Spin pumping of current in non-uniform conducting magnets

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Using irreversible thermodynamics, we show that current-induced spin transfer torque within a magnetic domain implies spin pumping of current within that domain. This has experimental implications for samples both with conducting leads and that are electrically isolated. These results are obtained by deriving the dynamical equations for two models of non-uniform conducting magnets: (1) a generic conducting magnet, with net conduction electron density n and net magnetization $\vec{M}$, and (2) a two-band magnet, with up and down spins each providing conduction and magnetism. For both models, in regions where the equilibrium magnetization is non-uniform, voltage gradients can drive adiabatic and nonadiabatic bulk spin torques. Onsager relations then ensure that magnetic torques likewise drive adiabatic and nonadiabatic currents—what we call bulk spin pumping. For a given amount of adiabatic and nonadiabatic spin torques, the two models yield similar but distinct results for the bulk spin pumping, thus distinguishing the two models. As for recent spin-Berry phase work, we find that within a domain wall, the ratio of the effective electromotive force to the magnetic field is approximately given by $P(2\mu_B/e)$, where P is the spin polarization. The adiabatic spin torque and spin-pumping terms are shown to be dissipative.

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I. INTRODUCTION

A. Current-induced spin transfer torque

Current-induced spin transfer torque at both surfaces[1,2] and in bulk is by now a well-established phenomenon.[3] In surface spin transfer torque, when a polarized spin current from a nonmagnet crosses the interface with a magnet, it causes spin motion. In bulk spin transfer torque, when a polarized spin current crosses a domain wall (where the magnetization varies in direction), it causes spin motion. For a uniform magnet, there is no current-induced spin torque.

To be specific, we consider that the magnetization $\vec{M}$ satisfies the equation[4]

$$\partial_t \vec{M} + \partial_i \vec{Q}_i = -\gamma \vec{M} \times \vec{H} + \vec{N}, \tag{1}$$

where $\vec{Q}_i$ is the magnetization flux (or magnetization current density), $-\gamma \vec{M} \times \vec{H}$ is the Larmor-like torque, and $\vec{N}$ is the rest of the torque density acting on the magnetization. We take γ to be the magnitude of the (negative) gyromagnetic ratio. $\vec{H}$ is the net effective field, which can include an external field, crystalline anisotropy, the demagnetization field, and non-uniform exchange. $\vec{H}$ is given in SI units of T, although $\vec{H}$ is not the magnetic induction field $\vec{B}$, whose units are also in T; an H field in A/m, on multiplication by μ_0, becomes an H field in T.

We will call

$$\vec{N}' = \vec{N} - \partial_i \vec{Q}_i \tag{2}$$

the non-Larmor-like spin torque. When $\vec{N}'$ contains a term that is proportional to the current (or to the gradient of the electrochemical potential), one says that there is a spin transfer torque.

For a uniform system, $\vec{N}'$ will contain only damping terms. However, for a non-uniform system, $\vec{N}'$ also contains terms of the form[5]

$$-\xi \partial_i \tilde{\mu} [\partial_i \vec{M} - \beta \partial_i \hat{M} \times \partial_i \vec{M}], \tag{3}$$

where $\tilde{\mu}$ the electrochemical potential, $-\xi \partial_i \tilde{\mu}$ has units of a velocity, and β is dimensionless. The term in $\partial_i \vec{M}$ is called the *adiabatic spin transfer torque* and the term in $\hat{M} \times \partial_i \vec{M}$ is called the *nonadiabatic spin transfer torque*.[6] This usage of adiabatic and nonadiabatic refers to the extent to which spatial variations in the magnetic structure (e.g., a domain wall width) compare with the spatial variations in the electronic structure (e.g., a Fermi wavelength).

In the literature, the current j_i is usually written (with a suitable conductivity) in place of $\partial_i \tilde{\mu}$. However, when one employs irreversible thermodynamics, $\partial_i \tilde{\mu}$, a thermodynamic driving force that is even under time reversal, is the more natural quantity to employ. Use of $\partial_i \tilde{\mu}$ unambiguously leads to the conclusion that, because of their time-reversal properties, the adiabatic spin transfer torque is irreversible (dissipative) and the nonadiabatic spin transfer torque is reversible (reactive). This will also be seen from a calculation of the volume rate of dissipation R, where the equivalent of ξ appears, but the equivalent of β is absent. Except for a superfluid (not considered here), preparation of a thermodynamic state must be done with the scalar chemical potential rather than with a vector potential that couples to the current.

B. Spin pumping of current

Spin pumping of current at surfaces is also a well-established phenomenon.[7] Here, spin dynamics at an interface transfers a spin-polarized current to an adjacent material. The first indication of spin pumping was provided by experiments on a thin magnetic film adjacent to both a vacuum and an ordinary conductor.[8] The present work studies two models for non-uniform conducting magnets, one a generic conducting magnet that is related to but distinct from the s-d model,[5] and one based on a two-band magnet.[9]

For both models, Onsager relations between transport coefficients imply a bulk version of spin pumping, to an extent related to the amount of bulk spin transfer torque. (This is analogous to how, if temperature gradients can cause an electric current, then electrochemical potential gradients can cause a heat current, with the size of these effects related by Onsager relations.) However, the effects are somewhat different for the two models, permitting them to be distinguished.

For the band model, there are two currents (from up and down spins) and two effective electrochemical potentials. Each of these currents can be "spin pumped" by disequilibrium of $\vec{M}$. For the generic conducting magnet, there is only one current, but it is taken to be spin polarized. For the generic conducting magnet, we study the number current density j_i^n [see Eq. (3)]. It contains terms with the form

$$j_i^n = -L_{nM1}(\partial_i\vec{M}) \cdot \vec{H} - L'_{nM}\partial_i\vec{M} \cdot (\vec{M} \times \vec{H}) + \cdots . \quad (4)$$

We consider the L_{nM1} term, proportional to $\partial_i\vec{M}$, to represent *adiabatic spin pumping,* and the L'_{nM1} term, proportional (on rewriting) to $\hat{m} \times \partial_i\vec{M}$, to represent the nonadiabatic spin-pumping term. Bulk spin pumping of current requires a non-uniform magnetization; it occurs within domains.

Barnes and Maekawa have recently studied the effective electric field associated with the spin-Berry phase induced by the domain structure.[10] The present results are remarkably similar to theirs, despite the vastly different methods. In particular, both works find that the ratio of the effective electromotive force (emf) to the magnetic field is given by $P(2\mu_B/e)$, where P is the spin polarization, although our result does not appear to be exact. We also note similar works by Duine[11] and Yang *et al.*[12] In these three works, a microscopic Berry-phase viewpoint is taken. As in Ref. 4, bulk spin transfer follows from the time dependence of the wave function; in addition, what we and Ref. 11 call bulk spin pumping follows from the spatial dependence of the wave function.

The spin-pumping terms are related to the spin transfer torque terms by Onsager relations. (In surface spin pumping, there is also a spin transfer torque in proportion to the spin pumping.[7]) Thus, not only does the fact that spin transfer torque has been observed tell us that spin pumping must exist (if we understand the correct thermodynamic description of our system), it also tells us how large the spin-pumping terms must be. For the two-band magnet, analogous terms appear in each of the two currents. However, the two models have forms that permit them to be distinguished. Thus, the theory may also provide a means to distinguish between the two models.[13]

C. Experimental implications

The most important prediction of this work, based on irreversible thermodynamics, is that bulk spin pumping occurs if there is spin transfer torque (and vice versa); moreover, the amount of spin pumping can be determined from appropriate Onsager relations.

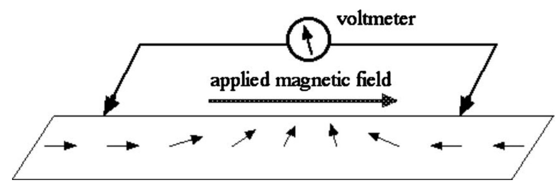

FIG. 1. Experimental geometry to observe spin pumping of current associated with a head-to-head domain wall. An applied magnetic field drives the domain wall to the right. A spin-pumped current goes to the left. One expects an associated voltage pulse when the domain wall crosses either voltage lead.

