Biordered superconductivity and strong pseudogap state

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Interrelation between the two-particle and mean-field problems is used to describe the strong pseudogap and superconducting states in cuprates. We present the strong pseudogap state as an off-diagonal short-range order (ODSRO) originating from quasistationary states of the pair of repulsing particles with large total momentum (K pair). Phase transition from the ODSRO state into the off-diagonal long-range ordered (ODLRO) superconducting state is associated with Bose-Einstein condensation of the K pairs. A checkerboard spatial order observable in the superconducting state in the cuprates is explained by a rise of the K-pair density wave. A competition between the ODSRO and ODLRO states leads to the phase diagram typical of the cuprates. Biordered superconducting state of coexisting condensates of Cooper pairs with zero momentum and K pairs explains some properties of the cuprates observed below T_c: Drude optical conductivity, unconventional isotope effect, and two-gap quasiparticle spectrum with essentially different energy scales.

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I. INTRODUCTION

Most commonly, it is believed that all essential physics of high-T_c cuprate superconductor, treated as a doped two-dimensional (2D) antiferromagnetic (AF) insulator with strong on-site Coulomb repulsion,[1] can be understood within the simplest one-band model of a strong correlated system such as the Hubbard or t-J model. It seems impossible to obtain analytic solutions for the ground states of these 2D models. Therefore, a variational approach based on a choice of an appropriate trial wave function can be considered as a natural way to solve the cuprate problem.[2] A great many of the fundamental ideas underlying the physics of the cuprates were introduced in this way.

Some results, obtained within the resonating valence bond (RVB) approach,[1] can be considered as a ground of the theory of high-T_c superconductivity. The wave function of the RVB ground state derived from Gutzwiller projected[3] d-wave ground state of the Bardeen-Cooper-Schrieffer (BCS) model[4] eliminates the possibility of double occupancy of a site (*no double-occupancy constraint*) and corresponds to extremely strong on-site correlations.

However, the physics of strong correlated systems is complicated by the fact that several of the low-energy ordered states have nearby ground-state energies[5]: d-wave superconductor (dSC), staggered flux phase,[6] spin and charge density waves (SDW and CDW, respectively), and some others.[7] Taking account of the competition and coexistence of such ordered states within extremely simplified models by numerical tools leads to a wide variety of phase diagrams. It is clear that, among them, one can always find diagrams similar to those typical of the cuprates.

Various approximations within the on-site Coulomb repulsion models often lead to antipodal conclusions concerning the possibility of the superconducting (SC) state itself.[8] The complete suppression of the double occupancy under Gutzwiller's projection promotes a rise of an insulating rather than a SC order. In this connection, Laughlin[8] has proposed an alternative approach to clarify the problem of the superconductivity of the cuprates. Instead of a numerical study of a highly simplified Hamiltonian, he suggested selecting a reasonable ground state in order to determine the Hamiltonian leading to such a state.

To take into account a realistic on-site repulsion, the ground state of Laughlin's gossamer superconductor[8] is chosen in the form of an incomplete projected BCS d-wave state with partially suppressed double occupancy. A Hamiltonian with such an exact forethought ground state, along with strong on-site repulsion, should manifest an attractive term which can lead to the dSC state.[8]

Gossamer superconductivity of the underdoped compound can be associated with a band of states with relatively low spectral weight inside a pronounced insulating forbidden band so that the chemical potential turns out to be pinned near the middle of this band.[8]

The repulsion-induced dSC state is highly sensitive to electron dispersion.[9] The simplest tight-binding approximation taking into account only the nearest neighbors (t model) seems to be insufficient, therefore, it is necessary to consider more complicated dispersion with the next-nearest-neighbor terms (t-t' model). Numerical study shows[9] that the stable dSC state corresponds to a hopping integral ratio t'/t within a narrow range near $t'/t=-0.3$. Just the same value of t'/t is consistent with the available angle resolved photoemission spectroscopy (ARPES) data.[10]

As follows from the SU(2) approach to the RVB problem,[9] it is necessary to consider doublets of fermions and bosons to realize the spin-charge separation correctly.[11] Two minima of the SU(2) boson dispersion are relative to the points (0, 0) and (π,π) in the 2D Brillouin zone,[11] therefore, the SC pairing channel corresponding to a large pair momentum should be taken into account along with the Cooper channel corresponding to zero pair momentum.

Geshkenbein *et al.*[12] have assumed that an enhancement of the scattering between the saddle points of electron dispersion results in the fact that the electron-electron interaction with large momentum transfer can be "less repulsive" with respect to small transfer. Therefore, in the vicinities of the saddle points, fermions pair into bosons. Such noncoherent preformed pairs arising near the antinodal arcs of the

Fermi contour (FC) might exist in the pseudogap state of underdoped cuprates as a normal Bose liquid.[12]

The FC outside of the arcs corresponds to unpaired fermions coexisting with the preformed pairs. Bose-Einstein condensation (BEC) of the preformed pairs with a large momentum due to their interaction with unpaired particles results in the SC gap on the whole of the FC. The SC state that arises in such a way describes reasonably a rather wide (intermediate with respect to BCS and BEC limiting cases) fluctuation region above T_c.

Instabilities in a 2D strong correlated electron system were investigated within the t-t' model at small t' by renormalization-group (RG) methods[13,14] using a discretization of the FC into a finite number of patches. The singularity in the Cooper channel exhibits a squared logarithmic divergence at low energies. For insulating Peierls channel with electron-hole pair momentum $\boldsymbol{Q}_\pi=(\pi,\pi)$, the singularity also exhibits a squared logarithm in the particular case $t'=0$ when nested FC has the form of a square coinciding with the boundary of the magnetic Brillouin zone of the parent compound. At $t'\neq 0$ and low doping, that is, in the case of a deviation of the FC from perfect nesting, the divergence is found weaker with logarithmic enhancement of the order of $\ln|t/t'|$ under the condition that $|t'|\ll|t|$. The singularities in the insulating and SC channels corresponding to zero and $\boldsymbol{Q}_\pi$ pair momenta, respectively, are found to be logarithmic in the case of small but nonzero $|t'/t|$. Such a case corresponds to approximately nested FC disposed close to saddle-point van Hove singularity. RG approach, involving the nesting effects,[15] gives a possibility to select singular contributions into pairing channels but corresponding preexponential factors remains undetermined.

General symmetry consideration, based on Zhang's SO(5) theory[16,17] or the SU(4) theory by Guidry *et al.*,[18] shows that one should take into account a closed set of competing ordered states to describe key features of correlated electron system. In this sense, singlet SC pairing channel with large momentum, incorporating singlet orbital insulating long-range (possibly, hidden[19,20]) or short-range (fluctuating between dSC and staggered flux states[21]) order, is naturally connected with the Cooper channel. Thus, there should be two SC gap parameters related to large and zero pair momenta, respectively.

The SC gap, which determines T_c and corresponds to a rise of the coherence in the system of electron pairs, can be directly extracted from experiments on Andreev reflection[22] or Josephson tunneling.[23] The observation of two SC gaps of about 10 and 50 meV, respectively, in tunnel experiment[24] in Bi2212 (in particular, a suppression of the lesser gap in high magnetic field at temperatures 30–50 mK) presents an indirect evidence in favor of two SC energy scales in the cuprates.

