Nonperturbative interaction effects in the thermodynamics of disordered wires

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We study nonperturbative interaction corrections to the thermodynamic quantities of multichannel disordered wires in the presence of the Coulomb interactions. Within the replica nonlinear σ-model (NLσM) formalism, they arise from nonperturbative soliton saddle points of the NLσM action. The problem is reduced to evaluating the partition function of a replicated classical one-dimensional Coulomb gas. The state of the latter depends on two parameters: the number of transverse channels in the wire N_{ch} and the dimensionless conductance $G(L_T)$ of a wire segment of length equal to the thermal diffusion length L_T. At relatively high temperatures, $G(L_T)\gtrsim \ln N_{ch}$, the gas is dimerized, i.e., consists of bound neutral pairs. At lower temperatures, $\ln N_{ch}\gtrsim G(L_T)\gtrsim 1$, the pairs overlap and form a Coulomb plasma. The crossover between the two regimes occurs at a parametrically large conductance $G(L_T)\sim \ln N_{ch}$ and may be studied independently from the perturbative effects. Specializing on the high-temperature regime, we obtain the leading nonperturbative correction to the wire heat capacity. Its ratio to the heat capacity for noninteracting electrons, C_0, is $\delta C/C_0 \sim N_{ch}G^2(L_T)e^{-2G(L_T)}$.

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I. INTRODUCTION

The interplay between disorder and electron-electron interactions in conductors influences their low-temperature properties in an essential way.[1,2] Depending on the disorder strength, the temperature, and other system parameters, a conductor may be either in the metallic or in the insulating regime. The manifestations of electron-electron interactions in the two regimes are quite different. In the insulating regime, the charge in a given localized site is quantized in the units of the electron charge, and charge discreteness effects dominate the system properties.[2] In the metallic regime, the charge in a given volume of the conductor can change continuously and charge discreteness effects are small. The two regimes can be distinguished by the value of the appropriately defined dimensionless conductance G, which is greater than unity in the metallic regime and smaller than unity in the insulating one. If the system crosses over from the metallic to the insulating regime due to a change in temperature or disorder strength, the Coulomb blockade effects are expected to gradually grow and become important at $G\sim 1$.

Theoretically, the transition between the metallic and the insulating regimes is typically approached from the metallic side, $G\gg 1$, where electron transport can be described semiclassically. Therefore, the study of incipient charge discreteness effects in the metallic regime is an important problem in the theory of disordered conductors. This problem has recently attracted much attention.[3–17] In the metallic regime, $1/G$ may be used as a small expansion parameter. For $G \gg 1$, the charge discreteness effects are exponentially small in G, and their analysis requires nonperturbative methods. To date, quantitative studies of nonperturbative interaction effects in the metallic regime have been limited to granulated systems, or to systems in which the electron-electron interaction is spatially separated from the disorder. The present paper is devoted to the study of nonperturbative effects in the thermodynamic properties of homogeneously disordered wires, in which electron-electron interactions and disorder spatially coexist.

The most promising technique to study this problem is the nonlinear σ model (NLσM), either in the replica[18] or Keldysh[19,20] formulation. We use Finkelstein's[18] replica formulation of the NLσM. We show that nonperturbative corrections to the thermodynamic quantities of the wire depend on two parameters: the number of channels N_{ch} in the wire and the dimensionless conductance $G(L_T)$ of the wire segment of length equal to the thermal diffusion length L_T. In contrast, the perturbative corrections[1] are controlled by a single parameter, $G(L_T)$. For example, the leading perturbation theory correction to the heat capacity is $\delta C_{PT}/C_0 \sim 1/G(L_T)$, where C_0 is the wire heat capacity in the noninteracting electron approximation.

Within the NLσM formalism, the nonperturbative effects are described by soliton saddle points of the NLσM action. The spatial extent of the solitons is given by the thermal diffusion length L_T, and their action is equal to $G(L_T)$. The nonperturbative contribution to the thermodynamic quantities is described by the partition function for a gas of these solitons. We map the problem onto a one-dimensional replicated Coulomb gas. At high temperatures, $G(L_T)\gtrsim \ln N_{ch}$, the Coulomb gas is dimerized, i.e., consists of widely separated neutral pairs (dimers). In the temperature range $\ln N_{ch} \gtrsim G(L_T)\gtrsim 1$, the dimers are ionized and form a Coulomb plasma. Since the crossover between the two regimes occurs at a parametrically large conductance, $G(L_T)\sim \ln N_{ch}$, it can be studied independently from the perturbative effects. In this paper, we specialize on the high-temperature regime, leaving consideration of the crossover to the low temperature one for future work.

The paper is organized as follows. In Sec. II, we describe the NLσM for multichannel wires. In Sec. III, we obtain the analytic solution for the saddle points of the NLσM action in the limit of the infinite number of channels N_{ch} and evaluate the functional integral over the fluctuations about the saddle points. In Sec. IV, we obtain the leading nonperturbative correction to the thermodynamic quantities of the wire for $N_{ch}\gg 1$. In Sec. V, we summarize our results.

II. NONLINEAR σ MODEL

We consider an infinitely long disordered wire with many transverse channels, $N_{ch} \gg 1$. The disorder is assumed to be weak, so that the elastic mean free path l satisfies the condition $k_F l \gg 1$, where k_F is the Fermi wave number. We consider the temperature T to be smaller than the Thouless energy for the transverse motion, $E_T \equiv D/d^2$, where d is the transverse wire dimension and D is the diffusion constant. In this regime, the wire is described by the one-dimensional NLσM.

Thermodynamic properties of the system can be extracted from the averaged over disorder realizations replicated partition function, $\langle Z^p \rangle = \langle \mathrm{Tr}\, e^{-p\hat{H}/T} \rangle$, with p being the number of replicas. We will be interested in the thermodynamic potential, which can be obtained using the replica trick:

$$\langle \Omega \rangle = -T \langle \ln Z \rangle = -T \lim_{p\to 0} \frac{\langle Z^p \rangle - 1}{p}. \tag{1}$$

In the diffusive regime, the replicated partition function $\langle Z^p \rangle$ has a functional integral representation in terms of NLσM, describing the low-energy physics of the problem. The derivation of the NLσM action has become a standard procedure.[18,21] Therefore, below, we only present its final form suitable for the problem under consideration. The NLσM action is a functional of two fields: the Q matrix, parametrizing the diffusive degrees of freedom of electron motion, and electric potential V. The former is a Hermitian matrix in the space of replicas and Matsubara frequencies, whose entries are 4×4 matrices in the space $S \otimes \mathcal{T}$, given by the product of spin, S, and time-reversal, $\mathcal{T}$, spaces.[21,22] The slowly varying in space electric potential V_a is introduced to treat the long range part of the Coulomb interaction in replica a. This part of the Coulomb interaction is of particular importance for the consideration below. It cannot be described by the Fermi-liquid interaction constants. Since the Fermi-liquid effects in disordered metals have been studied by Finkelstein[18] and are not essential for the phenomena discussed in this paper, we ignore them in order to keep the presentation more transparent. Then, the NLσM action can be written as

$$\langle Z^p \rangle = \int \mathcal{D}[Q,V] e^{-S_Q - S_C}, \tag{2a}$$

$$S_Q = A \frac{\pi\nu}{2} \int dx\, \mathrm{Tr}\left[\frac{D}{4}(\nabla Q)^2 - (\hat{\varepsilon} + \hat{V})Q\right] + A\nu \int d\tau dx \sum_a V_a^2(x,\tau), \tag{2b}$$

$$S_C = \frac{1}{2} \int d\tau dx dx' \sum_a V_a(x,\tau) K(x-x') V_a(x',\tau), \tag{2c}$$

where Tr denotes the trace over the replica, Matsubara, and $S\otimes\mathcal{T}$ spaces, ν is the density of states per spin at the Fermi level, and A is the wire cross section area. The matrices $\hat{\varepsilon}$ and $\hat{V}$ have the following structure in the replica and $S\otimes\mathcal{T}$ spaces: $\hat{\varepsilon} = i\delta^{ab}\tau_3\partial_\tau$, $\hat{V} = \delta^{ab}\tau_0 V_a$, with τ_i's defined as $\tau_i = t_i \otimes \sigma_0$, where σ_i, t_i are the Pauli matrices in the S and $\mathcal{T}$ spaces. The term S_Q, defined in Eq. (2b), represents the part of the action that describes electrons moving in the presence of the auxiliary fields V_a, whereas S_C, defined in Eq. (2c), is the bare Coulomb action. The kernel $K(x-x')$ describes the inverse effective Coulomb interaction in the wire and depends on the specific device geometry. For example, for a homogeneous wire in the absence of a nearby gate its Fourier transform is $K(q) = (1/e^2)\ln \frac{1}{q^2 d^2}$. We assume that the external magnetic field is absent. The action in Eqs. (2a)–(2c) constitutes the NLσM.

The Q matrix satisfies the nonlinear constraint $Q^2 = \mathbb{1}$. It also satisfies the charge conjugation condition,[21]

$$Q = CQ^T C^T,$$

$$C = \delta^{ab}\delta_{\varepsilon\varepsilon'} \otimes \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \equiv \delta^{ab}\delta_{\varepsilon\varepsilon'} \otimes [t_1 \otimes (-i\sigma_2)], \tag{3}$$

where a, b and ε, ε' denote the replica and Matsubara indices, respectively, and the superscript T denotes the transposition. In what follows, we restrict ourselves to the case of strong spin-orbit scattering.[25] In this case, the Q matrix belongs to the symplectic ensemble,[21] and its matrix elements are unit matrices in the spin space.

To resolve the nonlinear constraint $Q^2 = \mathbb{1}$, we will use the exponential parametrization of the Q matrix,

$$Q = e^{iW/2}\Lambda e^{-iW/2}, \quad \Lambda^{ab}_{\varepsilon\varepsilon'} = \delta^{ab}\delta_{\varepsilon\varepsilon'}\tau_0 \,\mathrm{sgn}\,\varepsilon,$$

$$\{W,\Lambda\} = 0, \quad W = W^+, \tag{4}$$

where $\{A,B\}$ denotes the anticommutator of A and B. The invariance of the Q matrix with respect to the operation of charge conjugation, Eq. (3), and its Hermiticity impose the following matrix structure on the rotation generators W in the $\mathcal{T}$ space:

$$W^{ab}_{\varepsilon\varepsilon'} = \begin{pmatrix} d & c \\ -c^* & -d^* \end{pmatrix}^{ab}_{\varepsilon\varepsilon'}, \quad \hat{d}^+ = \hat{d}, \quad \hat{c}^T = -\hat{c}. \tag{5}$$

The fields $(d,c)^{ab}_{\varepsilon\varepsilon'}$ represent the diffuson and Cooperon degrees of freedom, respectively, each being a unit matrix in the spin space.

The action in Eqs. (2a)–(2c) is characterized by two parameters. The first is $G(L_T) = 4\pi\hbar\nu DA/L_T$, where $L_T = \sqrt{\hbar D/2\pi T}$ is the thermal diffusion length. It has the meaning of the dimensionless conductance of the wire segment of length L_T. The other one is the number of transverse channels in the wire, $N_{ch}=k_F^2 A/4\pi$. We consider a multichannel metallic wire for which both parameters are large. From now on, the Planck's constant $\hbar$ is set to unity.

