Origin of critical temperature universal scaling relations in type-II superconductors

B. J. Taylor and M. B. Maple
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92093, USA

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Understanding the intricate relationship amongst fundamental properties of superconductors, in general, is paramount for unraveling the origin of high-temperature superconductivity. An analytical formula for the superconducting transition temperature T_c is derived from the $T \rightarrow 0$ limit of a recently proposed expression for the vortex glass melting line, $H_g(T)$, wherein the energy scale $k_B T_c$ is shown to be proportional to the product of the ground state condensation energy density $H_c^2/2\mu_0$, a Cooper pair coherence volume V_{coh}, and the Lindemann number c_L, which characterizes the stability of the solid vortex state to thermal and quantum fluctuations. This expression provides a fundamental starting point from which the empirical scaling relation $\rho_s \propto \sigma_{dc} T_c$, established by Homes *et al.* [Nature (London) **430**, 539 (2004)], as well as other recent observations, can be derived. Compatibility of the "bad metal" theory of superconductivity of Emery and Kivelson [Nature (London) **374**, 434 (1995)] with the results obtained herein is examined.

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I. INTRODUCTION

As there is to date no agreed upon theory for high-temperature superconductivity, many experimental studies over the past few decades have been focused on identifying universal trends amongst the various physical properties of high-temperature superconductors. Uemura *et al.*[1,2] first observed that the relation

$$T_c \propto \rho_s(0) \propto \lambda_0^{-2} \tag{1}$$

provided a good description of the relationship between the superconducting critical temperature T_c and the superfluid density $\rho_s(0)$ (or penetration depth λ_0) in underdoped layered cuprate superconductors. This simple linear scaling relationship breaks down, however, for optimally and overdoped materials. A few years later, Basov *et al.*[3] established a linear relation between the (*c*-axis) penetration depth and the (*c*-axis) normal-state conductivity at the critical transition temperature, $\sigma_{n,c}(T_c)$, with

$$\lambda_c^{-2} \propto \sigma_{n,c}(T_c). \tag{2}$$

Again, this simple relationship breaks down for overdoped samples.[4] The same relationship given in Eq. (2) was also later found by Pimenov *et al.*[5] between the *ab*-plane penetration depth and the *ab*-plane normal-state conductivity with the same breakdown of the linear correlation in overdoped samples.

Recently, Homes *et al.*[6] demonstrated (within experimental error) a truly universal scaling relationship (referred to as "Homes' law" in Ref. 7) for a number of high-T_c superconducting materials over their entire doping range from underdoped to overdoped, where

$$\rho_s(0) = (120 \pm 25)\sigma_n(T_c)T_c, \tag{3}$$

for both the *a*-*b* plane and *c*-axis conductivities (σ_n is referred to as σ_{dc} in Ref. 4). An interpretation of this scaling relationship was given by the authors for the *ab*-plane data which require the systems to be approaching the clean limit. The authors show that by combining the Drude model with the observation that the scattering rate $1/\tau$ scales linearly with T_c, it follows that $\rho_s(0) \propto \sigma_n(T_c)T_c$. However, a different reasoning was applied to the *c*-axis data in which it is assumed that the superconducting gap maximum $\Delta_0 \propto T_c$ so that the Josephson current density $J_c \propto \Delta_0/R_n \propto T_c/R_n$ ($R_n = d/\sigma_n$; d is the separation between planes). The final scaling relationship is arrived at from the expression for the *c*-axis penetration depth, $\lambda_c^2 \propto 1/J_c$, which is then equivalent to Eq. (3) above.

The above observation of Homes *et al.* has generated considerable debate as to what this universality means. The discussion has centered on the properties of the electrical conductivity σ_n. In particular, Zaanen[7] argues that Homes' law implies that in high-temperature superconductors, the dissipation time scale in the normal state is at the "Planckian limit," i.e., $\tau(T_c) \approx \hbar/k_B T_c$, and, as such, these materials are necessarily quantum critical. This reasoning has been criticized by Phillips and Chamon[8] on the basis that Zaanen's conclusion relies on the Drude formula, which, according to their investigation of the hyperscaling form of the frequency dependent conductivity, has nothing to do with quantum criticality. In spite of the considerable success of the expression above observed by Homes, criticism has also been raised as to whether or not the expression really has anything to do with fundamental universal properties. Tallon *et al.*[9] contend that the universality of Homes' law is coincidental in that many possible different and unrelated properties or phenomena in the various superconducting materials examined can independently exhibit behavior consistent with Eq. (3). Furthermore, in an apparent paradox, Zuev *et al.*[10] have observed for $YBa_2Cu_3O_{6+x}$ films that the superfluid density scales with the critical superconducting temperature as $\rho_s(0) \propto T_c^{2.3\pm0.4}$ and that their data also conform to the empirical scaling form of Homes' law. A further complication to the picture being developed within the context of Homes' law is the observation of a different "universal scaling law" by Pratt and Blundell[11] for molecular superconductors, which also involves $\rho_s(0)$, $\sigma_n(T_c)$, and T_c but includes an additional factor η which characterizes the coupling strength in a BCS-type

system. Clearly, it is important to understand what fundamental properties within the cuprate superconductors (and possibly those of other classes of superconductors, as well) lead to the universal scaling of Homes' law.

