However, higher correlation functions of the V operator, e.g., the VBS susceptibility $\langle V(x) V^\dagger(x')\rangle$, are still controlled by the OPE [Eq. (1.12)]. Such correlators are invariant under the $U(1)_\Phi$ symmetry, so as we argued above, their short distance properties are not sensitive to the location of the distant charge and, hence, to screening physics.

III. EASY PLANE MODEL AT $N=1$

This section, and the next, will examine a simplified version of the theory $\mathcal{S}+\mathcal{S}_{\mathrm{imp}}$ in which the non-Abelian global $SU(N)$ symmetry is reduced to an Abelian $U(1)^{N-1}$ symmetry. This enables us to use the tools of Abelian particle-vortex duality[35,36] to obtain a theory expressed in terms of fields which are locally related to the monopole/VBS operator V. The present section will consider the simplest case[34] with $N=1$. This model describes the onset of VBS order in a $S=1/2$ quantum antiferromagnet in the presence of a staggered magnetic field[16] and is useful in resolving a number of key technical questions in their simplest setting. For $N=1$, the theory $\mathcal{S}$ does not have any global continuous symmetry and becomes equivalent to scalar electrodynamics. With the results for the $N=1$ theory obtained in the present section, we will be able to rapidly analyze the general N case in the next section.

A. Duality and Wilson loops

It is well known that in three space-time dimensions, near its critical point, noncompact $N=1$ scalar electrodynamics is dual to a theory of a complex (pseudo)scalar field with a global U(1) symmetry.[35,36] The Lagrangians of these two theories are as follows:

$$L_{QED} = \frac{1}{2e^2} F_\mu^2 + |(\partial_\mu - iA_\mu) z|^2 + m^2 |z|^2 + \frac{g}{2}|z|^4, \quad (3.1)$$

$$L_{XY} = |\partial_\mu V|^2 + \tilde{m}^2 |V|^2 + \frac{\tilde{g}}{2}|V|^4. \quad (3.2)$$

Here, z and V are complex one component fields. The duality is understood as being true for the range of parameters where L_{QED} has a second order phase transition (which at weak coupling is believed to occur for g/e^2 sufficiently large). One way to understand the duality is by noting that the phase transition in scalar QED is driven by spontaneous breaking of flux symmetry $U(1)_\Phi$, which is precisely the global symmetry of L_{XY}. The order parameter for the flux symmery is the monopole operator $V(x)$—that is, the dynamical field of L_{XY}.[37] As we know, to each continous symmetry, there corresponds a conserved current. In the case of flux symmetry of QED, this pseudovector current is just the magnetic field F_μ, which is trivially conserved in the absence of monopoles, $\partial_\mu F_\mu=0$. Let us introduce an external field H_μ that would couple to this current,

$$\delta L_{QED} = i H_\mu F_\mu. \quad (3.3)$$

Suppose we are calculating some correlation function with the insertion of a string of monopole operators of charge q_i at points x_i. The gauge field A_μ in the path integral is then subject to the condition $\partial_\mu F_\mu = \Sigma_i 2\pi q_i \delta(x-x_i)$. Then, under the transformation,

$$H_\mu \to H_\mu + \partial_\mu \alpha, \tag{3.4}$$

$$S_{QED} \to S_{QED} + i \int dx \partial_\mu \alpha F_\mu = S_{QED} - i \int dx \alpha \partial_\mu F_\mu = S_{QED} - 2\pi i \sum_i q_i \alpha(x_i). \tag{3.5}$$

Hence, by introducing the field H_μ, we can enlarge the global $U(1)_\Phi$ symmetry to a fictitious local symmetry, provided that the monopole operators transform as

$$V^q(x) \to e^{2\pi i q \alpha(x)} V^q(x). \tag{3.6}$$

The dual Lagrangian L_{XY} has to possess this local symmetry. Hence, to introduce the field H_μ into the dual Lagrangian, we simply have to covariantize the derivative of the dynamical monopole field V,

$$\partial_\mu V \to D_\mu V = (\partial_\mu - 2\pi i H_\mu) V, \tag{3.7}$$

in Eq. (3.2). Other "gauge invariant" operators can also be added to L_{XY}, e.g., $H_{\mu\nu}^2$; however, their contribution will generally either cancel out in correlation functions or be less singular near the critical point.

Thus, the dual Lagrangian in the presence of a background source field H_μ is given by

$$L_{XY} = |(\partial_\mu - 2\pi i H_\mu) V|^2 + \tilde{m}^2 |V|^2 + \frac{\tilde{g}}{2} |V|^4. \tag{3.8}$$

The covariantization procedure [Eq. (3.7)] was explicitly written down in Ref. 38. Similar arguments for the case of a constant imaginary H_μ, which physically represents an external magnetic field in the QED language and translates into a chemical potential for the flux symmetry in the XY language, have been given in Ref. 39. In a companion paper,[40] we shall also give an argument based on an exact duality transformation on the lattice, which will support Eq. (3.8).

Having learned how to incorporate the source field H_μ into the dual Lagrangian, it is now trivial to dualize Wilson loops. Indeed, the insertion of a Wilson loop $W(\mathcal{C})$ into a correlation function is equivalent to adding into the Lagrangian the source term

$$\delta L = iQ \int_{\mathcal{C}} dx_\mu A_\mu = iQ \int_{\mathcal{S}} dS_\mu F_\mu = i \int dx H_\mu F_\mu, \tag{3.9}$$

where

$$H_\mu(x) = Q \int_{y \in \mathcal{S}} dS_\mu \delta(x-y). \tag{3.10}$$

That is, H_μ is a field that lives on the surface of the Wilson loop and is directed perpendicular to this surface.

Another benefit of introducing the source field H_μ is that by differentiating with respect to it, we can compute correlation functions of the magnetic field F_μ. For instance,

$$\langle -iF_\mu(x) \rangle_H = \frac{\delta \log Z[H]}{\delta H_\mu(x)} = -2\pi i \langle [V^\dagger D_\mu V - (D_\mu V)^\dagger V](x) \rangle_H. \tag{3.11}$$

That is, the topological flux current F_μ of QED gets mapped into Noether's current associated with the global U(1) symmetry of the dual model. Differentiating once again,

$$\langle F_\mu(x) F_\nu(y) \rangle_{H,conn} = -\frac{\delta Z[H]}{\delta H_\mu(x) \delta H_\nu(y)} \tag{3.12}$$

$$= (2\pi)^2 (\langle V^\dagger \overleftrightarrow{D}_\mu V(x) V^\dagger \overleftrightarrow{D}_\nu V(y) \rangle_{H,conn} + 2\delta_{\mu\nu} \delta(x-y) \langle V^\dagger V(x) \rangle_H). \tag{3.13}$$

The first term in Eq. (3.13) is the expected correlator of two $U(1)_\Phi$ currents, while the second term is a tadpole that ensures the overall transversality of the correlation function.

Having discussed the duality at length, we now return to our original problem. What is the influence of the external charge (Wilson line) on various physical observables? The observable of most interest to us is the monopole operator $V(x)$. However, this observable is physical only for integer-valued charge Q of the Wilson line (Dirac's condition). Indeed, recall that in the dual language, the field H depends on a choice of surface $\mathcal{S}$ of the Wilson loop. If we pick a different surface $\mathcal{S}'$, then the field H_μ undergoes a gauge transformation $H_\mu \to H'_\mu = H_\mu + \partial_\mu \alpha$, with $\alpha(x) = -Q 1_{x \in \mathcal{V}}$, where $\mathcal{V}$ is the volume bounded by the two surfaces $\mathcal{S}$ and $\mathcal{S}'$. Hence,

$$\langle V(x) \cdots \rangle_{H'} = e^{2\pi i \alpha(x)} \langle V(x) \cdots \rangle_H, \tag{3.14}$$

where the ellipses denote some other operators. Thus, the operator $V(x)$ is invariant under changing the surface of the Wilson loop if and only if Q is an integer. However, if the charge Q is a rational number, $Q = p/q$, where p and q are integers, then the flux $2\pi q$ monopole operator $V^q(x) \sim [V(x)]^q$ is physical. Moreover, a theory with arbitrary irrational Q is still sensible provided that we confine our attention to correlation functions of operators which are invariant under the fictitious $U(1)_\Phi$ local symmetry, e.g., the magnetic field operator $-iF_\mu = -2\pi i V^\dagger \overleftrightarrow{D}_\mu V$. In fact, if we are dealing with such gauge invariant operators, we do not necessarily have to use the precise form of H given by Eq. (3.10); defining γ_μ to be a field living on the perimeter of the Wilson loop and directed along it,

$$\gamma_\mu(x) = Q \int_{y \in \mathcal{C}} dy_\mu \delta(x-y), \tag{3.15}$$

we see that

$$\epsilon_{\mu\nu\lambda} \partial_\nu H_\lambda = \gamma_\mu. \tag{3.16}$$

Then, by performing a suitable gauge transformation on H_μ and V, we can choose H_μ to be any field with curl given by γ_μ. Thus, we see that the duality maps a Wilson loop of charge Q in the QED language to an external magnetic flux tube of flux $2\pi Q$ in the XY language. This correspondence has been noted in Ref. 41, but the consequences of this correspondence for the critical properties of Wilson loops were not discussed.

Now, we can address the problem that we originally posed in a dual language. Let us place an external charge Q at the spatial origin. For now, we do not insist that this charge be an integer. The dual source field H_μ must, therefore, satisfy

$$\nabla \times \vec{H} = Q\delta^2(\vec{x})\hat{\tau}. \tag{3.17}$$

Thus, we basically have to solve an Aharonov-Bohm problem with flux $2\pi Q$. One choice for the source field H_μ is

$$H_\mu(x) = Q\delta_{\mu,2}\theta(x)\delta(y). \tag{3.18}$$

This is the so-called string gauge, which corresponds to Eq. (3.10), with the surface of the Wilson loop being the plane $y=0$, $x>0$. As is well known, the string gauge is equivalent to $H_\mu=0$ and the boundary condition,

$$V(\theta=2\pi) = e^{-2\pi i Q}V(\theta=0), \tag{3.19}$$

where θ is the azimuthal angle. Thus, we have to solve the theory [Eq. (3.2)] with the twisted boundary condition [Eq. (3.19)]. We observe that the physics is, therefore, a periodic function of Q. For integer Q, the boundary condition [Eq. (3.19)] is trivial—there is no twist. So, our argument indicates that integral external charges do not affect correlation functions on distances of the order of the correlation length of the theory. The screening of integral charges takes place on distance scales of the order of microscopic UV cutoff. This surprising fact is discussed in more detail in a companion paper.[40]

The behavior at noninteger Q is less unexpected. One physical question that we may ask is, what is the magnetic (electric) field induced by the charge Q? (we define the electric field $E_i=F_{i3}=-\epsilon_{ij}F_j$ where Latin letters i, j, k run over spatial indices). Although this is a departure from our original goal, we will see that a lot of the results that we will obtain along the way will be useful when we return to discuss correlators of monopole field for the planar theory with N fields. Another question that we will adress for noninteger, rational, values of $Q=p/q$ is the behavior of higher flux monopole operators $V(x)^q$.