For a magnetic wire with head-to-head domains, and two voltage leads, if a domain wall is $\vec{H}$ field driven past one lead, there should be a voltage jump. Figure 1 shows an experimental geometry corresponding to a linear head-to-head domain wall that is driven by an external field. When the domain wall crosses the voltage lead on the right, the voltage difference between the leads will measure a pulse that drives current leftward. Likewise, for a magnetic dot with a vortex configuration, and two voltage leads, if the vortex structure is $\vec{H}$ field driven past one lead, there should be a voltage jump.

For Co, data on spin torque indicate that within a domain wall, a true E field $E_0 = 1.0 \times 10^4$ V/m can cause the same spin torque as direct application of a magnetic field $H_{ST} = 4 \times 10^{-3}$ T. From this, in Sec. VII, we estimate that within a domain wall a true H field $H_0 = 0.1$ T can cause the same spin-pumped current as direct application of an electric field $E_{SP} = 350$ V/m or, across the domain wall, can cause the same effect as a voltage difference of 7.0×10^{-6} V. In terms of the effective emf per magnetic field, given by $P(2\mu_B/e)$, this is 0.7×10^{-4} V/T. Note that when notational differences are accounted for, a similar result is given by Ref. 14 in the context of interfacial transport between two conductors, one magnetic and one nonmagnetic.

II. ON IRREVERSIBLE THERMODYNAMICS

A. Summary of methods

This paper employs the methods of irreversible thermodynamics. For an introduction to these methods that is directed to the magnetism community, see Ref. 15, which is available online. We summarize the approach as follows.[16–18]

(1) The first step is to determine the appropriate thermodynamic variables and the thermodynamic relation for the differential $d\varepsilon$ of the energy density as a sum of terms proportional to the various thermodynamic densities describing the system. This assumes local equilibrium even when the system has (slow) spatial and temporal variation. Moreover, spatial derivatives, but no time derivatives, can appear in $d\varepsilon$. For an ordinary one-component system, these densities are associated with the entropy and the particle number. For the generic magnetic conductor, these densities are the entropy density s, the density of carrier electrons n, and the magne-

tization $\vec{M}$. For a two-component nonmagnetic system such as a semiconductor, these densities are associated with the entropy and the particle numbers (electrons and holes).[19] For the two-band magnet, these densities are associated with the entropy and the particle numbers (up and down electrons, with particle densities $n_\uparrow, n_\downarrow$, and the magnetization direction $\hat{M}$). This is done in Sec. III.

(2) The second step is to require that all of the densities satisfy either a conservation law (with unknown fluxes), a source equation (with unknown sources), or both, and to consider the structure of the equation giving the rate of heat production. This is done in Secs. IV–VI.

(3) The third step is to construct the matrix of structure-dependent constants (the transport and dissipation coefficients) relating the sources and fluxes to the thermodynamic forces. This requires knowledge of the time-reversal properties of the fluxes, forces, and sources. Finally, the Onsager relations are used to reduce the number of independent coefficients. This section is in the Appendix, and its implications are discussed in Sec. VII. A brief summary and discussion follows in Sec. VIII.

B. Irreversible thermodynamics, magnetic damping, and spin transfer torque

In the context of magnetic damping without spin transfer or spin pumping, we note that two types of magnetic damping are favored in the literature. The first is by Landau and Lifshitz [a term in $\partial_t\vec{M}$ proportional to $\hat{M}\times(\vec{M}\times\vec{H})$].[20] The second is by Gilbert (a term in $\partial_t\vec{M}$ proportional to $\vec{M}\times\partial_t\vec{M}$).[21] The issue is not one on which the community has come to agreement, although the present author favors that by Landau and Lifshitz. Indeed, to our knowledge, all studies that employ standard irreversible thermodynamics (including the present one) obtain Landau-Lifshitz damping.[15,22,23] Moreover, Ref. 24 uses a Langevin approach with an energy-weighted thermal distribution and also obtains Landau-Lifshitz damping.

Reference 21, on the other hand, obtains Gilbert damping by using the Rayleigh dissipation function, developed to formally incorporate into Lagrange's equations the already-known damping of individual particles rather than, as for irreversible thermodynamics, to deduce the unknown damping in thermodynamic systems. Reference 25 employs the time derivative of the magnetization in the thermodynamic energy for a set of continuous magnetic conducting slabs. The use of such a time derivative places this work outside the framework of standard irreversible thermodynamics and may explain how this work obtains Gilbert damping.

A phenomenological theory such as provided by irreversible thermodynamics cannot specify the values of various parameters appearing in the theory. Hence, for example, it can make no statement about the relative value of the coefficients of adiabatic and nonadiabatic spin transfer torques. Reference 24 provides one particular viewpoint on the relative value of the coefficients of adiabatic and nonadiabatic spin transfer torques, given by β in Eq. (3), which is tied up with the issue of Landau-Lifshitz vs Gilbert damping. However, the issue is still considered unsettled.[4,5,26–32]

III. THERMODYNAMIC RELATIONS

A. Generic conducting magnet

We consider a system with magnetization $\vec{M}$ that is basically due to localized electrons, of gyromagnetic ratio $-\gamma$, and conduction electrons with electrochemical potential $\tilde{\mu}$ and number density n. We take the thermodynamics to be given by

$$d\varepsilon = Tds + \tilde{\mu}dn - \vec{H}^* \cdot d\vec{M}. \tag{5}$$

Here, $\vec{H}^*$ includes not only the field $\vec{H}$ that causes the Larmor torque but also a uniform exchange field that points along $\vec{M}$, and thus has no effect on the Larmor torque. The exchange field is chosen to make $\vec{H}^*_{eq}=\vec{0}$ in equilibrium, so that $\vec{H}^*$ will serve as a thermodynamic force.

To be specific, consider a uniform ferromagnet in a minor loop and in an external field $\vec{H}_0$. In the absence of anisotropy, in equilibrium it satisfies $\vec{M}=M_0\hat{H}_0+\chi\vec{H}_0$. There may also be an anisotropy field $\vec{H}_{an}$. Further, include a non-uniform exchange term $A'(\partial_i\vec{M})^2$ in the energy density, which will yield a non-uniform exchange field $\vec{H}_{ex}=2A'\nabla^2\vec{M}$. We assert that

$$\begin{aligned}\vec{H}^* &= -\frac{\delta\varepsilon}{\delta\vec{M}} = -\frac{\partial\varepsilon}{\partial\vec{M}} + \partial_i\frac{\partial\varepsilon}{\partial(\partial_i\vec{M})} \\ &= [\vec{H}_0 + \vec{H}_{an} + 2A'\nabla^2\vec{M}] - \left[\frac{\vec{M}-M_0\hat{M}}{\chi}\right] \\ &\equiv \vec{H} - \vec{H}_{int}\end{aligned} \tag{6}$$

has the desired form. The first bracket, with three terms, constitutes $\vec{H}$, which drives Larmor precession. The last bracket, with two terms, represents an internal field $\vec{H}_{int}$ due to exchange.

Setting $\vec{H}^*=\vec{0}$ gives $\vec{M}$ along $\vec{H}$, as desired for no Larmor torque. Moreover, it gives $\vec{M}=M_0\hat{H}+\chi\vec{H}$, as expected. Even out of equilibrium $\vec{H}_{int}$, which is along $\hat{M}$, does not contribute to $\vec{M}\times\vec{H}$, so that $\vec{M}\times\vec{H}^*=\vec{M}\times\vec{H}$. In the absence of anisotropy and non-uniform exchange, Eq. (6) is basically the form employed by Ref. 14.

In the s-d model as often used, there is an exchange field that couples to the s electrons, giving them a weak magnetization. If one wants to incorporate this idea in the present framework, then one may consider that $\vec{M}$ includes a contribution from the polarized conduction electrons. However, because no such specification is made, our generic conducting magnet is distinct from the s-d model.