One more energy scale, observed in ARPES and tunnel spectra of underdoped cuprates,[10,25,26] can be associated with the strong pseudogap state.[27] To describe this state, one can start from a reasonably chosen one-particle Green function. Recently, Yang *et al.*[28] developed the RVB phenomenology of the pseudogap state based on the assumption that this state can be viewed as a liquid formed by an array of weakly coupled two-leg Hubbard ladders. The coherent part of the Green function obtained within the random phase approximation is consistent with the Luttinger theorem[29] and describes the evolution of the FC (from small pockets to closed contour) with doping. Similar results follow from both the spin-charge separation approach[30] and the phenomenological account of short-range insulating order above T_c.[31]

In this paper, we develop the concept of Coulomb pairing,[32] based on the ordinary Hamiltonian with screened Coulomb repulsion, that results in the *biordered* state originating from two SC pairing channels, with large and zero pair momenta. The all-sufficient conditions of repulsion-induced superconductivity in these two channels are discussed in Sec. II. Section III deals with the strong pseudogap state arising from incoherent quasistationary states of pairs with large momenta that are inherent in the screened Coulomb pairing potential. In Sec. IV, we consider the symmetry and two-gap spectrum of the biordered state. Finally, some possible manifestations of the strong pseudogap and biordered SC states are briefly discussed in Sec. V.

II. COMPETING PAIRING CHANNELS

Screening of Coulomb repulsion in three-dimensional isotropic degenerate electron gas results in a momentum-dependent interaction energy of two electrons,

$$U(k) = 4\pi e^2/[k^2\epsilon(k)], \tag{1}$$

where static permittivity has the form[33]

$$\epsilon(k) = 1 + \frac{k_0^2}{2k^2}\left(1 + \frac{1-x^2}{4x}\ln\left|\frac{1+x}{1-x}\right|\right). \tag{2}$$

Here, $x=k/2k_F$ and k_F and $k_0^{-1}=(4\pi e^2 ng)^{1/2}$ are the Fermi momentum and screening length, respectively; n and g are the electron concentration and density of states on the Fermi level, respectively.

Kohn singularity at $k=2k_F$ leads to electron-electron interaction with damped Friedel oscillation in real space. At a distance $r\gg k_F^{-1}$, this interaction can be written as[33]

$$U(r) \simeq \frac{e^2}{2\pi}\frac{\cos 2k_F r}{r^3}. \tag{3}$$

Kohn and Luttinger[34] have argued that attractive contribution into screened Coulomb repulsion originating from the Friedel oscillation is sufficient to ensure Cooper pairing with nonzero angular momentum. Because of the weakness of the Kohn singularity, corresponding SC transition temperature turns out to be very low.[34]

In the case of nested FC, the Kohn singularity transforms into the Peierls one with strong anisotropy of $\epsilon(\boldsymbol{k})$. Therefore, effective pairing interaction can be enhanced both in particle-hole and particle-particle channels. In particular, this can give rise to CDW or SDW in singlet or triplet insulating pairing channels, respectively.

Peierls singularity in a particle-hole channel originates from the fact that momentum transfer turns out to be equal to nesting vector $\boldsymbol{Q}$ for any particle on a finite part of the FC [Fig. 1(a)]. For the sake of simplicity, in Fig. 1, the FC is presented as a square corresponding to the t model at half-

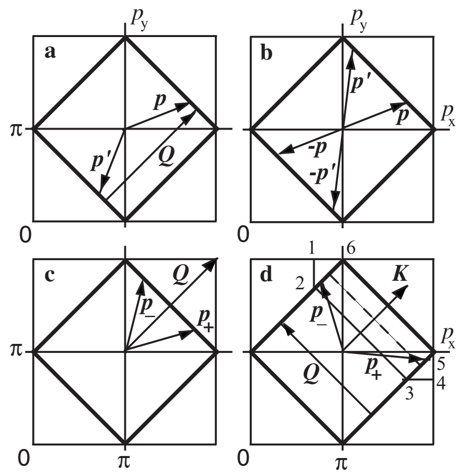

FIG. 1. Nested Fermi contour (bold line) in the form of a square coinciding with the 2D magnetic Brillouin zone of the parent compound with half-filled conduction band (within tight-binding model with nearest-neighbor interactions). (a) Electron-hole pairing with momenta $\boldsymbol{p}$ and $\boldsymbol{p}'=\boldsymbol{p}-\boldsymbol{Q}$ ($\boldsymbol{Q}$ is nesting momentum). (b) Cooper pairing with zero total momentum ($\boldsymbol{p}$ and $\boldsymbol{p}'$ are momenta before and after scattering, respectively). (c) SC pairing with nesting momentum ($\boldsymbol{p}_\pm=\boldsymbol{Q}/2\pm\boldsymbol{k}$, where $\boldsymbol{k}$ is the momentum of the relative motion of the pair). Domain of kinematic constraint is degenerated into a line coinciding with one of the sides of the square. (d) SC pairing with an incommensurate total momentum $\boldsymbol{K}$ ($\boldsymbol{p}_\pm=\boldsymbol{K}/2\pm\boldsymbol{k}$). Domain of kinematic constraint is bounded by the line 1-2-3-4-5-6. Parts 2-6 and 3-5 of this line are mirror nested pieces of the FC, resulting in a singular contribution into $\boldsymbol{K}$-pairing channel.

filling. In the case of low doping and under the condition that $|t'|\ll t$, the FC is close to this square.[13] Note that the FC of the underdoped cuprate compound has the form of a square with rounded corners as well. However, such a square is rotated by 45° with respect to the FC of the t model.[10,35] This rotation of the FC can be treated in terms of electron dispersion including, besides t and t', hopping integral t'' between second neighbors along the Cu–O bond direction ($tt't''$ model),[36]

$$\varepsilon(\boldsymbol{k}) = -2t(\cos k_x + \cos k_y) + 2t' \cos k_x \cos k_y + t''(\cos 2k_x + \cos 2k_y). \quad (4)$$

Here, k_x and k_y are coordinates of particle momentum $\boldsymbol{k}$. Dispersion Eq. (4) maps a set of isolines near the FC into those observable in underdoped cuprates if $t=0.5$ eV, $t'=-0.3$ eV, and $t''=0.2$ eV.

In the Cooper channel, Peierls enhancement of the Kohn singularity emerges appreciably weaker because momenta before and after scattering [$\boldsymbol{p}$ and $\boldsymbol{p}'$ in Fig. 1(b), respectively], giving rise to this enhancement, should be related as $\boldsymbol{p}-\boldsymbol{p}'\approx\boldsymbol{Q}$. Integration over momentum transfer $\boldsymbol{p}'-\boldsymbol{p}$ along the whole of nested FC smoothes down the Peierls singularity from the Cooper channel. Logarithmic singularity [$\ln(|\varepsilon_0/\varepsilon|)$, where ε_0 is an upper-limit cutoff energy] in the Cooper channel is ensured by a general feature of electron dispersion, $\varepsilon(-\boldsymbol{p})=\varepsilon(\boldsymbol{p})$, that holds for any momentum $\boldsymbol{p}$ (statistical weight of the Cooper pair is proportional to the length of the entire FC).

Density of states of a 2D system manifests logarithmic van Hove singularities originating from saddle-point vicinities with hyperbolic metric. Due to close proximity of the FC and the isoline connecting saddle points $(\pm\pi,\pm\pi)$, effective coupling constant w turns out to be logarithmically enhanced, $w\to w\ln(2tk_c^2/|\varepsilon-\varepsilon_s|)$.[13] Here, ε_s is the saddle-point energy (within the t model, $\varepsilon_s=0$) and momentum k_c is a scale of the part of the 2D Brillouin zone with hyperbolic metric.

