Anomalous Hall effect in two-dimensional systems with magnetization: The roles of electron-impurity and electron-phonon scatterings

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We present a kinetic equation approach to investigate the anomalous Hall effect in two-dimensional systems with magnetization considering both electron-impurity and electron-phonon scatterings. In our study, the spin-orbit interaction due to the external driving electric field as well as the extrinsic spin-orbit couplings induced by electron-impurity and electron-phonon scatterings are taken into account, while the intrinsic Rashba and Dresselhaus spin-orbit couplings are ignored. We derive the side-jump contributions to anomalous Hall current in terms of the distribution function and obtain the skew-scattering contribution by considering electron-impurity (and electron-phonon) scattering up to the second Born approximation. By performing a numerical calculation for InSb-based quantum wells, the temperature dependencies of the various components of anomalous Hall current are examined. We also discuss the roles of electron-impurity and electron-phonon scatterings in contributing to the total anomalous Hall current.

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I. INTRODUCTION

In the emerging field of spintronics, one of the central issues is how to effectively control the spin degree of freedom of carriers. Since the spin-orbit interaction (SOI) provides a promising way to manipulate spin without an external magnetic field, its physics has been extensively studied and many intriguing phenomena induced by SOI, such as spin-Hall effect,[1–7] spin relaxation,[8] etc., have been demonstrated. Depending on its origins, the spin-orbit coupling is usually classified into two categories: intrinsic and extrinsic. The intrinsic SOI arises from the rapid movement of carriers in the ionic field (Dresselhaus SOI[9]) or in a gate field (Rashba SOI),[10] while the extrinsic SOI is induced by the electron-impurity or electron-phonon scattering.

In systems with magnetization, the SOI may lead to an extraordinary contribution to the Hall effect.[11] This so-called anomalous Hall effect (AHE) has been investigated over a long period and has now become a powerful tool for characterizing the properties of ferromagnetic systems.[12,13] In 1954, Karplus and Luttinger proposed, for the first time, an AHE mechanism associated with the spin-orbit interaction due to the nuclear fields.[14] Recently, this intrinsic mechanism has been reformulated by Jungwirth *et al.* within the framework of a momentum-space Berry phase.[15] For the extrinsic spin-orbit coupling induced by electron-impurity scattering, two mechanisms responsible for AHE have been proposed: a side-jump (SJ) process[16] and a skew scattering (SS).[17] The side-jump AHE arises from a sidewise shift of the center of the electron wave packet and relates to an anomalous term in the current operator caused by the extrinsic spin-orbit (SO) coupling.[16,18] The skew-scattering AHE corresponds to an anisotropic enhancement of the wave packet due to electron-impurity scattering, and it can be accounted for by considering the electron-impurity scattering in the second Born approximation (SBA).[19] Recently, considering short-range electron-impurity scattering, Crépieux *et al.* presented a unified derivation of both the side-jump and skew scattering mechanisms within the framework of a formal Dirac equation for the electrons.[20] The effects of weak localization on both the side-jump and skew-scattering contributions to the extrinsic anomalous Hall current have also been examined.[21–24] However, the extrinsic AHE due to the SOI induced by *electron-phonon* scattering has received much less attention. In 1973, Lyo investigated the anomalous Hall effect within a Kubo formalism considering *only* the electron-phonon interaction.[25] It was found that the skew-scattering contribution to the anomalous Hall current is nonvanishing only up to the third Born approximation of the electron-phonon scattering. He also obtained the side-jump anomalous Hall current (AHC) by separating the carrier coordinate into nonlocal and local parts.

In this paper, we propose a kinetic equation approach to investigate the anomalous Hall effect in two-dimensional systems with magnetization considering both the electron-impurity and electron-phonon scatterings. The SOI directly induced by an external driving electric field and the extrinsic spin-orbit couplings associated with both the electron-impurity and the electron-phonon scatterings are taken into account, while the intrinsic Rashba and Dresselhaus spin-orbit couplings are ignored since the Rashba SOI can be controlled by altering the gate voltage and the Dresselhaus SOI in zinc-blende semiconductors may be suppressed by appropriate heterostructure growth protocols.[26] In our study, we derive the side-jump contributions from all these SOI's to anomalous Hall current in terms of a distribution function, while the skew-scattering AHC is obtained by considering electron-impurity scattering up to the second Born approximation. Performing a numerical calculation, we examine the temperature dependencies of the various components of the anomalous Hall current. The effects of electron-impurity and electron-phonon scatterings on AHC are also analyzed.

This paper is organized as follows. In Sec. II, we first derive the components of the side-jump anomalous Hall current contribution in terms of the distribution function and then obtain the kinetic equation for the distribution function considering long-range electron-impurity and electron-phonon scatterings up to the second Born approximation. In Sec. III, we perform a numerical calculation to investigate the anomalous Hall effect in InSb-based quantum wells. Finally, we review our results in Sec. IV.

II. FORMULATION

We consider a two-dimensional electron system in the x-y plane, driven by a uniform in-plane dc electric field $\mathbf{E}$ and by a homogeneous normal magnetization, $\mathbf{M} \equiv (0,0,M)$, which may be introduced by the injection of a spin-polarized electric current, by doping with magnetic impurities like Mn, and/or by photoinduced spin polarization. The Hamiltonian of an electron with momentum $\mathbf{p} \equiv (p_x, p_y) \equiv (p \cos \phi_{\mathbf{p}}, p \sin \phi_{\mathbf{p}})$ can be written as

$$\hat{h} = \hat{h}_0 + \hat{h}_{\rm imp} + \hat{h}_{\rm ph} + \hat{h}_E. \tag{1}$$

$\hat{h}_0$ is the free-electron Hamiltonian given by $\hat{h}_0 = \mathrm{diag}[\varepsilon_1(\mathbf{p}), \varepsilon_2(\mathbf{p})]$, with $\varepsilon_\mu(\mathbf{p}) = \varepsilon_{\mathbf{p}} + (-1)^\mu M$ and $\mu=1$ and 2 (or $\mu = \uparrow$ and $\downarrow$) as spin indices. $\varepsilon_{\mathbf{p}}$ is the dispersion relation of carriers in the absence of magnetization. In our study, $\varepsilon_{\mathbf{p}}$ may be nonparabolic, but it depends only on the magnitude of momentum. The Hamiltonian term $\hat{h}_{\rm imp}$ describes the electron-impurity interaction. It contains both the ordinary scattering potential term and the term related to the extrinsic SO coupling:

$$\hat{h}_{\rm imp} = \sum_i \{V_{\rm imp}(|\mathbf{r} - \mathbf{R}_i|) + \lambda[\boldsymbol{\sigma} \times \boldsymbol{\nabla} V_{\rm imp}(|\mathbf{r} - \mathbf{R}_i|)] \cdot \mathbf{p}\}, \tag{2}$$

where $\mathbf{r}$ and $\mathbf{R}_i$, respectively, are the coordinates of the carrier and impurity, $V_{\rm imp}(r)$ is the electron-impurity scattering potential, $\boldsymbol{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, and λ is the spin-orbit coupling constant depending on the intrinsic semiconductor parameters, such as energy gap E_0, spin-orbit splitting $\Delta_{\rm SO}$, and matrix element of the momentum operator between the conduction and valence bands P: $\lambda = [1/E_0^2 - 1/(E_0 + \Delta_{\rm SO})^2]P^2/3$.[27] $\hat{h}_{\rm ph}$ describes the electron-phonon interaction, and it takes the form

$$\hat{h}_{\rm ph} = \sum_\alpha \{V_{\rm ph}(\mathbf{r}, \alpha) + \lambda[\boldsymbol{\sigma} \times \boldsymbol{\nabla}_{\mathbf{r}} V_{\rm ph}(\mathbf{r}, \alpha)] \cdot \mathbf{p}\}. \tag{3}$$

Here, the electron-phonon scattering potential $V_{\rm ph}(\mathbf{r}, \alpha)$ is given by $V_{\rm ph}(\mathbf{r}, \alpha) = \frac{1}{2}\sum_{\mathbf{q}, q_z}[e^{i\mathbf{q}\cdot\mathbf{r}} M_{\mathbf{q}q_z\alpha} A_{\mathbf{Q}\alpha} + \mathrm{H.c.}]$, with $M_{\mathbf{q}q_z\alpha}$ as the electron-phonon scattering matrix and $A_{\mathbf{Q}\alpha} \equiv a^+_{-\mathbf{Q}\alpha} + a_{\mathbf{Q}\alpha}$ [$a^+_{\mathbf{Q}\alpha}$ and $a_{\mathbf{Q}\alpha}$, respectively, are the creation and annihilation operators for an α-branch three-dimensional phonon with momentum $\mathbf{Q} \equiv (\mathbf{q}, q_z)$]. $\hat{h}_E$ describes the application of the external electric field and, in the Coulomb gauge, it can be written as

$$\hat{h}_E = -e\mathbf{E} \cdot \mathbf{r} - \lambda[\boldsymbol{\sigma} \times \mathbf{E}] \cdot \mathbf{p}. \tag{4}$$

In Eq. (4), we consider the spin-orbit coupling directly induced by the external driving dc electric field. The anomalous Hall current due to this SOI has already been examined in Refs. 28–30.