B. Two-band magnet

The two-band magnet[9] considers a conducting magnetic system consisting of electrons of charge $-e$ and gyromagnetic ratio $-\gamma$, where $\gamma=|g|\mu_B/\hbar>0$ and $\mu_B=e\hbar/2m$ (for free electrons, we take $g=-2$). We assume that the electrons partially occupy two, spin-dependent, conduction bands, with number densities $n_\uparrow$ and $n_\downarrow$. For specificity, we assume that $n_\downarrow > n_\uparrow$ so that the magnetization will point along the "up" direction, determined either by spontaneous symmetry breaking or by an external field and anisotropy. The system also has an entropy density s. The total number density for the conducting electrons is

$$n = n_\uparrow + n_\downarrow. \tag{7}$$

Moreover, the magnetization is given by

$$\vec{M} = -\gamma \vec{S}, \tag{8}$$

where $\vec{S}$ is the spin density, of magnitude

$$S = (\hbar/2)(n_\downarrow - n_\uparrow). \tag{9}$$

Thus, the magnetization has magnitude $M=|\vec{M}|$ given by

$$M = \gamma(\hbar/2)(n_\downarrow - n_\uparrow) = (|g|\mu_B/2)(n_\downarrow - n_\uparrow). \tag{10}$$

With magnetization direction $\hat{M}$, we then have

$$\vec{M} = \gamma(\hbar/2)(n_\downarrow - n_\uparrow)\hat{M}. \tag{11}$$

Note that

$$d\vec{M} = \gamma(\hbar/2)\hat{M} d(n_\downarrow - n_\uparrow) + M d\hat{M}. \tag{12}$$

For the differential of the energy density, we take

$$d\varepsilon = T ds + \mu^*_\uparrow dn_\uparrow + \mu^*_\downarrow dn_{ow} - M(\vec{H} \cdot d\hat{M}). \tag{13}$$

(Note that $\vec{H} \cdot d\hat{M} = \vec{H}^* \cdot d\hat{M}$ because $d\hat{M}$ is normal to $\hat{M}$.) Here,

$$\mu^*_{\uparrow,\downarrow} = \tilde{\mu}_{\uparrow,\downarrow} \pm \gamma(\hbar/2)(\vec{H}^* \cdot \hat{M}), \tag{14}$$

where

$$\tilde{\mu}_{\uparrow,\downarrow} = \mu_{\uparrow,\downarrow} - eV \tag{15}$$

is the electrochemical potential in terms of the chemical potential, with V the electrical potential. Changes in the number density are affected both by the electrochemical potential and $\vec{H}^*$. In equilibrium, $\mu^*_\uparrow = \mu^*_\downarrow$ and $\hat{M} \times \vec{H} = \vec{0}$. If we further require that $\vec{H}^* = \vec{0}$ in equilibrium, then $\tilde{\mu}_\uparrow = \tilde{\mu}_\downarrow$ in equilibrium. Note that $\mu^*_\uparrow - \mu^*_\downarrow = \gamma\hbar \vec{H}^* \cdot \hat{M}$.

IV. ENERGY AND ENTROPY

Energy and entropy are treated the same way in both theories.

Consider the energy density ε which, being conserved, has only a flux:

$$\partial_t \varepsilon + \partial_i j^\varepsilon_i = 0. \tag{16}$$

Here, j^ε_i is the energy flux density. The intrinsic time-reversal signature of ε is even, so the intrinsic time-reversal signature of j^ε_i is odd.

Now, consider the entropy density s. It has both a flux j^s_i and a source R/T, where R is the volume rate of heat production:

$$\partial_t s + \partial_i j^s_i = \frac{R}{T} \geq 0. \tag{17}$$

Irreversible thermodynamics considers the time behavior of thermodynamic variables, which have definite signature under time reversal. As a consequence, in an equation of motion for that quantity that has both a flux and a source, because of the time derivative, the intrinsic time-reversal signatures of both the flux (here j^s_i) and the source (here R/T) are opposite to the intrinsic time-reversal signatures of the extensive density. Since s is even, the intrinsic time-reversal signatures of j^s_i and R/T are odd.

V. CONSERVATION LAWS AND EQUATIONS OF MOTION

A. Generic conducting magnet

We take the equations of motion and conservation laws for the four variables n and $\vec{M}$ to be

$$\partial_t n + \partial_i j^n_i = 0, \tag{18}$$

$$\partial_t \vec{M} + \partial_i \vec{Q}_i = -\gamma \vec{M} \times \vec{H} + \vec{N}. \tag{19}$$

Here, j^n_i is the number current density, $\vec{Q}_i$ is the magnetization flux density (i is the real space index), and $\vec{N}$ (a source) is the volume rate of change of magnetization due to torques associated with a lack of thermal equilibrium.[33]

B. Two-band magnet

We take the equations of motion and conservation laws for the four variables $n_\uparrow$, $n_\downarrow$, and $\hat{M}$ to be

$$\partial_t n_\uparrow + \partial_i j_{\uparrow i} = S, \tag{20}$$

$$\partial_t n_\downarrow + \partial_i j_{\downarrow i} = -S, \tag{21}$$

$$\partial_t \hat{M} = (\gamma \vec{H} + \vec{\Omega}) \times \hat{M}. \tag{22}$$

Here, $j_{\uparrow,i}$ and $j_{\downarrow,i}$ are the number current densities, S is the decay rate for up spins (by charge conservation, this is compensated by the decay rate $-S$ for down spins), and $\gamma\vec{H}$ and $\vec{\Omega}$ are the Larmor and non-Larmor parts of the rotation rate for $\hat{M}$. By definition, $\vec{\Omega}$ has only two components and is normal to $\vec{M}$.

C. Two-band magnet: Implied equations

The above equations imply certain equations of motion for n of Eq. (7), M of Eq. (10), and $\vec{M}$ of Eq. (11). These equations are not necessary, because the previous section is self-contained, but they are useful for comparison with previous work.

Continuity equation. Equations (7), (20), and (21) imply that

$$\partial_t n + \partial_i j^n_i = 0, \quad j^n_i \equiv j_{\uparrow i} + j_{\downarrow i}. \tag{23}$$

With the charge density $\rho=-en$ and current density (charge flux) given by

$$\rho = -en, \quad j_i = -ej_i^n = -e(j_{\uparrow i} + j_{\downarrow i}), \tag{24}$$

the continuity equation is automatically satisfied:

$$\partial_t \rho + \partial_i j_i = 0. \tag{25}$$

Magnitude of magnetization. Equations (10), (20), and (21) imply that

$$\partial_t M + \gamma(\hbar/2)\partial_i(j_{\downarrow i} - j_{\uparrow i}) = -2S\gamma(\hbar/2). \tag{26}$$

For magnetization along z, this is analogous to $\partial_t M_z$.

Vector magnetization. Equations. (11) and (20)–(22) imply that

$$\partial_t \vec{M} + \gamma(\hbar/2)\hat{M}\partial_i(j_{\downarrow i} - j_{\uparrow i}) = -\gamma \vec{M} \times \vec{H} + \vec{\Omega} \times \vec{M} - 2S\gamma(\hbar/2)\hat{M}. \tag{27}$$

This can be rewritten in the more conventional form

$$\partial_t \vec{M} + \partial_i \vec{Q}_i = -\gamma(\vec{M} \times \vec{H}) + \vec{N} \tag{28}$$

on setting

$$\vec{Q}_i = \gamma(\hbar/2)\hat{M}(j_{\downarrow i} - j_{\uparrow i}), \tag{29}$$

where $\vec{Q}_i$ is the magnetization flux density (i is the real space index) and

$$\vec{N} = \vec{\Omega} \times \vec{M} + \gamma(\hbar/2)(j_{\downarrow i} - j_{\uparrow i})\partial_i \hat{M} - 2S\gamma(\hbar/2)\hat{M}, \tag{30}$$

where $\vec{N}$ (a source) is the volume rate of change of magnetization due to torques associated with a lack of thermal equilibrium.

From Eqs. (29) and (30), the net non-Larmor spin transfer torque of Eq. (2) is given by

$$\vec{N}' = \vec{\Omega} \times \vec{M} - \gamma(\hbar/2)\hat{M}\partial_i(j_{\downarrow i} - j_{\uparrow i}) - 2S\gamma(\hbar/2)\hat{M}. \tag{31}$$

Thus, the only transverse part of $\vec{N}'$ comes from the $\vec{\Omega} \times \vec{M}$ term.