In the case of SC pairing with large total momentum $\boldsymbol{K}$ ($\boldsymbol{K}$ channel), the momenta of the particles composing a pair with given momentum ($\boldsymbol{K}$ pair), both being either inside or outside the FC at $T=0$, should belong to only a part of the Brillouin zone (domain of *kinematic constraint*) rather than the whole. In a general case, the kinetic energy of the $\boldsymbol{K}$ pair with relative motion momentum $\boldsymbol{k}$,

$$2\xi_{\boldsymbol{K}}(\boldsymbol{k}) = \varepsilon(\boldsymbol{K}/2+\boldsymbol{k}) + \varepsilon(\boldsymbol{K}/2-\boldsymbol{k}) - 2\mu, \quad (5)$$

vanishes only at some points of the FC inside this domain (μ is the chemical potential). Therefore, in contrast to the Cooper pairing when $\xi_0(\boldsymbol{k})=0$ on the whole of the FC, integration over $\boldsymbol{k}$ eliminates the logarithmic singularity in the $\boldsymbol{K}$ channel. However, if kinetic energies of the two particles, $\varepsilon(\boldsymbol{K}/2+\boldsymbol{k})$ and $\varepsilon(\boldsymbol{K}/2-\boldsymbol{k})$, coincide on finite pieces of the FC ["pair" Fermi contour (PFC)], logarithmic singularity $\ln(|\varepsilon_0/|\varepsilon|)$ survives and the $\boldsymbol{K}$ channel can result in the SC order.[32] *Mirror nesting* condition,

$$\varepsilon(\boldsymbol{K}/2+\boldsymbol{k}) = \varepsilon(\boldsymbol{K}/2-\boldsymbol{k}), \quad (6)$$

determines the locus in the momentum space that logarithmically contributes to the $\boldsymbol{K}$ channel. Statistical weight of $\boldsymbol{K}$ pair, proportional to the length of the PFC, should be less in comparison with the Cooper channel.

Mirror nesting is a necessary (not all-sufficient) condition of the SC pairing with large momentum. For example, this condition is perfectly satisfied within the t model at half-filling (when $\mu=0$) at pair momentum $\boldsymbol{K}=\boldsymbol{Q}_\pi$ [Fig. 1(c)]. However, it is obvious that, in such a case, the domain of kinematic constraint degenerates into a segment resulting in zero statistical weight of the paired state. Indeed, Eq. (5) is identically zero if momenta of the particles composing the $\boldsymbol{K}$ pair, $\boldsymbol{K}/2+\boldsymbol{k}$ and $\boldsymbol{K}/2-\boldsymbol{k}$, belong to the FC, whereas Eq. (6), which determines a locus in the momentum space, results in the mutual segment of two squares shifted by $(\pi/2,\pi/2)$ and $(-\pi/2,-\pi/2)$ with respect to the FC. Thus, the domain of kinematic constraint degenerates into this segment and the PFC consists of two points, $(-\pi/2,\pi/2)$ and $(\pi/2,-\pi/2)$.

To obtain finite statistical weight, one can choose incommensurate pair momentum $\boldsymbol{K}\neq\boldsymbol{Q}_\pi$. This results in the domain of kinematic constraint in the form of a relatively narrow strip containing the PFC as shown in Fig. 1(d). In the case of the t model and arbitrary pair momentum $\boldsymbol{K}$ along direction (π,π), FC mirror nesting condition Eq. (6) implicates $\sin k_x+\sin k_y=0$. Because particle momenta belong to the first Brillouin zone, projections k_x and k_y of the relative-motion momentum turn out to be restricted by a two-sided inequality,

$$-\pi + K'/2 \leq k_{x,y} \leq \pi - K'/2, \tag{7}$$

where $K' = K\sqrt{2}/2$. Thus, the momenta of the particles composing the $\boldsymbol{K}$ pair belong to two segments of finite length on the opposite FC sides, 2-6 and 3-5, shown in Fig. 1(d).

The coefficient of the logarithmic contribution to the $\boldsymbol{K}$ channel should be proportional to the length of the PFC. Considering the PFC as two patches connected by a nesting vector $\boldsymbol{Q}_\pi$ [Fig. 1(d)], one can conclude that this coefficient has to be logarithmically enhanced by umklapp scattering inherent in the Peierls channel.[13] Patch approximation[13] is relatively good, just as in the case of short PFC. The reason is that integration over momentum transfer $\boldsymbol{p}-\boldsymbol{p}' \approx \boldsymbol{Q}_\pi$ cannot completely eliminate the enhancement of the pairing interaction due to logarithmic singularities of the permittivity.

Another necessary condition of SC pairing under repulsion is connected with the existence of an oscillating attractive contribution into the pairing potential. It should be noted that an oscillation itself cannot ensure the rise of a bound state. For example, simple stepwise repulsive potential $U(\boldsymbol{k}) = U_0 > 0$ defined in a finite domain of the momentum space oscillates in real space. However, by analogy with the problem of a bound state in a one-dimensional asymmetric potential well,[37] such a potential cannot result in a bound state even under mirror nesting.

One can consider a screened Coulomb potential $U(\boldsymbol{k}-\boldsymbol{k}')$ as a kernel of a Hermite integral operator with a complete orthonormal system of eigenfunctions defined within the domain of kinematic constraint Ξ,

$$\varphi_s(\boldsymbol{k}) = \lambda_s \sum_{\boldsymbol{k}' \in \Xi} U(\boldsymbol{k}-\boldsymbol{k}')\varphi_s(\boldsymbol{k}'). \tag{8}$$

Here, a set of λ_s represents the spectrum of a pairing operator, which can be written in the form of a Hilbert-Schmidt expansion,

$$w(\boldsymbol{k},\boldsymbol{k}') = \sum_s \frac{\varphi_s(\boldsymbol{k})\varphi_s^*(\boldsymbol{k}')}{\lambda_s}. \tag{9}$$

The necessary (and sufficient, under mirror nesting) condition of the SC pairing under repulsion is the existence of at least one *negative eigenvalue* of the pairing operator.[32]

In the case of a comparatively small domain of kinematic constraint, one can replace the screened Coulomb potential by its expansion in powers of momentum transfer, $\boldsymbol{\kappa} = \boldsymbol{k}-\boldsymbol{k}'$, up to the term of the second order,

$$w(\kappa) = U_0 r_0^2 (1 - \kappa^2 r_0^2/2), \tag{10}$$

where U_0 and r_0 are the on-site repulsive energy and screening length, respectively. The simplest repulsive kernel (10), defined inside Ξ, has two even $[\varphi_1(\boldsymbol{k})$ and $\varphi_2(\boldsymbol{k})]$ and two odd $[\varphi_1'(\boldsymbol{k})$ and $\varphi_2'(\boldsymbol{k})]$ (with respect to inversion $\boldsymbol{k} \to -\boldsymbol{k}$) eigenfunctions.

Normalized odd eigenfunctions and corresponding eigenvalues, λ_1' and λ_2', have the form

$$\varphi_s'(\boldsymbol{k}) = \frac{r_0^2 k_s}{\sqrt{K_1'}}, \quad \lambda_s' = \frac{1}{U_0 K_1'}. \tag{11}$$

Here, $s=1$ and 2 is the number of coordinate axis along one of the symmetry axes of the domain of kinematic constraint,

$$K_s' = r_0^4 \int_\Xi k_s^4 d^2k. \tag{12}$$

Normalized even eigenfunctions can be written as

$$\varphi_s(\boldsymbol{k}) = a_s \left[1 - \frac{K_0 \pm \sqrt{(K_0-K_1)^2 + K^2}}{2K_1 - K_2} r_0^2 k^2 \right], \tag{13}$$

where $s=1$ and 2 corresponds to the upper and lower signs in Eq. (13), a_s is a normalizing factor,

$$K_n = r_0^{2n+2} \int_\Xi k^{2n} d^2k, \quad n = 0, 1, \text{ and } 2, \tag{14}$$

and, by virtue of the Cauchy-Bunyakovsky inequality, $K^2 \equiv K_0 K_2 - K_1^2$ is a positive quantity.