It should be noted that the spin-orbit coupling terms of $\hat{h}_{\rm imp}$, $\hat{h}_{\rm ph}$, and $\hat{h}_E$ are obtained from a decoupling of conduction and valence band states starting from an 8×8 Kane Hamiltonian in the framework of $\mathbf{k} \cdot \mathbf{p}$ theory.[27] These terms arise predominately from the induced effect of valence bands. Nevertheless, we examine here only the anomalous Hall effect of electrons in the conduction band.

From the Hamiltonian (1), it follows that the single-particle current operator $\hat{\mathbf{j}}(\mathbf{p})$ can be written as

$$\hat{\mathbf{j}}(\mathbf{p}) = \hat{\mathbf{j}}^{\rm f}(\mathbf{p}) + \hat{\mathbf{j}}^{\rm imp}(\mathbf{p}) + \hat{\mathbf{j}}^{\rm ph}(\mathbf{p}) + \hat{\mathbf{j}}^{E}(\mathbf{p}). \tag{5}$$

The term $\hat{\mathbf{j}}^{\rm f}(\mathbf{p})$ arises from the free-electron Hamiltonian $\hat{h}_0$: $\hat{\mathbf{j}}^{\rm f}_\mu(\mathbf{p}) = \mathbf{v}_\mu(\mathbf{p})$, with $\mathbf{v}_\mu(\mathbf{p}) \equiv \boldsymbol{\nabla}_{\mathbf{p}} \varepsilon_\mu(\mathbf{p})$. $\hat{\mathbf{j}}^{\rm imp}(\mathbf{p})$ and $\hat{\mathbf{j}}^{\rm ph}(\mathbf{p})$, respectively, arise from the SO coupling terms of $\hat{h}_{\rm imp}$ and $\hat{h}_{\rm ph}$, and they take the forms

$$\hat{\mathbf{j}}^{\rm imp}(\mathbf{p}) = ie\lambda \sum_{\mathbf{k}, i} V_{\mathbf{p}-\mathbf{k}} e^{i\mathbf{R}_i \cdot (\mathbf{k}-\mathbf{p})} [(\mathbf{k} - \mathbf{p}) \times \boldsymbol{\sigma}] \tag{6}$$

and

$$\hat{\mathbf{j}}^{\rm ph}(\mathbf{p}) = -ie\lambda \sum_{\mathbf{q}q_z\alpha} M_{\mathbf{q}q_z\alpha} A_{\mathbf{Q}\alpha} [\mathbf{q} \times \boldsymbol{\sigma}]. \tag{7}$$

The residual term in Eq. (5), $\hat{\mathbf{j}}^E(\mathbf{p})$, arises from the spin-orbit coupling directly induced by the external driving electric field, and it is given by

$$\hat{\mathbf{j}}^E(\mathbf{p}) = -e^2 \lambda \boldsymbol{\sigma} \times \mathbf{E}. \tag{8}$$

Taking the statistical ensemble average, we find that the observed net current $\mathbf{J}$ also consists of four components:

$$\mathbf{J} = \mathbf{J}^{\rm f} + \mathbf{J}^{\rm imp} + \mathbf{J}^{\rm ph} + \mathbf{J}^E. \tag{9}$$

$\mathbf{J}^{{\rm f},E}$ are determined by $\mathbf{J}^{{\rm f},E} = \sum_{\mathbf{p}} \mathrm{Tr}[\hat{\mathbf{j}}^{{\rm f},E}(\mathbf{p}) \hat{\rho}(\mathbf{p})]$, with trace being over the spin index and $\hat{\rho}(\mathbf{p}) = \langle \hat{\psi}^+_{\mathbf{p}} \hat{\psi}_{\mathbf{p}} \rangle$ as the distribution function ($\hat{\psi}^+_{\mathbf{p}}$ and $\hat{\psi}_{\mathbf{p}}$, respectively, are the carrier creation and annihilation operators). We note that, in systems with weak spin-orbit couplings induced by extrinsic scatterings and external electric field, the off-diagonal elements of the distribution function can be ignored in the linear response regime, and $\hat{\rho}(\mathbf{p})$ is essentially diagonal: $\hat{\rho}(\mathbf{p}) = \mathrm{diag}[\hat{\rho}_{11}(\mathbf{p}), \hat{\rho}_{22}(\mathbf{p})]$. This can be verified from the kinetic equation for the distribution function presented below. In this way, $\mathbf{J}^{{\rm f},E}$ can be further rewritten as

$$\mathbf{J}^{\rm f} = e \sum_{\mathbf{p}, \mu} \mathbf{v}_\mu(\mathbf{p}) \hat{\rho}_{\mu\mu}(\mathbf{p}) \tag{10}$$

and

$$\mathbf{J}^E = e^2 \lambda \sum_{\mathbf{p}} [\mathbf{E} \times \mathbf{n}][\hat{\rho}_{11}(\mathbf{p}) - \hat{\rho}_{22}(\mathbf{p})], \tag{11}$$

with $\mathbf{n}$ as a unit vector along the z axis.

$\mathbf{J}^{\rm imp}$ and $\mathbf{J}^{\rm ph}$, respectively, are associated with the single-particle current operators $\mathbf{j}^{\rm imp}(\mathbf{p})$ and $\mathbf{j}^{\rm ph}(\mathbf{p})$, and they take the forms

$$\mathbf{J}^{\text{imp}} = ie\lambda \sum_{\mathbf{p},\mathbf{k},i} V_{\mathbf{p}-\mathbf{k}} e^{i\mathbf{R}_i\cdot(\mathbf{k}-\mathbf{p})} \operatorname{Tr}\{\langle \hat{\psi}_{\mathbf{p}}^{+}\hat{\psi}_{\mathbf{k}}\rangle[(\mathbf{k}-\mathbf{p})\times\sigma]\} \tag{12}$$

and

$$\mathbf{J}^{\text{ph}} = -ie\lambda \sum_{\mathbf{p},\mathbf{q},q_z,\alpha} M_{\mathbf{q}q_z\alpha} \operatorname{Tr}\{\langle A_{\mathbf{Q}\alpha}\hat{\psi}_{\mathbf{p}}^{+}\hat{\psi}_{\mathbf{p}-\mathbf{q}}\rangle[\mathbf{q}\times\sigma]\}. \tag{13}$$

From Eqs. (12) and (13), we see that to determine $\mathbf{J}^{\text{imp}}$ and $\mathbf{J}^{\text{ph}}$, one has to analyze the functions $\langle\hat{\psi}_{\mathbf{p}}^{+}\hat{\psi}_{\mathbf{k}}\rangle$ and $\langle A_{\mathbf{Q}\alpha}\hat{\psi}_{\mathbf{p}}^{+}\hat{\psi}_{\mathbf{p}-\mathbf{q}}\rangle$. Within the framework of the nonequilibrium Green's function approach, the statistical average $\langle\ldots\rangle$ is related to the statistical average for the noninteracting system via $\langle\hat{B}(t)\rangle \equiv \langle\mathcal{T}_C S_C \hat{B}(t)\rangle_0$ [$\hat{B}(t)$ may be the operator $\hat{\psi}_{\mathbf{p}}^{+}(t)\hat{\psi}_{\mathbf{k}}(t)$ or the operator $A_{\mathbf{Q}\alpha}(t)\hat{\psi}_{\mathbf{p}}^{+}(t)\hat{\psi}_{\mathbf{p}-\mathbf{q}}(t)$], with $\mathcal{T}_C$ as time ordering operator, S_C as the time-loop S matrix, C as a contour running on the real time axis from $-\infty$ to t and back again, and $\langle\,\rangle_0$ representing the thermal average for the noninteracting system. By expanding the S matrix up to the first order of electron-impurity and electron-phonon interactions, $\mathbf{J}^{\text{imp}}$ and $\mathbf{J}^{\text{ph}}$ can be rewritten as

$$\mathbf{J}^{\text{imp}} = -iN_i e\lambda \sum_{\mathbf{p},\mathbf{k}} |V_{\mathbf{p}-\mathbf{k}}|^2 \times \int_C dt' \operatorname{Tr}\{\hat{\mathrm{G}}_{\mathbf{p}}(t',t)[(\mathbf{k}-\mathbf{p})\times\boldsymbol{\sigma}]\hat{\mathrm{G}}_{\mathbf{k}}(t,t')\} \tag{14}$$

