Fractional Shapiro steps in a triangular single-plaquette Josephson-junction array

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We study the dynamics of a triangular single-plaquette Josephson-junction array in the development of the fractional Shapiro steps. We show that synchronization on fractional steps can happen due to an intricate interplay of the three junctions as the plaquette is made dynamically unsymmetric, either by applying an external magnetic field or by changing the configuration of external currents. We propose a mechanism for synchronization when the asymmetry is only due to the frustration induced by the magnetic field.

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I. INTRODUCTION

A sufficiently large dc current through a Josephson junction causes the phase difference across the junction to grow with time (we will say "the phase rotates"). The rate of increase can be detected as a voltage difference across the junction. In the ac Josephson effect, an ac component to the applied current (or an external rf field) can synchronize with the rotation rate of the phase, leading to the dc part of the voltage across the junction to be proportional to a multiple of the frequency of the current or field.[1] This state is stable against small perturbations, so that as system parameters (such as the dc current through the device) are varied, the system exhibits constant voltage steps. This effect was predicted by Josephson and verified by Shapiro experimentally, subsequently taking on the name Shapiro steps.[1,2] A variation on this scheme uses two rf signals producing a device for frequency mixing with unconventional properties. Constant voltage steps can now appear that correspond to linear combinations of the rf frequencies; for example, steps appear at the difference frequency or the multiples of the difference frequency.[3]

In general, we can index the steps by integers m and n according to

$$2e\langle V\rangle = \frac{n}{m}\hbar\omega, \tag{1.1}$$

where $\langle V\rangle$ denotes the time average voltage across the junction, ω is the frequency of the applied field or the frequency of the ac part of the driving current, and the integers n and m are mutually prime. The cases where $m=1$ are called "integer steps;" otherwise, we have "fractional steps."[4]

For a single overdamped junction, only integer steps are observed. The explanations[5,6] of the absence of fractional steps are specific to the model: in particular, the relationship between supercurrent and phase difference must be exactly $I_{super}=I_c \sin(\Delta\phi)$ and not some more general periodic function. This suggests that fractional steps can occur in more general models.

Arrays of Josephson junctions can present several new effects. If the N junctions in series evolve in synchrony, the voltage is N times larger, giving an effect called integer giant Shapiro steps.[7] A perpendicular magnetic field will induce vortices and for special values of the field, there will be a vortex crystal present; the coherent motion of the vortex crystal can produce fractional giant Shapiro steps.[8,9] The question of the response of a single plaquette and how it compares with that of the array in the latter case has also been studied.[10,11] It was concluded that a single plaquette exhibits predominantly half-integer steps, for different reasons than those for the large arrays.[11]

The intrinsic nonlinearity of the Josephson equations will couple the dynamics of the array to an external rf signal. For a single junction, steps are observed when the period of rotation in the absence of the signal is close to the period of the external signal or an integer multiple of it.

Generalizing this rule to a network of junctions leads to other possibilities because the dynamics of the network is not necessarily periodic, and when it is periodic the period may require several phase slips of a junction, so that the period is not given by $2\pi/\langle\dot{\theta}\rangle$.

A triangular array of Josephson junctions, as shown in Fig. 1, is the simplest system of its type to study such interplay of the internal degrees of freedom, both amongst themselves and with the external frequency drive. This is the simplest Josephson-junction array to exhibit fractional Shapiro steps as well as integer Shapiro steps.

The main purpose of this paper is to explore how the dynamics of the array in the absence of an rf signal is related to its behavior when the rf signal is added. In particular, for

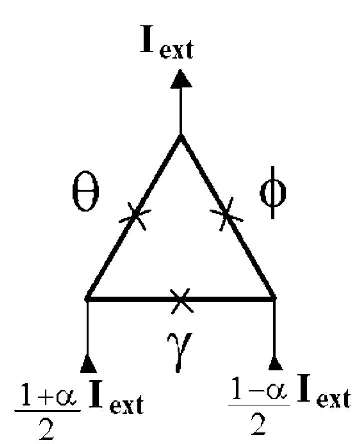

FIG. 1. The model circuit.

some choices of the system parameters, the array will exhibit periodic behavior and the imposed signal can entrain these, giving Shapiro steps.