Singlet SC order parameter should be determined by only even eigenfunctions belonging to eigenvalues λ_1 and λ_2 of opposite sign:

$$\lambda_s = \frac{-2}{U_0 K^2} [(K_0 - K_1) \pm \sqrt{(K_0-K_1)^2 + K^2}]. \tag{15}$$

Scattering between nested pieces of the FC leads to a strong anisotropy of the permittivity. Therefore, expansion of $w(\boldsymbol{k},\boldsymbol{k}')$ in powers of momentum transfer close to nesting momentum $\boldsymbol{Q}$ is anisotropic as well. The resulting pairing interaction kernel, analogous to Eq. (10), preserves its eigenvalue feature $\lambda_1\lambda_2 < 0$. It should be noted that the nesting momentum $\boldsymbol{Q} \neq \boldsymbol{Q}_\pi$; therefore, $\boldsymbol{Q}$ is the new nesting momentum, inherent in the real FC, which can result in the Peierls enhancement of the SC pairing.

The matrix of pairing operator Eq. (9) between its eigenfunctions is diagonal, $w_{ss'} = \lambda_s^{-1}\delta_{ss'}$. The necessary condition for the existence of a nontrivial solution to the self-consistency equation with kernel (10) has the form $\lambda_1\lambda_2 < 0$. Written in arbitrary basis, it takes the form of the Suhl inequality,

$$w_{11}w_{22} - w_{12}w_{21} < 0, \tag{16}$$

introduced as a necessary condition for superconductivity within a two-band model.[38]

The effective pairing potential oscillates in real space and, in agreement with Laughlin's proposal,[8] manifests a repulsive core at a small distance (corresponding to incomplete no double-occupancy constraint) and an attractive contribution outside of the core. Thus, such a pairing interaction gives rise not only to a bound state (with negative energy, $E<0$) but also to a quasistationary (with $E>0$) paired state with large momentum (Fig. 2).

The singular contribution into the SC order parameter is determined by a relatively small vicinity of the PFC with energy scale ε_0. In this respect, repulsion-induced $\boldsymbol{K}$ pairing seems to be similar to phonon-mediated pairing arising from attraction with negative coupling constant V. Rough

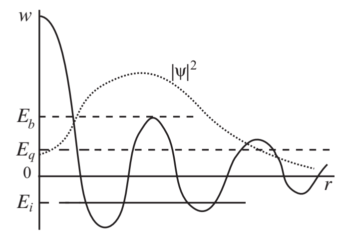

FIG. 2. Repulsive pairing potential, $w(r)$, and bound state, $|\psi|^2$ (dotted line), distributions in real space (schematically). Energies E_i and E_q correspond to bound and quasistationary states, respectively. Barrier height E_b corresponds to a break of the pair without tunneling through the barrier.

estimation[39] of Coulomb repulsion within the phonon-mediated mechanism of superconductivity leads to the fact that, to ensure SC pairing, $|V|$ must exceed a threshold value,

$$|V| > \frac{U_c}{1 + gU_c \ln(E_F/\varepsilon_D)}. \tag{17}$$

Here, E_F and U_c are Fermi and average Coulomb energies, respectively. Phonon-mediated attraction is defined inside a narrow layer (domain of dynamic constraint with energy scale of the order of Debye energy ε_D) enveloping the FC.

A deviation from perfect mirror nesting condition (6) outside of the PFC does not eliminate the logarithmic singularity in the $\boldsymbol{K}$ channel. Typical of high-T_c cuprates, nearly nested FC in the form of a square with rounded corners[10,35] results only in a weak suppression of the singularity under screened Coulomb pairing interaction. The SC gap turns out to be preserved on the PFC (by analogy with the case of insulating pairing studied by Losovik and Yudson).[40]

If mirror nesting condition Eq. (6) can be satisfied on a piece of the FC only approximately, there arises a cutoff on the low limit of integration in the self-consistency equation which determines the SC order parameter Δ. This cutoff δ can be treated as a mean square deviation of the FC from perfect mirror nesting. Then, setting $\Delta=\Delta_0$ when $\delta=0$, one can estimate the order parameter as[41]

$$\Delta = \sqrt{\Delta_0(\Delta_0 - \delta)} \tag{18}$$

by analogy with the order parameter of the Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) state.[42,43]

The Cooper and $\boldsymbol{K}$ channels can be found as approximately equally enhanced by both the Peierls singularity of screening and the proximity of the FC to van Hove singularity of the density of states. Therefore, it is the kinematic constraint that can ensure the preference of the $\boldsymbol{K}$ channel over the Cooper channel. As a result, the singular contribution to Peierls enhanced Coulomb $\boldsymbol{K}$ pairing can dominate the Cooper channel in doped cuprates.

III. STRONG PSEUDOGAP STATE

Taking into account the ground-state instability due to a rise of pairs, the mean-field approach to the problem of superconductivity excludes fluctuations of paired states from consideration. Within the mean-field theory,[4] the SC gap is directly relative to the binding energy of a pair resulted from the two-particle Cooper problem.[44] In the case of the $\boldsymbol{K}$ pairing, such a problem admits a more complicated solution as compared with the attraction-induced pairing.[32]

The integral equation which determines a wave function of the relative motion of two interacting particles (holes) above (below) the FC can be written as

$$\psi(\boldsymbol{k}) = G^{(0)}(\omega;\boldsymbol{k}) \sum_{\boldsymbol{k}' \in \Xi} w(\boldsymbol{k},\boldsymbol{k}')\psi(\boldsymbol{k}'). \tag{19}$$

Here, ω and $\boldsymbol{k}$ ($\boldsymbol{k}'$) are the energy and the momentum of the relative motion before (after) scattering, respectively, and

$$G^{(0)}(\omega;\boldsymbol{k}) = [\omega - \xi(\boldsymbol{k}) + i\gamma]^{-1} \tag{20}$$

is a one-particle Green function corresponding to the free relative motion of the $\boldsymbol{K}$ pair, $\gamma \to +0$. In contrast with one-particle Landau Fermi-liquid Green function, the condition that $[G^{(0)}(0,\boldsymbol{k})]^{-1}=0$ does not determine a closed FC. Indeed, within the t model at half-filling, a locus on the entire FC resulting from this condition, written in equivalent form $\xi_{\boldsymbol{K}}(\boldsymbol{k})=0$, coincides with finite pieces which present the PFC shown in Fig. 1(d).

In the case of mirror nested FC, one can separate a singular contribution to the Green function originating from a relatively small (with energy scale ε_0) part Ξ_s of the domain of kinematic constraint Ξ. The rest of the domain, including an energy range from ε_0 up to a cutoff value of about E_F, results in a regular contribution into $G^{(0)}$. One can consider this contribution in a way similar to taking account of the Coulomb repulsion within a phonon-mediated pairing attraction scenario[39] and renormalize kernel Eq. (9) to a kernel, defined inside Ξ_s, without a change of the spectrum.