For large $G(L_T)$, we can evaluate the replicated partition function in the saddle point approximation. In this approximation, the partition function is written as a sum of the contributions arising from all the saddle points:

$$\langle Z^p \rangle = \sum_{\substack{saddle\\ points}} e^{-S_{sp}} \int \mathcal{D}[\delta Q, \delta V] e^{-\delta S[\delta Q, \delta V]}. \tag{6}$$

Here, S_{sp} denotes the NLσM action evaluated at the saddle point, and $\delta Q, \delta V$ describe fluctuations of the Q matrix and electric potentials V_a around a particular saddle point. Finally, $\delta S[\delta Q, \delta V]$ denotes the action change due to these fluctuations. In the next section, we obtain the exact form of the saddle points of the action in Eqs. (2a)–(2c) in the $G(L_T)=$const, $N_{ch}\to\infty$ limit, which is referred to below as the "$N_{ch}\to\infty$ limit" for brevity. Then, in Sec. IV, we use these results to construct approximate solutions to the saddle point equations for a wire with large but finite number of channels.

III. SADDLE POINTS IN THE $N_{ch}\to\infty$ LIMIT

If the number of channels in the wire is sufficiently large, $e^2\nu A \gg 1$, one may neglect the Coulomb action S_C in Eqs. (2a)–(2c) when looking for the saddle points. This corresponds to the charge neutrality limit,[15] which can be seen by noting that formally such procedure corresponds to the limit $e\to\infty$, which clearly enforces electroneutrality. The saddle point equations in this limit are obtained by minimizing S_Q in Eq. (2a) with respect to V_a and Q and read

$$D\nabla(Q\nabla Q) - [\hat{\varepsilon}+\hat{V}, Q] = 0, \tag{7a}$$

$$V_a - \frac{\pi}{4}\mathrm{tr} Q^{aa}_{\tau\tau}(x) = 0, \tag{7b}$$

where tr is the trace in the $S\otimes\mathcal{T}$ space only. Equation (7a) is the Usadel equation, and Eq. (7b) represents the charge neutrality condition.

By direct substitution, one can check that Eqs. (7a) and (7b) possess a set of stationary spatially uniform solutions, $Q^{ab}_{\varepsilon\varepsilon'}=\delta^{ab}\delta_{\varepsilon\varepsilon'}\tau_0\,\mathrm{sgn}(\varepsilon+2\pi T w_a)$, $V_a=2\pi T w_a$, which are characterized by a set of integer winding numbers in each replica, w_a. All these solutions represent degenerate minima of action (2b). The sum $4\sum_a w_a \equiv \mathcal{W}$ (the factor 4 here arises from the 4×4 matrix structure of $Q^{ab}_{\varepsilon\varepsilon'}$ in the $S\otimes\mathcal{T}$ space) defines the trace of the Q matrix, $\mathrm{Tr}\,Q=2\mathcal{W}$. The Q matrices corresponding to the minima with different w_a but the same $\mathcal{W}$ can be transformed into each other via continuous rotations in the replica and Matsubara spaces, Eq. (4). Therefore, the NLσM action contains soliton minima in which the Q matrix and the potentials V_a smoothly interpolate between their values in different uniform minima.[15] Such solitons are similar to those first found in Ref. 8.

A. Single soliton solution

In this section, we find an analytic solution to the saddle point equations [Eqs. (7a) and (7b)] that correspond to a single soliton. To be specific, we construct a soliton that connects the following degenerate minima: $Q=\Lambda$, with all the winding numbers $w_a=0$ at $x=-\infty$, and $Q^{ab}_{\varepsilon\varepsilon'}=\delta^{ab}\delta_{\varepsilon\varepsilon'}\tau_0\,\mathrm{sgn}(\varepsilon+2\pi T w_a)$ at $x=\infty$, with $w_{1,2}=\mp 1$, all the other w_a being zero. This corresponds to a gradual change in the electric potential in replicas 1 and 2, $V_{1,2}$, from zero at negative spatial infinity to $\mp 2\pi T$ at positive infinity.

For such a soliton, the generator W_0 parametrizing the saddle point Q matrix via Eq. (4) corresponds to a rotation between Matsubara frequencies πT in replica 1 and $-\pi T$ in replica 2. In this subspace, W_0 has the following structure:

$$W_0 = \begin{pmatrix} 0 & \hat{\lambda} \\ \hat{\lambda}^+ & 0 \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} \theta_d e^{i\phi} & \theta_c e^{i\chi} \\ -\theta_c e^{-i\chi} & -\theta_d e^{-i\phi} \end{pmatrix}, \tag{8}$$

where θ_d, θ_c, ϕ, and χ are real parameters. In this equation, the matrix element of W_0 in the upper-left corner corresponds to $(W)^{11}_{\pi T,\pi T}\equiv 0$, the one in the upper-right corner to $(W)^{12}_{\pi T,-\pi T}=\hat{\lambda}$, and so on. All the other matrix elements of W_0 are zero.

Substituting the rotation generator (8) into Eq. (4), we obtain the matrix elements of the Q matrix that participate in the rotation:

$$\left(\begin{array}{c|c} Q^{11}_{\pi T,\pi T} & Q^{12}_{\pi T,-\pi T} \\ \hline Q^{21}_{-\pi T,\pi T} & Q^{22}_{-\pi T,-\pi T} \end{array}\right) = \left(\begin{array}{cc|cc} \cos\theta_d\cos\theta_c & e^{i(\phi+\chi)}\sin\theta_d\sin\theta_c & -ie^{i\phi}\sin\theta_d\cos\theta_c & -ie^{i\chi}\cos\theta_d\sin\theta_c \\ e^{-i(\phi+\chi)}\sin\theta_d\sin\theta_c & \cos\theta_d\cos\theta_c & ie^{-i\chi}\cos\theta_d\sin\theta_c & ie^{-i\phi}\sin\theta_d\cos\theta_c \\ \hline ie^{-i\phi}\sin\theta_d\cos\theta_c & -ie^{i\chi}\cos\theta_d\sin\theta_c & -\cos\theta_d\cos\theta_c & e^{-i(\phi-\chi)}\sin\theta_d\sin\theta_c \\ ie^{-i\chi}\cos\theta_d\sin\theta_c & -ie^{i\phi}\sin\theta_d\cos\theta_c & e^{i(\phi-\chi)}\sin\theta_d\sin\theta_c & -\cos\theta_d\cos\theta_c \end{array}\right). \tag{9}$$

All the other matrix elements are those of the Λ matrix.

The action for such a Q matrix is independent of the angles ϕ and χ and depends only on $(\nabla\phi)^2$ and $(\nabla\chi)^2$ with positive coefficients. Therefore, the action minimum corresponds to coordinate independent angles ϕ and χ. It can be shown that the soliton solutions with the minimum action correspond to either $\theta_d \neq 0$, $\theta_c=0$ (diffusonlike rotation), or $\theta_d=0$, $\theta_c \neq 0$ (Cooperon-like rotation). In these cases, substitution of Eq. (9) into Eq. (7b) gives $V_{1,2}(x) = \mp \pi T[1 - \cos\theta_{d,c}(x)]$ for the diffusonlike and Cooperon-like rotations, respectively. Then, Eq. (7a) yields

$$\nabla^2\theta_{d,c} - \frac{1}{2L_T^2}\sin 2\theta_{d,c} = 0. \tag{10}$$

The solution that corresponds to the sought soliton is

$$\theta_{d,c}(x) = 2\arctan(e^{(x-x_0)/L_T}) \equiv \theta_0(x-x_0), \tag{11}$$

giving for the electric potentials

$$V_{1,2}(x) = \mp V^0(x-x_0) \equiv \mp \pi T\{1 + \tanh[(x-x_0)/L_T]\}, \tag{12}$$

which clearly satisfies $V_{1,2}(x\to-\infty)=0$ and $V_{1,2}(x\to\infty) = \mp 2\pi T$. Here, x_0 denotes the soliton position.

Substituting the saddle point values of Q and V_a, Eqs. (9) and (12), into action (2b), we obtain the action for a single soliton,

$$S_0 = G(L_T).$$

We note that this action does not depend on the soliton position x_0 and the angles ϕ and χ in Eq. (9). However, for the diffusonlike $(\theta_c=0)$ soliton, the different values of the angle χ correspond to the same Q matrix, and similarly different values of ϕ correspond to the same Q matrix for the Cooperon-like $(\theta_d=0)$ soliton. Therefore, the action for the fluctuations about the soliton has only two zero modes. One is associated with a translation of the soliton (change in x_0). The other corresponds to a rotation of the Q matrix in the replica and Matsubara space caused by a uniform change in either ϕ or χ, depending on whether we consider a diffusonlike or a Cooperon-like soliton. The presence of these zero modes needs to be borne in mind when integrating over the fluctuations about the soliton configurations.

B. Fluctuations around a single soliton

In this section, we evaluate the single soliton contribution to the replicated partition function, Eq. (6), in the $N_{ch}\to\infty$ limit. This requires evaluating the functional integral over the fluctuations of the Q matrix and the potentials V_a around the single soliton saddle point.

As was explained at the end of Sec. III A, the fluctuation spectrum has two zero modes. We show below that all the other fluctuations are massive and integrate over them in the Gaussian approximation. The resulting fluctuation determinant is convergent and is evaluated below. The integration over the zero modes is reduced to the integration over the soliton position and the rotation angle.

The translational zero mode represents a simultaneous spatial shift of the saddle point solution for the Q matrix and the static (zero Matsubara frequency) component of the potentials V_a. In order to simplify the treatment of this zero mode, we first integrate over the latter. This step involves no approximations since action (2b) is quadratic in V_a. The resulting action depends only on the nonzero Matsubara components of V_a and on the Q matrix. In this representation, the zero modes involve only the Q matrix degrees of freedom, whereas all fluctuations of the nonzero Matsubara components of V_a are massive. Then, the single soliton contribution to the partition function, Eq. (6), in the $N_{ch}\to\infty$ limit can be written as $\exp[-G(L_T)]\Gamma_p$, where Γ_p is the functional integral over the fluctuations about the soliton and is given by

$$\Gamma_p = \alpha^p \int \mathcal{D}[W,\delta V] e^{-S^{(2)}[W,\delta V]}. \tag{13}$$

Here, the fluctuations of the nonzero Matsubara components of the electric potential are denoted by δV_a, the matrix W parametrizes the deviation of the Q matrix from the saddle point, and α^p is the factor coming from integration over the static components of V_a. We will see later that in order to obtain the physical observables, we will only need to evaluate Γ_p at $p=0$. Therefore, the value of α is of no importance. Finally, the quadratic fluctuation action $S^{(2)}[W,\delta V]$ is obtained by integrating over the fluctuations of the static component of V_a in Eq. (2b) and expanding the resulting action to the second order in W. Its form depends on the Q-matrix parametrization.

In the remainder of this section, we show that the fluctuation integral Γ_p can be expressed as

$$\Gamma_p = \alpha^p G(L_T)\Upsilon_p \int \frac{dx_0}{L_T}, \tag{14}$$

where x_0 is the position of the soliton and Υ_p is a numerical factor independent of the system parameters. In order to evaluate the thermodynamic quantities, we need only the $p=0$ value of this quantity, which is calculated below, $\Upsilon \equiv \Upsilon_{p=0} \approx 8$.