The main result of this paper is the *derivation* of a scaling relationship which relates the critical temperature T_c to the energy associated with the suppression of the order parameter by a magnetic field H within a superconducting unit volume V_{coh} and to the stability of the solid vortex state. This expression provides another viewpoint from which to examine the phenomenon of superconductivity. We show that the scaling form of Homes' law, the scaling observed by Zuev *et al.*, and the scaling equation of Pratt and Blundell can *all* be arrived at from our expression. Furthermore, we examine possible implications for the T-x phase diagram of the high-T_c cuprates by inserting an expression for the value of $\sigma_n(T_c)$ from the "bad metal" model of superconductivity proposed by Emery and Kivelson[12,13] into the scaling relationship found herein and then quantitatively comparing the result to Homes' law. However, we emphasize that our expression on its own makes no statement as to the mechanism of superconductivity and as such is not limited in application to the high-temperature superconducting cuprate-based compounds.

II. VORTEX SOLID MELTING LINE AT $T \to 0$, GINZBURG NUMBER, AND COOPER PAIR COHERENCE VOLUME

In two recent communications,[14,15] we developed two similar universal expressions for the vortex lattice (glass) melting line, $H_g(T)$, which were shown to describe the vortex solid melting line data of numerous high-T_c and conventional superconducting systems exceptionally well. These expressions are based upon the incorporation of an empirically observed temperature-field dependence of the relaxation time τ_r^v of a single vortex flux line in the region of the vortex solid melting transition into the two expressions arrived at in the quantum fluctuation based model of Blatter and Ivlev.[16,17] The first expression of Blatter and Ivlev is given by[16]

$$H_g(t) = \frac{4H_{c2}(0)\theta^2}{(1+\sqrt{1+4Q\theta})^2}, \tag{4}$$

where θ is a reduced temperature given by $\theta=(\pi c_L^2/\sqrt{G_i})(1-t)$, $Q=[\tilde{Q}_u/(\pi^2\sqrt{G_i})]\Omega\tau_r$ is a parameter measuring the relative strength of quantum to thermal fluctuations, $t \equiv T/T_c$, $\tilde{Q}_u=\frac{e^2}{\hbar}\frac{\rho_n}{d}$ is the dimensionless quantum of resistance, ρ_n is the normal-state resistivity, c_L is the Lindemann number, $G_i=\frac{1}{2}(k_BT_c/\{8\pi[H_c^2(0)/2\mu_0]\epsilon\xi^3(0)\})^2$ is the Ginzburg number (in mks units), H_c is the thermodynamic critical field, $\epsilon \equiv (m/M)^{1/2}$ is the anisotropy parameter, Ω is a cutoff frequency, and the single vortex relaxation time τ_r^v is *assumed* in their model to be equal to the scattering relaxation time of the quasiparticles in the vortex core given by the Drude formula $\rho_n^{-1}=\sigma_n=e^2n\tau_r/m$, i.e., $\tau_r^v \equiv \tau_r = m\sigma_n/e^2n$.

The second expression, arrived at by the inclusion of compressional modes of the vortex lattice, $c_{11}(\mathbf{k})$, which were not in the initial model, is given by[17]

$$H_g(t) = \frac{4H_{c2}(0)\theta^2}{(1+\sqrt{1+4S\theta/t})^2}, \tag{5}$$

where the reduced temperature θ is now given by $\theta=c_L^2\sqrt{\frac{\beta_{th}}{G_i}}(T_c/T-1)$, $S=q+c_L^2\sqrt{\frac{\beta_{th}}{G_i}}$, $q=\frac{2\sqrt{\beta_{th}}}{\pi^3}\frac{Q_u}{\sqrt{G_i}}\Omega\tau_r$ is the new quantum-thermal fluctuation parameter, and $\beta_{th}\approx 5.6$. It is observed that both of the modified expressions of Eqs. (4) and (5) given in Refs. 14 and 15, respectively, provide equivalently good descriptions of $H_g(T)$ over the entire reduced temperature range of data for all systems examined ($0.03 \leqslant t \leqslant 1$), with the only significant difference being the values extracted for the quantum parameters Q and q, with $q \approx Q/10$. This difference is attributed to the more accurate accounting of thermal fluctuations via the inclusion of compressional modes of the vortex lattice.[14,15] We emphasize that with identical values of the parameters c_L and s (defined below), the two lines given by the modified expressions of Eqs. (4) and (5) nearly coincide (and are experimentally indistinguishable) for all temperatures over which the vortex glass melting line exists, and, in particular, they both extrapolate to the same field value at $T=0$.