The one-particle Green function $G(\omega;\boldsymbol{k})$ corresponding to the relative motion of the $\boldsymbol{K}$ pair of particles (holes) excited above (below) the FC can be represented in the basis formed by the eigenfunctions of the renormalized pairing operator $w(\boldsymbol{k},\boldsymbol{k}')$,

$$G_{ss'}(\omega) = \sum_{\boldsymbol{k} \in \Xi} \varphi_s^*(\boldsymbol{k}) G(\omega;\boldsymbol{k}) \varphi_{s'}(\boldsymbol{k}). \tag{21}$$

Matrix elements (21) are the solutions to the Dyson equation,

$$\sum_{s''} \{\delta_{ss''} - \lambda_{s''}^{-1} G_{ss''}^{(0)}(\omega)\} G_{s''s'}(\omega) = G_{ss'}^{(0)}(\omega), \tag{22}$$

in which matrix elements $G_{ss'}^{(0)}(\omega)$ of the free Green function (20) are defined similar to Eq. (21).

The pairing operator with two even eigenfunctions[45] results in a 2×2 matrix (21). One can resolve Eq. (22) with respect to $G_{ss'}(\omega)$ and then obtain $G(\omega;\boldsymbol{k})$ in the form

$$G(\omega;\boldsymbol{k}) = D^{-1}(\omega)[G^{(0)}(\omega;\boldsymbol{k}) - B(\omega;\boldsymbol{k})], \tag{23}$$

where

$$B(\omega;\boldsymbol{k}) = \lambda_1^{-1}\lambda_2^{-1} B(\omega) \sum_{s=1}^{2} \lambda_s |\varphi_s(\boldsymbol{k})|^2, \tag{24}$$

$$B(\omega) = G_{11}^{(0)}(\omega)G_{22}^{(0)}(\omega) - G_{12}^{(0)}(\omega)G_{21}^{(0)}(\omega), \qquad (25)$$

and

$$D(\omega) = 1 - \frac{G_{11}^{(0)}(\omega)}{\lambda_1} - \frac{G_{22}^{(0)}(\omega)}{\lambda_2} + \frac{B(\omega)}{\lambda_1\lambda_2}. \qquad (26)$$

In the case of mirror nested FC, Green function (23) manifests a pole resulting in a bound state with negative energy $\omega = E_i$ determined from equation $D(\omega)=0$, in which all functions $G_{ss'}^{(0)}(\omega)$ are real. This pole is related to the instability of the ground state with respect to a rise of pairs. Within a small vicinity of the pole, Green function (23) can be represented as

$$G(\omega;\boldsymbol{k}) = \frac{[G^{(0)}(\omega;\boldsymbol{k}) - B(\omega;\boldsymbol{k})]}{D'(E_i)} \frac{1}{\omega - E_i}, \qquad (27)$$

where $D' = dD/d\omega$.

At $\omega > 0$, Green functions (21) are complex; therefore, equation $D(\omega)=0$ can lead to a complex solution $\omega = E_q - i\Gamma$, where E_q and Γ can be considered as the energy and decay of the quasistationary state (QSS) of the relative motion of the $\boldsymbol{K}$ pair, respectively.[45] Near this complex pole, Green function (23) has the form of Eq. (27), where E_i should be replaced by $E_q - i\Gamma$.

Wave functions of the relative motion of the $\boldsymbol{K}$ pair corresponding to both bound state and QSS, are localized, in main, in a wide region of the real space outside of the repulsive core as shown schematically in Fig. 2.

The $\boldsymbol{K}$ pairs can exist above T_c as long-living QSS due to a considerable increase of the density of states in a narrow vicinity of E_q. To overcome the potential barrier before tunnel decay, such a noncoherent pair should accumulate an energy exceeding the barrier height E_b. Thus, the energy $E_q - E_i$ is sufficient to destroy SC coherence, whereas corresponding pair-break energy should exceed $E_b - E_i$. A temperature range between the SC transition temperature $T_c \sim E_q - E_i$ and a crossover one, $T_{str}^* \sim E_b - E_i$, can be interpreted as a strong pseudogap state observable above T_c in underdoped cuprates.[27] If density-of-states peak at $\omega \approx E_q$ turns out to be smoothed due to Γ being large enough, the strong pseudogap state becomes unobservable. In such a case, the SC transition from coherent into noncoherent state should be assisted with a break of pairs at energies $\approx E_b - E_i$, similar to that in the BCS theory.

By analogy with the interrelation between the Cooper two-particle problem[44] and BCS theory,[4] the pair-break energy $E_b - E_i$ due to direct excitation of particles from a bound state into a continuous spectrum should be transformed into a momentum-dependent energy gap $\Delta(\boldsymbol{k})$ in the quasiparticle spectrum. In the strong pseudogap state, this gap, due to a noncoherence of QSS, can be presented as $\Delta = \sqrt{\Delta_c^2 + \Delta_p^2}$. Here, $\Delta_c \sim E_q - E_i$ corresponds to a transition from the coherent bound state into the noncoherent QSS, and $\Delta_p \sim E_b - E_q$ can be related to a break of the $\boldsymbol{K}$ pair as a result of a transition between two noncoherent states.

Microscopically, SC gap Δ_c and strong pseudogap Δ_p emerge with random phases. Therefore, the mean-field value Δ_p vanishes at any temperature, whereas Δ_c becomes nonzero below T_c due to Bose condensation of $\boldsymbol{K}$ pairs from QSS into the bound state and vanishes only above T_c. However, nonzero mean square strong pseudogap, $|\Delta_p|^2 \neq 0$, becomes apparent well above T_c. In this sense, the pseudogap parameter Δ_p, corresponding to decay of QSS of $\boldsymbol{K}$ pairs, reminds us of the RVB spin liquid pseudogap introduced by Yang *et al.*[28] However, it has a different physical meaning.

Green function $G^{(0)}(0;\boldsymbol{k})$ changes sign on the PFC from positive to negative through infinity. Therefore, Green function (23) at $\omega=0$ manifests the same feature. It should be noted that, in the case of Cooper pairing, the Green function changes sign on the whole of the FC.[28] In addition, Green function $G(0;\boldsymbol{k})$ changes sign on a zero line determined by the equation $G^{(0)}(0;\boldsymbol{k}) = B(0;\boldsymbol{k})$. This line does not coincide with the PFC.

Green function (23) of the two-particle problem has a pole corresponding to a bound state of the relative motion of the pair. Therefore, one can suppose, in line with Yang *et al.*,[28] a phenomenological BCS-like form of the coherent contribution to the normal (diagonal) Gor'kov Green function of the mean-field problem[46]

$$G(\omega;\boldsymbol{k}) = z_k \left[\frac{u_+^2(\boldsymbol{k})}{\omega - E(\boldsymbol{k}) + i\Gamma} + \frac{u_-^2(\boldsymbol{k})}{\omega + E(\boldsymbol{k}) - i\Gamma} \right], \qquad (28)$$

where $E = \sqrt{\xi_K^2 + |\Delta|^2}$ and $2u_\pm^2 = 1 \pm \xi_K / E$ are quasiparticle energy and coherence factors, respectively. In accordance with Eq. (23), momentum-dependent quasiparticle weight z_k vanishes on the line of zeroes and corresponds to a finite value $z(0 < z < 1)$ on the PFC. Two terms in Eq. (28) can be referred to pairs above and below the FC, respectively.