In the remainder of the present section, we derive Eq. (14). The presentation is organized as follows. In Sec. III B 1, we give the expression for the fluctuation action. In Sec. III B 2, we carry out the integration over the Q-matrix fluctuations. Section III B 3 deals with integration over the electric potential fluctuations. The reader not interested in the derivation of Eq. (14) may wish to proceed directly to Sec. IV, where we use it to evaluate nonperturbative corrections to the thermodynamic quantities.

1. Fluctuation action

We parametrize the deviations of the Q matrix from the saddle point in terms of the matrix W, whose structure is described by Eq. (5), as follows:

$$Q = e^{iW_0/2}e^{iW/2}\Lambda e^{-iW/2}e^{-iW_0/2}. \tag{15}$$

Here, the matrix W_0 parametrizes the saddle point Q matrix. For the soliton described in Sec. III A, it is given by

$$W_0(x)=\begin{pmatrix}0 & i\theta_0(x)\tau_i\\ -i\theta_0(x)\tau_i & 0\end{pmatrix}. \tag{16}$$

Here, $i=0,1$ corresponds to diffusonlike and Cooperon-like rotations, θ_0 is defined in Eq. (11), and we set $\phi=\chi=\pi/2$ and $x_0=0$ for convenience.

In the following, we use dimensionless coordinate $\xi=x/L_T$, dimensionless fermionic Matsubara frequencies, $\epsilon=\varepsilon/2\pi T$, and dimensionless Matsubara components of the electric potential $\mathcal{V}^{\omega}=\delta V^{\omega}/2\pi T$, where ω is an integer defining the bosonic Matsubara frequency, such that the latter is written as $2\pi T\omega$. In these variables, the quadratic action in Eq. (13) can be written as

$$S^{(2)}[W,\mathcal{V}]=S_{\mathcal{V}\mathcal{V}}+S_{WW}+S_{W\mathcal{V}}. \tag{17}$$

Here, $S_{\mathcal{V}\mathcal{V}}$ denotes the part of the action that is quadratic in the potentials $\mathcal{V}_a$,

$$S_{\mathcal{V}\mathcal{V}}=\frac{G(L_T)}{2}\sum_a\sum_{\omega\neq 0}\int d\xi\,\mathcal{V}_a^{\omega}(\xi)\mathcal{V}_a^{-\omega}(\xi'), \tag{18a}$$

S_{WW} denotes the part of the action that is quadratic in W,

$$\begin{aligned}S_{WW}=\frac{G(L_T)}{16}\int d\xi\Bigg(&\sum_{ab}\sum_{\varepsilon>0,\varepsilon'<0}\mathrm{tr}[(\epsilon-\epsilon')W^{ab}_{\varepsilon\varepsilon'}(W^{ab}_{\varepsilon\varepsilon'})^{\dagger}+\nabla W^{ab}_{\varepsilon\varepsilon'}\,\nabla(W^{ab}_{\varepsilon\varepsilon'})^{\dagger}]\\
&+\left\{-\frac{3}{4}\sin^2\theta_0+\frac{(1-\cos\theta_0)}{2}\right\}\sum_{a\varepsilon}\mathrm{tr}[W^{1a}_{\pi T,\varepsilon}(W^{1a}_{\pi T,\varepsilon})^{\dagger}+W^{a2}_{\varepsilon,-\pi T}(W^{a2}_{\varepsilon,-\pi T})^{\dagger}]\\
&+\frac{(\cos\theta_0-1)}{2}\sum_{a\varepsilon\varepsilon'}\mathrm{sgn}\,\varepsilon\;\mathrm{tr}[W^{1a}_{\varepsilon\varepsilon'}(W^{1a}_{\varepsilon\varepsilon'})^{\dagger}-W^{2a}_{\varepsilon\varepsilon'}(W^{2a}_{\varepsilon\varepsilon'})^{\dagger}]\\
&-\frac{\sin^2\theta_0}{4}\left\{\mathrm{tr}[(\tau_iW^{12}_{\pi T,-\pi T})^2+(\tau_iW^{21}_{-\pi T,\pi T})^2]-\frac{1}{4}(\mathrm{tr}[\tau_i(W^{12}_{\pi T,-\pi T}-W^{21}_{-\pi T,\pi T})])^2\right\}\Bigg),\end{aligned} \tag{18b}$$

and $S_{W\mathcal{V}}$ denotes the part of the action that is linear in W and $\mathcal{V}$,

$$\begin{aligned}S_{W\mathcal{V}}=&\,i\frac{G(L_T)}{8}\int d\xi\sum_a\sum_{\omega\varepsilon}\mathcal{V}_a^{\omega}\,\mathrm{sgn}\,\varepsilon\;\mathrm{tr}\,W^{aa}_{\varepsilon,\varepsilon+2\pi T\omega}\\
&+i\frac{G(L_T)}{8}\int d\xi\sum_{\omega>0}\Bigg[\mathcal{V}_1^{\omega}\left\{-\left(\cos\frac{\theta_0}{2}-1\right)\mathrm{tr}[W^{11}_{\pi T,2\pi T(1/2-\omega)}]^{\dagger}+\sin\frac{\theta_0}{2}\mathrm{tr}[\tau_iW^{12}_{2\pi T(1/2+\omega),-\pi T}]^{\dagger}\right\}\\
&+\mathcal{V}_1^{-\omega}\left\{-\sin\frac{\theta_0}{2}\mathrm{tr}[\tau_iW^{12}_{2\pi T(1/2+\omega),-\pi T}]+\left(\cos\frac{\theta_0}{2}-1\right)\mathrm{tr}\,W^{11}_{\pi T,2\pi T(1/2-\omega)}\right\}\\
&+\mathcal{V}_2^{\omega}\left\{-\left(\cos\frac{\theta_0}{2}-1\right)\mathrm{tr}[W^{22}_{-2\pi T(1/2-\omega),-\pi T}]^{\dagger}-\sin\frac{\theta_0}{2}\mathrm{tr}[\tau_iW^{12}_{\pi T,-2\pi T(1/2+\omega)}]^{\dagger}\right\}\\
&+\mathcal{V}_2^{-\omega}\left\{\sin\frac{\theta_0}{2}\mathrm{tr}[\tau_iW^{12}_{\pi T,-2\pi T(1/2+\omega)}]+\left(\cos\frac{\theta_0}{2}-1\right)\mathrm{tr}W^{22}_{-2\pi T(1/2-\omega),-\pi T}\right\}\Bigg].\end{aligned} \tag{18c}$$

Here, "†" denotes the Hermitian conjugation in the $S\otimes\mathcal{T}$ space, i.e., corresponds to complex conjugation and transposition within a 4×4 block, without interchanging replica or Matsubara indices. The diffusonlike soliton corresponds to $\tau_i=\tau_0$, and $\tau_i=\tau_1$ for the Cooperon-like one. To be specific, in what follows, we consider the case of a diffusonlike soliton, i.e., we set $\tau_i=\tau_0$. In the Cooperon-like case, the treatment exactly parallels the one presented below.

Introducing the notation

$$\Gamma_W=\int\mathcal{D}W\exp(-S_{WW}) \tag{19}$$

and

$$\Gamma_V=\int\mathcal{D}[\mathcal{V}]e^{-S_{\mathcal{V}\mathcal{V}}}\langle e^{-S_{W\mathcal{V}}}\rangle_W=\int\mathcal{D}[\mathcal{V}]e^{-S_{\mathcal{V}\mathcal{V}}+(1/2)\langle S^2_{W\mathcal{V}}\rangle_W}, \tag{20}$$

TABLE I. Potentials $U^{ab}_{\varepsilon\varepsilon'}$ appearing in the operators $\hat{L}^{ab}_{\varepsilon\varepsilon'}$, Eq. (23). Each entry gives the potential specific to particular replica and Matsubara indices in terms of the potentials $v_{1,2}$ and u defined in Eq. (26). (j,k) denote replica indices not equal to 1 or 2.

	ab								
$\varepsilon\varepsilon'$	jk	$1j$	$2j$	$j1$	$j2$	11	12	21	22
$\varepsilon>\pi T,\ \varepsilon'<-\pi T$	0	v_2-1	v_1	v_2	v_2-1	0	$2v_2-2$	$2v_1$	0
$\varepsilon=\pi T,\ \varepsilon'<-\pi T$	0	u	v_1	v_2	v_2-1	v_1+u	v_2+u-1	$2v_1$	0
$\varepsilon>\pi T,\ \varepsilon'=-\pi T$	0	v_2-1	v_1	v_2	u	0	v_2+u-1	$2v_1$	v_1+u
$\varepsilon=\pi T,\ \varepsilon'=-\pi T$	0	u	v_1	v_2	u	v_1+u	Excluded	$2v_1$	v_1+u

where $\langle\cdots\rangle_W$ denotes the Gaussian average with respect to the action S_{WW}, we can write Eq. (13) as

$$\Gamma_p = \alpha^p \Gamma_W \Gamma_V. \tag{21}$$

We evaluate the quantities Γ_W and Γ_V in Secs. III B 2 and III B 3.

2. Integration over W

We now evaluate the functional integral over the fluctuations of the Q matrix, Γ_W in Eq. (19). Examination of the quadratic action in Eq. (18b) shows that the variables $W^{ab}_{\varepsilon\varepsilon'}$ with different replica or Matsubara indices fluctuate independently. Moreover, with the exception of $W^{12}_{\pi T,-\pi T}$, for each $W^{ab}_{\varepsilon\varepsilon'}$, the actions for the diffusons and Cooperons constituting it are identical. The term containing $W^{12}_{\pi T,-\pi T}$ is special because it has the same replica and Matsubara indices as the rotation generator W_0 parametrizing the saddle point. The fluctuations of the diffuson and Cooperon components of $W^{12}_{\pi T,-\pi T}$ are also independent, but their propagators are different. In particular, we will see that for a soliton represented by a diffusonlike rotation, only the diffuson part of $W^{12}_{\pi T,-\pi T}$ has zero modes, and vice versa for a Cooperon-like rotation.

In terms of the diffuson and Cooperon variables, see Eq. (5), action (18b) can be written as

$$S_{WW} = \sideset{}{'}\sum_{\substack{ab \\ \varepsilon>0,\varepsilon'<0}} \int d\xi\big((d^{ab}_{\varepsilon\varepsilon'})^* \hat{L}^{ab}_{\varepsilon\varepsilon'} d^{ab}_{\varepsilon\varepsilon'} + (c^{ab}_{\varepsilon\varepsilon'})^* \hat{L}^{ab}_{\varepsilon\varepsilon'} c^{ab}_{\varepsilon\varepsilon'}\big) + \int d\xi(d_s^* \hat{L}_d d_s + c_s^* \hat{L}_c c_s), \tag{22}$$

where the primed sum means that the term with $a=1$, $b=2$, $\varepsilon=\pi T$, and $\varepsilon'=-\pi T$ is excluded, and $(d,c)_s \equiv (d,c)^{12}_{\pi T,-\pi T}$. The operators $\hat{L}^{ab}_{\varepsilon\varepsilon'}$, $\hat{L}_{d,c}$ are all of the Schrödinger type and have the form

$$\hat{L}_{d,c} = \frac{G(L_T)}{4}(\hat{L}_{\omega=1} + u_{d,c}(\xi)),$$

$$\hat{L}^{ab}_{\varepsilon\varepsilon'} = \frac{G(L_T)}{4}(\hat{L}_{\epsilon-\epsilon'} + U^{ab}_{\varepsilon\varepsilon'}(\xi)), \tag{23}$$

with the operator $\hat{L}_\omega$ defined as

$$\hat{L}_\omega = -\frac{d^2}{d\xi^2} + \omega, \tag{24}$$

with ω and ϵ being the appropriate dimensionless Matsubara frequencies. The potentials $u_{d,c}$ for d_s, c_s are given by

$$u_d(\xi) = -\frac{2}{\cosh^2(\xi)}, \quad u_c(x) = -\frac{1}{\cosh^2(\xi)}. \tag{25}$$

The potentials $U^{ab}_{\varepsilon\varepsilon'}$ depend on the replica and Matsubara indices involved and can be expressed in terms of the following potentials:

$$v_{1,2}(\xi) = \frac{1}{2}[1 \pm \tanh(\xi)], \quad u(\xi) = -\frac{3}{4\cosh^2(\xi)}. \tag{26}$$

The expressions for the potentials $U^{ab}_{\varepsilon\varepsilon'}$ in terms of $v_{1,2}(\xi)$ and $u(\xi)$ are summarized in Table I.