and

$$\mathbf{J}^{\text{ph}} = ie\lambda \sum_{\mathbf{p},\mathbf{q},q_z,\alpha} |M_{\mathbf{q}q_z\alpha}|^2 \int_C dt' D_{\mathbf{Q}\alpha}(t,t') \times \operatorname{Tr}\{\hat{\mathrm{G}}_{\mathbf{p}}(t',t)[\mathbf{q}\times\boldsymbol{\sigma}]\hat{\mathrm{G}}_{\mathbf{p}-\mathbf{q}}(t,t')\}, \tag{15}$$

with N_i as the impurity density, and $\hat{\mathrm{G}}_{\mathbf{p}}(t_1,t_2)$ and $D_{\mathbf{Q}\alpha}(t_1,t_2)$ as the carrier and phonon contour-ordered Green's functions, respectively. Note that to obtain Eq. (14), the average over the impurity configuration was taken. Applying the Langreth theorem[31] taken jointly with the generalized Kadanoff-Baym ansatz,[32,33] we finally arrive at the forms of $\mathbf{J}^{\text{imp}}$ and $\mathbf{J}^{\text{ph}}$ in terms of the distribution function:

$$\mathbf{J}^{\text{imp}} = \pi N_i e\lambda \sum_{\mathbf{p},\mathbf{k},\mu} (-1)^{\mu} |V_{\mathbf{p}-\mathbf{k}}|^2 [(\mathbf{k}-\mathbf{p})\times\mathbf{n}] \times \delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{p})][\hat{\rho}_{\mu\mu}(\mathbf{p})-\hat{\rho}_{\mu\mu}(\mathbf{k})], \tag{16}$$

$$\begin{aligned}\mathbf{J}^{\text{ph}} = \pi e\lambda \sum_{\mathbf{p},\mathbf{q},q_z,\mu,\alpha,\pm} (-1)^{\mu+1} |M_{\mathbf{q}q_z\alpha}|^2 [\mathbf{q}\times\mathbf{n}]\{N_{\mathbf{Q}\alpha}^{\pm}\delta[\varepsilon_\mu(\mathbf{p}) \\ -\varepsilon_\mu(\mathbf{p}-\mathbf{q})\mp\Omega_{\mathbf{Q}\alpha}]\hat{\rho}_{\mu\mu}(\mathbf{p})[1-\hat{\rho}_{\mu\mu}(\mathbf{p}-\mathbf{q})] \\ -N_{\mathbf{Q}\alpha}^{\pm}\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})\pm\Omega_{\mathbf{Q}\alpha}]\hat{\rho}_{\mu\mu}(\mathbf{p}-\mathbf{q}) \\ \times[1-\hat{\rho}_{\mu\mu}(\mathbf{p})]\},\end{aligned} \tag{17}$$

with $\Omega_{\mathbf{Q}\alpha}$ as phonon energy, $N_{\mathbf{Q}\alpha}^{\pm}=N(\Omega_{\mathbf{Q}\alpha})+\frac{1}{2}\pm\frac{1}{2}$, and $N(\Omega_{\mathbf{Q}\alpha})\equiv 1/[\exp(\Omega_{\mathbf{Q}\alpha}/k_BT)-1]$ as thermal phonon distribution (T is lattice temperature). Note that to derive Eqs. (16) and (17), we used the fact that, in our study, the distribution function is diagonal.

In order to carry out the calculation of anomalous Hall current, it is necessary to determine the carrier distribution function. Under homogeneous and steady-state conditions in the quasiclassical regime (in which quantum effects and collisional broadening as well as the intracollisional field effect, etc., are ignored), the distribution function $\hat{\rho}(\mathbf{p})$ obeys a kinetic equation taking the form

$$e\mathbf{E}\cdot\nabla_{\mathbf{p}}\hat{\rho}(\mathbf{p}) - ie\lambda\mathbf{E}\cdot(\mathbf{p}\times\mathbf{n})[\sigma_z,\hat{\rho}(\mathbf{p})] + i[\hat{h}_0,\hat{\rho}(\mathbf{p})] = -\hat{I}. \tag{18}$$

Here, $\hat{I}$ is a scattering term determined by

$$\hat{I} = \int \frac{d\omega}{2\pi}[\hat{\Sigma}^r(\mathbf{p},\omega)\hat{\mathrm{G}}^<(\mathbf{p},\omega) + \hat{\Sigma}^<(\mathbf{p},\omega)\hat{\mathrm{G}}^a(\mathbf{p},\omega) - \hat{\mathrm{G}}^r(\mathbf{p},\omega)\hat{\Sigma}^<(\mathbf{p},\omega) - \hat{\mathrm{G}}^<(\mathbf{p},\omega)\hat{\Sigma}^a(\mathbf{p},\omega)], \tag{19}$$

with $\hat{\mathrm{G}}^{r,a,<}(\mathbf{p},\omega)$ and $\hat{\Sigma}^{r,a,<}(\mathbf{p},\omega)$ as the retarded, advanced and "lesser" nonequilibrium Green's functions and self-energies, respectively. In Eqs. (18) and (19), the electron-impurity and electron-phonon scatterings are embedded in $\hat{\Sigma}^{r,a,<}(\mathbf{p},\omega)$.

In the present paper, we restrict our considerations to the linear response regime. In connection with this, all the functions, such as the nonequilibrium Green's functions, self-energies, and distribution function, can be expressed as sums of two terms: $A=A_0+A_1$, with A representing the Green's functions, self-energies, or distribution function. A_0 and A_1, respectively, are the unperturbed part and the linear electric field part of A. In these terms, the kinetic equation for the linear electric field part of the distribution, $\hat{\rho}_1(\mathbf{p})$, can be written as

$$e\mathbf{E}\cdot\nabla_{\mathbf{p}}\hat{\rho}_0(\mathbf{p}) + i[\hat{h}_0,\hat{\rho}_1(\mathbf{p})] = -\hat{I}^{(1)}, \tag{20}$$

with $\hat{I}^{(1)}$ as the linear electric field part of the collision term $\hat{I}$:

$$\hat{I}^{(1)} = \int \frac{d\omega}{2\pi}[\hat{\Sigma}_1^<(\mathbf{p},\omega)\hat{\mathrm{G}}_0^a(\mathbf{p},\omega) - \hat{\mathrm{G}}_1^<(\mathbf{p},\omega)\hat{\Sigma}_0^a(\mathbf{p},\omega) + \hat{\Sigma}_0^r(\mathbf{p},\omega)\hat{\mathrm{G}}_1^<(\mathbf{p},\omega) - \hat{\mathrm{G}}_0^r(\mathbf{p},\omega)\hat{\Sigma}_1^<(\mathbf{p},\omega)]. \tag{21}$$

Note that the second term on the left-hand side of Eq. (18) comes from the SOI induced directly by an external electric field. This term is eliminated in Eq. (20) in the linear response regime since the equilibrium distribution $\hat{\rho}_0(\mathbf{p}) \equiv \mathrm{diag}[n_F(\varepsilon_1(\mathbf{p})),n_F(\varepsilon_2(\mathbf{p}))]$ [$n_F(\omega)$ is the Fermi function] is diagonal and $[\sigma_z,\hat{\rho}_0(\mathbf{p})]=0$. Hence, in our study, the effect of SOI induced directly by an external electric field on the distribution function vanishes.

From Eq. (20), it is clear that the driving-force terms of the kinetic equation for $\hat{\rho}_1(\mathbf{p})$ are diagonal in the linear response regime. Consequently, the linear electric field part of the distribution $\hat{\rho}_1(\mathbf{p})$ is also diagonal. Physically, the off-diagonal elements of $\hat{\rho}_1(\mathbf{p})$ remain finite only during a transverse relaxation time, and they tend to vanish in the steady state if the off-diagonal driving forces vanish. Accordingly, the kinetic equation reduces to

$$e\mathbf{E}\cdot\nabla_{\mathbf{p}}\hat{\rho}_0(\mathbf{p}) = -\hat{I}^{(1)}, \tag{22}$$

since $[\hat{h}_0,\hat{\rho}_1(\mathbf{p})]$, standing on the left-hand side of Eq. (20), contains only off-diagonal elements of the distribution function and, hence, it vanishes.