In our earlier work, from the empirically determined expression for τ_r^v

$$\tau_r^v = \tau_0\left(\frac{T}{T_c}\right)^s\left(1-\frac{T}{T_c}\right)^{-s}, \tag{6}$$

it is seen that as $T \to 0$, τ_r^v and, in turn, both Q and $q \to 0$. This leads to the following conditions at $T=0$, for Eqs. (4) and (5), respectively:

$$\frac{H_g(0)}{H_{c2}(0)} = \frac{\pi^2 c_L^4}{G_i^{1/2\alpha}} \tag{7}$$

and

$$\frac{H_g(0)}{H_{c2}(0)} = 1, \tag{8}$$

where the exponent α characterizes the approximate form of the melting line at high temperatures, $H_g \sim (1-T/T_c)^\alpha$, via the field dependent expression for the Ginzburg number,[15] $G_i(H) \approx G_i^{1/2\alpha}\left(\frac{H}{H_{c2}(0)}\right)^{1/\alpha}$. Combining these two results, Eqs. (7) and (8), gives

$$G_i = (\pi^2 c_L^4)^{2\alpha}. \tag{9}$$

Before continuing, we return to the definition of the Ginzburg number, $G_i=\frac{1}{2}\{k_BT_c/[(8\pi H_c^2/2\mu_0)V_{coh}]\}^2$, which is a measure of the relative size of the energy density $H_c^2/2\mu_0$ associated with the condensation energy of a unit volume, with respect to the critical temperature, T_c. Combining Eq. (9) and the above definition of the Ginzburg number, a straight forward relationship between the critical temperature T_c and the condensation energy is found, wherein

$$k_BT_c = 8\pi\sqrt{2}(\pi^2 c_L^4)^\alpha \frac{H_c^2}{2\mu_0}V_{coh}. \tag{10}$$

For convenience, we define $\mathbb{C}_L \equiv 8\pi\sqrt{2}(\pi^2 c_L^4)^\alpha$, which is then a measure of the fraction of the condensation energy con-

tained within a unit volume associated with the pairing of a single Cooper pair.

The unit volume is conventionally given as $V_{coh}=\xi_a\xi_b\xi_c$ ($=\epsilon\xi^3$ in most cases). We argue, however, that, while the above unit volume is appropriate for a type-I superconductor, a different geometry applies for a type-II superconductor. A more accurate determination of the unit volume V_{coh} is that over which the order parameter is suppressed within a unit length of a vortex flux line (aligned along the c axis), $V_{\Phi_\ell}\approx\pi r_{rms}^2\xi_c$, where $r_{rms}=\sqrt{\xi_a\xi_b}$. This argument is arrived at by examining the expression for the thermodynamic critical field H_c,

$$H_c=\frac{\Phi_0}{2\sqrt{2}\pi\lambda\xi}=\frac{\Phi_0}{2\sqrt{2}A}, \tag{11}$$

where, in this case, $A=\pi r_{rms}^2$ is the area of a circle given by the length scales λ and ξ with an rms radius $r_{rms}=\sqrt{\lambda\xi}$. We reason that since we are concerned with the expulsion of the flux Φ_0 from within a cross sectional area associated with the energy scale set by the thermodynamic critical field H_c, the same geometric factor, $2\sqrt{2}$, ought to apply so that $V_{\Phi_\ell}=2\sqrt{2}\pi\epsilon\xi^3$.

Using the unit volume V_{ϕ_ℓ}, the Ginzburg number for a type-II superconductor is then

$$G_i=\left(\frac{\mu_0 k_B T_c\lambda_0^2}{2\Phi_0^2\epsilon\xi_0}\right)^2.$$

Returning to Eq. (9), we arrive at the universal expression

$$T_c=\frac{\mathbb{C}_L}{4\sqrt{2}\pi}\left[\frac{\Phi_0^2}{\mu_0 k_B}\right]\frac{\epsilon\xi_0}{\lambda_0^2}\approx(1.39\times10^{-2}\ \text{K m})\mathbb{C}_L\frac{\epsilon\xi_0}{\lambda_0^2}. \tag{12}$$

Using the relation $\xi_0=(\hbar v_F)/(\pi\Delta_0)$, we can rewrite Eq. (12) as

$$T_c=\frac{\mathbb{C}_L}{4\sqrt{2}\pi}\left[\frac{\Phi_0^2}{\mu_0 k_B}\right]\frac{\hbar v_F}{\pi}\frac{\epsilon}{\lambda_0^2\Delta_0}. \tag{13}$$