Diagonal Green function (28) describes a nonsuperconducting state with ODSRO that arises both in time and real space, and corresponds to the existence of noncoherent $\boldsymbol{K}$ pairs above T_c. The coherence time directly follows from Eq. (28): $\tau = \Gamma^{-1}$. Thus, τ is determined by the decay of the QSS of $\boldsymbol{K}$ pairs. The problem of defining the coherence length seems to be more complicated because it has need of the study of the real-space behavior of Green function Eq. (28). Below T_c, the ODSRO transforms into the ODLRO introduced by Yang.[47] It should be emphasized that ODLRO and ODSRO can be associated with bound $(E_i < 0)$ and quasistationary $(E_q > 0)$ states, respectively, arising, under repulsive pairing, within two-particle problem Eq. (19).

Excitation with a transition from the bound paired state into long-living QSS corresponds to quite small but finite decay $\Gamma = \Gamma(\omega;\boldsymbol{k})$. The transitions into stationary states above the barrier energy E_b should be associated with an infinitesimal decay, $\gamma \to +0$, leading to a conventional Fermi-liquid behavior of the diagonal Gor'kov function (28) above T_{str}^*. Thus, a rise of QSS results in a non-Fermi-liquid behavior of the diagonal Green function (23) that can be manifested in a rather wide temperature range $T_c < T \lesssim T_{str}^*$ relating to a strong pseudogap state. This range corresponds to transitions between bosonlike bound and quasistationary states. Therefore, Eq. (28) can be considered as a bridge between the BCS and BEC approaches to the problem of superconductivity, in accordance with the assumption by Geshkenbein *et al.*[12]

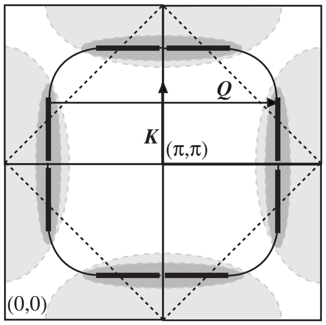

FIG. 3. Schematic representation of the Fermi contour of underdoped cuprate superconductor, in accordance with Ref. 35. Nested (with nesting momentum $\boldsymbol{Q}$) and mirror nested (corresponding to total pair momentum $\boldsymbol{K}$) pieces of the FC are shown by bold lines. The dotted line shows the magnetic Brillouin zone boundary. Shaded: dark narrow ovals correspond to vicinities of Peierls singularity of screening; light half-ovals designate extended saddle-point vicinities.

IV. SUPERCONDUCTING STATE WITH LARGE MOMENTUM

Nearly nested FC of underdoped (up to optimum doping) cuprate compound in the form of a square with rounded corners is shown schematically in Fig. 3. If it results in the fact that the $\boldsymbol{K}$ channel corresponding to an incommensurate momentum $\boldsymbol{K}$ dominates the Cooper channel, it is a rise of coherence in the system of $\boldsymbol{K}$ pairs that determines the SC transition temperature T_c. Therefore, there is a temperature range well below T_c in which mean-field SC order parameter $\Delta_c(\boldsymbol{k})$, relating to the $\boldsymbol{K}$ channel, can be approximately considered as governed by the only self-consistent equation,

$$\Delta_c(\boldsymbol{k}) = -\frac{1}{2}\sum_{\boldsymbol{k}'} \frac{w(\boldsymbol{k},\boldsymbol{k}')\Delta_c(\boldsymbol{k}')}{E(\boldsymbol{k}')} \tanh\left(\frac{E(\boldsymbol{k}')}{2T}\right). \quad (29)$$

Quasiparticle energy,

$$E(\boldsymbol{k}) = \sqrt{\xi_K^2(\boldsymbol{k}) + |\Delta_c(\boldsymbol{k})|^2 + |\Delta_p(\boldsymbol{k})|^2}, \quad (30)$$

aside from $\Delta_c(\boldsymbol{k})$, includes a strong pseudogap parameter, $\Delta_p(\boldsymbol{k})$, associated with QSS that can exist above T_c. Thus, Eq. (29) reflects the fact that SC order arises from a state other than the normal Fermi liquid. Therefore, Eq. (29) differs from the conventional BCS self-consistency equation. Pseudogap parameter $\Delta_p(\boldsymbol{k})$ is small above optimum doping and gradually increases with underdoping. This leads to smoothing of the singularity in Eq. (29) and, as a result, to a gradual decrease in T_c with underdoping.

One can reduce the summation in Eq. (29) to a small part Ξ_s of the domain of kinematic constraint similar to that in the two-particle problem considered above. Under repulsive interaction, a nontrivial solution to Eq. (29) can arise due to a competition of positive and negative contributions to the right-hand side of this equation. Such a competition determines the effective coupling constant and is strongly affected by electron-hole asymmetry.[45] Thus, $\Delta_c(\boldsymbol{k})$, arising due to repulsive pairing interaction, should have a line of zeroes [nodal line (NL)] inside Ξ_s.

In the case of phonon-mediated pairing with account of Coulomb repulsion,[39] the NL of the order parameter coincides with the boundary enclosing the domain of dynamic constraint. In this domain, attraction dominates logarithmically weakened repulsion in accordance with Eq. (17). This NL is disposed everywhere outside of the FC; therefore, mirror nesting of the FC can be considered as the only condition of the $\boldsymbol{K}$ pairing under attraction.

Peierls enhancement of the pairing interaction results in a strong anisotropy of the NL disposed close to the FC inside Ξ_s. This corresponds to an increase in the dominant part of Ξ_s that mainly contributes to the logarithmic singularity in the self-consistency equation and, thus, increases by a magnitude of Δ_c (it is clear that $|\Delta_c|$ should be much lesser than ε_0).

Since the SC order parameter arising in the $\boldsymbol{K}$ channel is essentially momentum dependent, there are three characteristic lines of zeroes: (1) the PFC on which kinetic energy of the pair equals zero, $2\xi_K(\boldsymbol{k})=0$; (2) the NL of the order parameter determined by $\Delta_c(\boldsymbol{k})=0$; and (3) the curve on which quasiparticle group velocity changes sign, $\nabla_{\boldsymbol{k}}E(\boldsymbol{k})=0$.

These three lines may have common points of intersection inside Ξ_s; therefore, the NL can be disposed both above and below the PFC. This results in a qualitatively different nonmonotonic momentum dependence of coherence factors $u_\pm^2(\boldsymbol{k})$ for two kinds of directions in $\boldsymbol{k}$ space intersecting at first the PFC and then the NL, and vice versa. Under pairing repulsion, the scattering across the NL turns out to be dominating in comparison with scattering inside or outside the NL, in accordance with the Suhl inequality Eq. (16).

Due to the fact that $|\Delta_p|^2 \neq 0$ in the strong pseudogap state, coherence factors in the diagonal Gor'kov function (28) may overlap each other near the PFC even above T_c. On the contrary, the BCS coherence factors are stepwise functions without an overlap in the normal Fermi-liquid state.

The SC state that arises below T_c should be described by both diagonal and off-diagonal (anomalous) Gor'kov functions. Taking into account the fact that mean-field (averaged over random phases) pseudogap parameter Δ_p vanishes whereas mean-field SC condensate parameter $\Delta_c \neq 0$ below T_c, one can introduce an off-diagonal Gor'kov function $F^+(\omega;\boldsymbol{k})$ in a phenomenological way similar to what we used to obtain the diagonal Gor'kov function (28). This function describes the ODLRO state[47] and can be written as

$$F^+(\omega;\boldsymbol{k}) = -\frac{z_{\boldsymbol{k}}\Delta_c^*}{[\omega - E(\boldsymbol{k}) + i\Gamma][\omega + E(\boldsymbol{k}) - i\Gamma]}. \quad (31)$$

Factor $z_{\boldsymbol{k}}$ is defined inside each of the crystal equivalent domains of kinematic constraint Ξ_j, where $j=1$, 2, 3, and 4 in the case of tetragonal symmetry of cuprate planes. Paired states with large total momenta $\boldsymbol{K}_j$, both coherent and noncoherent, arise exactly inside these domains. Parameters Δ_{cj}, Δ_{pj}, and Γ_j are identical for any of Ξ_j differing only by the domain of definition of $\boldsymbol{k}$.