To further simplify Eq. (22), we employ a two-band generalized Kadanoff-Baym ansatz (GKBA).[32,33] This ansatz, which expresses the lesser Green's function through the Wigner distribution function, has been proven sufficiently accurate to analyze transport and optical properties in semiconductors.[31] To first order in the dc field strength, the GKBA reads

$$\hat{G}_1^<(\mathbf{p},\omega) = -\hat{G}_0^r(\mathbf{p},\omega)\hat{\rho}_1(\mathbf{p}) + \hat{\rho}_1(\mathbf{p})\hat{G}_0^a(\mathbf{p},\omega), \tag{23}$$

where the retarded and advanced Green's functions are diagonal matrices: $\hat{G}_0^{r,a}(\mathbf{p},\omega)=\mathrm{diag}[(\omega-\varepsilon_1(\mathbf{p})\pm i\delta)^{-1},(\omega-\varepsilon_2(\mathbf{p})\pm i\delta)^{-1}]$. Note that, in our treatment, the effect of $\hat{G}_1^{r,a}(\mathbf{p},\omega)$ on the distribution function has been ignored because these linear electric parts of the retarded and advanced Green's functions lead to a collisional broadening effect on $\hat{\rho}_1(\mathbf{p})$, which plays a secondary role in transport studies.[31]

Further, as in all previous studies of extrinsic AHE, we consider the anomalous Hall current only to the first order of the spin-orbit coupling constant, λ. Thus, the scattering term $\hat{I}^{(1)}$ may be considered only in the lowest and first orders of λ. On the other hand, to account for the skew-scattering contribution to anomalous Hall current, the scattering term $\hat{I}^{(1)}$, and hence the self-energies $\hat{\Sigma}_1^{r,a,<}(\mathbf{p},\omega)\equiv\hat{\Sigma}_{1\mathrm{imp}}^{r,a,<}(\mathbf{p},\omega)+\hat{\Sigma}_{1\mathrm{ph}}^{r,a,<}(\mathbf{p},\omega)$, should be considered up to the second Born approximation of the electron-impurity and electron-phonon scatterings. Such an electron-impurity scattering part of the self-energy, $\Sigma_{1\mathrm{imp}}^{r,a,<}(\mathbf{p},\omega)$, has already been studied by Liu *et al.* using generalized T matrices.[30] To evaluate the electron-phonon interaction part of the self-energies, $\hat{\Sigma}_{1\mathrm{ph}}^{r,a,<}(\mathbf{p},\omega)$, one has to analyze the operators $\hat{h}_{\mathrm{ph}}(t_1)\hat{h}_{\mathrm{ph}}(t_2)$ and $\hat{h}_{\mathrm{ph}}(t_1)\hat{h}_{\mathrm{ph}}(t_2)\hat{h}_{\mathrm{ph}}(t_3)$. However, here, we study the electron-phonon scattering caused by *equilibrium* phonons. Hence, the operator $\hat{h}_{\mathrm{ph}}(t_1)\hat{h}_{\mathrm{ph}}(t_2)\hat{h}_{\mathrm{ph}}(t_3)$, containing $A_{\mathbf{Q}\alpha}(t_1)A_{\mathbf{Q}\alpha}(t_2)A_{\mathbf{Q}\alpha}(t_3)$, vanishes after taking the statistical average. Thus, the skew-scattering contribution due to electron-phonon interaction vanishes in the SBA. As a result, $\hat{I}^{(1)}$ can be written as

$$\hat{I}^{(1)} = \hat{I}_{\mathrm{imp}}^{(1)} + \hat{I}_{\mathrm{ph}}^{(1)}, \tag{24}$$

with $\hat{I}_{\mathrm{imp}}^{(1)}$ and $\hat{I}_{\mathrm{ph}}^{(1)}$ arising from the electron-impurity and electron-phonon scatterings, respectively. $\hat{I}_{\mathrm{imp}}^{(1)}$ can be further expressed as a sum of two terms: $\hat{I}_{\mathrm{imp}}^{(1)}=(\hat{I}_{\mathrm{imp}}^{(1)})^{\mathrm{FBA}}+(\hat{I}_{\mathrm{imp}}^{(1)})^{\mathrm{SBA}}$, with the first-Born-approximation term, $(I_{\mathrm{imp}}^{(1)})^{\mathrm{FBA}}$, and the SBA term, $(I_{\mathrm{imp}}^{(1)})^{\mathrm{SBA}}$, taking the forms

$$(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{FBA}}[\hat{\rho}_1] = 2\pi N_i\sum_{\mathbf{k}}|V(\mathbf{p}-\mathbf{k})|^2\delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{p})] \times[(\hat{\rho}_1)_{\mu\mu}(\mathbf{p})-(\hat{\rho}_1)_{\mu\mu}(\mathbf{k})] \tag{25}$$

and

$$(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{SBA}}[\hat{\rho}_1] = (-1)^{\mu+1}4\pi^2\lambda N_i\sum_{\mathbf{k},\mathbf{q}}V_{\mathbf{p}-\mathbf{k}}V_{\mathbf{k}-\mathbf{q}}V_{\mathbf{q}-\mathbf{p}}\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]\delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{q})][pq\sin(\phi_{\mathbf{p}}-\phi_{\mathbf{q}})+qk\sin(\phi_{\mathbf{q}}-\phi_{\mathbf{k}})+kp\sin(\phi_{\mathbf{k}}-\phi_{\mathbf{p}})]\times(\hat{\rho}_1)_{\mu\mu}(\mathbf{k}), \tag{26}$$

while $\hat{I}_{\mathrm{ph}}^{(1)}$ is given by

$$(\hat{I}_{\mathrm{ph}}^{(1)})_{\mu\mu}[\hat{\rho}_1] = 2\pi\sum_{\mathbf{q},q_z,\alpha}|M_{\mathbf{q}q_z\alpha}|^2\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})-\Omega_{\mathbf{Q}\alpha}]\times[\{N(\Omega_{\mathbf{Q}\alpha})+1-n_F[\varepsilon_\mu(\mathbf{p}-\mathbf{q})]\}(\hat{\rho}_1)_{\mu\mu}(\mathbf{p})-\{N(\Omega_{\mathbf{Q}\alpha})+n_F[\varepsilon_\mu(\mathbf{p})]\}(\hat{\rho}_1)_{\mu\mu}(\mathbf{p}-\mathbf{q})]-\{\mathbf{p}\leftrightarrow\mathbf{p}-\mathbf{q}\}. \tag{27}$$

In Eqs. (25)–(27), $[\hat{\rho}_1]$ standing on the left-hand sides denotes that $(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{FBA}}$, $(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{SBA}}$, and $(\hat{I}_{\mathrm{ph}}^{(1)})_{\mu\mu}$ depend on the specific form of the distribution function $\hat{\rho}_1$. Note that the off-diagonal elements of $\hat{I}^{(1)}$ vanish, since, in our study, the distribution function is essentially diagonal.

Up to first order of the spin-orbit coupling constant, we can assume that the solution of Eq. (22), $\hat{\rho}_1(\mathbf{p})$, takes the form $\hat{\rho}_1(\mathbf{p})=\hat{\mathcal{R}}_0(\mathbf{p})+\hat{\mathcal{R}}_1(\mathbf{p})$, with $\hat{\mathcal{R}}_0(\mathbf{p})$ and $\hat{\mathcal{R}}_1(\mathbf{p})$ as the lowest- and first-order terms of $\hat{\rho}_1(\mathbf{p})$ in the λ expansion. Substituting $\hat{\rho}_1(\mathbf{p})$ into the scattering term, we find that the lowest-order distribution function $\hat{\mathcal{R}}_0(\mathbf{p})$ satisfies the equation

$$e\mathbf{E}\cdot\nabla_{\mathbf{p}}\hat{\rho}_{0\mu\mu}(\mathbf{p}) = -(\hat{I}_{\mathrm{imp}}^{(1)})_\mu^{\mathrm{FBA}}[\hat{\mathcal{R}}_0]-(\hat{I}_{\mathrm{ph}}^{(1)})_\mu[\hat{\mathcal{R}}_0], \tag{28}$$

while the first-order term $\hat{\mathcal{R}}_1(\mathbf{p})$ is related to the leading term $\hat{\mathcal{R}}_0(\mathbf{p})$ by

$$(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{FBA}}[\hat{\mathcal{R}}_1]+(\hat{I}_{\mathrm{ph}}^{(1)})_{\mu\mu}[\hat{\mathcal{R}}_1]+(\hat{I}_{\mathrm{imp}}^{(1)})_{\mu\mu}^{\mathrm{SBA}}[\hat{\mathcal{R}}_0]=0. \tag{29}$$

In the present paper, we assume that the energy of free carriers depends only on the magnitude of momentum and it is independent of the direction of momentum. In connection with this, the momentum-angle dependencies of $\hat{\mathcal{R}}_0(\mathbf{p})$ and $\hat{\mathcal{R}}_1(\mathbf{p})$ can be evaluated explicitly. From Eq. (28), it follows that the elements of $\hat{\mathcal{R}}_0(\mathbf{p})$, $[\hat{\mathcal{R}}_0]_{\mu\mu}(\mathbf{p})$, can be written as

$$[\hat{\mathcal{R}}_0]_{\mu\mu}(\mathbf{p}) = -e\mathbf{E}\cdot\frac{\mathbf{v}_\mu(\mathbf{p})}{v_\mu(\mathbf{p})}n_F'[\varepsilon_\mu(\mathbf{p})]\Lambda_\mu[\varepsilon_\mu(\mathbf{p})], \tag{30}$$

with $n_F'(x)$ as the derivative of the Fermi function and $\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]$ given by

$$v_\mu(\mathbf{p})n_F'[\varepsilon_\mu(\mathbf{p})] = K_\mu[\Lambda_\mu]. \tag{31}$$