The operators $\hat{L}^{ab}_{\varepsilon\varepsilon'}$ and $\hat{L}_c$ are positive definite. The operator $\hat{L}_d$, Eq. (23), with the potential u_d, defined in Eq. (25), has one zero eigenvalue, with all the other ones being positive and separated by a finite gap. The integration over the zero modes requires a special consideration. We therefore defer the integration over the variables d_s in Γ_W, Eq. (19), to the end of this section and begin by integrating over all the other variables first. To this end, we introduce an auxiliary quantity Γ'_W as

$$\Gamma_W = \frac{\int \mathcal{D}[d_s]\exp\left(-\int d\xi d_s^* \hat{L}_d d_s\right)}{\int \mathcal{D}[d_s]\exp\left(-\frac{G(L_T)}{4}\int d\xi d_s^* \hat{L}_{\omega=1} d_s\right)}\Gamma'_W \equiv \Gamma_d \Gamma'_W. \tag{27}$$

Calculation of Γ'_W reduces to the evaluation of Gaussian integrals. Since $(d,c)^{ab}_{\varepsilon\varepsilon'}$ are complex fields, the integration over each of them gives a factor of an inverse determinant of the corresponding operator in the quadratic action, Eq. (22), and we obtain the following expression for Γ'_W:

$$\Gamma'_W = \frac{\alpha^p}{\det\left(\frac{G(L_T)}{4}\hat{L}_{\omega=1}\right)\det \hat{L}_c} \sideset{}{'}\prod_{\substack{ab \\ \varepsilon>0,\varepsilon'<0}} (\det \hat{L}^{ab}_{\varepsilon\varepsilon'})^{-2}.$$

The prime indicates that the product does not include the contribution from $a=1$, $b=2$, $\varepsilon=\pi T$, and $\varepsilon'=-\pi T$. The operators $\hat{L}^{ab}_{\varepsilon\varepsilon'}$ in the expression for Γ'_W can be classified according to whether their replica indices correspond to the replicas participating in the soliton rotation. In particular, for $a,b>2$, the operators $L^{ab}_{\varepsilon\varepsilon'}$ are insensitive to the presence of a soliton. Denoting each of these operators as $\hat{L}^{jk}_{\varepsilon\varepsilon'}$, we see that the product over the replicas with $a,b>2$ contributes a factor $(\Pi_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{jk}_{\varepsilon\varepsilon'})^{-(p-2)^2}$ to the fluctuation determinant. Analogously, for $a=1,2$ and $b>2$, we have $p-2$ identical operators $L^{ab}_{\varepsilon\varepsilon'}$ for each of $a=1$ and $a=2$, which we denote as $\hat{L}^{1j}_{\varepsilon\varepsilon'}$ and $\hat{L}^{2j}_{\varepsilon\varepsilon'}$, respectively. Finally, there are $p-2$ equal operators for $a>2$ and each of $b=1$ and $b=2$, denoted as $\hat{L}^{j1}_{\varepsilon\varepsilon'}$ and $\hat{L}^{j2}_{\varepsilon\varepsilon'}$. Using these observations, we rewrite the previous equation as

$$\Gamma'_W=\frac{\alpha^p}{\det\left(\frac{G(L_T)}{4}\hat{L}_{\omega=1}\right)\det\hat{L}_c}\left(\prod'_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{12}_{\varepsilon\varepsilon'}\det\hat{L}^{21}_{\varepsilon\varepsilon'}\det\hat{L}^{11}_{\varepsilon\varepsilon'}\det\hat{L}^{22}_{\varepsilon\varepsilon'}\right)^{-1}\left(\prod_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{jk}_{\varepsilon\varepsilon'}\right)^{-(p-2)^2}$$
$$\times\left(\prod_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{1j}_{\varepsilon\varepsilon'}\det\hat{L}^{2j}_{\varepsilon\varepsilon'}\det\hat{L}^{j1}_{\varepsilon\varepsilon'}\det\hat{L}^{j2}_{\varepsilon\varepsilon'}\right)^{-(p-2)}.$$

In the above expression, the prime means that $\det\hat{L}^{12}_{\pi T,-\pi T}$ is excluded from the product. To compute the thermodynamic quantities, we will need only the value of Γ'_W at $p=0$, for which we use the same notation,

$$\Gamma'_W=\frac{1}{\det\left(\frac{G(L_T)}{4}\hat{L}_{\omega=1}\right)\det\hat{L}_c}\left(\prod'_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{12}_{\varepsilon\varepsilon'}\det\hat{L}^{21}_{\varepsilon\varepsilon'}\det\hat{L}^{11}_{\varepsilon\varepsilon'}\det\hat{L}^{22}_{\varepsilon\varepsilon'}[\det\hat{L}^{jk}_{\varepsilon\varepsilon'}]^4\right)^{-1}$$
$$\times\left(\prod_{\varepsilon>0,\varepsilon'<0}\det\hat{L}^{1j}_{\varepsilon\varepsilon'}\det\hat{L}^{2j}_{\varepsilon\varepsilon'}\det\hat{L}^{j1}_{\varepsilon\varepsilon'}\det\hat{L}^{j2}_{\varepsilon\varepsilon'}\right)^{2}. \quad (28)$$

Using Eq. (18b), definitions (23)–(26), and the identity $\ln\det\hat{O}=\mathrm{tr}\ln\hat{O}$, we can write for Eq. (28)

$$\ln\Gamma'_W=2\sum_{\omega=1}^{\infty}\left(4\omega\,\mathrm{tr}_\xi\ln\frac{(L_\omega+v_1)(L_\omega+v_2)}{L_\omega(L_\omega+1)}-\omega\,\mathrm{tr}_\xi\ln\frac{(L_\omega+2v_1)(L_\omega+2v_2)}{L_\omega(L_\omega+2)}+4\,\mathrm{tr}_\xi\ln\frac{L_\omega+u}{L_\omega}-2\,\mathrm{tr}_\xi\ln\frac{(L_\omega+v_1+u)(L_\omega+v_2+u)}{L_\omega(L_\omega+1)}\right)-\mathrm{tr}_\xi\ln\frac{L_{\omega=1}+u_c}{L_{\omega=1}}, \quad (29)$$

where tr_ξ denotes the trace in the coordinate space, $\mathrm{tr}_\xi\hat{O}=\int d\xi O(\xi,\xi)$. The terms in Eq. (29) are grouped in such a way that each is finite at a given ω, i.e., does not diverge with the length of the system.

In the Appendix, it is shown that each term in Eq. (29) can be evaluated using the formula

$$\mathrm{tr}_\xi\ln\frac{(L_\omega+U_1)(L_\omega+U_2)}{L_\omega(L_\omega+h)}=\ln tt'=\ln\sqrt{\frac{\omega+h}{\omega}}t^2, \quad (30)$$

where the potentials $U_1(\xi)$ and $U_2(\xi)$ satisfy $U_1(\xi)=U_2(-\xi)$, $U_1(-\infty)=0$, $U_1(\infty)=h$ [for the potentials from Eq. (29), the parameter h takes on the values 0, 1, or 2]. The quantities t, t' describe the $\xi\to\infty$ asymptotics of the two independent solutions of the equation

$$[\hat{L}_\omega+U_1(\xi)]\psi=0. \quad (31)$$

Namely, if we find the two solutions $\psi_{1,2}$ whose asymptotics at $\xi\to\pm\infty$ are given by $\psi_1(\xi\to-\infty)\approx\exp(\sqrt{\omega}\xi)$ and $\psi_2(\xi\to+\infty)\approx\exp(-\sqrt{\omega+h}\xi)$, the parameters t and t' are given by the coefficients in front of the growing exponentials in the asymptotics of these solutions at opposite infinities,

$$\psi_1(\xi)\approx t\exp(\sqrt{\omega+h}\xi),\quad \xi\to\infty,$$
$$\psi_2(\xi)\approx t'\exp(-\sqrt{\omega}\xi),\quad \xi\to-\infty. \quad (32)$$

The last equality in Eq. (30) holds since $\sqrt{\omega+h}t=\sqrt{\omega}t'$; see the Appendix for details. The case of a potential vanishing at spatial infinities is recovered from Eq. (30) by setting $h=0$, $U_1=U_2=U$, and $t=t'$:

$$\mathrm{tr}_\xi\ln\frac{L_\omega+U}{L_\omega}=\ln t. \quad (33)$$

In order to find the parameters t and t' corresponding to the potentials in Eq. (29), we note that for each potential, Eq. (31) has the general form

$$-\frac{d^2\psi}{d\xi^2}+\left[\omega-\frac{\alpha}{\cosh^2\xi}+\frac{\beta}{2}(1+\tanh\xi)\right]\psi=0, \tag{34}$$

where the values of the parameters α, β depend on the specific potential. For example, the potentials $u(\xi)$ and $v_1(\xi)$ in Eq. (26) correspond to $\alpha=3/4$, $\beta=0$ and $\alpha=0$, $\beta=1$, respectively.