Once the difference in units is accounted for, $\vec{Q}_i$ above is equivalent to Eq. (8) of Ref. 9; there, both the magnetization and the magnetization flux densities are measured in units of γ, with $|g|=2$, and $\hat{u}$ is employed for the direction of the magnetization. Moreover, the second term of $\vec{N}$ in Eq. (30) is the same as the adiabatic spin torque density of Ref. 24. This adiabatic spin torque is enforced by the condition that the magnetization and the spin quantization axis track with one another. Note that Ref. 9 does not include the adiabatic spin torque density.

In Eq. (30), the first and second terms give the transverse components in spin space, and the third term gives the longitudinal component in spin space. Of course, we have yet to determine $\vec{\Omega}$, S, $j_{\downarrow i}$, or $j_{\uparrow i}$. Below, we show that $j_{\downarrow i}$ and $j_{\uparrow i}$ each have five terms, so that $Q_{\alpha i}$ has ten terms. Moreover, $\vec{\Omega}$ has seven terms and S has one term, so that $\vec{N}$ has 18 terms.

Note that spin angular momentum is not conserved; however, total angular momentum is conserved once one accounts for the crystal lattice angular momentum. With $\vec{H} = \vec{H}_0 + \vec{H}_{an} + \vec{H}_{ex}$, the angular momentum associated with $\vec{H}_0$ is transferred to the source of $\vec{H}_0$ (an external magnet or an external current-carrying circuit), the angular momentum associated with $\vec{H}_{an}$ (either lattice or dipolar anisotropy) is transferred to the lattice, and the angular momentum associated with $\vec{H}_{ex}$ integrates to zero, because it involves the spin system interacting with itself. $\vec{N}$ is associated with the lattice.

VI. RATE OF HEAT PRODUCTION

Irreversible thermodynamics accomplishes its task by combining the equations of motion and the thermodynamics to obtain an expression for the non-negative quantity R as a sum of products of fluxes (or sources) and thermodynamic "forces" (or their gradients).

A. Generic conducting magnet

Note that in Eq. (19), we may replace $\vec{H}$ by $\vec{H}^*$. With this substitution, Eqs. (16)–(19) placed in the time derivative of Eq. (5), where $\vec{H}^*$ is essential, yield

$$0 \le R = -\partial_i(j_i^\varepsilon - Tj_i^s - \tilde{\mu}j_i^n + \vec{H}^* \cdot \vec{Q}_i) - j_i^s\partial_i T - j_i^n\partial_i\tilde{\mu} + \vec{Q}_i \cdot \partial_i \vec{H}^* + \vec{N} \cdot \vec{H}^*. \tag{32}$$

Here j_i^s, j_i^n, and $\vec{Q}_i$ are thermodynamic fluxes and $\partial_i \vec{H}^*$, $\partial_i T$, and $\partial_i \tilde{\mu}$ are thermodynamic forces; further, $\vec{N}$ is a thermodynamic source and $\vec{H}^*$ is a thermodynamic force. In equilibrium, all of the thermodynamic forces are zero, and thus there is no entropy production.

Each of the four nondivergence terms in Eq. (32) has a clear physical interpretation as a source of heating: the first term to thermal conduction, the second to electrical conduction, the third to magnetic diffusion (or conduction), and the fourth to (local) spin damping.

B. Adiabatic (nonadiabatic) spin transfer torque is dissipative (nondissipative)

At this point, we can make specific statements about the time-reversal properties of the adiabatic and nonadiabatic spin torque terms. In Eq. (32), consider only the transverse part of $\vec{N}$, which has no component along $\vec{M}$, so that only $\vec{N}_\perp \cdot \vec{H}$ is relevant. Using the proper thermodynamic variable $\partial_i \tilde{\mu}$, we find that the adiabatic spin transfer torque component of $\vec{N}_\perp$ is proportional to $\partial_i \tilde{\mu} \partial_i \vec{M}$. Dotted with $\vec{H}$, as in Eq. (32), we see that this gives a term that is even under time reversal $\mathcal{T}$, since $\tilde{\mu}$ is even and both $\vec{M}$ and $\vec{H}$ are odd under $\mathcal{T}$. Since R is even under $\mathcal{T}$, the adiabatic term will contribute to R, and thus corresponds to dissipation. Likewise, the nonadiabatic spin transfer torque component of $\vec{N}_\perp$ is proportional to $\partial_i \tilde{\mu} \hat{M} \times \partial_i \vec{M}$. Dotted with $\vec{H}$, as in Eq. (32), we see that this gives a term that is odd under time reversal $\mathcal{T}$. Since R is even under $\mathcal{T}$, the nonadiabatic term cannot contribute to R and will be canceled by another thermodynamic flux cross term.

If our system were superconducting, the thermodynamic variables would be different. Then, $\partial_i\tilde{\mu}$ would be replaced by a superfluid velocity v_{si}, because such a metastable thermodynamic state can be set up by an effective vector potential in a thermal distribution. On the other hand, when the current can decay, the thermal weighting that drives the current is not a vector potential, but rather is a non-uniform chemical potential.

C. Two-band magnet

Equations (16), (17), and (20)–(22) placed in the time derivative of Eq. (13) yield

$$0 \le R = T(\partial_t s + \partial_i j_i^s) = -\partial_i(j_i^\epsilon - Tj_i^s - \mu_\uparrow^* j_{\uparrow i} - \mu_\downarrow^* j_{\downarrow i}) - j_i^s\partial_i T - j_{\uparrow i}\partial_i\mu_\uparrow^* - j_{\downarrow i}\partial_i\mu_\downarrow^* - (\mu_\uparrow^* - \mu_\downarrow^*)S + \vec{\Omega}\cdot(\vec{M}\times\vec{H}). \tag{33}$$

Here, j_i^s, $j_{\uparrow i}$, and $j_{\downarrow i}$ are thermodynamic fluxes and $\partial_i T$, $\partial_i\mu_\uparrow^*$, and $\partial_i\mu_\downarrow^*$ are thermodynamic forces; further, S and $\vec{\Omega}$ are thermodynamic sources and $(\mu_\uparrow^* - \mu_\downarrow^*) = \gamma\hbar\vec{H}^*\cdot\hat{M}$ and $\vec{M}\times\vec{H}$ are thermodynamic forces.

Each of the five nondivergence terms in Eq. (33) has a clear physical interpretation as a source of heating: the first term to thermal conduction, the second and third to (spin-dependent) electrical conduction, the fourth to (local) longitudinal magnetic damping, and the fifth to (local) transverse magnetic damping.

VII. CURRENT-INDUCED SPIN TORQUE AND SPIN PUMPING OF CURRENT

For details of how the various fluxes are determined in proportion to the thermodynamic forces, using the Onsager reciprocity relations, see the Appendix. In principle, Kubo-type formulas can be determined for the Onsager coefficients that appear below, on the phenomenological grounds of irreversible thermodynamics.

We are now prepared to discuss both the current-induced spin torque and the spin-pumping of current. For the two models studied, we will restrict ourselves to the appropriate terms in the net current density and the spin torque. For simplicity, we will consider only the subset of terms that are relevant to spin torque and spin pumping, and for that reason, we use the symbol $\supset$ to indicate that the appropriate quantities contain certain terms, but are not restricted to those terms.

A. Spin torque and spin-pumping results

1. Generic conducting magnet

Equation (A3) contains the spin pumping of number current terms

$$j_i^n \supset -L_{nM1}(\partial_i\vec{M})\cdot\vec{H}^* - L_{nM2}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*) - L'_{nM}\partial_i\vec{M}\cdot(\vec{M}\times\vec{H}). \tag{34}$$

The second of these is small if the magnetization is nearly saturated, as we will assume.

Equation (A5) contains the current-driven spin torque terms

$$\vec{N} \supset L'_{Mn}(\hat{M}\times\partial_i\vec{M})\partial_i\tilde{\mu} + L_{Mn1}\partial_i\vec{M}\partial_i\tilde{\mu} + L_{Mn2}\hat{M}(\hat{M}\cdot\partial_i\vec{M})\partial_i\tilde{\mu}. \tag{35}$$

Equation (A4) contains the current-driven spin torque terms

$$\vec{Q}_i \supset L_{Qn}\vec{M}\partial_i\tilde{\mu}. \tag{36}$$

The total non-Larmor spin torque $\vec{N}'$ thus contains the current-driven terms

$$\vec{N}' \supset L'_{Mn}(\hat{M}\times\partial_i\vec{M})\partial_i\tilde{\mu} + (L_{Mn1} - L_{Qn})\partial_i\vec{M}\partial_i\tilde{\mu} + L_{Mn2}\hat{M}(\hat{M}\cdot\partial_i\vec{M})\partial_i\tilde{\mu}. \tag{37}$$

For the generic conducting magnet, Eqs. (34) and (37) provide the basis of our later discussion of the relationship between spin torque and spin pumping.