B. Perturbative expansion of the dual theory for $Q\to 0$

The magnetic field $-iF_\mu$ is a conserved current and receives no renormalizations and, thus, has conformal dimension 2. Therefore, at the critical point, we expect

$$\langle -i\vec{E}\rangle = C(Q)\frac{1}{r^2}\hat{r}. \tag{3.20}$$

The electric field is imaginary as we are working in Euclidean space. The coeficient $C(Q)$ is a universal number that is a periodic function of charge Q. We shall be interested in determining this function.

For $Q\to 0$, we can perform a perturbative expansion in $H_\mu \sim \mathcal{O}(Q)$,

$$\langle -iF_\mu(x)\rangle_H = \frac{\delta \log Z[H]}{\delta H_\mu(x)} \approx \int dy \frac{\delta^2 \log Z}{\delta H_\mu(x)\delta H_\nu(y)}H_\nu(y)$$
$$= -\int dy\langle F_\mu(x)F_\nu(y)\rangle H_\nu(y). \tag{3.21}$$

As we have learned, the correlation function of the magnetic field F_μ dualizes to

$$K_{\mu\nu}(x-y) = \langle F_\mu(x)F_\nu(y)\rangle$$
$$= (2\pi)^2(\langle V^\dagger \overleftrightarrow{\partial}_\mu V(x)V^\dagger \overleftrightarrow{\partial}_\nu V(y)\rangle + 2\delta_{\mu\nu}\delta(x-y)\langle V^\dagger V\rangle). \tag{3.22}$$

By transversality,

$$K_{\mu\nu}(p) = K(p)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right). \tag{3.23}$$

By RG, $K(p)$ should have the form

$$K(p) = Mg(p/M), \tag{3.24}$$

where M is some physical scale in the theory [e.g., in the $\mathrm{U}(1)_\Phi$ disordered phase, the mass of the monopole field V]. At the critical point,

$$K(p) = A|p|, \tag{3.25}$$

where A is some universal number. On the XY side of the theory, this universal number has been computed before using both ϵ expansion[43] and large M expansion.[44] The large M expansion is obtained by replacing the complex scalar V in the action for the XY theory [Eq. (3.2)] by an M component complex field. In the large M expansion, the coefficient A is found to be at the next to leading order in M,

$$A = (2\pi)^2\frac{M}{16}\left(1 - \frac{1}{M}\frac{32}{9\pi^2}\right) \overset{M=1}{\approx} 1.6, \tag{3.26}$$

while in the ϵ expansion, one obtains $A\approx 2.0$ at $\mathcal{O}(\epsilon^2)$. Monte Carlo simulations on the XY model[44] indicate $A \approx 1.8$.[45] The coefficient A can also be computed by performing a large N expansion in the original QED, whereby the field z is promoted to have N components. At leading order, one obtains $A=16/N \overset{N=1}{=} 16$ (as usual, a direct large N expansion in QED produces results, which are numerically notoriously inaccurate for $N\sim 1$).

For completeness, we also discuss the behavior of $K(p)$ at small momenta on both sides of the critical point. In the phase where the $\mathrm{U}(1)_\Phi$ symmetry is spontaneously broken, the spectrum of the theory should contain a goldstone, which can be created out of the vacuum by the $\mathrm{U}(1)_\Phi$ current,

$$\lim_{p\to 0}\langle p|F_\mu(x)|0\rangle = 2\pi \lim_{p\to 0}\langle p| - iV^\dagger \overleftrightarrow{\partial}_\mu V(x)|0\rangle = 2\pi i f p_\mu e^{ipx}, \tag{3.27}$$

where in three dimensions f^2 defines a physical energy scale. Note that Eq. (3.27) is written in Minkowski space. We see that the goldstone is nothing but the photon of the original QED. Then, $K_{\mu\nu}(p)$ should have a pole at $p^2=0$ and using spectral decomposition,

$$\lim_{p\to 0} K(p) = (2\pi f)^2. \tag{3.28}$$

On the other hand, in the phase where the $U(1)_\Phi$ symmetry is unbroken (that is, in the "superconducting" phase of QED) the V field is massive, and all the excitatations have a gap. Therefore, $K_{\mu\nu}(p)$ cannot have a pole at $p^2=0$ and

$$\lim_{p\to 0} K(p) \sim \frac{p^2}{M}. \tag{3.29}$$

Having discussed the expected form of $K_{\mu\nu}$ in different phases, we can go back to Eq. (3.21) for electric field induced by the charge Q. Introducing the kernel $\mathcal{D}(p) = K(p)/p^2$ and using Eq. (3.17),

$$\begin{aligned}\langle -iF_\mu(x)\rangle &= -\int dy K_{\mu\nu}(x-y)H_\nu(y)\\ &= -Q\int d\tau' \epsilon_{\mu\nu 3}\partial_\nu^x \mathcal{D}(\vec{x},\tau').\end{aligned} \tag{3.30}$$

Hence,

$$\langle -i\vec{E}(\vec{x})\rangle = Qh(|\vec{x}|)\hat{r}, \tag{3.31}$$

where

$$h(|\vec{x}|) = -\frac{\partial}{\partial|\vec{x}|}\int d\tau' \mathcal{D}(\vec{x},\tau'). \tag{3.32}$$

Substituting expression (3.25) for $K(p)$ at the critical point, we obtain

$$\langle -i\vec{E}(\vec{x})\rangle = Q\frac{A}{2\pi|\vec{x}|^2}\hat{r}. \tag{3.33}$$

Hence, we identify

$$C(Q) \approx QA/(2\pi), \quad Q\to 0. \tag{3.34}$$

Similarly, in the $U(1)_\Phi$ ordered phase,

$$\langle -i\vec{E}(\vec{x})\rangle = Q\frac{2\pi f^2}{|\vec{x}|}\hat{r}. \tag{3.35}$$

So, in this phase, as expected, the external electric charge produces the usual Coulomb-like electric field, $\vec{E}=\frac{e_{\rm eff}^2 Q}{2\pi r}$, as appropriate to two spatial dimensions with the identification $e_{\rm eff}=2\pi f$.

C. Peculiarities of the free theory

So far, we have only discussed the leading term in $C(Q)$ for $Q\to 0$. In principle, we could continue the expansion in Q to higher orders. Then, the problem reduces to finding correlators of current operators $-iF_\mu = iV^\dagger \overleftrightarrow{\partial}_\mu V$. These correlators can be found by performing either ϵ or $1/M$ expansion of the XY model. In either case, going beyond the leading order in Q is not simple. So, instead, we choose to return to the formulation of the problem involving the twisted boundary condition [Eq. (3.19)]. In the next section, we will use this formulation to compute $C(Q)$ for all Q (albeit numerically) at $M=\infty$. However, before we do so, we will solve a slightly simpler problem: Namely we find the form of $C(Q)$ at the Gaussian fixed point $\tilde{g}=0$, $\tilde{m}^2=0$ of the Lagrangian [Eq. (3.2)]. The reason for studying the free theory is that the calculations in it are, technically, very similar to those in the strongly coupled $M=\infty$ theory addressed in the next section (even though the physical results are quite different).

In the free theory, $C(Q)$ can be determined exactly and, surprisingly, turns out to be a nonanalytic function of Q at $Q=0$. We have not been able to see any hints of this nonanalyticity from the perturbative expansion of the free theory in Q (perhaps because we could go perturbatively only to linear order in Q, whereas the nonanalyticity of $C(Q)$ starts only at order $|Q|^2$). On the other hand, once we go in the next section to the strongly interacting fixed point (obtained in the $M=\infty$ limit), the theory cures itself of all IR divergences and $C(Q)$ becomes analytic in Q.

So, let us compute

$$\begin{aligned}\langle -iF_\mu(x)\rangle &= \langle -2\pi i V^\dagger \overleftrightarrow{\partial}_\mu V(x)\rangle\\ &= -2\pi i \lim_{x\to y}(\partial_\mu^x - \partial_\mu^y)\langle V(x)V^\dagger(y)\rangle\end{aligned} \tag{3.36}$$

in the free theory $L=|\partial_\mu V|^2$ subject to the boundary condition [Eq. (3.19)]. As Eq. (3.36) shows, to find the $U(1)_\Phi$ current, it is sufficient to determine the propagator, $D(x-y) = \langle V(x)V^\dagger(y)\rangle$. The propagator will also determine the correlation function of operators $[V(x)]^q$ for rational $Q=p/q$,

$$\langle [V(x)]^q [V^\dagger(y)]^q\rangle = q!D(x-y)^q. \tag{3.37}$$

We note that our problem is invariant under translations along the temporal direction, so

$$D(\vec{x},\vec{x}',\tau-\tau') = \int \frac{d\omega}{2\pi} D_2(\vec{x},\vec{x}',\omega^2)e^{i\omega(\tau-\tau')}, \tag{3.38}$$

where $D_2(\vec{x},\vec{x},\omega^2)$ denotes the two-dimensional propagator with mass $m^2=\omega^2$ and twisted B.C. [Eq. (3.19)]. We use spectral decomposition to find D_2,

$$D_2(\vec{x},\vec{x}',m^2) = \sum_l \frac{e^{il\theta}}{2\pi}\int_0^\infty dE \frac{1}{m^2+E}\phi_{l,E}(\vec{r})\phi_{l,E}^*(\vec{r}'), \tag{3.39}$$

where we sum over states with fixed azimuthal angular momentum $l=n-Q$, $n\in\mathbb{Z}$. Note that the angular momenta are not integral due to the twisted B.C. [Eq. (3.19)]. The radial eigenfunctions $\phi_{l,E}(r)$ satisfy