III. QUANTITATIVE COMPARISON OF CALCULATED AND MEASURED CRITICAL TEMPERATURES

In Fig. 1, we have plotted calculated values of the critical temperature T_c vs the experimentally determined values of T_c for samples of $YBa_2Cu_3O_x$ presented in the work of Tallon *et al.*[18] using Eq. (13) (with $\alpha=3/2$ for all samples) and the data they give for λ_0 and Δ_0. Note that the value of Δ_0 in Ref. 18 is the maximum spectral gap and not the order parameter. The inset to Fig. 1 shows the same calculation but with the Lindemann number allowed to vary such that the calculated and experimental values of T_c are equal. The value of $c_L=0.30$ is in close agreement with the values of 0.31 and 0.28 found for the pure $YBa_2Cu_3O_{7-\delta}$ and $YBa_2Cu_3O_{6.5}$ samples, respectively, in Ref. 14 by fitting the melting line data with the modified vortex glass melting line expression of Blatter and Ivlev. We find that the variation is only $\Delta c_L=0.015$. We could have instead allowed the values of

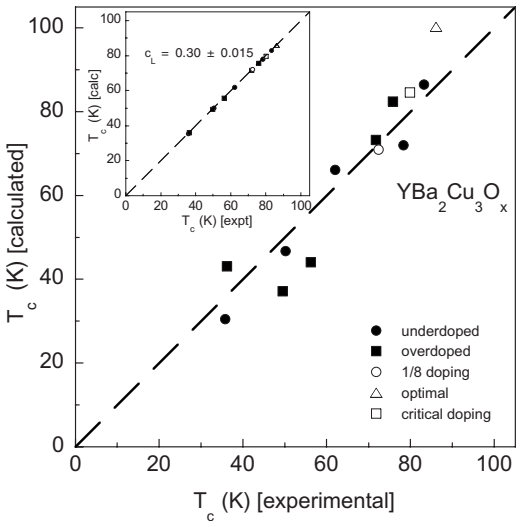

FIG. 1. Calculated value of T_c vs the experimentally determined value of T_c. The value of T_c is calculated from Eq. (13) using the values of Δ_0 and λ_0 from Ref. 18, ϵ from Ref. 19 a constant value of $c_L=0.30$, and a constant value of $v_F=9.3\times10^4$ m/s, derived from the relation $\xi_0=\hbar v_F/\pi\Delta_0$, with $\xi_0(x=6.96)=13$ Å and $\Delta_0(x=6.96)=15$ meV (Ref. 20). The experimentally determined value of T_c is taken from Ref. 18. Inset: The same calculation with c_L allowed to vary for each sample so that the calculated value of T_c is equal to the experimental value. The variation of c_L is found to be $\Delta c_L=\pm0.015$, well within the experimental error, about an average value of $c_L=0.30$.

v_F, λ_0, or Δ_0 to vary individually by ~10% to achieve the same result. In any case, it is demonstrated that Eq. (12) [and Eq. (13)] provides a quantitatively accurate description of the values of T_c in $YBa_2Cu_3O_x$. Further examination of the energies k_BT_c and $H_c^2/2\mu_0$ as a function of doping in Fig. 1 indicates that $\mathbb{C}_L\approx1.0$ in the heavily under- and overdoped regions, with somewhat smaller values down to ≈0.9 in the optimally doped region. This result indicates that the coherence volumes of Cooper pairs, at low temperatures, in $YBa_2Cu_3O_x$ overlap very little throughout the entire doping range.

In Fig. 2, we have plotted the experimentally determined values of k_BT_c and the calculated values of the volume energy $(H_c^2/2\mu_0)V_{\Phi_\ell}$ vs. hole concentration p for the same $YBa_2Cu_3O_x$ samples in Fig. 1 in the main panel and $(H_c^2/2\mu_0)$ and V_{Φ_ℓ} vs hole concentration p in the inset. It is readily seen that the fundamental property to which the superconducting transition temperature T_c or correlation energy k_BT_c is related is the volume energy defined by the thermodynamic critical field H_c and the correlation volume V_{Φ_ℓ}. The final piece of the picture is provided by the Lindemann number which characterizes the stability of the system in the solid vortex state to quantum and thermal fluctuations. It follows that a system in which vortices are more stable to fluctuations will have a higher critical temperature. However, as elaborated upon below, the correlation between the stability of the solid vortex ensemble and the critical temperature T_c is most likely a reflection of the stability of the superfluid condensate to fluctuations.