Here, $K_\mu[\Lambda_\mu]$ takes the form

$$K_\mu[\Lambda_\mu]=\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]\frac{n_F'[\varepsilon_\mu(\mathbf{p})]}{\tau_\mu^{\mathrm{imp}}(p)}-\frac{2\pi}{k_BT}\sum_{\mathbf{q}q_z\alpha}|M_{\mathbf{q}q_z\alpha}|^2N(\Omega_{\mathbf{Q}\alpha})\{\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})-\Omega_{\mathbf{Q}\alpha}](\hat{\rho}_0)_{\mu\mu}(\mathbf{p}-\mathbf{q})[1-(\hat{\rho}_0)_{\mu\mu}(\mathbf{p})]$$
$$+\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})+\Omega_{\mathbf{Q}\alpha}](\hat{\rho}_0)_{\mu\mu}(\mathbf{p})[1-(\hat{\rho}_0)_{\mu\mu}(\mathbf{p}-\mathbf{q})]\}\{\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]-\cos(\phi_{\mathbf{p}-\mathbf{q}}-\phi_{\mathbf{p}})\Lambda_\mu[\varepsilon_\mu(\mathbf{p}-\mathbf{q})]\}, \quad (32)$$

$\tau_\mu^{\mathrm{imp}}(p)$ is the relaxation time due to electron-impurity scattering: $[\tau_\mu^{\mathrm{imp}}(p)]^{-1}\equiv 2\pi N_i\Sigma_{\mathbf{k}}|V(\mathbf{p}-\mathbf{k})|^2[1-\cos(\phi_{\mathbf{p}}-\phi_{\mathbf{k}})]\delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{p})]$, and $\phi_{\mathbf{p}-\mathbf{q}}$ is the angle of momentum $\mathbf{p}-\mathbf{q}$. Substituting $\hat{\mathcal{R}}_0(\mathbf{p})$ into Eq. (29), we find that $\hat{\mathcal{R}}_1(\mathbf{p})$ can be written as

$$[\hat{\mathcal{R}}_1]_{\mu\mu}(\mathbf{p})=-e\mathbf{n}\cdot\mathbf{E}\times\frac{\mathbf{v}_\mu(\mathbf{p})}{v_\mu(\mathbf{p})}n_F'[\varepsilon_\mu(\mathbf{p})]\Phi_\mu[\varepsilon_\mu(\mathbf{p})], \quad (33)$$

with $\Phi_\mu[\varepsilon_\mu(\mathbf{p})]$ determined by

$$K_\mu[\Phi_\mu]+(-1)^{\mu+1}4\pi^2\lambda N_i n_F'[\varepsilon_\mu(\mathbf{p})]\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]\sum_{\mathbf{k},\mathbf{q}}V_{\mathbf{p}-\mathbf{k}}V_{\mathbf{k}-\mathbf{q}}V_{\mathbf{q}-\mathbf{p}}\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]\delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{q})]$$
$$\times[pq\sin(\phi_{\mathbf{p}}-\phi_{\mathbf{q}})+qk\sin(\phi_{\mathbf{q}}-\phi_{\mathbf{k}})+kp\sin(\phi_{\mathbf{k}}-\phi_{\mathbf{p}})]\sin(\phi_{\mathbf{p}}-\phi_{\mathbf{k}})=0. \quad (34)$$

Obviously, to obtain $\mathcal{R}_0(\mathbf{p})$ and $\mathcal{R}_1(\mathbf{p})$, one has to solve the integral equations (32) and (34).

Substituting Eq. (30) into Eqs. (16) and (17), the currents $\mathbf{J}^{\mathrm{imp}}$ and $\mathbf{J}^{\mathrm{ph}}$ in the first order of λ can be expressed as

$$\mathbf{J}^{\mathrm{imp}}=\frac{\pi}{2}N_ie^2\lambda[\mathbf{E}\times\mathbf{n}]\sum_{\mathbf{p},\mathbf{k},\mu}(-1)^{\mu+1}|V_{\mathbf{p}-\mathbf{k}}|^2\left[(\mathbf{k}-\mathbf{p})\cdot\frac{\mathbf{v}_\mu(\mathbf{p})}{v_\mu(\mathbf{p})}\right]\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]\delta[\varepsilon_\mu(\mathbf{k})-\varepsilon_\mu(\mathbf{p})][1-\cos(\phi_{\mathbf{p}}-\phi_{\mathbf{k}})]n'[\varepsilon_\mu(\mathbf{p})] \quad (35)$$

and

$$\mathbf{J}^{\mathrm{ph}}=\frac{\pi e^2\lambda}{2k_BT}[\mathbf{E}\times\mathbf{n}]\sum_{\mathbf{p},\mathbf{q},q_z\mu\pm\alpha}(-1)^{\mu+1}|M_{\mathbf{q}q_z\alpha}|^2N(\Omega_{\mathbf{Q}\alpha})\{\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})-\Omega_{\mathbf{Q}\alpha}](\hat{\rho}_0)_{\mu\mu}(\mathbf{p}-\mathbf{q})[1-(\hat{\rho}_0)_{\mu\mu}(\mathbf{p})]$$
$$+\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{p}-\mathbf{q})+\Omega_{\mathbf{Q}\alpha}](\hat{\rho}_0)_{\mu\mu}(\mathbf{p})[1-(\hat{\rho}_0)_{\mu\mu}(\mathbf{p}-\mathbf{q})]\}\left\{\left[\mathbf{q}\cdot\frac{\mathbf{v}_\mu(\mathbf{p})}{v_\mu(\mathbf{p})}\right]\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]-\left[\mathbf{q}\cdot\frac{\mathbf{v}_\mu(\mathbf{p}-\mathbf{q})}{v_\mu(\mathbf{p}-\mathbf{q})}\right]\Lambda_\mu[\varepsilon_\mu(\mathbf{p}-\mathbf{q})]\right\}. \quad (36)$$

From Eqs. (35) and (36), it follows that the directions of $\mathbf{J}^{\mathrm{imp}}$ and $\mathbf{J}^{\mathrm{ph}}$ are perpendicular to the directions of the external electric field and of the spin polarization. Actually, these two currents are just two components of the side-jump anomalous Hall current. This can be seen from the fact that the single-particle current operators of $\mathbf{J}^{\mathrm{imp}}$ and $\mathbf{J}^{\mathrm{ph}}$ involve the factor $\mathbf{p}\times\boldsymbol{\sigma}$, which reflects the shift of the electron wave-packet center toward the direction transverse to the driving electric field.

There is another component of the side-jump anomalous Hall current, $\mathbf{J}^E$, which arises from the spin-orbit coupling directly induced by the external electric field. From Eq. (11), we see that, in the linear response regime, $\mathbf{J}^E$ takes the form

$$\mathbf{J}^E=e^2\lambda\sum_{\mathbf{p}}[\mathbf{E}\times\mathbf{n}]\{n_F[\varepsilon_1(\mathbf{p})]-n_F[\varepsilon_2(\mathbf{p})]\}. \quad (37)$$

Substituting Eq. (33) into Eq. (10), we find that $\hat{\mathcal{R}}_1(\mathbf{p})$ also gives rise to a nonvanishing Hall current, $\mathbf{J}^{\mathrm{SS}}$:

$$\mathbf{J}^{\mathrm{SS}}=\frac{e^2}{2}\sum_{\mathbf{p}}[\mathbf{E}\times\mathbf{n}]v_\mu(\mathbf{p})n_F'[\varepsilon_\mu(\mathbf{p})]\Phi_\mu[\varepsilon_\mu(\mathbf{p})]. \quad (38)$$

This component of Hall current arises from an anisotropy of the carrier distribution due to electron-impurity scattering, and hence, it is just a component of skew-scattering AHC.

Obviously, to determine the total side-jump anomalous Hall current, $\mathbf{J}^{\mathrm{SJ}}\equiv\mathbf{J}^{\mathrm{imp}}+\mathbf{J}^{\mathrm{ph}}+\mathbf{J}^E$, one has to first obtain the function $\Lambda_\mu[\varepsilon_\mu(p)]$ from the integral equation (31) and then insert $\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]$ into Eqs. (35) and (36) [$\mathbf{J}^E$ can be obtained directly from Eq. (37) by integration over electron momentum]. The skew-scattering AHC is determined by Eq. (38) after obtaining $\Phi_\mu[\varepsilon_\mu(\mathbf{p})]$ from Eq. (32).