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{l^2}{r^2}\right]\phi_{l,E}(r) = E\phi_{l,E}(r) \tag{3.40}$$

and are normalized as

$$\int_0^\infty dr r\phi_{l,E}^*(r)\phi_{l,E'}(r) = \delta(E-E'). \tag{3.41}$$

The solution to ODE [Eq. (3.40)] is

$$\phi_{l,E}(r) = \frac{1}{\sqrt{2}} J_{|l|}(\sqrt{E}r), \qquad (3.42)$$

where $J_n(u)$ is the nth order Bessel function. Hence,

$$D(r,r',\theta-\theta',\tau-\tau') = \sum_l e^{il(\theta-\theta')} \int \frac{d\omega}{2\pi} e^{i\omega(\tau-\tau')} \int_0^\infty \frac{du}{2\pi} \frac{u}{u^2+\omega^2} J_{|l|}(ur) J_{|l|}(ur'), \qquad (3.43)$$

where we made the substitution $u=\sqrt{E}$. Integrating over ω,

$$D(r,r',\theta,\tau) = \frac{1}{4\pi r'} \sum_l e^{il\theta} \int_0^\infty dv J_{|l|}\left(\frac{r}{r'}v\right) J_{|l|}(v) \exp\left(-\frac{|\tau|}{r'}v\right). \qquad (3.44)$$

Now, we can ask, what is the behavior of the propagator $D(r,r',\theta,\tau)$ for $r\to 0$, i.e., for $r \ll r'$. Recalling $J_{|l|}(r) \approx \frac{1}{2^{|l|}\Gamma(|l|+1)} r^{|l|}$,

$$\int_0^\infty dv J_{|l|}\left(\frac{r}{r'}v\right) J_{|l|}(v) \exp\left(-\frac{|\tau|}{r'}v\right) \approx \left(\frac{r}{r'}\right)^{|l|} B_l\left(\frac{|\tau|}{r'}\right), \qquad (3.45)$$

with

$$B_l(u) = \frac{1}{2^{|l|}\Gamma(|l|+1)} \int dv\, v^{|l|} J_{|l|}(v) \exp(-uv) = \frac{\Gamma\left(|l|+\frac{1}{2}\right)}{\Gamma(|l|+1)} \left(1+\frac{\tau^2}{r'^2}\right)^{-|l|-1/2}. \qquad (3.46)$$

Thus, for $r\to 0$, the contribution of states with angular momentum l to the propagator scales as $r^{|l|}$. So, the largest contribution comes from the smallest $|l|=|n-Q|$. For $-\frac{1}{2}<Q<\frac{1}{2}$, the smallest $|l|$ is given by setting $n=0$ and $l=-Q$. Hence, for $|Q|<1/2$ and $r/r' \ll 1$,

$$D(r,r',\theta,\tau) \approx \frac{1}{4\pi r'} \left(\frac{r}{r'}\right)^{|Q|} e^{-iQ\theta} B_Q\left(\frac{\tau}{r'}\right). \qquad (3.47)$$

For values of $|Q|>1/2$, we simply periodize Eq. (3.47) since all physics in the XY model is periodic in Q with period 1 (see discussion in the previous section). From here on, we therefore confine our attention to $|Q|<1/2$.

Thus, if we were to perform the OPE in Eq. (1.12) in the XY model,

$$V(\vec{x},\tau) \sim |\vec{x}|^{\Delta^V_{\text{imp}}} e^{-iQ\theta} V_{\text{imp}}(\tau) \quad \text{for } |\vec{x}| \to 0, \qquad (3.48)$$

we would obtain the following for $|Q|<\frac{1}{2}$ in the free XY model:

$$\Delta^V_{\text{imp}} = |Q|. \qquad (3.49)$$

We immediately see that the free theory is nonanalytic in Q at $Q=0$. By periodizing in Q, we also see that Δ^V_{imp} is nonanalytic at $Q=\pm 1/2$. However, this later nonanalyticity appears only after we take the $r\to 0$ limit of the propagator, while we expect the nonanalyticity at $Q=0$ to persist in the propagator for arbitrary r,r'.

In fact, $Q=\frac{1}{2}$ is a very special point. At this point, the $n=0$, $l=-Q$ and $n=1$, $l=1-Q$, i.e., $l=\pm 1/2$ terms in the sum of Eq. (3.44), become equally important for $r/r'\to 0$. Thus, for $Q\to 1/2$, it makes sense to keep both terms in the asymptotic expansion of the propagator,

$$D(r,r',\theta,\tau) \approx \frac{1}{4\pi r'} \left[\left(\frac{r}{r'}\right)^{Q} e^{-iQ\theta} B_Q\left(\frac{\tau}{r'}\right) + \left(\frac{r}{r'}\right)^{1-Q} e^{-i(Q-1)\theta} B_{Q-1}\left(\frac{\tau}{r'}\right) \right], \qquad (3.50)$$

and we may hypothesize the impurity OPE for $Q\to 1/2$,

$$V(\vec{x},\tau) \sim c_Q |\vec{x}|^{\Delta^V_Q} e^{-iQ\theta} V_Q(\tau) + c_{Q-1} |\vec{x}|^{\Delta^V_{Q-1}} e^{-i(Q-1)\theta} V_{Q-1}(\tau) \quad \text{for } |\vec{x}| \to 0, \qquad (3.51)$$

where V_Q and V_{Q-1} are two impurity operators, with impurity anomalous dimensions Δ^V_Q and Δ^V_{Q-1}. In the free theory, $\Delta^V_Q=Q$ and $\Delta^V_{Q-1}=1-Q$. Hence, for $Q<1/2$, $\Delta^V_Q<\Delta^V_{Q-1}$ and the operator V_Q is the most relevant as $|\vec{x}|\to 0$, while the operator V_{Q-1} provides a subleading correction. For $Q>1/2$, the roles of these two operators are reversed. Finally, for $Q=1/2$, the two operators have degenerate anomalous dimensions, $\Delta^V_{1/2}=\Delta^V_{-1/2}$ and

$$V(\vec{x},\tau) \sim c_{1/2} |\vec{x}|^{\Delta^V_{1/2}} e^{-i\theta/2} V_{1/2}(\tau) + c_{-1/2} |\vec{x}|^{\Delta^V_{-1/2}} e^{i\theta/2} V_{-1/2}(\tau) \quad \text{for } |\vec{x}| \to 0. \qquad (3.52)$$

Physically, the $Q=1/2$ point is special because the CP symmetry is effectively restored at it.[46] Indeed, under CP, $Q\to -Q$. However, as already discussed, the universal physics is periodic in Q, so the points $Q=\pm 1/2$ are identified. Thus, the two impurity operators $V_{\pm 1/2}$ are just CP conjugates of each other and must have the same impurity anomalous dimensions. Hence, although our original analysis was performed for the case of the free theory, we expect the conclusions to remain valid in the strongly interacting theory.

We remind the reader that even though the operator $V(x)$ is mathematically well defined by specifying the surface $\mathcal{S}$ of the Wilson loop for arbitrary Q, it is not physical for nonintegral Q. Indeed, a physical operator cannot obey twisted boundary conditions. However, for rational $Q=p/q$, the flux $2\pi q$ monopole operator $V^q(x) \sim [V(x)]^q$ is well defined on both sides of the duality. Using Eqs. (3.37) and (3.47), we obtain the OPE,

$$V^q(\vec{x},\tau) \sim |\vec{x}|^{\Delta^V_{\text{imp}}(q)} e^{-iqQ\theta} V^q_{\text{imp}}(\tau) \quad \text{for } |\vec{x}| \to 0, \qquad (3.53)$$

with

$$\Delta^V_{\text{imp}}(q) = q|Q| \qquad (3.54)$$

in the free XY theory for $|Q|<1/2$. Since $qQ=p$ is an integer, the OPE [Eq. (3.53)] is invariant under $\theta\to\theta+2\pi$, making the operator $V^q(x)$ single valued, as required.

Having discussed the impurity OPEs, let us return to the calculation of the electric field. Since we know that the electric field will be radial, we only need the $\hat{\theta}$ component of the magnetic field,

$$\langle -iF_\theta\rangle = -2\pi i\frac{1}{r}\lim_{\theta\to\theta'}(\partial_\theta-\partial_{\theta'})D(r=r',\theta-\theta',\tau=\tau')$$
$$= -4\pi i\frac{1}{r}\lim_{\theta\to 0}\partial_\theta D(r=r',\theta,\tau=\tau'). \tag{3.55}$$

For this purpose, we do not need the propagator with $r/r' \ll 1$, but rather with $r\to r'$, $\tau\to\tau'$. We denote $D(r,\theta)=D(r=r',\theta,\tau=\tau')$. Unfortunately, if we plug $r=r'$ and $\tau=\tau'$ into the expression for the propagator [Eq. (3.44)], the integral over v diverges. We expect that if we instead first keep $r-r'$ and $\tau-\tau'$ finite, perform the integration over v, sum over angular momenta l, and only then take $r=r'$ and $\tau=\tau'$, the divergence disappears. There are also other ways to regularize the propagator: e.g., make the integral over ω in Eq. (3.38) run over $D-2$ dimensions. This would correspond to the XY model in D dimensions coupled to an external flux tube (the flux tube is a defect in two dimensions, so its world volume is $D-2$ dimensional). One then takes the limit $D\to 3$ at the end of the calculation. We have successfully used this method to compute the electric field (see Appendix A). The result for the coefficient $C(Q)$ of Eq. (3.20) is

$$C(Q)=\frac{1}{8}(1-2|Q|)^2\tan(\pi Q),\quad |Q|<1. \tag{3.56}$$

Thus, we see that the function $C(Q)$ is nonanalytic at $Q=0$. This analyticity occurs at nonleading order in Q,

$$C(Q)\approx\frac{\pi}{8}Q(1-4|Q|),\quad Q\to 0. \tag{3.57}$$

The leading order term, $C(Q)\approx\frac{\pi}{8}Q$, is the one which would have been predicted by expanding the free theory perturbatively in Q.

One can also derive the result [Eq. (3.56)] in a different way, which can be more easily generalized from the free theory to the $1/M$ expansion in a strongly interacting theory. This calculation is based on the integral representation of the propagator of the twisted theory derived in Ref. 47. We repeat the calculations of Ref. 47 in Appendix B since in the next section we will need to generalize them for application in $1/M$ expansion. The result is

$$D(r,\theta)=\frac{1}{4\pi r}\int_0^\infty d\nu\,\tanh(\pi\nu)U_\nu(\theta), \tag{3.58}$$

with

$$U_\nu(\theta)=\frac{e^{-2\pi iQ\,\mathrm{sgn}(\theta)}\sinh(\nu|\theta|)+\sinh[\nu(2\pi-|\theta|)]}{\cosh(2\pi\nu)-\cos(2\pi Q)} \tag{3.59}$$

from which one recovers Eq. (3.56) by using Eq. (3.55) (see Appendix B).