2. Two-band magnet

In the two-band magnet, each spin component has, in principle, its own electrochemical potential. In some cases, such a distinction can be made experimentally. (For example, by a suitable combination of electric field and of magnetic field gradient, it may be possible to produce a net spin current but zero net electric current.) For simplicity, however, let us consider that $\mu_\uparrow^* = \mu_\downarrow^*$. Then, by Eq. (A22), j_i^n contains the field-driven terms

$$j_i^n \supset -(\vec{M}\times\partial_i\vec{M})\cdot(\vec{M}\times\vec{H})(L_{\downarrow M} + L_{\uparrow M}) - \partial_i\vec{M}\cdot(\vec{M}\times\vec{H})(L'_{\downarrow M} + L'_{\uparrow M}). \tag{38}$$

For the non-Larmor spin torque in the two-band magnet, by Eq. (31) we need only the transverse terms, due to $\vec{\Omega}\times\vec{M}$. If we set $\mu_\uparrow^* = \mu_\downarrow^* = \tilde{\mu}$, then Eq. (A17) contains the current-driven terms

$$\vec{\Omega} \supset (L_{M\uparrow} + L_{M\downarrow})(\vec{M}\times\partial_i\vec{M})\partial_i\tilde{\mu} + (L'_{M\uparrow} + L'_{M\downarrow})\partial_i\vec{M}\partial_i\tilde{\mu}, \tag{39}$$

so that, by Eq. (31),

$$\vec{N}' \supset (L_{M\uparrow} + L_{M\downarrow})M^2\partial_i\vec{M}\partial_i\tilde{\mu} - (L'_{M\uparrow} + L'_{M\downarrow})M^2(\hat{M}\times\partial_i\vec{M})\partial_i\tilde{\mu}. \tag{40}$$

For the two-band magnet, Eqs. (38) and (40) provide the basis of our later discussion of the relationship between spin torque and spin pumping.

3. Comparison with previous work

Equation (7) of Ref. 9, using a phenomenology based on both up and down bands, has the form

$$j_i^{n*} = \frac{\sigma_\uparrow + \sigma_\downarrow}{e}E_i - \frac{D_\uparrow + D_\downarrow}{2}\partial_i n - \frac{D_\uparrow - D_\downarrow}{\hbar}\hat{M}\cdot\partial_i\vec{M}. \tag{41}$$

Equation (9) of Ref. 9 has the form

$$Q_i^* = \frac{\hbar}{2}\frac{\sigma_\uparrow - \sigma_\downarrow}{e}\hat{M}E_i - \frac{\hbar}{2}\frac{D_\uparrow - D_\downarrow}{2}\hat{M}\partial_i n - \frac{D_\uparrow + D_\downarrow}{2}\partial_i \vec{M}. \quad (42)$$

These forms are very similar to what we have derived in that there are three thermodynamic forces in play. For Ref. 9, they are the gradient of the voltage, the gradient of the density, and the longitudinal gradient of the magnetization. In the two-band magnet, they are the gradient of the electrochemical potentials, which depend on the voltage and on the densities of the up and down spins.

We now turn to the net torque. The sum of Eqs. (2) and (4) of Ref. 24 has the form

$$\vec{N}^* = -\lambda[\hat{M}\times(\vec{M}\times\vec{H})] - v_i[\partial_i\vec{M} - \beta\hat{M}\times\partial_i\vec{M}], \quad (43)$$

where β is dimensionless and we expect that $\beta \ll 1$.[24]

Microscopic theories for the adiabatic spin torque give, with P the polarization of the current (e.g., 0.6 for 60% in the up band),[24]

$$v_i = -\frac{Pj_i\mu_B}{eM} \quad \text{(model)}. \quad (44)$$

To make a proper comparison with this form, we must replace the current density j_i (a flux) by the form it takes when driven by the "force" $\partial_i\tilde{\mu}$. We also employ a less model-dependent form by introducing the constant ξ, with units of a diffusion constant divided by energy, and let $j_i \to (\sigma/e)\partial_i\tilde{\mu}$. Thus, we write

$$v_i = -\xi\partial_i\tilde{\mu}, \quad (45)$$

so Eq. (43) becomes

$$\vec{N}^* = -\lambda[\hat{M}\times(\vec{M}\times\vec{H})] + \xi\partial_i\tilde{\mu}[\partial_i\vec{M} - \beta\hat{M}\times\partial_i\vec{M}]. \quad (46)$$

In this form, which is appropriate to irreversible thermodynamics, the adiabatic spin torque (proportional to $\partial_i\vec{M}$) is odd under time reversal, opposite the even signature of a nondissipative spin torque. Therefore, the adiabatic spin torque is dissipative, as can be seen from its contribution to the rate of heating R above. On the other hand, the nonadiabatic spin torque (proportional to $\hat{M}\times\partial_i\vec{M}$) is even under time reversal, signifying that it is nondissipative, as can be seen by its absence from R.

For the above model, we then have

$$\xi = \frac{P\sigma\mu_B}{e^2M} \quad \text{(model)}. \quad (47)$$

To close this section, we note that Ref. 7 uses very different methods to show that, at surfaces, spin transfer torque and spin pumping are related. We also note that Refs. [10–12] use a Berry phase and a spin-Berry phase to predict that, for ferromagnetic conductors, there is an effective spin-dependent emf that drives an ordinary electric current and a spin emf that drives a spin current.

4. Comparison with the two models

Generic conducting magnet. Comparison of Eq. (46) and the generic conducting magnet result [Eq. (37)] shows that the two versions of the spin torque have the same form if

$$(L_{Mn1} - L_{Qn}) = \xi, \quad L'_{Mn} = -\beta\xi. \quad (48)$$

Two-band magnet. Comparison of Eq. (46) and the two-band magnet result [Eq. (40)] shows that the two versions of the spin torque have the same form if

$$(L_{M\uparrow} + L_{M\downarrow})M^2 = \xi, \quad (L'_{M\uparrow} + L'_{M\downarrow})M^2 = \beta\xi. \quad (49)$$

Note that the generic (g) conducting magnet and the two-band ($2b$) magnet have transport coefficients with different dimensionalities. By Eqs. (48) and (49), for later purposes, we write

$$L_g \approx L_{2b}M^2 \approx \xi. \quad (50)$$

B. Estimates

For purposes of estimation, we will employ Table I of Ref. 9, which for Co gives $P=0.6$ and $M=1.45\times10^6$ A/m and implies that $\sigma=\sigma_\uparrow+\sigma_\downarrow=3.4\times10^7/\Omega$ m. With standard values of $e=1.6\times10^{-19}$ C and $\mu_B=9.3\times10^{-24}$ A m^2, we estimate that, for Co, $\xi\approx5.1\times10^{15}$ m^2/J s. (As indicated above, this has units of a diffusion constant over an energy. Indeed, use of $\sigma=ne^2\tau/m$ and $M=Pn\mu_B$ gives $\xi=\tau/m$. For a bare electron mass m, this corresponds to the somewhat short time of $\tau\approx4.5\times10^{-15}$ s.) We take $\gamma\approx1.9\times10^{11}$/T s.