We will also show that to have fractional steps, the triangular plaquette needs to be made asymmetric. This can be achieved by either applying an external magnetic field or by dividing up the input current unevenly.

In Sec. II, we will introduce the model and the equations governing the dynamics. We also introduce the integer and fractional Shapiro steps for a special set of parameters, to be discussed in detail further on. Three interesting special solutions in the absence of an ac signal are introduced in Sec. III; the stability of each solution is also discussed in this section.

Looking at the dynamics of the system in the absence of an ac signal, in Sec. IV, we argue for the presence of internal resonances as the dc current is varied. We will make good use of these arguments in Secs. V and VI.

Sections V and VI give details of the mechanisms bringing about the fractional Shapiro steps: Sec. V considers the case that the current is divided unevenly (in the absence of a magnetic field) and Sec. VI shows what happens when there is an applied magnetic field and the current is divided equally. The discussion hinges on the role of the junction denoted as γ in Fig. 1. We will show how this junction can be looked upon as a coupling for the other two junctions in describing the dynamics of the system. Section VII is devoted to our conclusions.

II. MODEL

The circuit consists of three identical overdamped Josephson junctions. The character of the junction is determined by the McCumber parameter, $\beta=2eI_cR^2C/\hbar$, where I_c denotes the critical current of the Josephson junction, R its resistance, and C its capacitance. A small β means that the system is strongly damped and that the transients may be neglected.[12]

The junctions are placed on a triangular plaquette; external currents are supplied and extracted, as shown in Fig. 1. The external current may have both a dc and an ac component, so that $I_{ext}=I_{dc}+I_{ac}\sin\omega t$. The parameter α sets the asymmetry of the current division, with $\alpha=0$ being the symmetrical case and $\alpha=\pm 1$ the fully unsymmetric case. We define the superconducting island forming the top node to have the phase and voltage zero and let the other two nodes have phases and voltages θ and $\dot{\theta}$ (for the bottom left node) and ϕ and $\dot{\phi}$ (for the bottom right).

In the presence of a magnetic field perpendicular to the plane of the plaquette, the resistively shunted junction model for this circuit is given by

$$I_{ext}=\sin(\theta-\pi f)+\sin(\phi+\pi f)+\dot{\theta}+\dot{\phi}, \qquad (2.1)$$

$$\frac{1}{2}(1+\alpha)I_{ext}=\sin(\theta-\pi f)+\sin(\gamma)+\dot{\theta}+\dot{\gamma}, \qquad (2.2)$$

where $\gamma=\theta-\phi$. The frustration parameter $f=\Phi/\Phi_0$ is the magnetic flux through the plaquette in units of the flux quantum, $\Phi_0=hc/2e$. The equations are written in a dimensionless form wherein I_{ext} is scaled by the critical current for the junction I_c (that is, we set $I_c=1$ in what follows), voltage is scaled by I_cR, and I_{ext} is scaled by $\hbar/2eI_cR$. We will refer to the three junctions as θ, ϕ, and γ (Fig. 1). We note that the equations have a symmetry (corresponding to the geometrical symmetry of the diagram): replacing $\alpha\to-\alpha$, $f\to-f$, and $\phi\to\theta$ (and of course $\gamma\to-\gamma$) leaves the equations unchanged.

Depending on the value of I_{dc} and other parameters, the system can exhibit several different behaviors.

When I_{dc} is small (so that all junction currents are less than I_c), there is no need for a normal current. The phases are constant (except for overdamped oscillations driven by I_{ac}) and the junction voltages (proportional to the time averages of $\dot{\phi}$ and $\dot{\theta}$) are zero.

At large values of I_{dc}, the dynamics is nonperiodic. The phases increase with time at different rates, so that the junction voltages are different and not rationally related. If the ac component of the external current has a frequency that matches one of these voltages [as described by Eq. (1.1)], integer Shapiro steps might be observed.