D. 1/*M* expansion of the dual theory

We now progress from the free XY model to the $1/M$ expansion of the strongly interacting theory. We take the Lagrangian to be

$$L=|\partial_\mu V|^2+i\lambda\left(|V|^2-\frac{1}{g}\right). \tag{3.60}$$

Here, V is an M component complex scalar and λ is a Lagrange multiplier, which enforces the local constraint,

$$|V|^2=\frac{1}{g}. \tag{3.61}$$

This hard constraint replaces the self-interaction of the V field. In the presence of an external charge in the direct theory, we take V to satisfy the twisted boundary conditions [Eq. (3.19)]. In principle, we would like to solve the theory [Eq. (3.60)] in the limit $M\to 1$. However, practically, we will only be able to perform computations at $M=\infty$.

We will be interested in the properties of the theory [Eq. (3.19)] at its critical point $g=g_c$. As is well known from standard $1/M$ expansion techniques, at $M=\infty$, the critical coupling is given by

$$\frac{1}{Mg_c}=\frac{1}{M}\langle V^\dagger V\rangle = D(x=x'), \tag{3.62}$$

where D is the usual massless three-dimensional (3D) propagator,

$$D(x,x')=\frac{1}{4\pi|x-x'|}. \tag{3.63}$$

Of course, the propagator with $x=x'$ in Eq. (3.62) is UV singular and has to be regularized. Since we will perform calculations of the propagator in position space, it is convenient for us to use point-splitting regularization.

In the absence of the twisted boundary condition [Eq. (3.19)] and at the critical point, we perform the expansion around $\langle i\lambda\rangle=0$ (so that the effective mass for the V particles vanishes). However, once Q is finite, $\lambda=0$ is no longer sufficient to make the constraint [Eq. (3.61)] satisfied. Instead, the Lagrange muliplier aquires a spatial dependence,

$$\langle i\lambda(\vec{x},\tau)\rangle=\frac{a(Q)}{|\vec{x}|^2}. \tag{3.64}$$

Here, a is a universal function of the charge Q. The dependence on $\vec{x}$ is determined from the canonical dimension of λ (λ aquires a nontrivial anomalous dimension only at order $1/M$). Thus, at finite Q, the propagator of V field satisfies

$$\left(-\partial^2+\frac{a(Q)}{|\vec{x}|^2}\right)D(x,x',Q)=\delta(x-x') \tag{3.65}$$

and $a(Q)$ should be determined self-consistently from the equation

$$\frac{1}{Mg_c}=\frac{1}{M}\langle V^\dagger V\rangle_Q = D(x=x',Q). \tag{3.66}$$

Combining Eqs. (3.62) and (3.66),

$$\lim_{x\to x'}[D(x,x',Q)-D(x,x',Q=0)]=0. \quad (3.67)$$

Thus, the problem is reduced to finding the propagator $D(x,x',Q)$. Just as in the free case, we use spectal decomposition [Eq. (3.39)], and the radial functions $\phi_{l,E}(r)$ now satisfy

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{l^2+a}{r^2}\right]\phi_{l,E}(r)=E\phi_{l,E}(r), \quad (3.68)$$

where, again, due to the twisted boundary conditions $l=n-Q$, $n\in\mathbb{Z}$. The solution to Eq. (3.68) is

$$\phi_{l,E}(r)=\frac{1}{\sqrt{2}}J_{\sqrt{l^2+a}}(\sqrt{E}r). \quad (3.69)$$

Comparing the result above to the free theory [Eq. (3.42)], we see that the only difference is in the replacement of the indices of Bessel functions $|l|\to\sqrt{l^2+a}$. Going from a two-dimensional to 3D propagator as in the free case [Eq. (3.44)],

$$D(r,r',\theta,\tau)=\frac{1}{4\pi r'}\sum_l e^{il\theta}\int_0^\infty dvJ_{\sqrt{l^2+a}}\left(\frac{r}{r'}v\right)\times J_{\sqrt{l^2+a}}(v)\exp\left(-\frac{|\tau|}{r'}v\right). \quad (3.70)$$

Finally, expanding the propagator [Eq. (3.70)] for $r\ll r'$, we obtain the equivalent of Eq. (3.47),

$$D(r,r',\theta,\tau)\approx\frac{1}{4\pi r'}\left(\frac{r}{r'}\right)^{\sqrt{Q^2+a(Q)}}e^{-iQ\theta}B_{\sqrt{Q^2+a(Q)}}\left(\frac{\tau}{r'}\right),\quad |Q|<1/2. \quad (3.71)$$

Thus, we recover the OPE [Eq. (3.48)], but the impurity exponent now becomes some nontrivial function of Q,

$$\Delta^V_{\text{imp}}=\sqrt{Q^2+a(Q)},\quad |Q|<1/2. \quad (3.72)$$

We note that similar to the free case, as Q passes $1/2$, the most relevant angular momentum l in the sum [Eq. (3.70)] changes from $l=-1/2$ to $l=1/2$, and at $Q=1/2$ we have the OPE [Eq. (3.52)] with two degenerate impurity operators.

To find the nontrivial impurity exponent, we need to solve Eq. (3.67) for $a(Q)$. We are, therefore, after the propagator $D(x,x',Q)$, with $x\to x'$. We could, in principle, proceed as in the free case. Namely, make our flux tube uniform along $D-2$ spatial dimensions [introducing a convergence factor v^{D-3} into Eq. (3.70)], perform the integrals in Eq. (3.70) with $r=r'$ and $\tau=0$, perform the sum over the angular momenta l, and take $\theta\to 0$ and $D\to 3$. However, unlike in the free case, the sums over angular momenta cannot be performed now analytically in terms of hypergeometric functions (with nice analytic continuation for $\theta\to 0$). The sum over l can still be performed numerically; however, the convergence is rather slow. Nevertheless, we have been able to determine $a(Q)$ numerically using this method. However, this method is less suitable for finding the electric field coefficient $C(Q)$, which requires us to differentiate the propagator at $\theta=0$, making the convergence properties of the series even worse.

Instead, we shall use a different method, generalizing the integral form of the propagator [Eq. (3.58)] derived in Ref. 47 to the present problem. As shown in Appendix B, the twisted propagator at $M=\infty$ is given by

$$D(r,\theta)=\frac{1}{4\pi r}\int_0^\infty d\nu\tanh(\pi\nu)\frac{\nu}{\sqrt{\nu^2+a}}U_{\sqrt{\nu^2+a}}(\theta), \quad (3.73)$$

with $U_\nu(\theta)$ still given by Eq. (3.59).

Now, $a(Q)$ can be determined from Eq. (3.67),

$$0=\lim_{\theta\to 0}[D(r,\theta,Q)-D(r,\theta,Q=0)] \quad (3.74)$$

$$=\frac{1}{4\pi r}\int_0^\infty d\nu\tanh(\pi\nu)\times\left(\frac{\nu}{\sqrt{\nu^2+a}}\frac{\sinh(2\pi\sqrt{\nu^2+a})}{\cosh(2\pi\sqrt{\nu^2+a})-\cos(2\pi Q)}-\frac{\sinh(2\pi\nu)}{\cosh(2\pi\nu)-1}\right). \quad (3.75)$$

Equation (3.74) can be solved numerically for $a(Q)$. However, before we do this, let us verify our claim that $\langle i\lambda\rangle=0$ (i.e., $a=0$) is not sufficient to satisfy Eq. (3.66) for finite Q. Indeed, from Eq. (3.74), we obtain

$$\lim_{x\to x'}[D(x,x',Q,a=0)-D(x,x',Q=0)]=\frac{1}{M}(\langle V^\dagger V(x)\rangle_Q-\langle V^\dagger V(x)\rangle_{Q=0})=\frac{1}{4\pi r}\int_0^\infty d\nu\frac{\cos(2\pi Q)-1}{\cosh(2\pi\nu)-\cos(2\pi Q)}=-\frac{1}{8\pi r}(1-2|Q|)\tan(\pi|Q|),\quad |Q|<1, \quad (3.76)$$

where expectation values in the second line of Eq. (3.76) are computed in the free theory. The precise value of expression (3.76) is not very important for our purposes (although it is interesting to note that like many quantities in the free theory, it is nonanalytic in Q at $Q=0$). What is important for us is that expression (3.76) is negative. This means that the twisted boundary condition effectively creates a repulsive barrier, leading to a decrease in $V^\dagger V$ compared to the untwisted theory. To compensate for this decrease in the strongly interacting theory, we need $\langle i\lambda(x)\rangle$ to provide an attractive potential for V particles. Hence, we conclude that $a(Q)<0$ for Q finite. One may be concerned that the square roots in expressions (3.73) and (3.74) are ambigious for $a<0$ and $\nu^2<|a|$. However, it turns out that these expressions do not depend on our choice of the sign for the square root as long as it is consistent. The numerical solution for $a(Q)$ is shown in Fig. 4. We note that this solution agrees with the one obtained using the spectral form of the propagator [Eq. (3.70)].

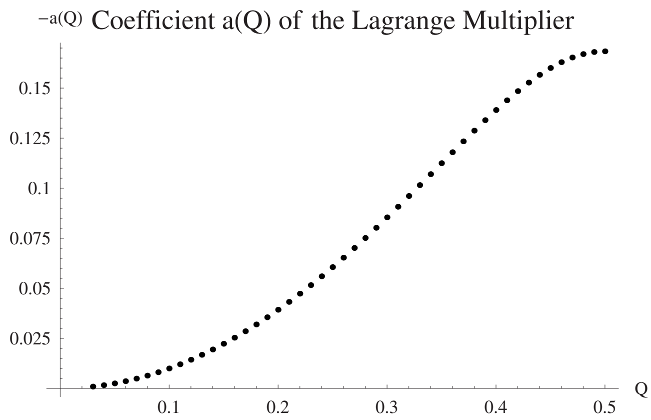

FIG. 4. Coefficient $a(Q)$ of the Lagrange multiplier $\langle i\lambda(x)\rangle$ [see Eq. (3.64)] in the $M=\infty$ generalization of the dual theory.

One can also attempt to use Eq. (3.74) to find a series solution for $a(Q)$ near $Q=0$. It is easy to convince oneself that

$$a(Q) \approx -Q^2, \quad Q \to 0. \tag{3.77}$$

Unfortunately, the integrand in Eq. (3.74) is quite singular at $\nu \to 0$ for $a \to 0$ and $Q \to 0$, so that a systematic series expansion beyond the leading order is not straightforward. Nevertheless, we believe that such an expansion exists and $a(Q)$ is an analytic function of Q near $Q=0$. Assuming such analyticity and using charge conjugation symmetry, $a(Q)=a(-Q)$, one obtains, $a(Q) \approx -Q^2 + c_4 Q^4$ for $Q \to 0$. Here, c_4 is a positive constant as the integral [Eq. (3.74)] diverges for $a < -Q^2$.