Let δ be a characteristic domain wall dimension, which for purpose of estimation (in Co) we will take to be 10 nm. Within the domain walls, we may take $|\partial_i\vec{M}|\approx2M/\delta$ (2 because the magnetization reverses). In order to make estimates, we will neglect the vector nature of various quantities and not consider signs. We will consider only the adiabatic spin torque, which we believe dominates experimentally.[24]

On neglecting spin-pumping terms, a field E yields a current density j proportional to the conductivity σ via

$$j \approx \sigma E. \quad (51)$$

For non-uniform magnets, there is also a current-induced spin torque N_{ST} that is proportional to j. We define the equivalent spin torque field H_{ST} via

$$H_{ST} = \frac{N_{ST}}{\gamma M}. \quad (52)$$

From Eq. (46), with L_g the appropriate Onsager coefficient [by Eq. (48), for estimation purposes L_g is like ξ, with units of a diffusion constant divided by an energy], we have

$$N_{ST} = L_g\frac{eM}{\delta}E. \quad (53)$$

Now, define

$$R_{ST} = \frac{H_{ST}}{E}, \quad (54)$$

which is in T m/V. Then, by Eqs. (51) and (52), Eq. (54) yields

$$R_{ST} = \frac{N_{ST}}{\gamma ME} = L_g \frac{e}{\gamma\delta}. \quad (55)$$

From the "velocity" of Eqs. (44) and (51), we can write the spin torque terms of Eq. (46) as

$$N_{ST} = P\sigma E\mu_B \frac{1}{eM}\frac{M}{\delta} = \frac{P\sigma\mu_B}{e\delta}E. \quad (56)$$

Comparison of Eqs. (53) and (56) gives the result

$$L_g = \frac{P\sigma\mu_B}{e^2 M}. \quad (57)$$

This permits the estimate that, for Co, L_g=5.1 $\times 10^{15}$ m^2/J s. Then, again for Co, application of Eq. (55) gives R_{ST}=4$\times 10^{-7}$ T m/V. Therefore, within a domain wall a true E field E_0=1.0$\times 10^4$ V/m can cause the same torque as direct application of a magnetic field H_{ST}=4$\times 10^{-3}$ T. This merely restates what is already known.

From Eq. (34), an applied field H_0 that produces a torque

$$N = \gamma M H_0 \quad (58)$$

also produces a spin-pumping driven current within a domain wall. With $|\partial_i \vec{M}| \approx 2M/\delta$, this current is given by

$$j_{SP} = eL_g \frac{2M}{\delta} H_0. \quad (59)$$

We define the equivalent spin-pumping field E_{SP} via

$$E_{SP} = \frac{j_{SP}}{\sigma}. \quad (60)$$

With the definition

$$R_{SP} = \frac{E_{SP}}{H_0}, \quad (61)$$

we find that

$$R_{SP} = \frac{j_{SP}}{\sigma H_0} = \frac{2eM}{\delta\sigma} L_g. \quad (62)$$

Note that R_{SP} and R_{ST} have units that are inverse to one another.

For Co, this gives R_{SP}=7.0$\times 10^3$ V/T m. Thus, within a domain wall a true H field H_0=0.1 T can cause the same current as direct application of an electric field E_{SP} =700 V/m. The corresponding voltage difference across the domain wall is on the order of $E_{SP}\delta$, or 7.0$\times 10^{-6}$ V.

Consider the situation depicted in Fig. 1, where $\vec{H}$ points rightward and $\partial_x \vec{M}$ points leftward. With $L_{nM1} \approx \xi > 0$, by Eq. (34) the number current density points rightward, so that (within the domain wall) the electric current density points leftward.

C. Comparison with Barnes and Maekawa

For comparison with Ref. 10, we define a spin-pumping emf

$$\mathcal{E}_{SP} = E_{SP}\delta. \quad (63)$$

Then, the emf divided by the field, on using Eqs. (61), (62), and (57), is given by

$$\frac{\mathcal{E}_{SP}}{H_0} = \frac{E_{SP}}{H_0}\delta = \frac{2P\mu_B}{e}. \quad (64)$$

Reference 10 finds a quantity $\mathcal{E}_s$ to be given by $2\mu_B H_0/e$, and that $P\mathcal{E}_s$ drives a current density. On division by H, this is precisely Eq. (64). We believe that this exact agreement, but not the parameter dependence, is accidental.

VIII. SUMMARY AND DISCUSSION

Using irreversible thermodynamics, we have shown, for both a two-band magnet and a generic conducting magnet, that current-induced spin transfer torque within a magnetic domain leads, by Onsager relations, to spin pumping of current within that domain. For a given amount of adiabatic and nonadiabatic spin torques, the two models yield similar but distinct results for the bulk spin pumping, thus distinguishing the two models.

This has experimental implications both for samples with conducting leads and that are electrically isolated. For Co, we estimate that within a domain wall a true H field H_0 =0.1 T can cause the same spin-pumped current as direct application of an electric field E_{SP}=350 V/m, or across the domain wall can cause the same effect as a voltage difference of 7.0$\times 10^{-6}$ V. Correspondingly, the ratio of the effective emf to the field H_0 is, for Co, about 0.7$\times 10^{-4}$ V/T.

The similarity between our results and those of Barnes and Maekawa is likely not an accident. In the present case, we have shown that the "off-diagonal" current-induced adiabatic spin transfer torque implies a similar off-diagonal adiabatic spin pumping of current. Barnes and Maekawa[10] show that, in addition to being able to generate a spin transfer torque,[4] a spin-dependent Berry phase can generate what we have called spin pumping. Their approach also gives a natural way to understand the associated emf-to-field ratio, $2\mu_B/e$. Both Refs. 11 and 12 make similar spin-pumping predictions, with specific application to its generation by domain wall motion.

Using irreversible thermodynamics, from the time-reversal properties of the thermodynamic fluxes we have shown that both the adiabatic spin transfer torque and the adiabatic spin-pumping emf correspond to irreversible processes. The irreversible nature of these quantities is not clear from Berry-phase approaches, where one employs currents (fluxes) as primary variables, as if they were driven by a vector potential. However, in considering experimental quantities, which correspond to thermal averaging, these terms must be considered to be driven by non-uniform chemical potentials. This establishes the irreversible nature of the adiabatic spin transfer torque and the adiabatic spin-pumping emf.

Although we did not emphasize the point, the present theory predicts that heat flow can also produce spin transfer torque, and that spin pumping can also produce entropy currents. A related work on heat flow producing spin transfer torque has recently been published.[34] Comparison with that work is not obvious, because it considers transfer across surfaces, but it seems likely that the same physics is in play there as we have considered in the present work.

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APPENDIX: SOURCES AND FLUXES

1. Generic conducting magnet

This model has been treated in the absence of terms associated with the conducting electrons.[15]

Energy flux j_i^ϵ. The energy flux is given by constraining the divergence to be zero (up to an arbitrary curl), which leads to

$$j_i^\epsilon = T j_i^s + \tilde{\mu} j_i^n + \vec{H}^* \cdot \vec{Q}_i. \tag{A1}$$

We now must express each flux and source as the sum over the suitably weighted "forces" $\partial_i T$, $\partial_i \tilde{\mu}$, $\vec{H}^*$, and $\partial_i \vec{H}^*$, all of which are zero in equilibrium. The coefficients may be constructed from the "order parameters" of the equilibrium state, $\vec{M}$ and $\partial_i \vec{M}$. The vector nature of the fluxes must be respected (including their properties under both real space and spin space rotations).

Entropy flux j_i^s. The entropy flux, a vector in real space whose nondissipative part is odd under time reversal $\mathcal{T}$, takes the form

$$j_i^s = -\frac{\kappa}{T}\partial_i T - L_{sn}\partial_i \tilde{\mu} - L_{sQ}\vec{M}\cdot\partial_i \vec{H}^* - L_{sM1}(\partial_i \vec{M})\cdot\vec{H} - L_{sM2}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*) - L'_{sM}\partial_i\vec{M}\cdot(\vec{M}\times\vec{H}). \tag{A2}$$

There are six terms. The terms with unprimed coefficients are even under time reversal $\mathcal{T}$, signifying dissipation. The term with a primed coefficient is odd under $\mathcal{T}$, signifying no dissipation. κ is the usual thermal conductivity. L_{sn} has units of [diffusion constant × density/temperature]. L_{sQ}, L_{sM1}, and L_{sM2} have units of [velocity × length/temperature] or [diffusion constant/temperature]. L'_{sM} has units of [diffusion constant × density / temperature × magnetization].