If one introduces the variable $z=(1+\tanh\xi)/2$, and $y(z)=z^{-\sqrt{\omega}/2}(1-z)^{-\sqrt{\omega+\beta}/2}\psi(z)$, the above equation reduces to the hypergeometric equation for $y(z)$:

$$z(1-z)\frac{d^2y}{dz^2}+[c-(a+b+1)z]\frac{dy}{dz}-aby=0, \tag{35}$$

where the parameters a, b, and c are given by the following expressions:

$$c=1+\sqrt{\omega},$$

$$a=\frac{1}{2}(1+\sqrt{\omega}+\sqrt{\omega+\beta}-\sqrt{1+4\alpha}),$$

$$b=\frac{1}{2}(1+\sqrt{\omega}+\sqrt{\omega+\beta}+\sqrt{1+4\alpha}). \tag{36}$$

Using the properties of the hypergeometric functions $F(a,b,c,z)$ (Ref. 23) and switching back to the original variable $\xi=\text{arctanh}(2z-1)$, it is easy to show that the two independent solutions $\psi_{1,2}$ of Eq. (34) satisfying the desired asymptotics, $\psi_1(\xi\to-\infty)\to\exp(\sqrt{\omega}\xi)$ and $\psi_2(\xi\to\infty)\to\exp(-\sqrt{\omega+h}\xi)$, are given by

$$\psi_1(z)=z^{\sqrt{\omega}/2}(1-z)^{\sqrt{\omega+\beta}/2}F(a,b,c,z),$$

$$\psi_2(z)=z^{\sqrt{\omega}/2}(1-z)^{\sqrt{\omega+\beta}/2}F(a,b,a+b-c+1,1-z). \tag{37}$$

The asymptotic behavior of $\psi_1(\xi)$ at $\xi\to+\infty$ is

$$\psi_1(\xi\to+\infty)\approx\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}e^{\sqrt{\omega+\beta}\xi}, \tag{38}$$

where $\Gamma(x)$ is the Euler gamma function. Comparing Eq. (38) with Eq. (32), we find that the value of the coefficient t entering Eq. (30) is given by

$$t=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{39}$$

with a, b, c defined in Eq. (36). Using Eqs. (30), (33), and (39) and the identity $\Gamma(x+1)=x\Gamma(x)$, we obtain the following for Γ'_W, Eq. (29):

$$\ln\Gamma'_W=2\sum_{\omega=1}^{\infty}\Bigg(4\omega\ln\left[\frac{4\sqrt{\omega}\sqrt{\omega+1}}{(\sqrt{\omega}+\sqrt{\omega+1})^2}\frac{\Gamma^2(\sqrt{\omega})\Gamma^2(\sqrt{\omega+1})}{\Gamma^4(\sqrt{\omega}/2+\sqrt{\omega+1}/2)}\right]-\omega\ln\left[\frac{4\sqrt{\omega}\sqrt{\omega+2}}{(\sqrt{\omega}+\sqrt{\omega+2})^2}\frac{\Gamma^2(\sqrt{\omega})\Gamma^2(\sqrt{\omega+2})}{\Gamma^4(\sqrt{\omega}/2+\sqrt{\omega+2}/2)}\right]$$
$$-2\ln\left[\frac{16\sqrt{\omega}\sqrt{\omega+1}}{[(\sqrt{\omega}+\sqrt{\omega+1})^2-1]^2}\frac{\Gamma^2(\sqrt{\omega})\Gamma^2(\sqrt{\omega+1})}{\Gamma^4(\sqrt{\omega}/2+\sqrt{\omega+1}/2-1/2)}\right]+4\ln\left[\frac{\sqrt{\omega}}{(\omega-1/4)}\frac{\Gamma^2(\sqrt{\omega})}{\Gamma^2(\sqrt{\omega}-1/2)}\right]\Bigg)$$
$$+\ln\left(\Gamma\left(\frac{3-\sqrt{5}}{2}\right)\Gamma\left(\frac{3+\sqrt{5}}{2}\right)\right)\approx\ln 0.08. \tag{40}$$

One can check that the sum over ω above converges, since the summand behaves like $\omega^{-3/2}$ for large ω. The last equality was obtained by performing the summation numerically.

We now complete the evaluation of Γ_W by computing the functional integral Γ_d, defined in Eq. (27). The operator $\hat{L}_d$ defined by Eqs. (23) and (25) has one zero eigenvalue. The corresponding eigenfunction is $1/\cosh\xi$. The fluctuations of $\text{Re}\, d_s$ and $\text{Im}\, d_s$ in the numerator of Eq. (27) along this mode correspond to the rotational and translational zero modes of the soliton discussed at the end of Sec. III A.

Indeed, in the parametrization [Eq. (15)], a soliton displacement, $Q_0(\xi)\to Q_0(\xi-\xi_0)$, by a small amount, $\xi_0=x_0/L_T$, is described by the generator W_{ξ_0} that can be obtained from the condition

$$\delta Q\approx-ie^{iW_0/2}\Lambda W_{\xi_0}e^{-iW_0/2}=-\frac{\partial Q_0}{\partial\xi}\xi_0,$$

where Q_0 is given by Eq. (15) with $W=0$ and W_0 from Eq. (16) with $i=0$. From this equation, it follows that W_{ξ_0} has the same structure as W_0, Eq. (8), with matrix $\hat{\lambda}$ replaced by $(W_{\xi_0})^{12}_{\pi T,-\pi T}$ defined as

$$(W_{\xi_0})^{12}_{\pi T,-\pi T}=-i\tau_0\frac{d\theta_0(\xi)}{d\xi}\xi_0.$$

Comparing this expression with Eq. (5), we see that the soliton translation corresponds to the diffuson fluctuation of the form $d_s(\xi)=-i\frac{d\theta_0(\xi)}{d\xi}\xi_0=-(i/\cosh\xi)\xi_0$. Along the same lines of reasoning, it can be shown that the soliton rotation by the angle ϕ_0, $\phi \to \pi/2+\phi_0$ in Eq. (9), corresponds to $d_s(\xi) = (1/\cosh \xi)\phi_0$ and represents the other zero mode.

We separate the functional integral over d_s in the numerator of Eq. (27) into a product of integrals over the zero and massive modes:

$$\Gamma_d = \frac{J\int d\xi_0 \int d\phi_0 \int \mathcal{D}[\tilde{d}_s]\exp\left(-\frac{G(L_T)}{4}\int d\xi \tilde{d}_s^{*}(\hat{L}_{\omega=1}+u_d)\tilde{d}_s\right)}{\int \mathcal{D}[d_s]\exp\left(-\frac{G(L_T)}{4}\int d\xi d_s^{*}\hat{L}_{\omega=1}d_s\right)},$$

where $\tilde{d}_s$ contains the massive modes only and J denotes the Jacobian for the change of variables $\{d_s\} \to \{\tilde{d}_s, \xi_0, \phi_0\}$. The product of the Jacobian J and the ratio of the functional integrals in this expression can be evaluated using the following trick. We introduce a regularized ratio $\Gamma_d(\eta)$ of the functional integrals over d_s in the last equation by infinitesimally shifting the frequency ω from unity, $\omega \to 1+\eta$, where η is positive. As a result, the zero modes acquire a finite mass and $\Gamma_d(\eta)$ can be written as

$$\Gamma_d(\eta) \equiv \frac{J\int d\xi_0 \int d\phi_0 \exp\left[-\frac{\eta G(L_T)}{4}\int \frac{d\xi}{\cosh^2 \xi}(\xi_0^2+\phi_0^2)\right]\int \mathcal{D}[\tilde{d}_s]\exp\left(-\frac{G(L_T)}{4}\int d\xi \tilde{d}_s^{*}(\hat{L}_{\omega=1}+u_d)\tilde{d}_s\right)}{\int \mathcal{D}[d_s]\exp\left(-\frac{G(L_T)}{4}\int d\xi d_s^{*}\hat{L}_{\omega=1}d_s\right)}. \quad (41)$$

On the other hand, this ratio of Gaussian integrals can be calculated using Eqs. (33), (39), and (36). In the limit of $\eta \to 0$, we obtain

$$\Gamma_d(\eta) = \det \frac{\hat{L}_{\omega=1+\eta}}{\hat{L}_{\omega=1+\eta}+u_d} = \frac{\Gamma(\sqrt{1+\eta}-1)\Gamma(\sqrt{1+\eta}+2)}{\Gamma(\sqrt{1+\eta})\Gamma(\sqrt{1+\eta}+1)} \approx \frac{4}{\eta}. \quad (42)$$

To arrive at this expression, we set $\omega=1+\eta$ in Eq. (36) and used the fact that the potential u_d corresponds to $\alpha=2$ and $\beta=0$. Integrating over ξ_0 and ϕ_0 in Eq. (41) and comparing the result with Eq. (42), we conclude that Γ_d can be written as

$$\Gamma_d = \frac{2G(L_T)}{\pi}\int d\phi_0 \int d\xi_0.$$

Substituting this expression into Eq. (27) and integrating over ϕ_0, we obtain the following expression for Γ_W:

$$\Gamma_W = 4\Gamma_W' G(L_T)\int d\xi_0, \quad (43)$$

with Γ_W' given by Eq. (40).

3. Integration over the electric potential fluctuations

We now turn to the evaluation of the functional integral over the potential fluctuations, Γ_V in Eq. (20). The action for the potential fluctuations is obtained by evaluating the Gaussian average $\langle S_{W\mathcal{V}}^2\rangle_W$ in Eq. (20) with respect to the action S_{WW} in Eq. (18b). The result of this tedious but straightforward calculation can be expressed in the form

$$S_{\mathcal{V}\mathcal{V}} - \frac{1}{2}\langle S_{W\mathcal{V}}^2\rangle_W = \frac{G(L_T)}{2}\sum_{\omega\neq 0}\int d\xi d\xi'\left[\sum_{a=1}^{p}\mathcal{V}_a^{\omega}(\xi)\Pi_0^{\omega}(\xi-\xi')\mathcal{V}_a^{-\omega}(\xi') - \sum_{a=1,2}\mathcal{V}_a^{\omega}(\xi)\delta\Pi^{\omega}(\xi,\xi')\mathcal{V}_a^{-\omega}(\xi')\right]. \quad (44)$$

Here, in the $p-2$ replicas not participating in the soliton rotation, the dimensionless polarization operator $\Pi_0^{\omega}(\xi-\xi')$ is given by the usual expression

$$\Pi_0^{\omega}(\xi-\xi') = \int \frac{dq}{2\pi}e^{iq(\xi-\xi')}\frac{q^2}{|\omega|+q^2}, \quad (45)$$

and in the remaining two replicas $(a=1,2)$, the dimensionless polarization operators acquire a correction $\delta\Pi^{\omega}(\xi,\xi')$ due to the presence of a soliton,

$$\delta\Pi^{\omega}(\xi,\xi') = \cos\frac{\theta(\xi)}{2}G_1^{\omega}(\xi,\xi')\cos\frac{\theta(\xi')}{2} + \sin\frac{\theta(\xi)}{2}G_2^{\omega}(\xi,\xi')\sin\frac{\theta(\xi')}{2} - G_0^{\omega}(\xi-\xi'). \quad (46)$$

In the last equation, we introduced the following Green's functions:

$$G_0^{\omega}(\xi-\xi') = \hat{L}_{|\omega|}^{-1},$$

$$G^{\omega}_{1,2}(\xi,\xi') = (\hat{L}_{|\omega|} + v_{1,2} + u)^{-1}, \tag{47}$$

where the operator L_ω and the potentials $v_1(\xi)$, $v_2(\xi)$, and $u(\xi)$ are defined in Eqs. (24) and (26).

We note that the polarization operator in the presence of the soliton, Eq. (44), is diagonal in Matsubara frequencies. This is a consequence of the fact that the soliton saddle point is static. We also note that no inter-replica couplings between the potential fluctuations are generated.