Having found $a(Q)$, we immediately obtain the impurity anomalous dimension of the operator V [given by Eq. (3.72)] (see Fig. 5). This anomalous dimension is no longer the trivial value $\Delta^V_{\text{imp}}=|Q|$ of the free theory [Eq. (3.49)]. Given the leading behavior of $a(Q)$ as $Q \to 0$ [Eq. (3.77)] and assuming analyticity of $a(Q)$, we conclude that Δ^V_{imp} will also be analytic at $Q=0$ (as opposed to the situation in the free theory). Moreover,

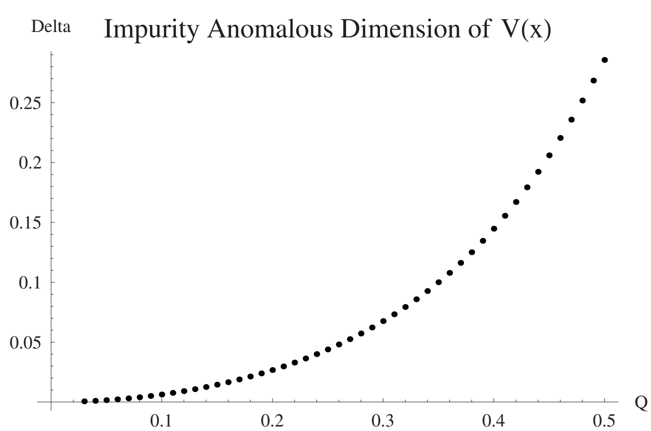

FIG. 5. Impurity anomalous dimension Δ^V_{imp} of the monopole operator $V(x)$ [see Eq. (3.48)] computed in the $M=\infty$ generalization of the dual theory.

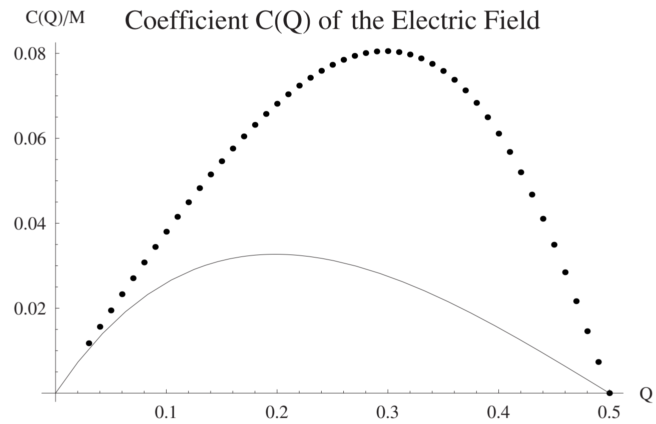

FIG. 6. Coefficient $C(Q)$ of the electric field [see Eq. (3.20)]. The doted and solid curves correspond to the strongly interacting theory at $M=\infty$ and the free theory, respectively.

$$\Delta^V_{\text{imp}} \approx \sqrt{c_4}Q^2, \quad Q \to 0. \tag{3.78}$$

Finally, we can now compute the coefficient of the electric field $C(Q)$. For the M-field generalization of the dual theory, we define the magnetic field by the same equation [Eq. (3.36)] as for $M=1$ theory; that is, we consider the current associated with the global U(1) symmetry,

$$\begin{aligned}\langle -iF_\mu(x)\rangle &= \langle -2\pi i V^\dagger_\alpha \overleftrightarrow{\partial}_\mu V_\alpha(x)\rangle \\ &= -2\pi i \lim_{x\to y}(\partial^x_\mu - \partial^y_\mu)\langle V_\alpha(x)V^\dagger_\alpha(y)\rangle \\ &= -2\pi i M \lim_{x\to y}(\partial^x_\mu - \partial^y_\mu)D(x,y).\end{aligned} \tag{3.79}$$

Due to our normalization of the U(1) current, the electric field induced will be of order M. Now, differentiating $D(r,\theta)$ in Eq. (3.73) and taking the symmetric limit as $\theta \to 0$,

$$-i\partial_\theta D(r,0) = -\frac{1}{4\pi r}\int_0^\infty d\nu\,\nu \tanh(\pi\nu)\frac{\sin(2\pi Q)}{\cosh(2\pi\sqrt{\nu^2+a}) - \cos(2\pi Q)}. \tag{3.80}$$

Using the values of $a(Q)$ found earlier (Fig. 4) and evaluating the integral [Eq. (3.80)] numerically, we obtain the coefficient $C(Q)$, shown in Fig. 6 (dotted curve). Figure 6 also shows the value of $C(Q)$ in the free theory [Eq. (3.56)] for comparison (solid line).

Alternatively, we can use Eq. (3.80) to expand $C(Q)$ in a series in Q. Using the leading behavior [Eq. (3.77)], we find

$$C(Q) \approx M\left(\frac{\pi Q}{8} + \mathcal{O}(Q^3)\right), \quad Q \to 0. \tag{3.81}$$

We see that the leading term in Eq. (3.81) agrees with the one, that would be obtained by perturbation theory in Q in the large M limit [Eqs. (3.26) and (3.34)]. It is also interesting to compare Eq. (3.81) to the asymptotic behavior of $C(Q)$ in the free theory [Eq. (3.57)]. We see that the leading term $C(Q)/M \approx \pi Q/8$ in both cases is the same; however, the subleading terms are different. The first subleading term in the free theory is nonanalytic $\sim|Q|Q$, as opposed to the strongly interacting theory's analytic $\mathcal{O}(Q^3)$. Thus, we have been able to verify that the leading nonanalyticity of $C(Q)$ in the free theory disappears in the interacting theory. We actually expect that the interacting theory cures itself of nonanalyticities in Q at all orders in Q.

Finally, let us discuss impurity anomalous dimensions of higher flux operators $V^q(x)$ for rational $Q=p/q$, as these are actual physical observables on the QED side of the duality. Once we go from the $M=1$ dual theory to its large M counterpart, there are many possible generalizations of the $V^q(x)$ operator. Indeed, we can form different SU(M) multiplets out of q instances of the SU(M) fundamental $V_\alpha(x)$. We expect that these multiplets will have different (impurity) anomalous dimensions for M finite. However, for $M=\infty$, all of these operators will have degenerate (impurity) anomalous dimensions. We can consider, for instance, the completely symmetric representation $V^q_S(x)=[V_\alpha(x)]^q$, where α is some fixed index (no summation over α). Then, for $M=\infty$,

$$\langle V^q_S(x)[V^q_S(y)]^\dagger\rangle = q![D(x-y)]^q. \tag{3.82}$$

Hence, just as in the free case, the operator $V^q_S(x)$ has the impurity OPE [Eq. (3.53)] with the corresponding impurity anomalous dimension,

$$\Delta^V_{\text{imp}}(q) = q\Delta^V_{\text{imp}}. \tag{3.83}$$

IV. EASY PLANE THEORY FOR GENERAL *N*

We now turn to the general case of the model $\mathcal{S}+\mathcal{S}_{\text{imp}}$ with a global U$(1)^{N-1}$ symmetry. The results of the previous section with $N=1$ can be rapidly generalized and will lead to a quantitative result for the scaling dimension of the monopole/VBS operator V near the impurity.

A. Duality in the easy plane theory

In this section, we consider a theory with N flavors of spinon fields z_α (N does not necessarily have to be large),

$$L = \frac{1}{2e^2}F_\mu^2 + |(\partial_\mu - iA_\mu)z_\alpha|^2 + U(z_\alpha). \tag{4.1}$$

Here, U is some potential with the global U$(1)^N$ symmetry under independent phase rotations of the z_α fields. The singlet component of this symmetry is actually gauged by the field A_μ,

$$\text{U}(1):\quad z_\alpha \to e^{i\theta(x)}z_\alpha,\quad A_\mu \to A_\mu + \partial_\mu\theta, \tag{4.2}$$

while the nonsinglet components are true global symmetries of the theory,

$$\text{U}(1)^{N-1}:\quad z_\alpha \to e^{i\theta^a t^a_\alpha}z_\alpha, \tag{4.3}$$

where t^a, $a=1,\ldots,N-1$, are the generators of the U$(1)^{N-1}$ symmetry satisfying $\Sigma_\alpha t^a_\alpha=0$. We require U to have a symmetry under the permutation of labels of z_α fields. We choose U in such a fashion that in the "condensed" phase of the theory, it favors nonzero expectation values of all components of the z_α field, so that the vacuum manifold of the theory is a torus $(S^1)^N$ (here, we temporarily forget that the singlet symmetry is gauged). For $N=2$, the theory under consideration is believed to describe the phase transition in the easy-plane antiferromagnet.

We would like to dualize the theory [Eq. (4.1)]. Similar theories were dualized in Refs. 11, 22, and 48–50, and here we will present a related discussion. An exact duality on the lattice appears in the companion paper,[40] but we can write down the form of the dual action from very general considerations. Let us first identify the dual degrees of freedom. We go to the condensed phase of the theory [Eq. (4.1)], where all $\langle z_\alpha\rangle \neq 0$. Then, we can have vortices in any component of the z_α field. Formally, the homotopy group, $\pi_1[(S^1)^N]=\mathbb{Z}^N$. So, we have N types of vortices, which become the degrees of freedom of the dual theory V_α, $\alpha=1,\ldots,N$.

These vortices are global rather than local. Indeed, let us consider a vortex in the first component z_1,

$$z_1(\vec{x}) \sim ve^{i\lambda(\vec{x})},\quad z_\alpha \sim v,\quad \alpha\neq 1,\quad |\vec{x}|\to\infty, \tag{4.4}$$

where $\lambda(\vec{x})$ winds from 0 to 2π as one goes around a contour out at infinity surrounding the vortex. Then, this vortex corresponds to a space-time dependent transformation of the vacuum [Eqs. (4.2) and (4.3)], with, $\theta(\vec{x})=\frac{1}{N}\lambda(\vec{x})$ and $\theta^a(\vec{x})t^a=(1-1/N,-1/N,\ldots,-1/N)\lambda(\vec{x})$. Thus, our vortex possesses a winding both in the local and in the global symmetry group. The winding in the local U(1) group will be canceled by the gauge field,

$$A_\mu(x) = \partial_\mu\theta(x) = \frac{1}{N}\partial_\mu\lambda(x). \tag{4.5}$$

Hence, our global vortices carry a magnetic flux $\Phi=2\pi/N$.[51] Therefore, under the flux symmetry [Eq. (3.4)], the fields V_α should transform as

$$V_\alpha(x) \to e^{2\pi i\alpha(x)/N}V_\alpha(x) \tag{4.6}$$

This fact will be crucial for the analysis to follow.