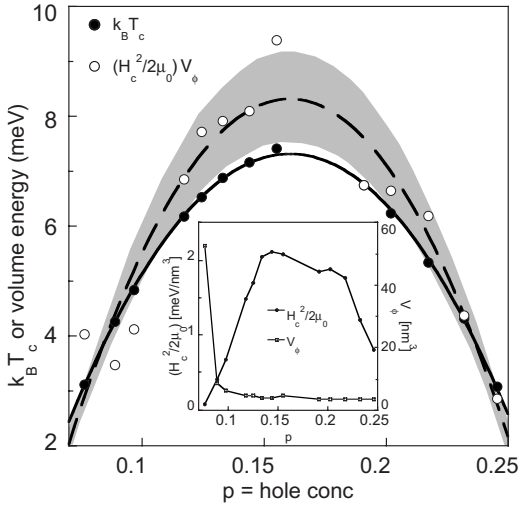

FIG. 2. $k_B T_c$ and volume energy $(H_c^2/2\mu_0)V_{\Phi_\ell}$ vs hole concentration p. H_c is calculated using the same values of λ_0, ξ_0, and ϵ used in Fig. 1. The shaded region represents the range of $(H_c^2/2\mu_0)V_{\Phi_\ell}$ with a 5% variation in the values used for λ_0. Inset: $(H_c^2/2\mu_0)$ and V_{Φ_ℓ} vs hole concentration p. Note that the coherence volume V_{Φ_ℓ} remains essentially constant over the optimally doped to overdoped region but begins to enlarge rapidly as the system becomes more underdoped.

IV. EQUIVALENCE WITH HOMES' LAW

Equation (12) can be rewritten in another useful form using the expression for the dimensionless in-plane quantum of resistance[21] $\tilde{Q}_u$,

$$\tilde{Q}_u \equiv \left(\frac{e^2}{\hbar}\right)\left(\frac{\rho_n}{\epsilon\xi}\right), \tag{14}$$

and the value of the flux quantum $\Phi_0 = \frac{h}{2e} = \pi\frac{\hbar}{e}$. With $\rho_n = 1/\sigma_n$, Eq. (12) then becomes

$$T_c = \frac{\mathbb{C}_L}{4\sqrt{2}\pi}\left[\frac{\pi^2\hbar}{\mu_0 k_B}\right]\frac{1}{\sigma_n\tilde{Q}_u}\frac{1}{\lambda_0^2}, \tag{15}$$

which, with the additional material dependent factor of $\mathbb{C}_L/\tilde{Q}_u$, is equivalent in form to, and provides an explanation for, the origin of the empirical scaling relation observed by Homes *et al.*,[6] given in Eq. (3).

Using the definition of the anisotropy parameter,

$$\epsilon \equiv \left(\frac{m}{M}\right)^{1/2} = \frac{\lambda_{ab}}{\lambda_c} = \sqrt{\frac{\sigma_c}{\sigma_{ab}}}, \tag{16}$$

we can readily see that $\sigma_{ab}\lambda_{ab}^2 = \sigma_c\lambda_c^2$, from which it follows that

$$T_c \propto \frac{\mathbb{C}_L}{\tilde{Q}_u}\frac{1}{\sigma_{ab}\lambda_{ab}^2} = \frac{\mathbb{C}_L}{\tilde{Q}_u}\frac{1}{\sigma_c\lambda_c^2}. \tag{17}$$

Equation (17) provides an explanation for the observation of Homes *et al.*, wherein both the *a*-*b* plane and *c*-axis conductivities scale with $\rho_{s[ab]}(0) \equiv \lambda_{ab}^{-2}(0)$ and $\rho_{s[c]}(0) \equiv \lambda_c^{-2}(0)$, respectively, onto the same universal line given by Eq. (3) above. Note also that for the case where there is anisotropy with respect to the *a*- and *b*-axis properties, λ_a, λ_b, etc., the anisotropy parameter is simply redefined so that $\lambda_c = \epsilon_a\lambda_a = \epsilon_b\lambda_b$ or, equivalently, $\lambda_c^2 = \epsilon_a\epsilon_b\lambda_a\lambda_b \equiv \epsilon_{ab}^2\lambda_{ab}^2$. This redefinition leaves Eqs. (12) and (13), etc., unchanged in the same fashion as shown in Eqs. (16) and (17) above.

TABLE I. Values of the Lindemann number c_L for a range of high-T_c cuprate compounds calculated via Eq. (12). Values for ξ_0, λ_0, and ϵ are taken from Refs. 6, 20, and 26–29.

	x	$T_c(K)$	c_L
$HgBa_2CuO_{4+\delta}$		96	0.27
$HgBa_2CuO_{4+\delta}$		52	0.21
$Tl_2Ba_2CuO_x$		86	0.30
$YBa_2Cu_4O_8$		80	0.29
$La_{2-x}Sr_xCuO_4$	0.075	19	0.31
$La_{2-x}Sr_xCuO_4$	0.08	26.5	0.34
$La_{2-x}Sr_xCuO_4$	0.1	36.9	0.35
$La_{2-x}Sr_xCuO_4$	0.125	34.8	0.42
$La_{2-x}Sr_xCuO_4$	0.15	39.5	0.31
$La_{2-x}Sr_xCuO_4$	0.17	36.2	0.30
$La_{2-x}Sr_xCuO_4$	0.2	33.9	0.29