As an important consistency check, let us prove that

$$\int_{-\infty}^{\infty} d\xi' \Pi_s^{\omega\neq 0}(\xi,\xi') = 0, \tag{48}$$

which must hold due to particle number conservation. The polarization operator Π_0^ω automatically satisfies this property, as its Fourier transform is proportional to q. To prove that $\delta\Pi^\omega$ satisfies the same condition, we note that $\left(-\frac{d^2}{d\xi^2}+v_1+u\right)\cos\frac{\theta_0}{2}=0$ and $\left(-\frac{d^2}{d\xi^2}+v_2+u\right)\sin\frac{\theta_0}{2}=0$. Thus, we can write

$$\begin{aligned}\int_{-\infty}^{\infty} d\xi'\, \delta\Pi^\omega(\xi,\xi') &= \int_{-\infty}^{\infty} d\xi' \cos\frac{\theta_0(\xi)}{2} G_1^\omega(\xi,\xi') \cos\frac{\theta_0(\xi')}{2} \\ &\quad + \int_{-\infty}^{\infty} d\xi' \sin\frac{\theta_0(\xi)}{2} G_2^\omega(\xi,\xi') \sin\frac{\theta_0(\xi')}{2} \\ &\quad - \int_{-\infty}^{\infty} d\xi' G_0^\omega(\xi-\xi') = \frac{1}{|\omega|}\cos^2\frac{\theta_0}{2} \\ &\quad + \frac{1}{|\omega|}\sin^2\frac{\theta_0}{2} - \frac{1}{|\omega|} = 0,\end{aligned} \tag{49}$$

as expected.

Performing the Gaussian integral over $\mathcal{V}$ in Eq. (20) and taking the number of replicas p to zero, we obtain

$$\begin{aligned}\Gamma_V &= \prod_{\omega>0} \frac{\det^2 \Pi_0^\omega}{\det^2(\Pi_0^\omega - \delta\Pi^\omega)} \\ &= \exp\left(-2\sum_{\omega>0} \mathrm{tr}_\xi \ln[1-\delta\Pi^\omega(\Pi_0^\omega)^{-1}]\right).\end{aligned} \tag{50}$$

Due to the complicated form of $\delta\Pi^\omega$, explicit evaluation of this quantity is a daunting task. In particular, the method of the previous section does not apply here because the polarization operators are not represented by Schrödinger-type operators. However, we note that the dimensionless polarization operators Π_0^ω and $\Pi_0^\omega-\delta\Pi^\omega$ are independent of the system parameters. Provided the sum over ω in Eq. (50) converges it is clear that Γ_V is a parameter-independent numerical factor. Below, we prove that the sum over ω in the exponent of Eq. (50) does converge and evaluate Γ_V numerically.

To this end, we obtain the large-ω asymptotics of the summand in Eq. (50). This can be done by expanding the logarithm in Eq. (50) to first order in $\delta\Pi^\omega(\Pi_0^\omega)^{-1}$,

$$\begin{aligned}\mathrm{tr}_\xi \ln[1-\delta\Pi^\omega(\Pi_0^\omega)^{-1}] &\approx -\mathrm{tr}_\xi\, \delta\Pi^\omega(\Pi_0^\omega)^{-1} \\ &= -\int d\xi d\xi'\, \delta\Pi^\omega(\xi,\xi')(\Pi_0^\omega)^{-1}_{\xi'-\xi}.\end{aligned} \tag{51}$$

Using the Fourier transform of Π_0^ω from Eq. (45), the above trace can be written as

$$\mathrm{tr}_\xi\, \delta\Pi^\omega(\Pi_0^\omega)^{-1} = \int \frac{dq}{2\pi}\frac{\omega+q^2}{q^2}\int d\xi d\xi' e^{-iq\xi}\delta\Pi^\omega(\xi,\xi')e^{iq\xi'}. \tag{52}$$

We note that each of the two terms in the expression for $\delta\Pi^\omega$, Eq. (46), can be written as $\psi_0(\xi)G(\xi,\xi')\psi_0(\xi')$, where $G(\xi,\xi')$ is the resolvent of the operator $\left(\omega-\frac{d^2}{d\xi^2}+U\right)$ and ψ_0 is the zero mode of $\left(-\frac{d^2}{d\xi^2}+U\right)$. The phase factors in the last integral in Eq. (52) can be interpreted as a gauge transformation of the Green's function $\tilde{G}(\xi,\xi')=e^{-iq\xi}G(\xi,\xi')e^{iq\xi'}=\left(\omega+\left(\frac{1}{i}\frac{d}{d\xi}+q\right)^2+U\right)^{-1}$. Therefore, the integral can be written as

$$\begin{aligned}\int d\xi d\xi' e^{-iq\xi}\delta\Pi^\omega(\xi,\xi')e^{iq\xi'} &= \int d\xi d\xi' \cos\frac{\theta_0(\xi)}{2}\left(\frac{1}{\omega+q^2+2q\left(\frac{1}{i}\frac{d}{d\xi}\right)-\frac{d^2}{d\xi^2}+v_1+u}\right)_{\xi,\xi'}\cos\frac{\theta_0(\xi')}{2} \\ &\quad + \int d\xi d\xi' \sin\frac{\theta_0(\xi)}{2}\left(\frac{1}{\omega+q^2+2q\left(\frac{1}{i}\frac{d}{d\xi}\right)-\frac{d^2}{d\xi^2}+v_2+u}\right)_{\xi,\xi'}\sin\frac{\theta_0(\xi')}{2} - \int d\xi\frac{1}{\omega+q^2}.\end{aligned} \tag{53}$$

Then, we expand each of the first two kernels in the right hand side of Eq. (53) in powers of $\left[-\frac{d^2}{d\xi^2}+v_{1,2}+u+2q\left(\frac{1}{i}\frac{d}{d\xi}\right)\right]/(\omega+q^2)$ to the order that gives first nonvanishing contribution to the entire integral. The zeroth order term vanishes in the same way as it happened in Eq. (49), and so does the first order one being proportional to $\cos\frac{\theta_0}{2}\frac{d}{d\xi}\cos\frac{\theta_0}{2}$

$+\sin\frac{\theta_0}{2}\frac{d}{d\xi}\sin\frac{\theta_0}{2}\equiv\frac{d}{d\xi}(1/2)=0$. Therefore, expansion to the second order gives the first nonzero contribution, and we arrive at

$$\mathrm{tr}_\xi\,\delta\Pi^\omega(K+\Pi_0^\omega)^{-1}\approx\int\frac{dq}{2\pi}\frac{\omega+q^2}{q^2}(-4)\frac{q^2}{(\omega+q^2)^3}\times\int d\xi\left\{\cos\frac{\theta_0}{2}\frac{d^2}{d\xi^2}\cos\frac{\theta_0}{2}+\sin\frac{\theta_0}{2}\frac{d^2}{d\xi^2}\sin\frac{\theta_0}{2}\right\}\approx\frac{1}{2\omega^{3/2}}. \quad (54)$$

Equation (54) proves convergence of the sum over frequencies in Eq. (50). Therefore, we do not have to introduce any additional regulators.

We can calculate Γ_V numerically by expanding the logarithm in Eq. (50) in $\delta\Pi^\omega(\Pi_0^\omega)^{-1}$ and calculating the corresponding traces. The explicit calculation shows that expansion to the third order yields a precision better than a percent, which is sufficient for our purposes. Proceeding this way, we obtain $\Gamma_V\approx 24$. Combining Γ_V with Γ_W, expressed via Γ'_W and Γ_d calculated in Eqs. (40) and (43), we obtain the final expression for Υ, determining the fluctuation integral $\Gamma_{p=0}$, Eq. (14), needed to calculate the thermodynamics quantities:

$$\Upsilon=4\Gamma'_W\Gamma_V\approx 8. \quad (55)$$

IV. NONPERTURBATIVE CORRECTIONS TO THE THERMODYNAMIC QUANTITIES

In the previous section, we found the soliton saddle points and showed that the functional integral over the fluctuations about a single soliton configuration can be expressed in terms of the integral over the soliton position, Eq. (14). In the present section, we use these results to obtain nonperturbative corrections to the thermodynamic quantities at relatively high temperatures, $G(L_T)\gg 1$. We begin by considering the $N_{ch}\to\infty$ limit in Sec. IV A and turn to the case of large but finite N_{ch} in Sec. IV B.

A. Infinite channel number

In the $N_{ch}\to\infty$ limit, the Coulomb action [Eq. (2c)] vanishes. In this case, the NLσM action has infinitely many degenerate saddle points with spatially uniform potentials characterized by the winding numbers w_a, $V_a(x)=2\pi Tw_a$, with the usual saddle point, $Q=\Lambda$, corresponding to all $w_a=0$. The single soliton solutions, studied in Sec. III A, represent exact inhomogeneous saddle points with a finite action $G(L_T)$ and correspond to a kinklike change of the electric potentials $V_a(x)$ by $\pm 2\pi T$, Eq. (12), in two of the replicas involved in the soliton rotation. The spatial size of the kink is given by the thermal diffusion length L_T. In the dilute soliton gas limit, which corresponds to $G(L_T)\gg 1$, multisoliton saddle points can be viewed as sets of such kinks separated by distances much larger than L_T. In this case, the action of a multisoliton saddle point is given by the sum of single soliton actions. Similarly, the functional integral over the massive modes factorizes into a product of fluctuational determinants for each soliton. Thus, in the dilute regime, the soliton gas is noninteracting. Noting that the sum over the saddle points in Eq. (6) factorizes into a product of a sum over the uniform saddle points and the sum over the soliton configurations, we can easily find the multisoliton contributions to the replicated partition function in the dilute soliton gas regime,

$$\langle Z^p\rangle=Z_0^p\sum_{n=0}^{\infty}\frac{[2p(p-1)\alpha^p\Upsilon_pG(L_T)e^{-G(L_T)}]^n}{n!}\prod_{i=1}^{n}\int d\xi_0^{(i)}=Z_0^p\exp[2p(p-1)\alpha^p\Upsilon_pG(L_T)e^{-G(L_T)}L/L_T]. \quad (56)$$

Here, Z_0^p denotes the contribution of the homogeneous saddle points to the replicated partition function, L/L_T is the dimensionless wire length, the factor of $p(p-1)$ arises from the number of ways the two replicas participating in the soliton rotation can be chosen from the p replicas available, $\xi_0^{(i)}$ denotes the position of the ith soliton, and the factor of 2 arises from taking the Cooperon-like and diffusonlike solitons into account. Substituting this result into Eq. (1), we obtain the leading nonperturbative correction to the average thermodynamic potential in the $N_{ch}\to\infty$ limit:

$$\delta\Omega_\infty=2\Upsilon G(L_T)e^{-G(L_T)}\frac{L}{L_T}T, \quad (57)$$

where Υ is defined in Eq. (55).

The correction to the heat capacity can be obtained as $\delta C_\infty=-T\frac{\partial^2\delta\Omega_\infty}{\partial T^2}$. Taking into account that the largest contribution comes from differentiating $G(L_T)$ in the exponential, we obtain the ratio of δC_∞ to the heat capacity of noninteracting electrons, $C_0=\frac{2\pi^2}{3}\nu ATL$,

$$\frac{\delta C_\infty}{C_0}=-24\Upsilon G^2(L_T)e^{-G(L_T)}. \quad (58)$$

The analysis above was restricted to the charge neutrality limit, $N_{ch}\to\infty$. In the next section, we consider the case of a large but finite number of channels in the wire. In this case, the Coulomb action [Eq. (2c)] may not be neglected. Its presence significantly modifies the behavior of the soliton gas.

B. Finite channel number

For $N_{ch}\gg 1$, the influence of the Coulomb action [Eq. (2c)] on the soliton shape and on the massive fluctuations about the multisoliton configurations is small and may be neglected. Therefore, each soliton configuration is still fully characterized by the soliton positions and the indices of the replicas participating in the soliton rotation. The Coulomb action for each configuration is given by the term S_C, Eq. (2c), evaluated for the specific potential profile $V_a(x)$ corresponding to such a configuration.