At intermediate values of I_{dc}, there can be periodic collective modes of the system. In the simplest case, θ and ϕ increase at the same average rate and the junction voltages are the same; however, there also can be more complicated periodicities, wherein θ and ϕ advance by different multiples of 2π. In this latter case, the junction voltages have rational ratios.

Equations (2.3) and (2.4) can be viewed as describing the motion of a particle on the two-dimensional $\phi-\theta$ plane. The solutions are described by trajectories which cannot intersect. Since the "force" functions are periodic in θ and ϕ, periodic solutions can be constructed that repeat (mod 2π) after m periods in ϕ and n periods in θ. The trajectories are conveniently classified by plotting how the system point evolves on the $\dot{\theta}-\dot{\phi}$ plane: periodic trajectories give a simple orbit, while aperiodic trajectories are space filling. The periodic trajectories describe the internal order that leads to the fractional Shapiro steps.

Another way to interpret the internal order is to introduce the concept of a vortex. The presence of a vortex can be detected by first adding or subtracting 2π from the three phase differences $\phi-\pi f$, $\pi f-\theta$, and $\theta-\phi$ so that they are all within the range $(-\pi,\pi)$ and then adding them up. In the absence of a vortex, they sum to $2\pi f$; if they sum to $2\pi(f-1)$, there is a vortex present. In a larger planar array of Josephson junctions, the vortex is a topological singularity of the phase field. Then, we can interpret the event where $\cos(\phi-2\pi f)$ passes through the value -1 as the entry of a vortex into the system. For the triangular circuit, the configuration with a vortex present is energetically unstable, and so the vortex immediately leaves through one of the sides. This can occur in a nonperiodic fashion or according to a pattern of finite period. The latter case gives rise to the fractional Shapiro steps.

A useful diagnostic for the presence of these collective modes is the Josephson energy

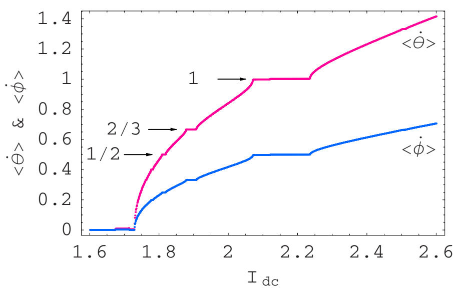

FIG. 2. (Color online) The average voltage drops across θ and ϕ junctions, with $f=0$, $\omega=1$, and the ac signal I_{ac} amplitude $a=0.5$, for the unsymmetric case $\alpha=1$.

$$E_{Josephson} = -\cos(\phi+\pi f) - \cos(\theta-\pi f) - \cos(\theta-\phi). \tag{2.3}$$

The Josephson energy averaged over a trajectory is close to zero for chaotic trajectories but less than zero for the entrained cases. Thus, it is very sensitive to these internal resonances.[19] In an array of interacting oscillators, such an order parameter represents the fraction of the locked oscillators.[20]

Using Eqs. (2.1) and (2.2), we can write the coupled equations for the ϕ and θ junctions, respectively, as

$$\dot{\phi} = \frac{3-\alpha}{6}I_{ext} - \frac{2}{3}\sin(\phi+\pi f) - \frac{1}{3}\sin(\theta-\pi f) + \frac{1}{3}\sin(\theta-\phi), \tag{2.4}$$

$$\dot{\theta} = \frac{3+\alpha}{6}I_{ext} - \frac{1}{3}\sin(\phi+\pi f) - \frac{2}{3}\sin(\theta-\pi f) - \frac{1}{3}\sin(\theta-\phi). \tag{2.5}$$

In some cases, it is useful to make a change of variables to $\xi=\theta+\phi$ and $\gamma=\theta-\phi$. Then, Eqs. (2.4) and (2.5) become

$$3\frac{d\gamma}{dt} + 2\sin(\gamma) + 2\sin\left(\frac{\gamma-2\pi f}{2}\right)\cos\left(\frac{\xi}{2}\right) = \alpha I_{ext}, \tag{2.6}$$

$$\frac{d\xi}{dt} + 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\gamma-2\pi f}{2}\right) = I_{ext}. \tag{2.7}$$

For most values of the parameters, this model exhibits both integer and fractional Shapiro steps. For example, the choices $I_{ext}=I_{dc}+I_{ac}\sin\omega t$ with $\omega=1$, $I_{ac}=0.5$, $f=0$, and $\alpha=1$ gives Fig. 2.