By inverting Eq. (12) and using a fixed value of $\alpha = 3/2$, we have calculated the Lindemann numbers c_L for other high-T_c compounds based on reported values of ξ_0, λ_0, and ϵ [see Table I]. We urge caution in any interpretation of the values of c_L obtained at this point since the parameter values used to calculate c_L for each sample were not all taken from measurements on the same sample, and the proper value of the exponent α is not known. In particular, from Eq. (10) and the related discussion, we must have $\mathbb{C}_L \leqslant 1$. With $\alpha = 3/2$, this requires $c_L \lesssim 0.31$, which is not the case for some of the compounds listed in Table I. However, having demonstrated that use of Eq. (12) gives reasonable and consistent values of c_L for these materials as well, the main point to be made is that the equivalence of Eq. (12) to the universal scaling demonstrated by Homes *et al.* implies that the same relationship between T_c, $H_c^2 V_{\Phi_\ell}$, and the carrier concentration, displayed in Fig. 2 for $YBa_2Cu_3O_x$, is valid for all systems which conform to Homes' law. Furthermore, from the calculated values of c_L and the corresponding value of $\mathbb{C}_L \approx 1$, it appears that the Cooper pair coherence volumes also overlap very little in these compounds, as illustrated for the $YBa_2Cu_3O_x$ samples in Fig. 2. Of further interest, the larger variation of the values of c_L for $La_{2-x}Sr_xCuO_4$ samples occurs on the underdoped side corresponding to the doping range associated with the smaller of the "two domes" of the T_c vs x phase diagram. Confirmation of this result, with a more accurate determination of c_L (i.e., by measuring the relevant quantities on the same sample for each doping), would seem to indicate an evolution in the underlying intervortex behavior, implying fundamental differences of the superfluid properties in each dome.

V. RESOLUTION OF "ZUEV'S PARADOX"

As mentioned above, recent measurements performed by Zuev *et al.*[10] on severely underdoped to optimally doped

$YBa_2Cu_3O_{6+x}$ films ($3\ \mathrm{K} \lesssim T_c \lesssim 50\ \mathrm{K}$, $T_c = 90\ \mathrm{K}$) reveal a relationship between the superfluid density and the superconducting critical temperature wherein $\rho_s(0) \propto 1/\lambda^2 \propto T_c^{2.3\pm0.4}$, in contrast to the results of Uemura *et al.*,[1,2] i.e., $\rho_s(0) \propto T_c$. It is readily seen from Fig. 2 of Ref. 10 that the latter (linear) power law relationship, inferred from data spanning much less than a decade in either temperature ($50\ \mathrm{K} \lesssim T_c \lesssim 90\ \mathrm{K}$) or length scale, ($20\ \mu\mathrm{m}^{-2} \lesssim 1/\lambda^2 \lesssim 80\ \mu\mathrm{m}^{-2}$), does not accurately represent a universal relationship between $\rho_s(0)$ and T_c in the underdoped to optimally doped regions as originally concluded.

Again, returning to Eq. (12), we can also rewrite this expression using the relations between the penetration depth and the superfluid density, $\lambda_0^{-2} = \rho_s(0)/c^2 = \mu_0 \epsilon_0 \rho_s(0)$, and between the Cooper pair coherence length and the superconducting critical temperature, $\xi_0 = a\hbar v_F / k_B T_c$, where $a \approx 0.12$–0.18, giving the result

$$T_c^2 = \left(\frac{\mathbb{C}_L}{4\sqrt{2}\pi} \right) \frac{a \epsilon \Phi_0^2 \hbar v_F \epsilon_0}{k_B^2} \rho_s(0), \tag{18}$$

over all doping regimes, in agreement with the relation observed by Zuev *et al.*,[10] $\rho_s(0) \propto T_c^{2.3\pm0.4}$, for heavily underdoped to lightly overdoped samples (including data from Uemura *et al.*).[2] Thus, Eqs. (15) and (18) show that the finding of Zuev *et al.*, wherein their data conform to both Homes' law and their empirical scaling relation, is, in fact, compatible since both equations can be derived from the relationship given in Eq. (10).

VI. MOLECULAR SUPERCONDUCTORS

Next, we turn our attention to the universal scaling law of Pratt and Blundell[11] observed for molecular superconductors, wherein they find that $\rho_s(0)$, $\sigma_n(T_c)$, and T_c scale as

$$\epsilon_0 \rho_s = \frac{2k_B}{\pi\hbar} \eta \sigma_n(T_c) T_c, \tag{19}$$

where $\eta = 2\Delta/k_B T_c$ [and $\sigma_n(T_c)$ is denoted as $\sigma_0(T_c)$]. Using $\lambda_0^{-2} = \mu_0 \epsilon_0 \rho_s(0)$, Eq. (19) can be rearranged

$$\frac{1}{\lambda_0^2} = \frac{2\mu_0 k_B}{\pi\hbar} \eta \sigma_n(T_c) T_c. \tag{20}$$

By comparing Eq. (20) to Eq. (15) (or any of the equivalent forms derived above), we can see that the scaling relationships of Pratt and Blundell are identical in form, implying that, while there are certainly fundamental differences in the behavior exhibited by molecular and high-T_c cuprate superconductors, ultimately the class of molecular superconductors are also described by the fundamental relationship given in Eq. (10).