In previous studies, it was found that in the absence of electron-phonon scattering, $\mathbf{J}^{\mathrm{imp}}$ is equal to $\mathbf{J}^E$ for a parabolic dispersion relation.[29,30] Although the form of $\mathbf{J}^{\mathrm{imp}}$ that we have obtained significantly differs from the form of $\mathbf{J}^E$, we can prove that $\mathbf{J}^{\mathrm{imp}}=\mathbf{J}^E$ if the only scattering mechanism is due to the electron-impurity interaction. A detailed proof is presented in the Appendix. We note that the equality of $\mathbf{J}^{\mathrm{imp}}$ and $\mathbf{J}^E$ is also valid for a nonparabolic $\varepsilon_\mu(\mathbf{p})$, provided that $\varepsilon_\mu(\mathbf{p})$ depends only on the magnitude of momentum, so that the delta function $\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]$ is nonvanishing only when $p=k$.

It should be noted that, in our study, the kinetic equation (20) is derived in the Coulomb gauge. Actually, when a gauge transformation is applied, this equation is also valid in the quasiclassical linear response regime: ignoring quantum effects and collisional broadening in the quasiclassical regime leads to vanishing of the extra driving terms in the kinetic equation, which arise from the transformation.[31]

III. NUMERICAL RESULTS AND DISCUSSIONS

We have carried out a numerical calculation to investigate the anomalous Hall effect in InSb-based quantum wells. It is well known that the InSb semiconductor is a good material for AHE observation because its band gap, $E_0=0.235$ eV, spin-orbit splitting, $\Delta_{SO}=0.81$ eV, and $P=9.63$ eV Å result in a pronounced spin-orbit coupling constant $\lambda=5.31$ nm^2 (for GaAs, $\lambda=0.053$ nm^2).[34]

In our numerical calculation, an *attractive* interaction between the electrons and the background impurities in the quantum wells is considered (the attractive and repulsive interactions lead to differing anomalous Hall effects because their contributions to AHC in the second Born approximation have opposite signs). Thus, the scattering potential $V_{\mathbf{q}}$ can be written as[35]

$$V_{\mathbf{q}} = U(q)F(q)/\kappa(q,0), \tag{39}$$

with $U(q)=-e^2/2\varepsilon_0\kappa q$ (κ is the static dielectric constant) and the form factor $F(q)$ determined by ($u=qa$)

$$F(q) = \frac{8\pi^2}{(4\pi^2+u^2)u}\left[1+\frac{u^2}{4\pi^2}-\frac{1-\exp(-u)}{u}\right]. \tag{40}$$

$\kappa(q,0)$ is a static dielectric function and, in the random phase approximation, it takes the form

$$\kappa(q,0) = 1 + \frac{q_s}{q}H(q), \tag{41}$$

with $q_s=m^* e^2/2\pi\epsilon_0\kappa$ (m^* is the effective mass of electrons) and $H(q)$ given by[36]

$$H(q) = 3\frac{1-\exp(-u)}{u^2+4\pi^2} + \frac{u}{u^2+4\pi^2} - \frac{1-\exp(-u)}{(u^2+4\pi^2)^2}(u^2-4\pi^2) + \frac{2}{u}\left[1-\frac{1-\exp(-u)}{u}\right]. \tag{42}$$

In regard to the electron-phonon interaction, the scatterings considered are those due to polar longitudinal optical (LO) phonons via the Fröhlich coupling and also due to the longitudinal acoustic phonons via the deformation potential. The corresponding matrix elements, $|M_{\mathbf{q}q_z,\mathrm{LO}}|^2$ and $|M_{\mathbf{q}q_z,\mathrm{AC}}|^2$, take the forms

$$|M_{\mathbf{q}q_z,\mathrm{LO}}|^2 = \frac{e^2}{2\varepsilon_0(q^2+q_z^2)}\left(\frac{1}{\kappa_\infty}-\frac{1}{\kappa}\right)\Omega_{\mathrm{LO}}|I(iq_z)|^2 \tag{43}$$

and

$$|M_{\mathbf{q}q_z,\mathrm{AC}}|^2 = \frac{\Xi^2(q^2+q_z^2)}{2dv_{sl}}|I(iq_z)|^2, \tag{44}$$

with Ω_{LO} as the energy of LO phonons, k_∞ as the optical dielectric constant, d as the mass density of crystal, v_{sl} as the velocity of longitudinal sound wave, and Ξ as the shift of the band edge per unit dilation. $|I(iq_z)|^2$ is the form factor for electron-phonon scattering and it takes the form[35] ($y=q_z a/2$)

$$|I(iq_z)|^2 = \frac{\pi^4 \sin^2 y}{y^2(y^2-\pi^2)^2}. \tag{45}$$

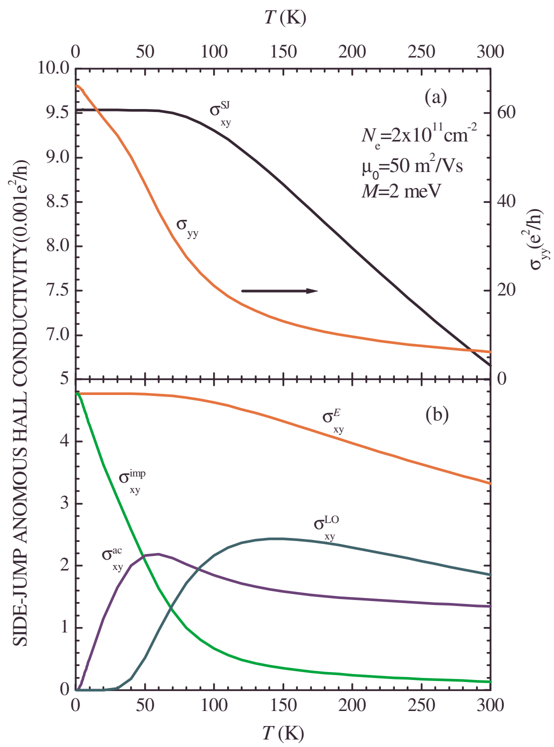

FIG. 1. (Color online) Temperature dependencies of (a) the total side-jump anomalous Hall conductivity, $\sigma_{xy}^{\mathrm{SJ}}$, and the longitudinal conductivity, σ_{yy}, and (b) the components of the side-jump Hall conductivity in an InSb-based quantum well with width $a=20$ nm. The electron density is $N_e=2\times10^{15}$ m^{-2}, the magnetization is $M=2$ meV, and the zero-temperature mobility in the absence of magnetization μ_0 is $\mu_0=50$ m^2/V s.

In our calculation, the other parameters are chosen as follows: $\kappa=17.54$, $\kappa_\infty=15.68$, $d=5.775$ g cm^{-3}, $v_{sl}=3.75\times10^3$ m s^{-1}, $\Omega_{\mathrm{LO}}=24.2$ meV, $\Xi=20$ eV, and $m^*=0.0138m_0$.[37] The width of the InSb-based quantum wells is assumed to be $a=20$ nm and the sheet density of electrons is taken as $N_e=2\times10^{15}$ m^{-2}. The impurity density is determined by assuming an electron mobility in the absence of magnetization at $T=0$ K, μ_0. Further, to take account of the nonparabolicity of the energy band of InSb, we use the Kane band model:

$$\varepsilon(1+\alpha\varepsilon) = p^2/2m^*, \tag{46}$$

with $\alpha=1/E_0$ as the nonparabolic parameter. In this calculation, without loss of generality, we specifically study the anomalous Hall current flow along the x direction when the electric field is applied along the y direction: $\mathbf{E}\equiv(0,E_y,0)$.

In Fig. 1, we plot the total side-jump anomalous Hall conductivity, $\sigma_{xy}^{\mathrm{SJ}}\equiv J_x^{\mathrm{SJ}}/E_y$, the longitudinal conductivity, $\sigma_{yy}\equiv J_y/E_y$, and the components of the side-jump Hall conductivity, $\sigma_{xy}^{\mathrm{imp},E}\equiv J_x^{\mathrm{imp},E}/E_y$ and $\sigma_{xy}^{\mathrm{ac,LO}}\equiv J_x^{\mathrm{ac,LO}}/E_y$ ($J_x^{\mathrm{ac,LO}}$ are the side-jump anomalous Hall currents due to the acoustic and LO phonons and their sum is just J_x^{ph}), as functions of temperature for $M=2$ meV and $\mu_0=50$ m^2/V s. We find that as temperature increases, $\sigma_{xy}^{\mathrm{ac}}$ and $\sigma_{xy}^{\mathrm{LO}}$ first increase and then decline. $\sigma_{xy}^{\mathrm{imp}}$ and σ_{xy}^{E} decrease monotonically with increasing