To construct this figure, we integrated[21] Eqs. (2.3) and (2.4) for many values of I_{dc}. The average value of $\dot{\theta}$ is proportional to the dc part of the voltage across this junction according to the Josephson equation $\langle V\rangle=\hbar/2e\langle\dot{\theta}\rangle$. As the figure shows, for certain ranges of I_{dc}, the voltage is constant and proportional to $\hbar\omega/2e$. The proportionality constant can take on both integer and fractional values.

III. SPECIAL SOLUTIONS AND THEIR STABILITIES

We begin our analysis by considering some special cases in the *absence* of an ac input current.

For the case $f=0$, $\alpha=0$, symmetry leads to the solution $\phi=\theta$; this solution decouples Eqs. (2.3) and (2.4). In order to check the stability of this solution, we assume a small perturbation around $\gamma=0$ in Eq. (2.5) and linearize it to obtain

$$3\dot{\gamma} = -\gamma[2+\cos(\xi/2)], \tag{3.1}$$

which shows that the solution is stable regardless of the form of ξ. This means that the initial conditions for γ are not important; the different orbits through θ and ϕ rapidly converge to a common stable orbit.

There is a corresponding special solution for other values of f: when $\alpha I_{ext}=2\sin 2\pi f$, $\theta=\phi+2\pi f$ (i.e., $\gamma=2\pi f$). For this special choice, the two symmetry-breaking factors cancel each other's effects, and ϕ and θ both satisfy the same "single junction" equation. Linearizing Eq. (2.6), we have

$$3\dot{\gamma} = -2(\gamma-2\pi f)[2\cos(2\pi f) + \cos(\xi/2)]. \tag{3.2}$$

A strong stability condition regardless of ξ is $f<1/6$; this makes the transients decay without any oscillations. A weaker stability condition is that the time integral of the terms in the brackets over a period be positive. Our numerical results show that this type of stability exists for f up to $1/5$. We will return to this special case in the presence of an ac signal in Sec. VI.

The most unsymmetrical case $\alpha=1$ is more interesting. Now, ϕ and γ are in series and carry the same current. So, we can consider the θ junction as a nonlinear load which couples the serial junctions. The numerical studies indicate that phase locking occurs, so that $\phi=\gamma$ and $\theta=2\phi+\pi f$. This leads to a decoupling of Eqs. (2.3) and (2.4),[13]

$$\dot{\phi} = \frac{1}{3}I_{ext} - \frac{1}{3}\sin(\phi) - \frac{1}{3}\sin(2\phi-2\pi f), \tag{3.3}$$

and similarly for θ. For $f=0$, we can study the stability of this solution by introducing a perturbation such that $\gamma=\phi+\eta$. This leads to the linearized equation

$$\dot{\eta} = -\eta\cos\phi, \tag{3.4}$$

which integrates to

$$\eta(t) = \exp\left[-\int^{t}\cos\phi(\tau)d\tau\right]. \tag{3.5}$$

The coefficient of η in Eq. (3.4) is an oscillating function; yet, as Eq. (3.5) indicates, if the time integral of this oscillating functions is positive, we still have a kind of stability. It can be shown that the right-hand side of Eq. (3.3) has its minimum for a positive value of $\cos(\phi)$ and its maximum for a negative value of $\cos(\phi)$, suggesting that the time average of $\cos(\phi)$ should be positive; our numerical solutions also show that the integral is always positive, but that with increasing I_{dc} it becomes small. Hence, for very large dc input currents, it takes a large time for perturbations to decay and the solution approaches neutral stability. For large dc inputs, the θ junction behaves as a linear resistor, and our result is consistent with the neutral stability of serial arrays with resistive loads.[14]

We checked serial arrays with more than two junctions in series, having a single junction as the load. The results showed instability for even more moderate dc inputs.