VII. COMPARISON WITH BAD METAL MODEL OF SUPERCONDUCTIVITY

While the expression we have arrived at in Eq. (12) does not specify a mechanism of superconductivity, we find a compelling result by turning to the bad metal theory of superconductivity described by Emery and Kivelson (EK).[12,13] This theory addresses the importance of phase fluctuations of the order parameter in superconductors with a small superfluid density, a class to which the cuprate-based high-temperature superconductors belong. The consequence of the small phase "stiffness" of the order parameter in these materials is that, unlike conventional metallic superconductors, the onset of long-range phase order can occur at a temperature $T_\theta^{max} < T^{MF}$, where T^{MF} is the BCS-Eliashberg mean-field pairing temperature.[22] When this is the case, the value of T_c is determined by $T_\theta^{max} \gtrsim T_c$, and T^{MF} is now the temperature below which pairing becomes significant locally. EK show that in bad metal superconductors, there is a connection between the critical temperature T_c the value of the conductivity at the critical temperature, $\sigma_n(T_c)$, and the value of the quantum of conductivity σ_Q [defined as $\sigma_Q \equiv (2e)^2/hb$ in Ref. 12, where b is an appropriate length scale of the order of $\xi_c(0)$]. This relationship is given by

$$\sigma_Q = \frac{\ln(T_\theta^{max}/T_c)}{\ln(\varepsilon/k_B T_c)} \sigma_n(T_c), \tag{21}$$

where ε is the energy scale associated with the coupling mechanism.

We can rearrange Eq. (12) again as

$$T_c = \mathbb{C}_L \left[\frac{\pi\hbar}{4\sqrt{2}\mu_0 k_B} \right] \rho_Q \frac{1}{\lambda_0^2}, \tag{22}$$

where we have defined the quantum of resistance, $\rho_Q = \sigma_Q^{-1}$ to agree with the dimensionless quantum of resistance Q_u used above in Eq. (14), i.e., $\rho_Q = \hbar \epsilon \xi_0 / e^2$.

Combining Eq. (21) with Eq. (22), we have

$$\frac{1}{\lambda_0^2} = \mathbb{C}_L^{-1} \left(\frac{4\sqrt{2}\mu_0 k_B}{\pi\hbar} \right) \left[\frac{\ln(T_\theta^{max}/T_c)}{\ln(\varepsilon/k_B T_c)} \right] \sigma_n(T_c) T_c. \tag{23}$$

Comparing Eq. (23) to Homes' law, Eq. (3), we must have

$$\mathbb{C}_L^{-1} \left(\frac{4\sqrt{2}\mu_0 k_B}{\pi\hbar} \right) \left[\frac{\ln(T_\theta^{max}/T_c)}{\ln(\varepsilon/k_B T_c)} \right] \approx 120\ \Omega\ \mathrm{cm}^{-1}\ \mathrm{K}^{-1}, \tag{24}$$

which gives a universal relationship of $\sigma_n(T_c) \approx (25\mathbb{C}_L^{-1})\sigma_Q$ for the conductivity of bad metal superconductors at T_c over the *entire* doping range. This result agrees well with the estimate of $\sigma_n(T_c) \approx 10\sigma_Q$ given by EK.[12] Solving for the ratio $\varepsilon/k_B T_c$ gives

$$\frac{\varepsilon}{k_B T_c} \approx \left(\frac{T_\theta^{max}}{T_c} \right)^{25\mathbb{C}_L^{-1}}. \tag{25}$$

The relationship given in Eq. (25) results in a modification of the phase diagram given by EK in Fig. 1 of Ref. 12. As indicated in Fig. 3, it appears that the boundary of T_θ^{max} must follow closely the critical temperature boundary, even in the overdoped region. Otherwise, the values of ε necessary to satisfy Eq. (25) become unphysical. This does not necessarily imply that the critical temperature T_c in the overdoped region is no longer determined by the mean-field pairing temperature T^{MF}, only that the upper bound in the over-