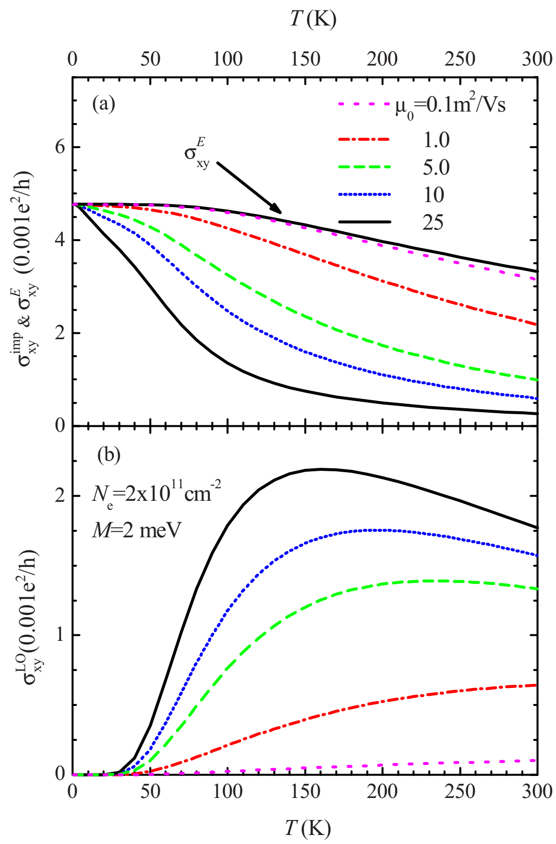

FIG. 2. (Color online) Dependencies of (a) $\sigma_{xy}^{\mathrm{imp}}$ and σ_{xy}^{E}, and (b) $\sigma_{xy}^{\mathrm{LO}}$ on temperature in InSb-based quantum wells for various impurity densities N_i. These N_i correspond to different zero-temperature mobilities in the absence of magnetization: μ_0=0.1, 1, 5, 10, and 25 $\mathrm{m}^2/\mathrm{V\,s}$. The other parameters are the same as in Fig. 1.

temperature. The total side-jump anomalous Hall conductivity $\sigma_{xy}^{\mathrm{SJ}}$ is practically a constant for $T<60$ K due to competition of $\sigma_{xy}^{\mathrm{ac,LO}}$ and $\sigma_{xy}^{\mathrm{imp},E}$, and it then decreases with further temperature increase.

It was known that when the electron-phonon scatterings can be ignored, $\mathbf{J}^{\mathrm{imp}}=\mathbf{J}^{E}$. From Fig. 1, we see that for $\mu_0=50\ \mathrm{m}^2/\mathrm{V\,s}$, $\mathbf{J}^{\mathrm{imp}}$ is approximately equal to $\mathbf{J}^{E}$ only at very low temperature since in this temperature regime the electron-impurity interaction dominates the electron relaxation process. Moving out of this regime, electron-phonon scattering can no longer be ignored and $\mathbf{J}^{\mathrm{imp}}$ is always less than $\mathbf{J}^{E}$ at a given temperature. It is to be expected that $\mathbf{J}^{\mathrm{imp}}$ may tend to coincide with $\mathbf{J}^{E}$ over the entire studied temperature regime, $0\leqslant T\leqslant 300$ K, if μ_0 decreases such that electron-impurity scattering becomes dominant in comparison with the electron-phonon interaction. This is evident in Fig. 2(a), where the temperature dependencies of $\sigma_{xy}^{\mathrm{imp}}$ and σ_{xy}^{E} are plotted for various impurity densities. From Fig. 2(a) we see that, at a given temperature, $\mathbf{J}^{\mathrm{imp}}$ increases with increasing impurity density. For $\mu_0\approx 0.1\ \mathrm{m}^2/\mathrm{V\,s}$, $\mathbf{J}^{\mathrm{imp}}$ and $\mathbf{J}^{E}$ practically are the same.

In Fig. 2(b), we also show $\sigma_{xy}^{\mathrm{LO}}$ as functions of temperature for various impurity densities. We see that, at a given temperature, $\sigma_{xy}^{\mathrm{LO}}$ decreases with increasing N_i. This arises from the fact that $\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]$ decreases with increasing impurity density.

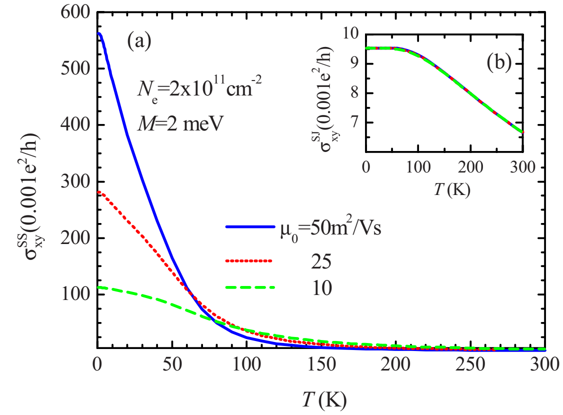

FIG. 3. (Color online) Temperature dependencies of (a) the skew-scattering anomalous Hall conductivity $\sigma_{xy}^{\mathrm{SS}}$ and (b) the total side-jump anomalous Hall conductivity $\sigma_{xy}^{\mathrm{SJ}}$ in InSb-based quantum wells for various impurity densities or zero-temperature mobilities in the absence of magnetization: μ_0=10, 25, and 50 $\mathrm{m}^2/\mathrm{V\,s}$. The other parameters are the same as in Fig. 1.

The temperature dependencies of skew-scattering AHC, $\sigma_{xy}^{\mathrm{SS}}\equiv J_x^{\mathrm{SS}}/E_y$, and total side-jump AHC, $\sigma_{xy}^{\mathrm{SJ}}$, for various impurity densities are also plotted in Fig. 3. We see that as temperature increases, $\sigma_{xy}^{\mathrm{SS}}$ decreases rapidly. With increasing impurity density, $\sigma_{xy}^{\mathrm{SS}}$ decreases at low temperature, but it increases in the relatively high temperature regime. On the other hand, $\sigma_{xy}^{\mathrm{SJ}}$ practically is independent of N_i. For $\mu_0\geqslant 10\ \mathrm{m}^2/\mathrm{V\,s}$, we see that $\sigma_{xy}^{\mathrm{SS}}$ is always much larger than $\sigma_{xy}^{\mathrm{SJ}}$.

In Fig. 4(a), we also plot the temperature dependencies of $\sigma_{xy}^{\mathrm{SJ}}$ and $\sigma_{xy}^{\mathrm{SS}}$ for relatively small μ_0. It is clear that $\sigma_{xy}^{\mathrm{SJ}}$ and $\sigma_{xy}^{\mathrm{SS}}$ are of the same order of magnitude for $\mu_0\sim 1\ \mathrm{m}^2/\mathrm{V\,s}$. In Fig. 4(b), the total anomalous Hall conductivities, $\sigma_{xy}^{\mathrm{AHE}}\equiv\sigma_{xy}^{\mathrm{SJ}}+\sigma_{xy}^{\mathrm{SS}}$, are shown as functions of temperature. We see that with increasing temperature, $\sigma_{xy}^{\mathrm{AHE}}$ decreases. At a given temperature, it increases with μ_0.

Our investigation indicates, in agreement with previous works, that $\sigma_{xy}^{\mathrm{SJ}}$ is important in relatively dirty samples, while

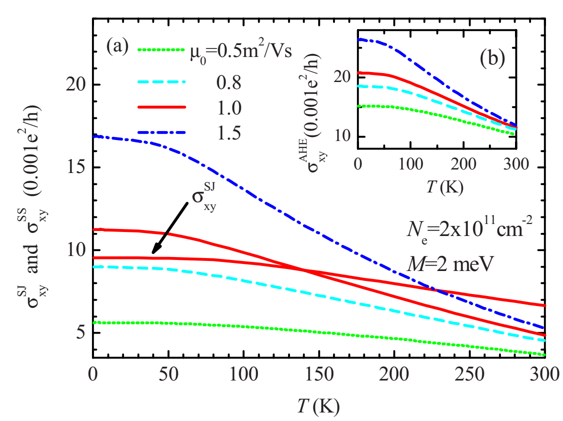

FIG. 4. (Color online) Temperature dependencies of (a) $\sigma_{xy}^{\mathrm{SS}}$ and $\sigma_{xy}^{\mathrm{SJ}}$, and (b) the total anomalous Hall conductivity $\sigma_{xy}^{\mathrm{AHE}}$ for various impurity densities or zero-temperature mobilities in the absence of magnetization: μ_0=0.5, 0.8, 1.0, and 1.5 $\mathrm{m}^2/\mathrm{V\,s}$.

σ_{xy}^{SS} is dominant in systems with high μ_0. Our investigation also makes it clear that the effects of electron-phonon scattering on the total anomalous Hall conductivity are twofold: electron-phonon scattering makes a nonvanishing contribution to side-jump AHC, and it also affects σ_{xy}^{SJ} and σ_{xy}^{SS} through the diagonal distribution function (or by $\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]$).