It is important for our later discussions to remark on the stability of the solutions when an ac input is added. Our general assumption is that the stability conditions are preserved, at least for small amplitudes of the ac drive. For the third case discussed above, this would mean that we still deal with Eq. (3.3). Indeed, if we solve this equation instead of the two coupled equations, we get identical results to those shown in Fig. 2 (which has $\alpha=1$ pertaining to this case).

We note in passing that Eq. (3.3), in the special case where $f=0$, undergoes a saddle-node bifurcation at $I_{ext}=I_{dc} \simeq 1.76$ [that is, for $\sin(\phi_0)+\sin(2\phi_0)$, where $\cos(\phi_0)=(\sqrt{33}-1)/8$]; for larger currents, ϕ is no longer constant in time.[15] This equation also comes up in the study of S/F/S Josephson junctions.[16,17]

IV. INTERNAL RESONANCES IN THE ABSENCE OF ac CURRENT

The two extremes for α resulted in two special solutions that led to qualitatively different behaviors for the array. When the current division is symmetric, θ and ϕ have the same time dependence as a single junction. So, we expect to see the integer Shapiro steps only in this case. At the other extreme, $\alpha=1$, we get a nonsinusoidal equation of state which is known to lead to fractional Shapiro steps.[11,18]

Assuming that $\langle\dot{\theta}\rangle$ and $\langle\dot{\phi}\rangle$ exist, we can define

$$\theta(t) = b\phi(t) + \epsilon h(t), \tag{4.1}$$

where b is a constant that depends on I_{dc}. The function $h(t)$ is not necessarily bounded (for example, it could perform a random walk), but in the case of greatest interest, it will be periodic. In these latter cases, the parameter b is rational. The mechanism that leads to this entrainment of the phases was discussed above.

Figure 3 shows the value of b and the Josephson energy as a function of I_{dc}, for $\alpha=0.9$ and $f=0$. For $I_{dc}<1.8$, the system is in a steady state. Above this value, θ and ϕ rotate. For $I_{dc}<2.03$, they rotate at the same rate, so that $b=1$. At higher values for I_{dc}, the rotation rates are different; some steps are visible in the graph of b, where the evolution is periodic. The dependence of $\langle\dot{\theta}\rangle$ and $\langle\dot{\phi}\rangle$ on I_{dc} is similar to what is shown in Fig. 4. Figure 3(b) shows that the Josephson energy is negative when θ and ϕ are entrained [at the steps in Fig. 3(a)], but zero otherwise.

In Fig. 3, many smaller steps are present besides 3/2 and 4/3 which are visible in larger scales.

V. JUNCTION γ IN ROTATING STATE

Figure 4 shows the time average voltages across the θ and ϕ junctions as a function of I_{dc} for $\alpha=0.9$, as determined from a numerical solution of Eqs. (2.3) and (2.4) when an ac with amplitude $a=0.5$ and frequency $\omega=1$ (in the normalized units) has been added.

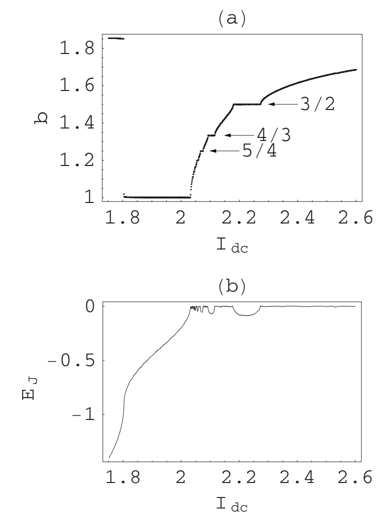

FIG. 3. (a) The average ratio of phases b and (b) the Josephson energy of the array, for $\alpha=0.9$ and $f=0$ in the absence of the ac signal. The signature for mutual entrainment due to the internal degrees of freedom is manifested as the steps on the ratio and minimums in energy.

For small currents, ϕ and θ assume time-independent values; there is no voltage across the junctions. As the current increases, there is a transition from the zero voltage state to the state in which θ and ϕ are rotating at the same rate, so that the voltages across θ and ϕ are the same (that is, the time averages $\langle\dot{\phi}\rangle$ and $\langle\dot{\theta}\rangle$ are equal) so $b=1$ and as can be seen no fractional steps exist in the characteristics. Further increase of current brings about a separation of voltages for the ϕ and θ junctions and sends the γ junction to a rotating state.