Number flux j_i^n. The number flux, like the entropy flux a vector in real space whose nondissipative part is odd under time reversal $\mathcal{T}$, takes the form

$$j_i^n = -\frac{\sigma}{e^2}\partial_i\tilde{\mu} - L_{ns}\partial_i T - L_{nQ}\vec{M}\cdot\partial_i\vec{H}^* - L_{nM1}(\partial_i\vec{M})\cdot\vec{H}^* - L_{nM2}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*) - L'_{nM}\partial_i\vec{M}\cdot(\vec{M}\times\vec{H}). \tag{A3}$$

There are six terms. The terms with unprimed coefficients are even under time reversal $\mathcal{T}$, signifying dissipation. The term with a primed coefficient is odd under $\mathcal{T}$, signifying no dissipation. σ is the usual electrical conductivity. We will later interpret the last three terms as *bulk spin-pumping* terms, and we will relate them to the *bulk spin torque* terms, to be discussed shortly. L_{ns} has units of [diffusion constant × density/temperature]. L_{nQ}, L_{nM1}, and L_{nM2} have units of [diffusion constant/energy]. L'_{nM} has units of [diffusion constant × density / energy × magnetization].

Magnetization flux $\vec{Q}_i$. The magnetization flux, whose nondissipative part is even under time reversal, takes the form

$$\vec{Q}_i = C_\parallel\vec{M}(\hat{M}\cdot\partial_i\vec{H}^*) - C_\perp\vec{M}\times(\hat{M}\times\partial_i\vec{H}^*) - C'\vec{M}\times\partial_i\vec{H}^* + L_{Qs}\vec{M}\partial_i T + L_{Qn}\vec{M}\partial_i\tilde{\mu} + L_{QN1}\partial_i\vec{M}(\hat{M}\cdot\vec{H}^*) + L_{QN2}(\hat{M}\cdot\partial_i\vec{M})\vec{H}^* + L_{QN3}\hat{M}(\partial_i\vec{M}\cdot\vec{H}^*) + L_{QN4}\hat{M}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*) + L'_{QN1}\partial_i\vec{M}\times\vec{H}^* + L'_{QN2}\hat{M}[(\hat{M}\times\partial_i\vec{M})\cdot\vec{H}^*] + L'_{QN3}(\hat{M}\times\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*) + L'_{QN4}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\times\vec{H}). \tag{A4}$$

There are 13 terms. The terms with unprimed coefficients are odd under time reversal $\mathcal{T}$, signifying dissipation. The terms with primed coefficients are even under $\mathcal{T}$, signifying no dissipation. The terms involving $\vec{H}$ are discussed in Ref. 15, as is the term involving $\partial_i T$. $C_\parallel$ and $C_\perp$ have dimensions of a diffusion constant/magnetization, and indeed they represent longitudinal and transverse diffusion. C' has the dimensions of [diffusion constant/magnetization]. L_{Qn} has units of [diffusion constant/energy]. The term associated with L_{Qn}, when the divergence is taken, will lead to a *bulk spin torque* term. All of the terms in L_{QN} and L'_{QN} have units of [diffusion constant/field], with field in tesla. Note that, for near saturation of the magnetization, the terms $\hat{M}\cdot\partial_i\vec{M}$ will be small, so that only L_{QN1} and L_{QN3} will yield possibly important new terms in the damping.

Spin torque density $\vec{N}$. The spin torque density, whose nondissipative part is even under time reversal, takes the form

$$\vec{N} = A_\parallel\vec{M}(\vec{M}\cdot\vec{H}^*) - A_\perp\vec{M}\times(\vec{M}\times\vec{H}) + L_{Ms1}\partial_i\vec{M}\partial_i T + L_{Ms2}\hat{M}(\hat{M}\cdot\partial_i\vec{M})\partial_i T + L'_{Ms}(\hat{M}\times\partial_i\vec{M})\partial_i T + L'_{Mn}(\hat{M}\times\partial_i\vec{M})\partial_i\tilde{\mu} + L_{Mn1}\partial_i\vec{M}\partial_i\tilde{\mu} + L_{Mn2}\hat{M}(\hat{M}\cdot\partial_i\vec{M})\partial_i\tilde{\mu} + L_{NQ1}\hat{M}(\partial_i\vec{M}\cdot\partial_i\vec{H}^*) + L_{NQ2}(\hat{M}\cdot\partial_i\vec{M})\partial_i\vec{H}^* + L_{NQ3}\partial_i\vec{M}(\hat{M}\cdot\partial_i\vec{H}^*) + L_{NQ4}\hat{M}(\hat{M}\cdot\partial_i\vec{M}) \times(\hat{M}\cdot\partial_i\vec{H}^*) + L'_{NQ1}\partial_i\vec{M}\times\partial_i\vec{H}^* + L'_{NQ2}(\hat{M}\times\partial_i\vec{M})(\hat{M}\cdot\partial_i\vec{H}^*) + L'_{NQ3}\hat{M}(\hat{M}\times\partial_i\vec{M})\cdot\partial_i\vec{H}^* + L'_{NQ4}(\hat{M}\cdot\partial_i\hat{M})\hat{M}\times\partial_i\vec{H}^*. \tag{A5}$$

There are 16 terms. The terms with unprimed coefficients are odd under time reversal $\mathcal{T}$, signifying dissipation. The terms with primed coefficients are even under $\mathcal{T}$, signifying no dissipation. The terms involving $\partial_i\vec{H}$ have been discussed in Ref. 15, as has the term involving $\partial_i T$. We interpret the terms in L_{Mn1}, L_{Mn2}, and L'_{Mn} as bulk spin torque terms. $A_\parallel$ and $A_\perp$ have units of $[1/\text{time}\times(\text{magnetization})^2]$ for $\vec{H}$ given in A/m, or they must include a factor of $1/\mu_0$ if $\vec{H}$ is given in T. They represent longitudinal and transverse damping; in terms of the Landau-Lifshitz parameter λ, one has $A_\perp=\lambda/M$. L_{Ms1} and L_{Ms2} have units of [velocity × length/temperature] or [diffusion constant/temperature]. L'_{Ms} has units of [diffusion constant × density/temperature × magnetization]. L_{Mn1} and L_{Mn2} have units of [diffusion constant/energy]. L'_{Mn} has units of [diffusion constant × density/energy × magnetization]. All of the terms in L_{NQ} and L'_{NQ} have units of [diffusion constant/field], with field in tesla.

Rate of entropy production R. The rate of entropy production is strictly even under time reversal. A total of 41 terms can contribute to R when the above equations are substituted to find R. The term involving C' is identically zero. There are six diagonal terms once the longitudinal and transverse parts of $\vec{Q}_i$ and $\vec{N}$ are accounted for. The remaining 34 terms are cross terms that occur in pairs. We find that

$$\begin{aligned}R={}&\frac{\kappa}{T}(\partial_iT)^2+\frac{\sigma}{e^2}(\partial_i\tilde{\mu})^2+(L_{sn}+L_{ns})\partial_iT\partial_i\tilde{\mu}+C_\parallel(\vec{M}\cdot\partial_i\vec{H}^*)^2+C_\perp(\vec{M}\times\partial_i\vec{H}^*)^2+A_\parallel(\vec{M}\cdot\vec{H}^*)^2+A_\perp(\vec{M}\times\vec{H})^2\\&+(L_{sQ}+L_{Qs})\vec{M}\cdot\partial_i\vec{H}^*\partial_iT+(L_{nQ}+L_{Qn})\vec{M}\cdot\partial_i\vec{H}^*\partial_i\tilde{\mu}+(L_{sM1}+L_{Ms1})\partial_i\vec{M}\cdot\vec{H}^*\partial_iT+(L_{nM1}+L_{Mn1})\partial_i\vec{M}\cdot\vec{H}^*\partial_i\tilde{\mu}+(L_{sM2}+L_{Ms2})\\&\times(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)\partial_iT+(L_{nM2}+L_{Mn2})(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)\partial_i\tilde{\mu}+(L'_{sM}+L'_{Ms})(\hat{M}\times\partial_i\vec{M})\cdot\vec{H}^*\partial_iT+(L'_{nM}+L'_{Mn})(\hat{M}\\&\times\partial_i\vec{M})\cdot\vec{H}^*\partial_i\tilde{\mu}+(L_{QN1}+L_{NQ1})(\hat{M}\cdot\vec{H}^*)(\partial_i\vec{M}\cdot\partial_i\vec{H}^*)+(L_{QN2}+L_{NQ2})(\hat{M}\cdot\partial_i\vec{M})(\vec{H}^*\cdot\partial_i\vec{H}^*)+(L_{QN3}+L_{NQ3})(\hat{M}\cdot\partial_i\vec{H}^*)\\&\times(\partial_i\vec{M}\cdot\vec{H}^*)+(L_{QN4}+L_{NQ4})(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)(\hat{M}\cdot\partial_i\vec{H}^*)+(L'_{QN1}+L'_{NQ1})(\partial_i\vec{M}\times\vec{H}^*\cdot\partial_i\vec{H}^*)+(L'_{QN2}+L'_{NQ2})(\hat{M}\\&\times\partial_i\vec{M}\cdot\vec{H}^*)(\hat{M}\cdot\partial_i\vec{H}^*)+(L'_{QN3}+L'_{NQ3})(\hat{M}\cdot\vec{H}^*)(\hat{M}\times\partial_i\vec{M}\cdot\partial_i\vec{H}^*)+(L'_{QN4}+L'_{NQ4})(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\times\vec{H}^*\cdot\partial_i\vec{H}^*).\end{aligned}\tag{A6}$$