For a single soliton situated at x_0, the potential profile in the two replicas participating in the rotation is represented by a kink, $V_0(x-x_0)$, Eq. (12), in one of the replicas and an antikink, $-V_0(x-x_0)$, in the other one. Thus, each soliton is characterized by its position $x_0^{(i)}$ and the index of the replica containing the kink, $a_+^{(i)}$, and the antikink, $a_-^{(i)}$. The potential profile for each soliton configuration is given by

$$V_a(x) = \sum_i [\delta_{a,a_+^{(i)}} - \delta_{a,a_-^{(i)}}] V_0(x - x_0^{(i)}). \tag{59}$$

Using this representation, the replicated partition function, Eq. (6), can be written as

$$\langle Z^p \rangle = \tilde{Z}_0^p \sum_{n=0}^{\infty} \frac{[2\alpha^p \Upsilon_p G(L_T) e^{-G(L_T)}]^n}{n!} \times \prod_{i=1}^{n} \int d\xi_0^{(i)} \sum_{a_\pm^{(i)}} \exp[-S_C(\{\xi_0^{(i)}, a_\pm^{(i)}\})], \tag{60}$$

where $S_C(\{\xi_0^{(i)}, a_\pm^{(i)}\})$ denotes the Coulomb action [Eq. (2c)] evaluated for a given soliton configuration $\{\xi_0^{(i)}, a_\pm^{(i)}\}$. Since the Coulomb action diverges for any uniform saddle point with $w_a \neq 0$, such saddle points are forbidden, and the factor $\tilde{Z}_0^p$, arising from the uniform saddle points, contains the contribution only from the usual saddle point, $Q=\Lambda$, $\{w_a=0\}$.

Equation (60) is valid in the dilute soliton gas regime, $G(L_T) \gg 1$, in which the typical intersoliton distance exceeds the thermal diffusion length L_T. In the following, we assume that at these length scales, the Coulomb interaction is screened due to the presence of a nearby gate, so that its Fourier transform is given by $K(q) \approx (1/e^2)\ln\left(\frac{d_g^2}{d^2}\right)$, where d_g is of the order of the distance to the gate. This assumption simplifies further calculations but does not reduce the generality of the results obtained below. Then, defining the kink density $\rho_a(\xi)$ in replica a,

$$\rho_a(\xi) = \sum_i \delta(\xi - \xi_0^{(i)}) [\delta_{a,a_+^{(i)}} - \delta_{a,a_-^{(i)}}], \tag{61}$$

we can express the Coulomb action in Eq. (60) in the dilute gas limit as

$$S_C(\{\xi_0^{(i)}, a_\pm^{(i)}\}) = -\frac{\pi v_F}{32 e^2 \ln \dfrac{d_g}{d}} \frac{G(L_T)}{N_{ch}} \times \sum_a \int d\xi d\xi' \rho_a(\xi) \rho_a(\xi') |\xi - \xi'|. \tag{62}$$

Equations (60)–(62) express the replicated partition function of the disordered wire as a partition function of a one-dimensional replicated neutral gas of kinks and antikinks interacting via a linear potential. Importantly, the positive and negative charges in this gas occur only in pairs, such that the appearance of a positive charge in one replica is accompanied by the appearance of a negative charge in a different replica at the same spatial position. This problem can be mapped onto a one-dimensional replicated sine-Gordon model.[24] Below, we will not use this mapping but work in the replicated kink gas representation.

Only the soliton configurations that correspond to a neutral kink gas in each replica give a nonvanishing contribution to the partition functions because all non-neutral configurations possess an infinite Coulomb action. The density of the kink gas is controlled by the fugacity, $\Upsilon_p G(L_T) e^{-G(L_T)}$. Depending on its value, the kink gas can be in two different regimes. At high temperatures, for $G(L_T) \gtrsim \ln N_{ch}$, the gas is dimerized. In other words, the kinks within each replica form a dilute gas of bound pairs of a kink and an antikink. At lower temperatures, $\ln(N_{ch}) \gtrsim G(L_T) \gtrsim 1$, the kink pairs overlap and form an ionized plasma. The dilute soliton gas approximation used to derive Eq. (60) is valid in both of these cases. We restrict our analysis below to the high-temperature regime, $G(L_T) > \ln N_{ch}$.

For $G(L_T) \gg 1$, Eq. (60) may be viewed as an expansion of the replicated partition function in the powers of the fugacity, $\Upsilon_p G(L_T) e^{-G(L_T)}$. In the presence of the Coulomb action, the single soliton contribution to the partition function vanishes, since the corresponding Coulomb action is infinite. Therefore, the leading term in this expansion is given by the contribution of two solitons which corresponds to two kink-antikink pairs in different replicas. We shall refer to this object as a dimer.

To evaluate the contribution of a single dimer into the replicated partition function we express the Coulomb action [Eq. (62)] in terms of the kink-antikink separation within the dimer, ξ_{rel}, and substitute the result into Eq. (60). Denoting the dimer center of mass coordinate by ξ_{cm}, summing over all possible pairs of replicas that can accommodate the dimer, and recalling that each soliton can be either Cooperon-like or diffusonlike, we obtain

$$\langle Z^p \rangle = \tilde{Z}_0^p \left(1 + 4p(p-1)\alpha^p \Upsilon_p^2 G^2(L_T) e^{-2G(L_T)} \times \frac{1}{2!} \int_0^{L/L_T} d\xi_{cm} \int_{-\infty}^{\infty} d\xi_{rel} e^{-|\xi_{rel}| L_T / L_N} \right), \tag{63}$$

where we introduced the notation

$$L_N = \frac{8e^2 \ln \dfrac{d_g}{d}}{\pi v_F} \frac{N_{ch}}{G(L_T)} L_T \tag{64}$$

that has the meaning of the typical kink-antikink separation within each dimer. Since $L_N \sim L_T/\sqrt{T\tau_{el}}$, where τ_{el} is the elastic mean free time, this length scale is much larger than L_T within the validity domain of the NLσM description. Therefore, the dilute soliton gas approximation is justified. Performing the integrals over ξ_{cm} and ξ_{rel} in Eq. (63), we get

$$\langle Z^p \rangle = \tilde{Z}_0^p \left(1 + 4p(p-1)\alpha^p \Upsilon_p^2 \frac{L L_N}{L_T^2} G^2(L_T) e^{-2G(L_T)} \right). \tag{65}$$

This expression shows that the single dimer contribution to the partition function diverges as the length of the wire L goes to infinity. From the second term, we infer that the spatial density of dimers is $\sim \frac{L_N}{L_T^2} G^2(L_T) e^{-2G(L_T)}$. In the regime $G(L_T) \gtrsim \ln N_{ch}$, this density is smaller than $1/L_N$, and multisoliton configurations appear as a dilute gas of dimers.

Since the dimers in the dilute limit do not interact, the integration over all dimer configurations results in exponentiation of the correction arising from a single dimer, second term in Eq. (65),

$$\langle Z^p\rangle = \tilde{Z}_0^p\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(4p(p-1)\alpha^p\Upsilon_p^2\frac{LL_N}{L_T^2}G^2(L_T)e^{-2G(L_T)}\right)^n\right] = \tilde{Z}_0^p e^{4p(p-1)\alpha^p\Upsilon_p^2(LL_N/L_T^2)G^2(L_T)e^{-2G(L_T)}}. \quad (66)$$

Using Eq. (1) and definition (64), we get the expression for the leading nonperturbative correction for the thermodynamic potential:

$$\delta\Omega = \frac{32}{\pi}\Upsilon^2\frac{e^2}{v_F}\ln\frac{d_g}{d}N_{ch}G(L_T)e^{-2G(L_T)}\frac{L}{L_T}T, \quad (67)$$

where Υ is defined in Eq. (55). Using this expression, we obtain the ratio of the nonperturbative correction to the heat capacity to that of noninteracting electrons,

$$\frac{\delta C}{C_0} = -\frac{384}{\pi}\Upsilon^2\frac{e^2}{v_F}\ln\frac{d_g}{d}N_{ch}G^2(L_T)e^{-2G(L_T)}. \quad (68)$$

Equations (67) and (68) are the main results of this paper. These results are drastically different from expressions (57) and (58) obtained by taking the formal $N_{ch}\to\infty$ limit. We note that the corrections for the thermodynamic potential for infinite and finite N_{ch}, Eqs. (57) and (67), become of the same order at $G(L_T)\sim \ln N_{ch}$, when the dimer gas crosses over into the ionized regime.

V. SUMMARY

We studied nonperturbative interaction corrections to the thermodynamic quantities of a multichannel disordered wire. Within the replica NLσM formalism, these corrections arise from soliton saddle points of the NLσM action. In the limit of infinite number of channels N_{ch} in the wire, we obtained the exact single soliton solution of the saddle point equations and evaluated the function integral over the fluctuation about the soliton configuration. We showed that for $G(L_T)\gg 1$ and $N_{ch}\gg 1$, nonperturbative corrections to the thermodynamic quantities of the system are described by a partition function for a dilute gas of solitons. The latter is equivalent to the partition function for a replicated classical one-dimensional Coulomb gas. As the temperature is lowered, this gas undergoes a crossover from the dimerized regime of neutral soliton pairs at $G(L_T)>\ln N_{ch}$ to the regime of ionized plasma for $G(L_T)<\ln N_{ch}$. The crossover $G(L_T)\sim \ln N_{ch}\gg 1$ occurs at temperatures that are parametrically larger than those corresponding to the transition from weak to strong localization, $G(L_T)\sim 1$. This enables one to study this crossover separately from the perturbative effects. We specialized on the high-temperature regime, $G(L_T)\gtrsim \ln N_{ch}$, and obtained the leading nonperturbative correction to the specific heat (relative to that of noninteracting electrons), $\delta C/C_0 \sim N_{ch}G^2(L_T)e^{-2G(L_T)}$, Eq. (68). We would like to emphasize that this correction is drastically different from the result obtained by taking the formal limit $N_{ch}\to\infty$, Eq. (58), $\delta C_\infty/C_0\sim G^2(L_T)e^{-G(L_T)}$. It is worth noting that these corrections are sensitive to the magnetic field. It can be shown[17] that the magnetic field suppresses the Cooperon-like solitons, thus decreasing the correction magnitude.

Although our treatment was specialized to the symplectic ensemble, we believe that the mapping of the nonperturbative corrections to the soliton gas and to the replicated Coulomb gas, described by Eqs. (60), holds for all three ensembles. Indeed, the existence of soliton minima is generic for all three ensembles.[15] The mapping to the replicated classical Coulomb gas relies only on the fact that the functional integral over the fluctuations about a single soliton configuration can be reduced to the integral over the soliton position, Eq. (14). This, in turn, is a consequence of the fact that the integral over the massive modes converges, which we expect to be true for all ensembles.

The generalization of our formalism to the treatment of nonperturbative corrections to the transport characteristic is left for future work.

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APPENDIX: DERIVATION OF EQUATION (30)

In Sec. III B 2, we encountered expressions containing determinants of Schrödinger-type operators of the form

$$\ln D_0 = \mathrm{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U}{\omega - \frac{d^2}{d\xi^2}}, \quad \text{(A1a)}$$

$$\ln D_h = \mathrm{tr}_\xi \ln \frac{\left(\omega - \frac{d^2}{d\xi^2} + U_1\right)\left(\omega - \frac{d^2}{d\xi^2} + U_2\right)}{\left(\omega - \frac{d^2}{d\xi^2} + h\right)\left(\omega - \frac{d^2}{d\xi^2}\right)}, \quad \text{(A1b)}$$

where $U(\xi)$ is a potential that vanishes at $\xi\to\pm\infty$ and $U_{1,2}(\xi)$ are steplike potentials, satisfying $U_1(\xi)=U_2(-\xi)$, $U_1(-\infty)=0$, and $U_1(\infty)=h$.