This qualitative change in the behavior of the system means that the junctions are no longer rotating with the same angular frequency. The parameter b is no longer fixed at unity and thus can change smoothly with I_{dc}. However, it can also become fixed at other values.

An interesting feature of Fig. 4 is that for some considerable ranges of I_{ext}, both $\langle\dot{\phi}\rangle$ and $\langle\dot{\theta}\rangle$ are on a step. The step where $\langle\dot{\theta}\rangle=1$ (near $I_{ext}=2.3$) is the integer Shapiro step and occurs because the rotation frequency of the θ junction

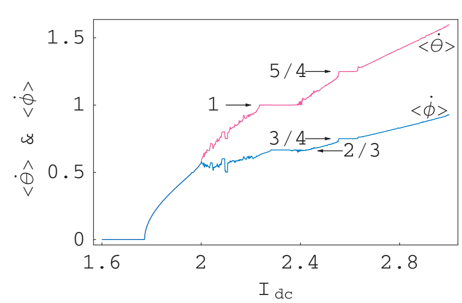

FIG. 4. (Color) IV characteristic for θ and ϕ junctions with $\alpha=0.9$, $\omega=1$, and $a=0.5$ in the normalized units and $f=0$.

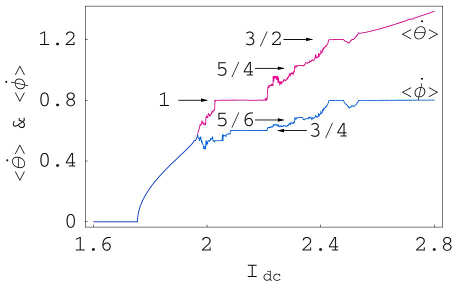

FIG. 5. (Color online) The *IV* characteristic for θ and ϕ junctions with the parameters similar to those of Fig. 4 except for $\omega = 0.8$ here.

matches the external ac frequency. The simultaneous locking of $\langle \dot{\phi} \rangle$ at the value $\frac{2}{3}$ is notable. This shows that the b ratio which was stable in the absence of ac drive (stability meaning with respect to I_{dc}) appears again and makes a fractional step in the characteristics of ϕ junction. The integer step can be considered as the response of θ junction to the ac input, whereas the corresponding fractional step for ϕ reveals the internal order which has been existing before imposing ac. The occurrence of a stable fractional step around $I_{dc}=2.6$ is more interesting. Here, none of the junctions is on an integer step and the ratio $b=5/3$ is obtained.

By varying the frequency of ac, it is possible to pick out the other steps in the b plot. For example, in Fig. 5, we exhibit the characteristic for $\omega=0.8$. By comparison with Fig. 3, the $b=4/3$ step (near $I_{ext}=2.1$) and the $b=3/2$ step (near $I_{ext}=2.45$) are now visible. Their presence can be explained by the argument in the previous paragraph since one of the junctions in each case is on an integer step.

In Fig. 6, we have plotted the characteristics of the θ junction in the presence of ac drive for different values of α. It can be seen that the transition from $\langle \dot{\phi} \rangle = \langle \dot{\theta} \rangle = 0$ to $\langle \dot{\phi} \rangle = \langle \dot{\theta} \rangle \neq 0$ is not so sensitive to the asymmetry in current division. However, this asymmetry does facilitate the transition to $\langle \dot{\theta} \rangle \neq \langle \dot{\phi} \rangle$ where γ is in the rotating state. This shows why in the cases studied in Secs. IV and V we took α close to unity. The signature of the latter transition is that the curves are noisy when the junction γ begins to rotate. For the values of α near zero, this transition occurs for larger values of I_{dc} which are beyond the range of our study.