In the whole of the Brillouin zone, the SC order parameter in the mixed representation can be presented as a superposition,

$$\Delta_c(\boldsymbol{R},\boldsymbol{k}) = \sum_{j=1}^{4} \gamma_j(\boldsymbol{k}) e^{i\boldsymbol{K}_j\boldsymbol{R}} \Delta_{cj}(\boldsymbol{k}), \tag{32}$$

where $\boldsymbol{R}$ is center-of-mass radius vector, $\Delta_{cj}(\boldsymbol{R},\boldsymbol{k})$ is the order parameter defined inside the jth domain of the kinematic constraint, and coefficients $\gamma_j(\boldsymbol{k})$ should be chosen in accordance with the symmetry of the order parameter.

As a function of $\boldsymbol{R}$, the order parameter (32) arising in the $\boldsymbol{K}$ channel turns out to be spatially modulated similar to the FFLO state.[42,43] Such a modulation with relatively short wavelength reflects a variation of $\boldsymbol{K}$-pair density and can be associated with a pair density wave with a checkerboard order in cuprate planes. Thus, the $\boldsymbol{K}$ pairing leads to a microscopic ground of the pair density wave concept introduced phenomenologically within the SO(5) theory by Zhang.[16,17]

The period of a checkerboard structure should be determined by pair momentum $\boldsymbol{K}$ and the symmetry of the cuprate plane. Because the pair momentum is, generally speaking, incommensurate, there are some possibilities to form commensurate structures with various periodicity as follows, for example, from the SO(5) theory.[16,17] In our view, the period close to $4a$ turns out to be more probable in the cuprates because of their special dispersion with mirror nesting of the Fermi contour at $\boldsymbol{K} \approx \pi/2a$, being only one of the possible among observable commensurate checkerboard structures.[48]

Even eigenfunctions of the pairing interaction kernel Eq. (9) form an intrinsic basis suitable to expand the order parameter over these functions,[32]

$$\Delta_c(\boldsymbol{R},\boldsymbol{k}) = \sum_s \Delta_s(\boldsymbol{R}) \varphi_s(\boldsymbol{k}). \tag{33}$$

In the simplest case, repulsion-induced SC pairing can be described by a two-component order parameter, so that two complex components, $\Delta_s(\boldsymbol{R})$, $s=1$ and 2 form the order parameter structure in the framework of the Ginzburg-Landau phenomenology. Absolute values and the relative phase of the components are connected with the relative motion of the $\boldsymbol{K}$ pair.

As follows from the Ginzburg-Landau equation system for a two-component order parameter,[49] two qualitatively different SC states become admissible. One of them corresponds to a constant value π of the relative phase of the components that is generic for repulsion-induced superconductivity. The other state, with the relative phase different from π, can be related to a change of the phase of the wave function of the $\boldsymbol{K}$ pair due to a rise of the internal magnetic field of spontaneous orbital currents. Such currents can be associated[49] with insulating orbital antiferromagnetic order, for example, in the form of d-density wave (DDW).[20]

V. BIORDERED SUPERCONDUCTING STATE

Superposition (32) mixes, in particular, two paired states with opposite momenta, $\boldsymbol{K}$ and $-\boldsymbol{K}$. It is clear that particles composing pairs with these momenta can compose pairs with zero momentum. In such a way, the Cooper channel is associated with the $\boldsymbol{K}$ channel in accordance with symmetry consideration.

In such a case, mean-field order parameters $\Delta_0(\boldsymbol{k})$ and $\Delta_c(\boldsymbol{k})$ corresponding to the Cooper and $\boldsymbol{K}$ channels, respectively, should be the solution to a self-consistency equation system. This system degenerates into two independent equations (each of them determines one of the order parameters) if one neglects the interconnection of the channels. Then, one would obtain temperatures, T_c and T_c', of transitions into the states with order parameters $\Delta_c(\boldsymbol{k})$ and $\Delta_0(\boldsymbol{k})$, respectively. Let us assume that $T_c' < T_c$ even in the case when attractive phonon-mediated pairing contributes to the Cooper channel. Then, SC transition temperature T_c can be obtained directly from Eq. (29).

At $T_c' < T < T_c$, there arises a SC order due to $\boldsymbol{K}$ pairing with order parameter $\Delta_c(\boldsymbol{k})$ defined in relatively small vicinities of the PFC, where factor $z_{\boldsymbol{k}}$ is close to unity. Inside this temperature range, Cooper pairing on the whole of the FC should be induced by $\boldsymbol{K}$ pairing. In this case, the magnitude of $\Delta_0(\boldsymbol{k})$ has to be small in comparison with the magnitude of $\Delta_c(\boldsymbol{k})$.

Thus, biordered SC state arising in such a way should be described by two order parameters, Δ_c and Δ_0, defined in the vicinities of the PFC and the entire FC, respectively. As temperature decreases from T_c down to $T \approx T_c'$, Cooper ordering with order parameter Δ_0 exists as induced by the $\boldsymbol{K}$ channel of SC pairing. In this case, the superfluid density turns out to be approximately proportional to the PFC length. Opening of the Cooper channel at $T \approx T_c'$ leads to a considerable increase in Δ_0 and, as a result, in the superfluid density which becomes proportional to the whole of the FC length at $T \lesssim T_c'$.

In the vicinities of the PFC, two branches ($m=1$ and 2) of a strong anisotropic quasiparticle spectrum of the biordered superconductor can be written in the form

$$E_m(\boldsymbol{k}) = \sqrt{\xi_K^2(\boldsymbol{k}) + |\Delta_p(\boldsymbol{k})|^2 + |\Delta_c(\boldsymbol{k}) \pm \Delta_0(\boldsymbol{k})|^2}. \tag{34}$$

Here, we take into account the fact that the kinetic energy of $\boldsymbol{K}$ pair (5) is equal to the kinetic energy of the Cooper pair,

$$2\xi_0(\boldsymbol{k}) = \varepsilon(\boldsymbol{K}/2 + \boldsymbol{k}) + \varepsilon(-\boldsymbol{K}/2 - \boldsymbol{k}) - 2\mu. \tag{35}$$

Two-gap spectrum (34), with the lesser gap $|\Delta_c - \Delta_0|$ observable at excitation energies up to the greater gap $|\Delta_c + \Delta_0|$, should be apparent at $T \lesssim T_c'$. Above the greater gap, the spectral weight transfers from the low- to high-energy branch of the quasiparticle spectrum.

Diagonal and off-diagonal Gor'kov functions of the biordered SC state preserve their form, Eqs. (28) and (31), respectively, with the exception of the fact that the SC order parameter Δ_c has to take into account both SC pairing channels. Thus, these two channels result in two coexisting ODLRO states.

Most likely, Cooper channel, including both Coulomb and phonon-mediated pairings, cannot result in a rise of QSS. Therefore, the SC gap parameter turns out to be BCS-like everywhere on the FC with the exception of the PFC on which the unconventional $\boldsymbol{K}$ channel is opened.