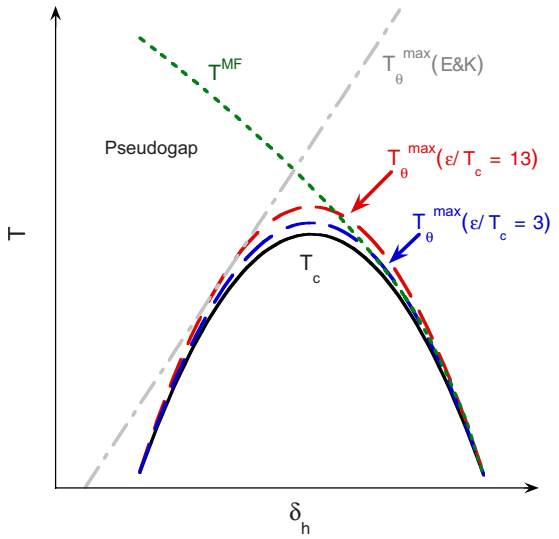

FIG. 3. (Color online) Modified sketch of the phase diagram of high-temperature superconductors as a function of temperature T and hole doping δ_h proposed by Emery and Kivelson in Ref. 12. The temperatures T_c (solid, black), T^{MF} (short dashed, green), and T_θ^{max} (EK) (long-short dashed, gray) are qualitatively shown as originally indicated in Ref. 12. The red and blue dashed lines are the locations of $T_\theta^{max} \approx 1.11 T_c$ and $T_\theta^{max} \approx 1.04 T_c$ for the energy ratios of $\varepsilon / k_B T_c$ equal to 13 and 3, respectively, in accordance with Eq. (25) (with $\mathbb{C}_L = 1$ over the entire doping range) in the text below. These values roughly correspond to the ratios of antiferromagnetic exchange energies and phonon energies to the critical temperature.

doped region lies lower than previously thought. The change with respect to T_θ^{max} in Fig. 3 does not alter the picture described by EK, wherein the phase diagram of the cuprate high-T_c superconductors consists of underdoped and optimally doped materials, where the onset of superconductivity is determined by phase coherence and overdoped materials where the order parameter amplitude and phase coherence are established close to or at the same temperature.[12,13,23]

VIII. DISCUSSION AND CONCLUSIONS

By considering the $T \rightarrow 0$ limit of a recently proposed expression for the vortex glass melting line,[14,15] $H_g(T)$, we arrive at an analytical formula for the superconducting transition temperature T_c wherein the energy scale $k_B T_c$ is found to be proportional to the condensation energy $E_c \equiv (H_c^2 / 2\mu_0) V_{coh}$. The proportionality is determined by the extent to which the Cooper pair coherence volumes overlap as characterized by the Lindemann number c_L, which, in turn, reflects the stability of the solid vortex state to thermal and quantum fluctuations.

As pointed out above, the expression we have arrived at in Eq. (12) does not specify a mechanism of superconductivity and is expected to be applicable to all classes of superconductors. Our expression does, however, identify three key physical properties that theories and experimental studies of superconductivity need to address: the energy scale H_c^2, the geometry of the volume over which Cooper pairs are correlated, V_{Φ_ℓ}, and the relationship between the stability of the solid vortex structure in the presence of fluctuations, as characterized by c_L, and the extent to which the coherence volumes of Cooper pairs overlap, as characterized by $\mathbb{C}_L$. It seems reasonable that, in accordance with our definition of $\mathbb{C}_L$, it ought to be possible to generalize the meaning of the value of the Lindemann number c_L from only characterizing the stability of the solid vortex state to a characterization of the stability of the order parameter to quantum and/or thermal fluctuations. Or, from another perspective, it should be possible to arrive at an expression for T_c in terms of $(H_c^2 / 2\mu_0) V_{coh}$ with a prefactor equivalent in meaning to that of $\mathbb{C}_L$ in Eq. (10) without the use of Eqs. (7) and (8).

Equation (12) (as well as the equivalent equations) provides a simple and potentially useful means of analyzing the change of the critical temperature T_c with respect to tuning by doping, pressure, etc., i.e.,

$$\frac{\partial T_c}{\partial X} = \left(\frac{\Phi_0^2}{4\sqrt{2}\pi\mu_0 k_B} \right) \frac{\partial}{\partial X} \left(\frac{\mathbb{C}_L \epsilon \xi_0}{\lambda_0^2} \right), \tag{26}$$

where X is a tuning parameter. We would anticipate that a fundamental change of the nature of the superconducting mechanism ought to be discernible in systems where the end members of a doping regime, such as $Pr(Os_{1-x}Ru_x)_4Sb_{12}$, consist of superconductors with distinctly different electronic ground states,[24] or in systems where there is (evidence for) a change of the symmetry of the superconducting order parameter.[25]

The results found in Eqs. (21)–(25) do not necessarily constitute a proof of the bad metal model of superconductivity proposed by EK for the high-T_c cuprate superconductors. However, the remarkable agreement between the value of the ratio $\sigma_n(T_c)/\sigma_Q$ estimated by EK and that found here with the use of Homes' law can, at least, be considered as support for the theory. Within this context, a significant implication is that the upper boundary for the temperature at which long-range phase order of the superconducting order parameter can be established, T_θ^{max}, is determined by the energy scale ε and the ratio $k_B T_c / E_c = \mathbb{C}_L$ in all regimes, underdoped, optimally doped, and overdoped.

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