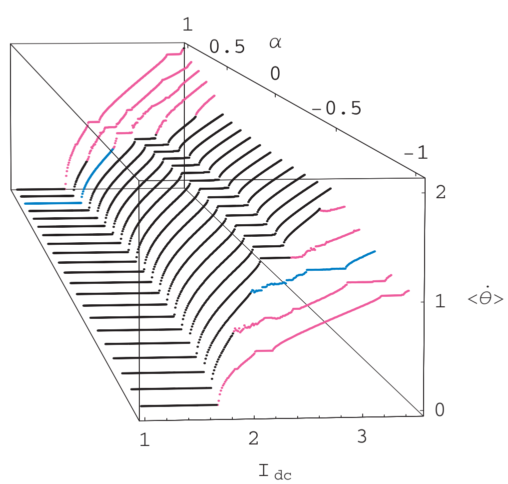

FIG. 6. (Color) The *IV* characteristic for θ for various values of current division with $\omega=1$, $a=0.5$, and $f=0$. Note that with the replacement $\alpha \rightarrow -\alpha$, this becomes the set of graphs for ϕ. In the regions colored red, $\langle \dot{\theta} \rangle$ and $\langle \dot{\phi} \rangle$ are unequal. The blue lines are the same data, as shown in Fig. 4.

The case of $\alpha \neq 1$ and identical junctions can be mimicked for $\alpha=1$ by allowing γ to have a higher critical current.[22] However, α serves as a better control parameter experimentally.

VI. JUNCTION γ IN OSCILLATING STATE

Fractional Shapiro steps can occur rather generally whenever the dynamics of θ, ϕ, and γ is nontrivial. Consider, for example, the case $\alpha=0$: in the absence of a magnetic field, this gives a totally symmetric circuit. In this case, θ and ϕ phase lock so that $\gamma=0$, and then each acts like a single junction, for which there are only integer steps. For nonzero f, however, fractional steps do occur at $\alpha=0$; half-integer steps are quite prominent. Unlike the situation above, $b=1$ for all values of f and I_{dc}, so the previous arguments about the origin of fractional steps do not work here. We intend to explain all these points in this section.

Let us reconsider Eqs. (2.6) and (2.7) with $\alpha=0$. With minor rearrangement, these read

$$3\frac{d\gamma}{dt} + 2\sin(\gamma/2)[2\cos(\gamma/2) + \cos \pi f \cos(\xi/2)] = 2\cos\left(\frac{\gamma}{2}\right)\cos\left(\frac{\xi}{2}\right)\sin \pi f, \tag{6.1}$$

$$\frac{d\xi}{dt} + 2\sin\left(\frac{\xi}{2}\right)\cos\left(\frac{\gamma}{2} - \pi f\right) = I_{ext}. \tag{6.2}$$

When the dc part of I_{ext} is greater than unity, $\xi = \theta + \phi$ will be in a rotating state, no matter what γ does. When $\alpha=0$, γ is uncoupled from ξ only in the special case $f=0$ (where $\gamma=0$ is the stable solution, even in the presence of an ac input). When f is not zero, γ cannot be stationary, except in the trivial case below the critical current. Regarding f and the oscillatory part of γ to be small, Eq. (6.2) can be looked upon as a single junction equation, with the average value of $\cos(\gamma/2 - \pi f)$ playing the role of the critical current. The oscillating factor $\cos(\xi)$ that appears on the right-hand side of Eq. (6.1) will cause oscillations in γ. The amplitude of this driving term is proportional to $\sin \pi f$; the numerical results support the expectation that the oscillations of γ have amplitude that scales with $\sin \pi f$.

As another feature of these equations, the frequency of oscillations of γ is determined by $\cos(\xi)$. These oscillations again feed back into Eq. (6.2) since $\cos(\gamma/2 - \pi f)$ has the role of critical current in this equation. In short, the equation is both externally and *parametrically* driven.[23] Now, the existence of higher harmonics in the spectrum of γ can be brought out by inserting the nonsinusoidal terms in the equation describing ξ, which results in the fractional steps, as can