The winding in the global group will lead to a long-range Coulombic interaction between our vortices. We will need dynamical gauge fields in the dual theory to give rise to this interaction. However, if we have a unit winding in each component of the z field, our vortex becomes completely local and carries a total flux of 2π. We can think of such a local vortex as a composite of N global vortices of different types. The creation operator for this flux tube, therefore, will be

$$\mathcal{V}(x) = \prod_\alpha V_\alpha(x). \tag{4.7}$$

Since the local vortex carries the flux 2π, we can also associate the operator [Eq. (4.7)] with the monopole operator of the direct theory. Indeed, given Eq. (4.6), under the flux symmetry [Eq. (3.4)],

$$\mathcal{V}(x) \to e^{2\pi i\alpha(x)}\mathcal{V}(x), \tag{4.8}$$

which is the correct transformation law for the monopole operator [Eq. (3.6)]. We expect local vortices to interact by short-range forces. Therefore, the operator [Eq. (4.7)] should not be charged under the emergent gauge fields of the dual theory.

We are now ready to write down the dual theory,

$$L = \frac{1}{2\tilde{e}^2}\sum_i (F^\alpha_\mu)^2 + \left|\left(\partial_\mu - iB^\alpha_\mu - \frac{2\pi i}{N}H_\mu\right)V_\alpha\right|^2 + \tilde{U}(V_\alpha). \tag{4.9}$$

Here, $B^\alpha_\mu = B^a_\mu t^a_\alpha$, $a=1,\ldots,N-1$, are emergent dual gauge fields, which couple to the nonsinglet currents. $F^\alpha = \epsilon_{\mu\nu\lambda}\partial_\nu B^\alpha_\lambda$ are the corresponding field strengths. The dual potential $\tilde{U}(V_\alpha)$ is chosen to have the same properties as the direct potential U: it has a $U(1)^N$ symmetry under independent phase rotations of the fields V_α and a symmetry under the permutation of the labels of V_α fields. Moreover, it favors $\langle V_\alpha\rangle \neq 0$ for all α in the condensed phase of the dual theory. Thus, the theory in Eq. (4.9) has a local $U(1)^{N-1}$ symmetry,

$$U(1)^{N-1}: \quad V_\alpha(x) \to e^{i\phi^a(x)t^a_\alpha}V_\alpha(x), \quad B^a_\mu \to B^a_\mu + \partial_\mu\phi^a, \tag{4.10}$$

as well as the global U(1) flux symmetry of the direct theory [Eq. (4.6)] (which we have promoted to a local symmetry by introducing a nondynamical source field H_μ). As required, the monopole operator [Eq. (4.7)] is invariant under the local $U(1)^{N-1}$ symmetry of the dual theory [Eq. (4.10)].

The theory [Eq. (4.9)] also has a global $U(1)^{N-1}$ symmetry associated with the conservation of fluxes of the $N-1$ emergent gauge fields. This topological symmetry can be identified with Noether's symmetry [Eq. (4.3)] of the direct theory.

B. Wilson loops in the easy plane theory

Now, we would like to apply the duality discussed in the previous sections to study the properties of Wilson loops in the $U(1)^{N-1}$ symmetric theory [Eq. (4.1)]. Recall that to represent Wilson loops, we must use a source field H_μ given by Eq. (3.10). As discussed for the case of $N=1$ theory, the effect of such a source field on the dual action [Eq. (4.9)] is to introduce a twisted boundary condition for the vortex fields,

$$V_\alpha(\theta = 2\pi) = e^{-2\pi iQ/N}V_\alpha(\theta = 0), \tag{4.11}$$

where Q is the charge of our Wilson line. The physical origin of the factor $1/N$ is the fractional charge $2\pi/N$ of the vortex fields V_α under the flux symmetry. Thus, we come to the amazing conclusion that the universal physics in the planar model is periodic in the charge Q of the Wilson line, with period $Q=N$. This is a generalization of the $Q=1$ periodicity of the single flavor QED discussed before. As explained in a companion paper,[40] we expect that this $Q\sim N$ periodicity is a feature of the easy plane theory and does not generalize to the case with the full SU(N) invariance.

Now, we would like to discuss more quantitative features of Wilson loops in the planar model. In particular, we would like to find the impurity anomalous dimension of the monopole operator [Eq. (3.48)] and the coefficient of the electric field [Eq. (3.20)] at the critical point of the theory. We note that as in the $N=1$ case, we can easily dualize the magnetic field by differentiating the dual action with respect to the source field H_μ,

$$\langle -iF_\mu\rangle = \frac{(-2\pi i)}{N}\langle V^\dagger_\alpha \overleftrightarrow{D}_\mu V_\alpha\rangle, \tag{4.12}$$

with $D_\mu V_\alpha = \left(\partial_\mu - iB^\alpha_\mu - \frac{2\pi i}{N}H_\mu\right)V_\alpha$.

To find Δ^V_{imp} and $C(Q)$, we follow the procedure established for the $N=1$ case in Sec. III D and perform a large M expansion of the dual theory [Eq. (4.9)]. Namely, we promote each field V_α to a SU(M) multiplet, V^i_α, $i=1,\ldots,M$. Moreover, we replace the soft potential $\tilde{U}(V_\alpha)$ by a hard constraint, $\sum_i |V^i_\alpha|^2 = 1/g$, for each $\alpha=1,\ldots,N$. This constraint will be enforced by a set of N Lagrange multipliers λ_α. Thus, our Lagrangian becomes

$$L = \sum_{\alpha,i}\left|\left(\partial_\mu - iB^\alpha_\mu - \frac{2\pi i}{N}H_\mu\right)V^i_\alpha\right|^2 + \sum_{\alpha,i} i\lambda_\alpha\left(|V^i_\alpha|^2 - \frac{1}{g}\right). \tag{4.13}$$

In Eq. (4.13), we have also dropped the kinetic term for the gauge fields, as near the critical point such operators will be irrelevant. In addition to the $U(1)_\Phi$ global flux symmetry and the $U(1)^{N-1}$ local symmetry of the original $M=1$ action, the theory in Eq. (4.13) also has a $SU(M)^N$ global symmetry under independent SU(M) rotations of the N M-tuplets V^i_α. We note that the various SU(M) multiplets talk to each other only through the gauge fields B^α_μ.

We would like to generalize the observables of the $M=1$ theory to the large M case. The magnetic field [Eq. (4.12)] is generalized trivially,

$$\langle -iF_\mu\rangle = \frac{(-2\pi i)}{N}\langle (V^i_\alpha)^\dagger \overleftrightarrow{D}_\mu V^i_\alpha\rangle. \tag{4.14}$$

The monopole operator [Eq. (4.7)], on the other hand, now carries indices under the $SU(M)^N$ group,

$$\mathcal{V}(x)_{i_1,\ldots,i_N} = \prod_\alpha V^{i_\alpha}_\alpha(x). \tag{4.15}$$

The insertion of the Wilson loop source H_μ is again equivalent to the twisted boundary condition [Eq. (4.11)].

We now perform a large M expansion of the theory [Eq. (4.13)] with the twisted boundary condition [Eq. (4.11)], keeping N fixed. We will be only able to make computations for $M=\infty$. We are interested in the physics at the critical point. We expand the theory about the saddle point $B^\alpha_\mu = 0$ (this is a saddle point as the twisted boundary condition [Eq. (4.11)] does not couple to the nonsinglet sectors of the theory[52]). As usual, the fluctuations of these gauge fields about the saddle point will be suppressed by powers of $1/M$. Thus, at $M=\infty$, we are left with N decoupled instances of the Lagrangian [Eq. (3.60)] that has been discussed at length for the case of the $N=1$ theory. The only difference is the replacement, $Q\to Q/N$ in the boundary condition [Eq. (3.19)]. Hence, we conclude that

$$\langle \mathcal{V}(x)_{i_1,\ldots,i_N}\mathcal{V}^\dagger(x')_{j_1,\ldots,j_N}\rangle \overset{M=\infty}{=} \prod_\alpha \langle V_\alpha^{i_\alpha}(x)(V_\alpha^{j_\alpha})^\dagger(x')\rangle = D(x,y,Q/N)^N \prod_\alpha \delta_{i_\alpha j_\alpha}, \quad (4.16)$$

where $D(x,x',Q)$ is the propagator in the $N=1$ theory [Eq. (3.60)] with the twisted boundary condition [Eq. (3.19)] at $M=\infty$. The asymptotic behavior of this propagator for $r \ll r'$ is given in Eq. (3.71). Thus, the asymptotic behavior of the correlation function [Eq. (4.16)] for $r\ll r'$ is

$$\langle \mathcal{V}(x)_{i_1,\ldots,i_N}\mathcal{V}^\dagger(x')_{j_1,\ldots,j_N}\rangle \approx \left(\frac{1}{4\pi r'}\right)^N \left(\frac{r}{r'}\right)^{N\sqrt{(Q/N)^2+a(Q/N)}} \times e^{-iQ\theta}G(\tau/r')\prod_\alpha \delta_{i_\alpha j_\alpha}, \quad |Q/N|<1/2, \quad (4.17)$$

where G is some (known) function. Hence, the monopole operator $\mathcal{V}(x)$ in the planar N component theory has the impurity OPE,

$$\mathcal{V}(\vec{x},\tau) \sim |\vec{x}|^{\Delta^{\mathcal{V}}_{\text{imp}}} e^{-iQ\theta}\mathcal{V}_{\text{imp}}(\tau) \quad \text{for } |\vec{x}|\to 0, \quad (4.18)$$

with

$$\Delta^{\mathcal{V}}_{\text{imp}} = N\sqrt{(Q/N)^2 + a(Q/N)} = N\Delta^V_{N=1}(Q/N), \quad |Q/N|<1/2, \quad (4.19)$$

where the monopole impurity anomalous dimenension $\Delta^V_{N=1}(Q)$ in the $N=1$, $M=\infty$ theory is given by Fig. 5.