Symmetry of the biordered SC state is determined by Eqs. (32) and (33), where Δ_s should be considered as the component of the momentum-dependent order parameter arising as

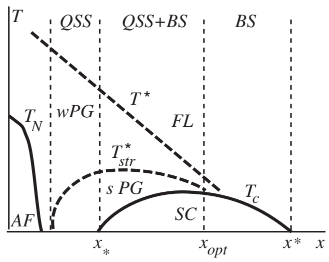

FIG. 4. Schematic phase diagram of underdoped cuprates. Here, T_N and T_c are phase transition temperatures corresponding to Néel AF and unconventional SC orders, respectively; T^*_{str} is strong pseudogap (sPG) crossover temperature and T^* bounds the region of weak pseudogap (wPG) state. High doping and high temperature state corresponds to normal Fermi liquid (FL). On top, there are doping ranges in which there is a rise of QSS, QSS together with BS, and BS.

a result of both Cooper and $\boldsymbol{K}$ pairings. One can approximately represent the gap parameter in the conventional form

$$\Delta(\boldsymbol{k}) = D(\boldsymbol{k})(\cos k_x \pm \cos k_y), \tag{36}$$

where momentum-dependent magnitude $D(\boldsymbol{k})$ reflects a strong anisotropy of the $\boldsymbol{K}$ pairing. The upper (lower) sign in Eq. (36) corresponds to extended $s(d)$-wave symmetry of the order parameter. It should be noted that, in the case of Peierls enhanced $\boldsymbol{K}$ pairing, the nodal line of the order parameter $\Delta(\boldsymbol{k})$ passes through the center of the corresponding domain of the kinematic constraint. This results in the fact that the order parameter $\Delta(\boldsymbol{k})$ differs in sign on opposite parts of the PFC.

VI. CONCLUSION

We believe that biordered superconductivity is generic for such superconductors as doped cuprate compounds. Unconventional features of these compounds, especially universality of their phase diagram (Fig. 4), can be associated with the evolution of the FC and pairing interaction with doping.

It is clear that a singular contribution to Coulomb pairing interaction is sensitive to a doping dependent form of the FC. Therefore, one can suppose that oscillating real-space pairing potential $w(r)$ varies with doping in such a way that only noncoherent QSS of SC pairs arise under extremely low doping. This corresponds to a strong pseudogap penetrating into the insulating region of doping below the onset of superconductivity at $x=x_*$. In the underdoped region $x_*<x<x_{opt}$, along with QSS, there is a bound state. Both bound state energy $|E_i|$ and QSS decay Γ increase with doping so that, near optimal doping x_{opt}, pair-break energy approximately coincides with the energy corresponding to the loss of phase coherence. Thus, in the overdoped regime $x_{opt}<x<x^*$, pairing interaction can result in only a bound state.

As doping increases, the $\boldsymbol{K}$ channel loses a preference over the Cooper channel; therefore, overdoped SC state can manifest properties inherent in conventional BCS state. When doping exceeds x_{opt}, a decrease in T_c down to $T_c=0$ at $x=x^*$ can also be associated with doping dependence of the pairing interaction. This interaction becomes more repulsive at $x>x_{opt}$ as the FC gradually leaves the vicinity of the extended van Hove saddle point.

A suppression of the phonon-mediated component of the SC pairing has the same origin. An effective increase in repulsion can result in the fact that inequality (17) can be reversed because of not too large ratio E_F/ε_D typical of cuprates.

There is rather strong evidence that, in underdoped cuprates, the dimensionless ratio $2\Delta^{(0)}/T_c$ considerably exceeds the universal BCS value of 3.52.[50] Here, $\Delta^{(0)}$ is the SC energy gap extrapolated down to $T=0$. In underdoped biordered superconductor, this parameter should be determined by both Cooper and $\boldsymbol{K}$ pairings; therefore, $\Delta^{(0)} = \sqrt{\Delta^2+\Delta_0^2+\Delta_p^2}$. Taking into account the fact that T_c is determined by $\boldsymbol{K}$ pairing only and Δ^2, Δ_0^2, and Δ_p^2 are, generally speaking, of the same order, one can easily conclude that the ratio $2\Delta^{(0)}/T_c$ may considerably exceed 3.52 (as observed in Ref. 50, values of $2\Delta^{(0)}/T_c \gtrsim 10$). In the overdoped regime, the strong pseudogap parameter $\Delta_p \to 0$ and the Cooper channel dominates $\boldsymbol{K}$ pairing. Therefore, $2\Delta^{(0)}/T_c$ should be close to 3.52 in accordance with the BCS theory.

Superfluid density ρ_s is determined by condensation of $\boldsymbol{K}$ pairs within a broad temperature range below T_c down to the onset of the Cooper channel. Below T_c', superfluid density increases considerably. Conversely, off-condensate particle density decreases. Drude-like behavior of the coherent contribution into optical conductivity $\sigma_1(\omega) \sim \omega^{-2}$, observed below T_c, can be connected with a rather high off-condensate density (experimental data available[51] show that the spectral weight of the off-condensate particles may exceed the spectral weight of the SC condensate below T_c). At $T \lesssim T_c'$, the Drude component of the optical conductivity of the biordered superconductor, $\sigma_1(\omega)$, should be suppressed due to shedding of the off-condensate particles into the condensate of Cooper pairs.

The two-gap excitation spectrum of biordered superconductor is consistent with tunnel conductance measurements[24] and quasilinear temperature dependence of heat capacity, $c_V = \gamma(T)T$.[52] We believe that low-temperature break-junction tunneling observation[24] of a small SC energy gap within the main SC gap provides an indirect evidence in favor of the *third* energy scale besides the *first* one, observable below T_c in the superconducting state, and the *second* one, below T^* in the pseudogap state. Scanning tunneling spectroscopy data,[26] evidencing the first and second scales, show that these scales are almost identical to each other, in full agreement with our conclusion relating to the nature and energy scales of the SC coherent and pseudogap quasistationary states of $\boldsymbol{K}$ pairs.

All of the investigated homologous cuprate series demonstrate universal dependence of T_c on the number of CuO_2 layers in the unitary cell, $T_c(n)$, with maximum at $n=3$.[53] Strong initial increase in $T_c(n)$ cannot be associated with local real-space pairing interaction. Weak interlayer tunneling can explain this feature qualitatively by a rather small effective enhancement of the coupling constant.[54] Coulomb pairing with finite screening length ensures strong correlation between electrons in the nearest-neighbor layers and results in a quantitative explanation of $T_c(n)$, leading to an almost triple increase of the coupling constant.[55]

Doping dependent isotope effect, observed in cuprate superconductors,[56] is highly sensitive to sample quality and reflects the contribution of phonon-mediated component to the SC pairing interaction. Depending on the interrelation between Coulomb and phonon-mediated contributions,[57] the exponent of the isotope effect on T_c can be close to both zero, in the case of dominating repulsion, and BCS limit in the opposite case of dominating phonon-mediated attraction. Relative isotope shift, negligible above optimum doping, increases with underdoping. We believe that this can be considered as an indirect evidence in behalf of the fact that, in the case of low doping, Coulomb correlation effects dominate phonon-mediated contribution to the $\boldsymbol{K}$ channel, which determines T_c in biordered superconductor.

The isotope effect on the London penetration depth λ_L, absent within the BCS theory, also turns out to be enhanced with underdoping.[58] The penetration length is weakly sensitive to isotope substitution in a wide temperature range well below T_c. Then, starting from $T \approx T_c'$, the isotope shift on λ_L increases gradually at $T \to 0$. As $\lambda_L^{-2} \sim \rho_s$, such a behavior of isotope effect on λ_L can be associated with temperature and doping dependence of superfluid density inherent in the biordered SC state.

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