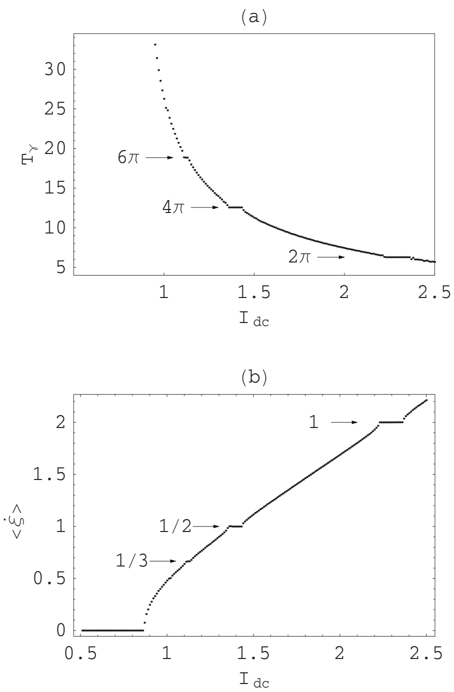

FIG. 7. (a) Average period of oscillations for γ as a function of I_{dc}. (b) The average time derivative of ξ for $\omega=1$, $a=0.5$, and $\alpha=0$ with $f=2/5$.

be seen in Fig. 7. Note that this feedback between Eqs. (6.1) and (6.2) is broken whenever $f=0$. The strength of the feedback increases with f monotonically.[24]

We note in passing that, in this case of $\alpha=0$, since the oscillations in γ keep a small amplitude, the triangular plaquette acts as an inductive superconducting quantum interference device, which is known to exhibit fractional Shapiro steps.[25,26] Also, when $\alpha=0$, the system is mathematically similar to the square plaquette studied by Sohn and Octavio,[11] in which there are two degrees of freedom.

VII. CONCLUSION

In conclusion, we have studied the intricate interplay of the Josephson junctions on a single triangular plaquette. This coupling brings about a mutual entrainment of the junctions for certain ranges of the dc current; mutual entrainment means synchronization in the absence of an ac master drive.

In the simpler case, the array is made asymmetric by unequal current division. The ratio of the average voltages across the junctions maintains constant fractional values for certain ranges of I_{dc}. This signals the presence of mutual entrainment—an internal rhythm or "internal clock."[27] When the ac signal is added on, this internal rhythm can lock onto the external frequency, bringing about the fractional Shapiro steps.

The fractional Shapiro steps serve as a probe for the internal rhythm amongst the components of a system. By contrast, the integer Shapiro steps signal an effectively one component system.

In the intriguing case of the symmetric current division, once the magnetic field is present, but with no master drive, the base junction oscillates and maintains a zero average voltage. This comes about as the other two junctions rotate with the same average rate, and each draws from the same set of harmonic components. It is this harmonic content that results in the fractional Shapiro steps once the ac signal is included.

The occurrence of fractional Shapiro steps is the normal case for the triangular array. Suppression of them requires special circumstances, such as symmetry of the circuit ($\alpha=0$ and $f=0$). Breaking the symmetry usually leads to fractional steps. However, we note that the two kinds of asymmetry can effectively cancel each other out, as in the case of the second special solution mentioned in Sec. III: when $\alpha I_{ext}=2\sin 2\pi f$, the junction equations reduce to a one-dimensional system and fractional steps do not appear. The magnetic and geometric asymmetries have canceled each other out.

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[17] If we choose the parameters of the ϕ and γ junctions such that their critical current is rI_c and their normal resistance is R/r, where I_c and R are the parameters of the θ junction, Eq. (2.5) changes to

$$\dot{\phi} = \frac{1}{r+2} I_{ext} - \frac{r}{r+2} \sin(\phi) - \frac{1}{r+2} \sin(2\phi).$$

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[21] The numerical integration was done using the fourth order Runge-Kutta algorithm. We checked several cases using the methods built into MATHEMATICA, with good agreement. To construct a curve such as those shown in Fig. 2, I_{dc} was varied from below the critical current increasing by small steps; for each such value, the time integration was carried out for several million time steps ($\Delta t = 0.001$). Averaging was performed after discarding the first few hundred thousand. Different values for the time increment were also tried to ensure the reproducibility of the results. We also tested the dependence of the averaged quantities on initial conditions and detected none.

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