From OPE [Eq. (4.18)], we observe that for integer Q, the monopole operator is single valued under $\theta\to\theta+2\pi$, even though the dynamical fields of the theory V_α obey twisted boundary conditions [Eq. (4.11)]. We also note that formulas (4.18) and (4.19) are correct only for $|Q/N|<1/2$; for other values of Q, they should be extended by periodicity $Q\sim Q+N$.

We can now take the $N\to\infty$, Q-fixed limit of Eq. (4.19). Using the asymptotic behavior [Eq. (3.78)], $\Delta^{\mathcal{V}}_{\text{imp}}\sim Q^2/N$. Thus, the impurity anomalous dimension of the monopole operator is of the order of $\mathcal{O}(1/N)$ for $N\to\infty$ in the easy plane theory. It is interesting to note that, as discussed in Sec. II, this is also true of the theory with a full SU(N) symmetry. At this point, it is not clear whether this is just a coincidence.

Finally, let us discuss the special point $Q/N=1/2$. Our interest in this point is not purely academic, as we expect $N=2$, $Q=1$ to correspond to the physical case of a single impurity in an easy plane antiferromagnet or superfluid. We recall that at this point the propagator $D(r,r',\theta,\tau)$ for $r\ll r'$ is dominated by two angular momenta, $l=\pm 1/2$,

$$D(r,r',\theta,\tau) \approx \frac{1}{4\pi r'}\left(\frac{r}{r'}\right)^{\sqrt{1/4+a(1/2)}} \times (e^{i\theta/2}+e^{-i\theta/2})B_{\sqrt{1/4+a(1/2)}}\left(\frac{\tau}{r'}\right), \quad (4.20)$$

so that

$$D(r,r',\theta,\tau)^N \approx \left(\frac{1}{4\pi r'}\right)^N\left(\frac{r}{r'}\right)^{N\sqrt{1/4+a(1/2)}}\sum_{m=0}^{2Q}\binom{2Q}{m}e^{i(m-Q)\theta}G(\tau/r'). \quad (4.21)$$

Hence, using Eq. (4.16), the correlation function of two monopole operators is dominated by angular momenta $l=-Q,-Q+1,\ldots,Q-1,Q$ for $r\ll r'$. So, we conjecture the operator product expansion,

$$\mathcal{V}(\vec{x},\tau) \sim \sum_{l=-Q}^{Q} c_l|\vec{x}|^{\Delta_l^{\mathcal{V}}}e^{-il\theta}\mathcal{V}_l(\tau) \quad \text{for } |\vec{x}|\to 0. \quad (4.22)$$

At $M=\infty$, all the operators $\mathcal{V}_l$ have degenerate impurity anomalous dimensions $\Delta_l^{\mathcal{V}}$. As discussed in Sec. III C, the anomalous dimensions of operators with oposite angular momenta are equal by CP symmetry emergent at the $Q/N=1/2$ point. However, there is no fundamental reason why anomalous dimensions of operators with different values of l should be equal. Thus, we expect the degeneracy to be lifted at higher orders in the $1/M$ expansion. Therefore, unfortunately, the question of whether the OPE [Eq. (4.22)] will be dominated by $l=0$ or by finite l is beyond the reach of our calculation. Nevertheless, our calculation at $M=\infty$ predicts for the physically relevant case of $N=2$ and $Q=1$,

$$\Delta^{\mathcal{V}}_{\text{imp}} \approx 0.57, \quad N=2, \quad Q=1. \quad (4.23)$$

The emergent CP symmetry at the point $Q/N=1/2$ means that quantum fluctuations manage to render the states of Figs. 2 and 3 degenerate in the long-wavelength limit. We remind the reader that the CP symmetry is due to the emergent $Q\sim N$ periodicity of the easy plane theory. No such periodicity is expected to occur in the full SU(N) symmetric theory, where the impurity OPE is dominated by a single operator with a definite angular momentum as in Eq. (1.12).

For completeness sake, we also discuss the coefficient $C(Q)$ of the electric field. From Eq. (4.14) at $M=\infty$, we obtain

$$C(Q) = C_{N=1}(Q/N), \quad (4.24)$$

where the coefficient $C_{N=1}(Q)$ in the $N=1$, $M=\infty$ theory is given by Fig. 6. We note that for $Q/N=1/2$, the electric field vanishes, as it should, by the emergent CP symmetry.

V. CONCLUSIONS

This paper began with the theory $\mathcal{S}$ in Eq. (1.1) for square lattice quantum antiferromagnets in the vicinity of Néel-VBS quantum phase transitions. We considered generic local deformations of the antiferromagnet and argued that they could be classified into two categories. The first category, illustrated in Fig. 1, is a modulated exchange impurity, for which we found an enhancement of the VBS order, characterized by the exponent in Eq. (1.9). The second category was realized by a missing or additional spin (e.g., Zn or Ni impurities on Cu sites), shown in Fig. 2. For this case, we found that the VBS order was suppressed by the appearance of a VBS pinwheel, as in Fig. 2, and characterized by the scaling properties discussed in Sec. I B.

The results of this paper should be useful in numerical studies of the quantum phase transition between the Néel and VBS states.[53,54] By enhancing an exchange constant as in Fig. 1 and measuring the decay of the average VBS order parameter away from the impurity, the exponent Δ^V can be estimated from Eqs. (1.6)–(1.9). There will be no mean VBS order in the vicinity of a missing spin impurity as in Fig. 2. However, the spatial dependence in the VBS susceptibility is fixed by Δ^V_{imp} in Eq. (1.12). The positive value of Δ^V_{imp} indicates that the VBS susceptibility should be suppressed near such an impurity.

In scanning tunneling microscopy (STM) studies of the cuprates, we have noted earlier the demonstration of bond-centered charge order in the local density of states by Kohsaka *et al.*[5] A numerical analysis of the pinning of such charge order by modulated exchange impurities (in the class in Sec. I A) has also been carried out.[6,7] However, it is also experimentally possible to induce "missing spin" impurities (in the class of Sec. I B) by replacing the Cu sites with Zn and Ni impurities. There have been STM studies of such impurities,[55–57] and it is of great interest to carefully examine the nature of the bond-centered modulations in the vicinity of such impurities. If we assume that the "stripe" instability is primarily associated with the appearance of magnetic order,[58–61] then the theory of the enhancement of magnetic order near such impurities[19,28] should apply. We should therefore expect an increase in the strength of the density of states modulations in this model. In contrast, if we assume a VBS theory of the modulations, then in the impurity model of Sec. I B, the bond-centered modulations should be suppressed. The experimental situation could well include both effects, complicating the interpretation. However, evidence for VBS pinwheel configurations like those in Fig. 2 would lend strong support to the VBS theory.

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APPENDIX A: ELECTRIC FIELD IN THE FREE THEORY

We wish to use Eq. (3.55) to compute the electric field in the free theory. We start from the equation for the propagator [Eq. (3.38)] and promote the ω integral to run over $D-2$ dimensions, as discussed in Sec. III C, obtaining

$$D(r,\theta) = \frac{\Gamma(2-D/2)}{(4\pi)^{(D-2)/2}} \sum_l \frac{e^{il\theta}}{2\pi} \int du u^{D-3} J_{|l|}(ur)^2 \tag{A1}$$

$$= \frac{\Gamma[(3-D)/2]}{(4\pi)^{(D-1)/2}} \frac{1}{r^{D-2}} \sum_l \frac{e^{il\theta}}{2\pi} \frac{\Gamma(|l|+D/2-1)}{\Gamma(|l|-D/2+2)}. \tag{A2}$$

We see that the prefactor diverges for $D=3$. However, at $D=3$, the sum over angular momenta becomes $\sum_l \frac{e^{il\theta}}{2\pi} = \delta(\theta)$. So, we have to first perform the sum over angular momenta and then take the limit $D \to 3$. The sum over angular momenta can be performed in terms of hypergeometric functions, giving the following for $0<Q<1$:

$$\begin{aligned} D(r,\theta) = \frac{\Gamma[(3-D)/2]}{2^D \pi^{(D+1)/2}} \frac{e^{-iQ\theta}}{r^{D-2}} \Bigg(& e^{i\theta} \frac{\Gamma(D/2-Q)}{\Gamma(3-D/2-Q)} \\ & \times F(\{1, D/2-Q\}, \{3-D/2-Q\}, e^{i\theta}) \\ & + \frac{\Gamma(D/2-1+Q)}{\Gamma(2-D/2+Q)} \\ & \times F(\{1, -1+D/2+Q\}, \{2-D/2+Q\}, e^{-i\theta}) \Bigg), \end{aligned} \tag{A3}$$

where F denotes the Barnes extended hypergeometric function. One can check that for $D=3$ the expression in brackets in Eq. (A3) vanishes, canceling the pole in the prefactor. Now, differentiating with respect to θ,

$$\begin{aligned} -i\partial_\theta D(r,\theta) = \frac{\Gamma[(3-D)/2]}{2^D \pi^{(D+1)/2}} \frac{e^{-iQ\theta}}{r^{D-2}} \Bigg(& \frac{(1-Q)\Gamma(D/2-Q)}{\Gamma(3-D/2-Q)} \\ & \times e^{i\theta} F(\{1, D/2-Q\}, \{3-D/2-Q\}, e^{i\theta}) \\ & + \frac{\Gamma(D/2+1-Q)}{\Gamma(4-D/2-Q)} \\ & \times e^{2i\theta} F(\{2, D/2-Q+1\}, \{4-D/2-Q\}, e^{i\theta}) \\ & - \frac{Q\Gamma(D/2-1+Q)}{\Gamma(2-D/2+Q)} \\ & \times F(\{1, D/2-1+Q\}, \{2-D/2+Q\}, e^{-i\theta}) \\ & - \frac{\Gamma(D/2+Q)}{\Gamma(3-D/2+Q)} \\ & \times e^{-i\theta} F(\{2, D/2+Q\}, \{3-D/2+Q\}, e^{-i\theta}) \Bigg). \end{aligned} \tag{A4}$$