As explained in the text above, Eq. (33), the trace in Eq. (A1a) can be obtained as a particular case of that in Eq. (A1b). Therefore, we concentrate our attention on the latter. We first rewrite Eq. (A1b) as

$$\ln D_h = \mathrm{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U_1}{\omega - \frac{d^2}{d\xi^2} + h\Theta} + \mathrm{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U_2}{\omega - \frac{d^2}{d\xi^2} + h(1-\Theta)} + \mathrm{tr}_\xi \ln \frac{\left(\omega - \frac{d^2}{d\xi^2} + h\Theta\right)\left(\omega - \frac{d^2}{d\xi^2} + h(1-\Theta)\right)}{\left(\omega - \frac{d^2}{d\xi^2} + h\right)\left(\omega - \frac{d^2}{d\xi^2}\right)}, \quad \text{(A2)}$$

where $\Theta(\xi)$ is the step function. The third term does not depend on the potentials and can be calculated explicitly, which is done at the end of this appendix. The first two terms are equal since $U_1(\xi)=U_2(-\xi)$. We denote each of them as $\ln D_{h1}$ and proceed to the calculation this quantity.

To compute

$$\ln D_{h1} = \mathrm{tr}_\xi \ln \frac{\omega - \dfrac{d^2}{d\xi^2} + U_1}{\omega - \dfrac{d^2}{d\xi^2} + h\Theta}, \quad \text{(A3)}$$

we represent the potential $U_1(\xi)$ as a sum $U_1(\xi)=h\Theta(\xi)+v(\xi)$, where $v(\xi)$ vanishes at spatial infinities and express the variational derivative of $\ln D_{h1}$ with respect to $v(\xi)$ in terms of the Green's function $G(\xi,\xi')=\left(\omega-\frac{d^2}{d\xi^2}+U_1\right)^{-1}\Big|_{\xi,\xi'}$,

$$\frac{\delta \ln D_{h1}}{\delta v(\xi)} = \frac{\delta \mathrm{tr}_\xi \ln G^{-1}}{\delta v(\xi)} = G(\xi,\xi). \quad \text{(A4)}$$

The Green's function $G(\xi,\xi')$ can be found by solving the differential equation

$$\left[\omega - \frac{d^2}{d\xi^2} + U_1(\xi)\right] G(\xi,\xi') = \delta(\xi-\xi'), \quad \text{(A5)}$$

with the boundary conditions that $G(\xi,\xi')$ vanishes at spatial infinities $\xi,\xi'\to\pm\infty$. It can be expressed[23] in terms of the two independent solutions of the homogeneous equation

$$\left[\omega - \frac{d^2}{d\xi^2} + U_1(\xi)\right]\phi_i(\xi) = 0, \quad \text{(A6)}$$

such that $\phi_1(\xi\to-\infty)\to 0$ and $\phi_2(\xi\to+\infty)\to 0$. In particular, at coinciding points, we have

$$G(\xi,\xi) = \frac{\phi_1(\xi)\phi_2(\xi)}{W[\phi_1(\xi),\phi_2(\xi)]}, \quad \text{(A7)}$$

where $W[\phi_1(\xi),\phi_2(\xi)]$ is the Wronskian of $\phi_1(\xi)$ and $\phi_2(\xi)$,

$$W[\phi_1(\xi),\phi_2(\xi)] = \frac{d\phi_1(\xi)}{d\xi}\phi_2(\xi) - \phi_1(\xi)\frac{d\phi_2(\xi)}{d\xi}. \quad \text{(A8)}$$

The Wronskian of the two independent solutions of Eq. (A6) does not depend on coordinate ξ and, therefore, may be expressed in terms of the $\xi\to\pm\infty$ asymptotics of $\phi_i(\xi)$. By appropriately normalizing the solutions, we can express the latter as

$$\phi_1(\xi) = \begin{cases} e^{k\xi}, & \xi\to-\infty \\ te^{\varsigma\xi}, & \xi\to\infty \end{cases}, \quad \phi_2(\xi) = \begin{cases} t'e^{-k\xi}, & \xi\to-\infty \\ e^{-\varsigma\xi}, & \xi\to\infty \end{cases}, \quad \text{(A9)}$$

where $k=\sqrt{\omega}$, $\varsigma=\sqrt{\omega+h}$, and t, t' depend on the specific form of the operator in Eq. (A6). Evaluating the Wronskian at $\xi\to\pm\infty$ using the asymptotics [Eq. (A9)], we obtain

$$W[\phi_1(\xi),\phi_2(\xi)] = 2\varsigma t = 2kt'. \quad \text{(A10)}$$

Next, we prove that

$$\frac{\delta \ln D_{h1}}{\delta v(\xi)} = \frac{\delta \ln t}{\delta v(\xi)}. \quad \text{(A11)}$$

To this end, we introduce an auxiliary construction

$$\tilde{W}[\phi_1(\xi),\tilde{\phi}_2(\xi)] = \frac{d\phi_1(\xi)}{d\xi}\tilde{\phi}_2(\xi) - \phi_1(\xi)\frac{d\tilde{\phi}_2(\xi)}{d\xi}. \quad \text{(A12)}$$

Here, ϕ_1 and $\tilde{\phi}_2$ are solutions of Eq. (A6) with the same ω, but for two different potentials $v(\xi)$ and $\tilde{v}(\xi)$, both of which vanish at $\xi\to\pm\infty$. The tilde denotes quantities corresponding to $\tilde{v}$. We assume that ϕ_1 and $\tilde{\phi}_2$ have the asymptotic form [Eq. (A9)], with $\tilde{\phi}_2$ characterized by $\tilde{t}'$.

In contrast to the Wronskian $W[\phi_1(\xi),\phi_2(\xi)]$, built out of the solutions of the same equation, the quantity $\tilde{W}[\phi_1(\xi),\tilde{\phi}_2(\xi)]$ depends on the coordinate and satisfies the differential equation

$$\frac{d\tilde{W}(\xi)}{d\xi} = \frac{d^2\phi_1(\xi)}{d\xi^2}\tilde{\phi}_2(\xi) - \phi_1(\xi)\frac{d^2\tilde{\phi}_2(\xi)}{d\xi^2} = [v(\xi)-\tilde{v}(\xi)]\phi_1\tilde{\phi}_2 \quad \text{(A13)}$$

that follows directly from Eq. (A6) for ϕ_1 and $\tilde{\phi}_2$.

Integrating Eq. (A13) with respect to ξ from $-\infty$ to ∞ and using the asymptotic form of ϕ_1 and $\tilde{\phi}_2$, Eq. (A9), we obtain

$$\tilde{W}(\infty) - \tilde{W}(-\infty) = \int_{-\infty}^{\infty} d\xi (v-\tilde{v})\phi_1\tilde{\phi}_2 = 2\varsigma t - 2k\tilde{t}'. \quad \text{(A14)}$$

Taking a variational derivative of this equation with respect to $v(\xi)$ at $v(\xi)=\tilde{v}(\xi)$, we obtain

$$\phi_1(\xi)\phi_2(\xi) = 2\varsigma\frac{\delta t}{\delta v(\xi)}. \quad \text{(A15)}$$

Plugging Eqs. (A10) and (A15) into Eq. (A7) for the Green's function and using Eq. (A4), we obtain Eq. (A11).

Integrating Eq. (A11) with respect to v from $v(\xi)=0$ to its final value, we obtain

$$\ln D_{h1} = \ln\frac{t}{t_0}, \quad \text{(A16)}$$

where t_0 is the coefficient in front of $e^{\varsigma\xi}$ in asymptotic form [Eq. (A9)] of ϕ_1 for $v(\xi)=0$. The latter can be easily found from the continuity of the logarithmic derivative $d\ln\phi_1(\xi)/d\xi$ at $\xi=0$ and is given by $t_0=(1+k/\varsigma)/2$.

Finally, the third term in Eq. (A2) can be calculated in the following manner. We denote this term by $T_3(\omega)$ and introduce the Green's functions g_0, g_h, and $g^\pm$ that vanish at $\xi,\xi'\to\pm\infty$ and satisfy the equations

$$\left(\omega - \frac{d^2}{d\xi^2}\right)g_0 = \delta(\xi-\xi'), \quad \left(\omega - \frac{d^2}{d\xi^2} + h\right)g_h = \delta(\xi-\xi'),$$

$$\left(\omega - \frac{d^2}{d\xi^2} + h\Theta\right)g^+ = \delta(\xi-\xi'),$$

$$\left(\omega - \frac{d^2}{d\xi^2} + h(1-\Theta)\right)g^- = \delta(\xi - \xi'). \tag{A17}$$

Taking the derivative of $T_3(\omega)$ with respect to ω, we obtain

$$\frac{\partial T_3}{\partial \omega} = \int_{-\infty}^{\infty} d\xi [g^+(\xi,\xi) + g^-(\xi,\xi) - g_0(\xi,\xi) - g_h(\xi,\xi)]. \tag{A18}$$

All Green's functions entering this equation are easily calculated using the method of Wronskian, as was done above for a general potential. Specifically, we obtain the following expressions for the Green's function at coinciding points:

$$g_0(\xi,\xi) = \frac{1}{2k}, \quad g_h = \frac{1}{2\varsigma}, \quad g^- = g^+(-\xi, -\xi'),$$

$$g^+(\xi,\xi) = \Theta(-\xi)\left(\frac{1}{2k} + \frac{k-\varsigma}{2k(k+\varsigma)}e^{2k\xi}\right) + \Theta(\xi)\left(\frac{1}{2\varsigma} + \frac{\varsigma - k}{2\varsigma(k+\varsigma)}e^{-2\varsigma\xi}\right). \tag{A19}$$

Keeping in mind that $T_3(\omega\to\infty)\to 0$, we can express it as $T_3(\omega) = -\int_\omega^\infty d\omega' \frac{\partial T_3(\omega')}{\partial \omega'}$. Substituting expressions (A17) for the Green's functions into Eq. (A18), we obtain

$$T_3 = \ln \frac{\varsigma t_0^2}{k}, \tag{A20}$$

with t_0 defined below Eq. (A16).

Substituting Eqs. (A20) and (A16) into Eq. (A2), we obtain the final expression for the trace in Eq. (A1b),

$$\mathrm{tr}_\xi \ln \frac{\left(\omega - \frac{d^2}{d\xi^2} + U_1\right)\left(\omega - \frac{d^2}{d\xi^2} + U_2\right)}{\left(\omega - \frac{d^2}{d\xi^2} + h\right)\left(\omega - \frac{d^2}{d\xi^2}\right)} = 2 \ln D_{h1} + T_3 = \ln \sqrt{\frac{\omega + h}{\omega}} t^2 = \ln tt', \tag{A21}$$

where in the last expression we used $\sqrt{\omega + h}t = \sqrt{\omega}t'$ to write a more symmetric expression, in which t and t' are defined in Eq. (A9). This proves Eq. (30).

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