It should be noted that, in our study, we have considered the SOI directly induced by the driving electric field and the spin-orbit coupling associated with the electron-impurity and electron-phonon scatterings, while the intrinsic spin-orbit couplings, including the Rashba and Dresselhaus spin-orbit interactions, have been ignored. Actually, such intrinsic SOI can also give rise to a nonvanishing contribution to the Hall conductivity.[30,38–40] However, the Rashba SOI can be controlled by altering the gate voltage and the Dresselhaus SOI in zinc-blende semiconductors may be suppressed by appropriate heterostructure growth protocols.[26] In this context, it is meaningful to examine the combined effects of electron-impurity and electron-phonon interactions on AHC while ignoring such intrinsic SOI.

In the present paper, we have investigated the anomalous Hall effect in two-dimensional systems with magnetization, which may be introduced by the injection of spin-polarized electric current, by doping with magnetic impurities like Mn, or by photoinduced spin polarization. In general, the magnetization may also be induced by a magnetic field. However, this magnetic field can also lead to an ordinary Hall conductivity which is much larger than the anomalous Hall conductivity in InSb-based quantum wells: the ordinary Hall conductivity is $34.6e^2/h$ for a magnetic field $B=0.34$ T (in an InSb-based quantum well with effective g factor $g=-51.4$, this magnetic field corresponds to a magnetization $M=1$ meV). Hence, it is difficult to distinguish the anomalous Hall effect from the total Hall conductivity observed in InSb-based quantum wells driven by magnetic fields. In contrast to this, the AHE is expected to be observed in quantum wells based on narrow-band ferromagnetic semiconductors, also in spin-injection structures involving narrow-band semiconductors, and/or in quantum wells based on narrow-band semiconductors with carriers induced by circularly polarized light, where the ordinary Hall conductivity vanishes.

IV. CONCLUSIONS

Considering electron-impurity and electron-phonon scatterings, we have presented a kinetic equation approach to investigate the anomalous Hall effect in two-dimensional systems with magnetization. The spin-orbit coupling induced directly by an external driving electric field, as well as the spin-orbit interaction due to the electron-impurity and electron-phonon scatterings, have been taken into account. The side-jump contributions from these SOI's to anomalous Hall current were expressed in terms of a distribution function, while the skew-scattering AHC was obtained by considering the electron-impurity (and electron-phonon) scattering up to the second Born approximation. Further, we performed a numerical calculation for InSb-based quantum wells to examine the temperature dependencies of the various components of the anomalous Hall current. The combined effects of electron-impurity and electron-phonon scatterings on the total anomalous Hall current have been discussed in detail.

ACKNOWLEDGMENTS

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APPENDIX: PROOF OF THE EQUALITY OF $\mathbf{J}^{\mathrm{imp}}$ AND $\mathbf{J}^{E}$ IN THE ABSENCE OF ELECTRON-PHONON SCATTERINGS

Since the carrier energy $\varepsilon_\mu(\mathbf{p})$ depends only on the magnitude of momentum, the velocity $\mathbf{v}_\mu(\mathbf{p})$ is parallel to momentum $\mathbf{p}$. By using the triangle relation for $\phi_\mathbf{p}-\phi_\mathbf{k}$, $\mathbf{J}^{\mathrm{imp}}$ presented in Eq. (35) can be rewritten as

$$\mathbf{J}^{\mathrm{imp}}=\frac{\pi}{2}N_i e^2\lambda[\mathbf{E}\times\mathbf{n}]\sum_{\mathbf{p},\mathbf{k},\mu}(-1)^\mu|V_{\mathbf{p}-\mathbf{k}}|^2\frac{\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]}{v_\mu(\mathbf{p})}(\mathbf{k}-\mathbf{p})\cdot[\mathbf{v}_\mu(\mathbf{k})-\mathbf{v}_\mu(\mathbf{p})]\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]n'[\varepsilon_\mu(\mathbf{p})]. \tag{A1}$$

The factor $(\mathbf{k}-\mathbf{p})$ can be replaced by $[\cos(\phi_\mathbf{k}-\phi_\mathbf{p})-1]\mathbf{p}$ or by $[1-\cos(\phi_\mathbf{p}-\phi_\mathbf{k})]\mathbf{k}$ since the term associated with $\sin(\phi_\mathbf{k}-\phi_\mathbf{p})$ vanishes after momentum summation, and the delta function standing on the right-hand side of Eq. (A1) ensures the equality of magnitudes of the momenta $\mathbf{p}$ and $\mathbf{k}$. Thus, $\mathbf{J}^{\mathrm{imp}}$ can be further rewritten as

$$\mathbf{J}^{\mathrm{imp}}=\frac{\pi}{2}N_i e^2\lambda[\mathbf{E}\times\mathbf{n}]\sum_{\mathbf{p},\mathbf{k},\mu}(-1)^\mu|V_{\mathbf{p}-\mathbf{k}}|^2[1-\cos(\phi_\mathbf{k}-\phi_\mathbf{p})]\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]\times\left\{\mathbf{k}\cdot\mathbf{v}_\mu(\mathbf{k})n'[\varepsilon_\mu(\mathbf{k})]\frac{\Lambda_\mu[\varepsilon_\mu(\mathbf{k})]}{v_\mu(\mathbf{k})}+\mathbf{p}\cdot\mathbf{v}_\mu(\mathbf{p})n'[\varepsilon_\mu(\mathbf{p})]\frac{\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]}{v_\mu(\mathbf{p})}\right\}. \tag{A2}$$

Here, we use the fact that $n'[\varepsilon_\mu(\mathbf{k})]\frac{\Lambda_\mu[\varepsilon_\mu(\mathbf{k})]}{v_\mu(\mathbf{k})}\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]=n'[\varepsilon_\mu(\mathbf{p})]\frac{\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]}{v_\mu(\mathbf{p})}\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]$. Considering that $\Sigma_\mathbf{p}|V_{\mathbf{p}-\mathbf{k}}|^2[1-\cos(\phi_\mathbf{k}-\phi_\mathbf{p})]\delta[\varepsilon_\mu(\mathbf{p})-\varepsilon_\mu(\mathbf{k})]=1/[2\pi\tau_\mu^{\mathrm{imp}}(k)]$ for the terms involving $\Lambda_\mu[\varepsilon_\mu(\mathbf{k})]$ and $\Lambda_\mu[\varepsilon_\mu(\mathbf{p})]$ in Eq. (A2), we can further simplify $\mathbf{J}^{\mathrm{imp}}$ as

$$\mathbf{J}^{\mathrm{imp}} = \frac{1}{2} N_i e^2 \lambda [\mathbf{E} \times \mathbf{n}] \sum_{\mathbf{p},\mu} (-1)^{\mu} \mathbf{p} \cdot \mathbf{v}_{\mu}(\mathbf{p}) \times n'[\varepsilon_{\mu}(\mathbf{p})] \frac{\Lambda_{\mu}[\varepsilon_{\mu}(\mathbf{p})]}{v_{\mu}(\mathbf{p}) \tau_{\mu}^{\mathrm{imp}}(p)}. \quad \text{(A3)}$$

When the electron-phonon scattering can be ignored, $\Lambda_{\mu}[\varepsilon_{\mu}(\mathbf{k})]$ takes a simple form: $\Lambda_{\mu}[\varepsilon_{\mu}(\mathbf{k})]=v_{\mu}(\mathbf{p})\tau_{\mu}^{\mathrm{imp}}(p)$. Considering $\mathbf{v}_{\mu}(\mathbf{p})n'[\varepsilon_{\mu}(\mathbf{p})]=\boldsymbol{\nabla}_{\mathbf{p}}n[\varepsilon_{\mu}(\mathbf{p})]$, we arrive at

$$\mathbf{J}^{\mathrm{imp}} = \frac{1}{2} N_i e^2 \lambda [\mathbf{E} \times \mathbf{n}] \sum_{\mathbf{p},\mu} (-1)^{\mu} \mathbf{p} \cdot \boldsymbol{\nabla}_{\mathbf{p}} n[\varepsilon_{\mu}(\mathbf{p})]. \quad \text{(A4)}$$

After replacing the momentum summation by an integral and performing the momentum integral by parts, we finally obtain $\mathbf{J}^{\mathrm{imp}}=\mathbf{J}^{E}$.

From Eq. (11), we see that $\mathbf{J}^{E}$ can be rewritten as

$$\mathbf{J}^{E} = e^2 \lambda [\mathbf{E} \times \mathbf{n}] \eta N_e, \quad \text{(A5)}$$

with $\eta \equiv (N_{\uparrow}-N_{\downarrow})/N_e$ as the spin polarization ($N_{\uparrow}$ and $N_{\downarrow}$ are the density of electrons with spins $\uparrow$ and $\downarrow$, respectively). Since $\mathbf{J}^{\mathrm{imp}}$ is equal to $\mathbf{J}^{E}$ in the absence of electron-phonon scattering, the total side-jump anomalous Hall current at zero temperature takes the form

$$\mathbf{J}^{\mathrm{SJ}} = 2e^2 \lambda [\mathbf{E} \times \mathbf{n}] \eta N_e, \quad \text{(A6)}$$

in agreement with that presented in Ref. 29.

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