According to Eq. (3.55), to compute the electric field we need to take the limit as $\theta \to 0$ of Eq. (A4). Strictly speaking, this limit does not exist as the hypergeometric functions blow up as $\theta \to 0$ (that is, when the last argument goes to 1). However, we note that only the imaginary part of Eq. (A4) becomes infinite as $\theta \to 0$, while the real part has a well-defined limit. The expectation value of the electric field $\langle -iE_r \rangle = -\langle -iF_\theta \rangle$ should be real. Thus, we can drop the infinite imaginary part. Moreover, the imaginary part is antisymmetric under $\theta \to -\theta$, so the "symmetrized" limit of Eq. (A4) exists. It turns out that this symmetrized limit can be obtained by the formal summation formulas

$$F(\{1,a\},\{b\},1) = \frac{1-b}{a-b+1}, \quad \text{(A5)}$$

$$F(\{2,a\},\{b\},1) = \frac{(b-1)(b-2)}{(a-b+1)(a-b+2)}. \quad \text{(A6)}$$

So, taking $\theta \to 0$, plugging Eq. (A5) into Eq. (A4) and performing a few manipulations,

$$-i\partial_\theta D(\theta=0,r) = \frac{(2Q-1)\Gamma((1-D)/2)}{2^{D+2}\pi^{(D+1)/2}} \frac{\Gamma(D/2+Q-1)}{\Gamma(1-D/2+Q)} \times \left(\frac{\sin[\pi(D/2+Q)]}{\sin[\pi(D/2-Q)]} - 1\right)\frac{1}{r^{D-2}}. \quad \text{(A7)}$$

Taking the limit $D \to 3$,

$$-i\partial_\theta D(\theta=0,r) = -\frac{1}{32\pi r}(2Q-1)^2 \tan(\pi Q). \quad \text{(A8)}$$

Finally, plugging into Eq. (3.55) we recover Eq. (3.20) with

$$C(Q) = \frac{1}{8}(1-2Q)^2 \tan(\pi Q), \quad 0 < Q < 1. \quad \text{(A9)}$$

We remind the reader that all the manipulations above have been performed for $0<Q<1$. The function $C(Q)$ can then be extended to other values of Q by periodicity. In particular, extending to the range $|Q|<1$,

$$C(Q) = \frac{1}{8}(1-2|Q|)^2 \tan(\pi Q), \quad |Q| < 1. \quad \text{(A10)}$$

APPENDIX B: INTEGRAL FORM OF THE TWISTED PROPAGATOR

In this section, we review the derivation of the integral form of the twisted propagator [Eq. (3.58)] given in Ref. 47. We use this integral form to compute the electric field [Eq. (3.55)] and show that it is in agreement with the result obtained using spectral representation of the propagator (see Appendix A). We also indicate how the free twisted propagator should be modified in the strongly interacting $M=\infty$ theory.

Recall that the free massive propagator in two dimensions (without any twisted B.C.) obeys

$$(-\partial^2 + m^2)D(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}') \quad \text{(B1)}$$

and is given by

$$D_2(\vec{x},\vec{x}') = \frac{1}{2\pi}K_0(m|\vec{x}-\vec{x}'|) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu K_{i\nu}(mr)K_{i\nu}(mr')e^{\pi\nu}e^{-\nu|\theta-\theta'|}, \quad \text{(B2)}$$

where the integral representation is valid for $|\theta-\theta'|<2\pi$. The Bessel K functions of an imaginary argument satisfy the equation

$$\left(-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{\nu^2}{r^2} + m^2\right)K_{i\nu}(mr) = 0. \quad \text{(B3)}$$

Hence, the functions $K_{i\nu}(mr)e^{\pm\nu\theta}$ are in the kernel of the operator $-\partial_2^2+m^2=-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+m^2$. Applying this operator to $D_2(\vec{x},\vec{x}')$, we learn

$$\frac{1}{\pi^2 r^2}\int_{-\infty}^{\infty} d\nu\nu K_{i\nu}(mr)K_{i\nu}(mr')e^{\pi\nu} = \frac{1}{r}\delta(r-r'). \quad \text{(B4)}$$

This identity will be useful to us later.

Now, we want to modify the propagator [Eq. (B2)] in such a way that it satisfies the twisted boundary conditions [Eq. (3.19)]. Let us first symmetrize Eq. (B2) with respect to ν by noting that $K_{i\nu}=K_{-i\nu}$. Then,

$$D_2(r,r',\theta-\theta') = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu K_{i\nu}(mr)K_{i\nu}(mr')\cosh[\nu(\pi-|\theta-\theta'|)]. \quad \text{(B5)}$$

Now, we can generalize

$$D_2(r,r',\theta,Q) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} d\nu K_{i\nu}(mr)K_{i\nu}(mr')\sinh(\pi\nu)U_\nu(\theta), \quad \text{(B6)}$$

where

$$U_\nu(\theta) = \frac{\cosh[\nu(\pi-|\theta|)]}{\sinh(\pi\nu)} + c(\nu)e^{\nu\theta} - c(-\nu)e^{-\nu\theta}. \quad \text{(B7)}$$

$D_2(r,r',\theta,Q)$ still satisfies Eq. (B1) since, as noted above, the functions $K_{i\nu}(mr)e^{\pm\nu\theta}$ are in the kernel of $-\partial_2^2+m^2$. It remains to find $c(\nu)$ such that the propagator [Eq. (B6)] obeys boundary conditions [Eq. (3.19)]. After a few manipulations, one arrives at

$$U_\nu(\theta) = \frac{e^{-2\pi iQ\,\mathrm{sgn}(\theta)}\sinh(\nu|\theta|) + \sinh[\nu(2\pi-|\theta|)]}{\cosh(2\pi\nu) - \cos(2\pi Q)}. \quad \text{(B8)}$$

Next, one uses the identity

$$\sinh(\pi\nu)K_{i\nu}(mr)K_{i\nu}(mr') = \frac{\pi}{2}\int_{\xi_2}^{\infty} du J_0(m(2rr')^{1/2}[\cosh(u)-\cosh\xi_2]^{1/2})\sin(\nu u), \quad \text{(B9)}$$

where $\xi_2>0$ is defined by

$$\cosh\xi_2 = \frac{r^2+r'^2}{2rr'}. \quad \text{(B10)}$$

Substituting this into Eq. (B6),

$$D_2(r,r',\theta,Q) = \frac{1}{2\pi}\int_{\xi_2}^{\infty} du J_0(m(2rr')^{1/2}[\cosh(u) - \cosh\xi_2)^{1/2}]\int_0^{\infty} d\nu U_\nu(\theta)\sin(\nu u). \quad \text{(B11)}$$

We are mostly interested in the propagator with $r=r'$,

$$D_2(r=r',\theta,Q) = \frac{1}{2\pi}\int_0^{\infty} du J_0[mr\sqrt{2}(\cosh u - 1)^{1/2}] \times \int_0^{\infty} d\nu U_\nu(\theta)\sin(\nu u). \quad \text{(B12)}$$

In principle, it is possible to perform the integral over ν analytically in Eq. (B11) (see Ref. 47). This, however, will not be very beneficial for our purposes. Instead, let us proceed directly to the three-dimensional massless propagator, obtained by integrating over the mass parameter of the two-dimensional propagator [Eq. (3.38)],

$$D(r,\theta) = \frac{1}{2\pi^2 r\sqrt{2}}\int_0^{\infty} du \frac{1}{(\cosh u - 1)^{1/2}}\int_0^{\infty} d\nu U_\nu(\theta)\sin(\nu u), \quad \text{(B13)}$$

where we have computed only the three-dimensional propagator with $r=r'$, $\tau=\tau'$. Now, performing the integral over u,

$$D(r,\theta) = \frac{1}{4\pi r}\int_0^{\infty} d\nu \tanh(\pi\nu) U_\nu(\theta). \quad \text{(B14)}$$

To find the electric field, we again use Eq. (3.55),

$$-i\partial_\theta D(r,\theta) = -\frac{1}{4\pi r}\int_0^{\infty} d\nu \nu \tanh(\pi\nu) \times \left(\frac{\sin(2\pi Q)\cosh(\nu\theta)}{\cosh(2\pi\nu) - \cos(2\pi Q)} + i\,\mathrm{sgn}(\theta) \times \frac{(\cos(2\pi Q)\cosh(\nu\theta) - \cosh[\nu(2\pi - |\theta|)])}{\cosh(2\pi\nu) - \cos(2\pi Q)}\right). \quad \text{(B15)}$$

Again, the real part of $-i\partial_\theta D(r,\theta)$ has a well-defined limit as $\theta \to 0$, while the imaginary part is antisymmetric under $\theta \to -\theta$ and diverges as $\theta \to 0$. So the "symmetrized" limit is given by

$$-i\partial_\theta D(r,\theta=0) = -\frac{1}{4\pi r}\int_0^{\infty} d\nu \nu \frac{\sin(2\pi Q)\tanh(\pi\nu)}{\cosh(2\pi\nu) - \cos(2\pi Q)} = -\frac{1}{32\pi r}(2|Q| - 1)^2 \tan(\pi Q), \quad \text{(B16)}$$

in agreement with an earlier computation [Eq. (A8)] based on spectral decomposition. Thus, $C(Q)$ is again given by expression (3.56).

Now, we generalize the above derivation of the twisted propagator to the strongly interacting $M=\infty$ theory. The strongly interacting theory differs from the free theory by the aditional space-varying potential $\langle i\lambda(\vec{x},\tau)\rangle$, so that the propagator satisfies

$$\left(-\partial^2 + \frac{a(Q)}{|\vec{x}|^2}\right) D(x,x',Q) = \delta(x - x'). \quad \text{(B17)}$$

We again rewrite $D(x,x',Q)$ in terms of the two-dimensional massive propagator $D_2(\vec{x},\vec{x},m^2,Q)$ as in Eq. (3.38). The two-dimensional propagator satisfies

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{a}{r^2} + m^2\right] D_2(r,r',\theta,\theta') = \delta(\vec{x} - \vec{x}'). \quad \text{(B18)}$$

We need to generalize the two-dimensional, massive, twisted, free propagator [Eq. (B6)] so that it obeys the above equation. We observe that the function $U_\nu(\theta)$ [Eqs. (B7) and (B8)] satisfies

$$\frac{\partial^2 U_\nu}{\partial\theta^2} = \nu^2 U_\nu(\theta) - 2\nu\delta(\theta). \quad \text{(B19)}$$

Now, combining Eqs. (B3), (B4), and (B19), we find that

$$D_2(r,r',\theta,Q) = \frac{1}{\pi^2}\int_0^{\infty} d\nu K_{i\nu}(mr)K_{i\nu}(mr')\sinh(\pi\nu)\frac{\nu}{\sqrt{\nu^2 + a}} U_{\sqrt{\nu^2+a}}(\theta) \quad \text{(B20)}$$

satisfies Eq. (B18) as needed. Proceeding as above from two- to three-dimensional propagator, and setting $r=r'$, $\tau=\tau'$,

$$D(r,\theta) = \frac{1}{4\pi r}\int_0^{\infty} d\nu \tanh(\pi\nu)\frac{\nu}{\sqrt{\nu^2 + a}} U_{\sqrt{\nu^2+a}}(\theta). \quad \text{(B21)}$$

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