Recall that $\vec{M}\times\vec{H}=\vec{M}\times\vec{H}^*$ and that $\vec{H}^*=\vec{0}$ in equilibrium, so that there are no dissipative terms associated with $\vec{H}^*$ when the magnetization is in equilibrium.

To ensure that R is invariant under time reversal, the nondissipative cross terms (which are odd under $\mathcal{T}$) must be eliminated. This is done by imposing the conditions

$$L'_{sM}=-L'_{Ms},\quad L'_{nM}=-L'_{Mn},$$

$$L'_{NQi}=-L'_{QNi},\quad i=1,4.\tag{A7}$$

In addition, in order to satisfy the Onsager relations for dissipative cross terms (which are even under $\mathcal{T}$), we must impose the conditions

$$L_{ns}=L_{ns},\quad L_{sQ}=L_{Qs},\quad L_{nQ}=L_{Qn},$$

$$L_{sM1}=L_{Ms1},\quad L_{sM2}=L_{Ms2},$$

$$L_{nM1}=L_{Mn1},\quad L_{nM2}=L_{Mn2},$$

$$L_{QNi}=L_{NQi},\quad i=1,4.\tag{A8}$$

We then have

$$\begin{aligned}R={}&\frac{\kappa}{T}(\partial_iT)^2+\frac{\sigma}{e^2}(\partial_i\tilde{\mu})^2+(L_{sn}+L_{ns})\partial_iT\partial_i\tilde{\mu}+C_\parallel(\vec{M}\cdot\partial_i\vec{H}^*)^2+C_\perp(\vec{M}\times\partial_i\vec{H}^*)^2+A_\parallel(\vec{M}\cdot\vec{H}^*)^2+A_\perp(\vec{M}\times\vec{H})^2\\&+2L_{sQ}\vec{M}\cdot\partial_i\vec{H}^*\partial_iT+2L_{nQ}\vec{M}\cdot\partial_i\vec{H}^*\partial_i\tilde{\mu}+2L_{sM1}\partial_i\vec{M}\cdot\vec{H}^*\partial_iT+2L_{nM1}\partial_i\vec{M}\cdot\vec{H}^*\partial_i\tilde{\mu}+2L_{sM2}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)\partial_iT\\&+2L_{nM2}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)\partial_i\tilde{\mu}+2L_{QN1}(\hat{M}\cdot\vec{H}^*)(\partial_i\vec{M}\cdot\partial_i\vec{H}^*)+2L_{QN2}(\hat{M}\cdot\partial_i\vec{M})(\vec{H}^*\cdot\partial_i\vec{H}^*)+2L_{QN3}(\hat{M}\cdot\partial_i\vec{H}^*)(\partial_i\vec{M}\cdot\vec{H}^*)\\&+2L_{QN4}(\hat{M}\cdot\partial_i\vec{M})(\hat{M}\cdot\vec{H}^*)(\hat{M}\cdot\partial_i\vec{H}^*).\end{aligned}\tag{A9}$$

Note that the terms in L_{Mn1} and L_{Mn2}, which are due both to adiabatic bulk spin torque and adiabatic bulk spin-pumping terms, produce dissipation. Likewise, the terms in L_{nQ}, which also produce a (longitudinal) bulk spin-pumping term and a longitudinal spin flux, produce dissipation.

2. Two-band magnet

Energy flux j_i^{ϵ}. The energy flux is given by constraining the divergence to be zero (up to an arbitrary curl), which leads to

$$j_i^{\epsilon} = T j_i^s - \mu_\uparrow^* j_{\uparrow i} - \mu_\downarrow^* j_{\downarrow i}. \tag{A10}$$

We now must express each flux and source as the sum over the suitably weighted forces $\tilde{\mu}_\uparrow^* - \tilde{\mu}_\downarrow^*$, $\partial_i T$, $\partial_i \tilde{\mu}_\uparrow^*$, $\partial_i \tilde{\mu}_\downarrow^*$, and $\vec{M} \times \vec{H}$, all of which are zero in equilibrium. The coefficients may be constructed from the order parameters of the equilibrium state, $\vec{M}$ and $\partial_i \vec{M}$. The vector nature of the fluxes must be respected (including their properties under both real space and spin space rotations).

Decay rate S. The intrinsic signatures under time reversal $\mathcal{T}$ of $n_\uparrow$ and $n_\downarrow$ are even, so that the intrinsic signatures under $\mathcal{T}$ of their time derivatives are odd. The decay rate S is associated with both $\partial_t n\uparrow$ and $\partial_t n\downarrow$; hence the nondissipative part of S is odd under $\mathcal{T}$.

For S, the only possible form is

$$S = -\alpha(\mu_\uparrow^* - \mu_\downarrow^*), \tag{A11}$$

with α a material-dependent constant having units of $(\mathrm{J\,m^3\,s})^{-1}$. No other form is allowed because S is a scalar in both real space and spin space. One might think that the order parameter $\vec{M}$ could be multiplied by the thermodynamic force $\vec{M} \times \vec{H}$ to obtain a scalar, but that dot product is identically zero. Equation (A11) is even under time reversal, and therefore is dissipative. Recall that $\mu_\uparrow^* - \mu_\downarrow^* = \gamma\hbar \vec{H}^* \cdot \hat{M}$.

For small deviations from equilibrium, we have

$$S \approx -\alpha\left(\frac{\partial\mu_\uparrow^*}{\partial n_\uparrow}\delta n_\uparrow - \frac{\partial\mu_\downarrow^*}{\partial n_\downarrow}\delta n_\downarrow\right) = -\frac{\delta n_\uparrow}{\tau_\uparrow} + \frac{\delta n_\downarrow}{\tau_\downarrow}, \tag{A12}$$

$$\frac{1}{\tau_\uparrow} \equiv \alpha\frac{\partial\mu_\uparrow^*}{\partial n_\uparrow}, \quad \frac{1}{\tau_\downarrow} \equiv \alpha\frac{\partial\mu_\downarrow^*}{\partial n_\downarrow}, \tag{A13}$$

a result that could have been expected on physical grounds. Note that if $\tau_\uparrow = \tau_\downarrow$, then the longitudinal part of the magnetization would have a decay rate proportional to the deviation in the longitudinal magnetization itself. However, since under most circumstances local electroneutrality enforces $\delta n_\uparrow = -\delta n_\downarrow$, so $\delta M \sim \delta n_\uparrow$; this normally will be the case anyway. In this case, we can think of S as determining T_1 processes.

Entropy flux j_i^s. The entropy flux, a vector in real space whose reversible part is odd under time reversal $\mathcal{T}$, takes the form

$$j_i^s = -\frac{\kappa}{T}\partial_i T - L_{s\uparrow}\partial_i \mu_\uparrow^* - L_{s\downarrow}\partial_i\mu_\downarrow^* - L_{sM}(\vec{M} \times \partial_i \vec{M}) \cdot (\vec{M} \times \vec{H}) - L'_{sM}\partial_i \vec{M} \cdot (\vec{M} \times \vec{H}). \tag{A14}$$

There are five terms. The terms with unprimed coefficients are even under time reversal, signifying dissipation. The term with a primed coefficient is odd under time reversal, signifying no dissipation. κ is the usual thermal conductivity. $L_{s\uparrow}$ and $L_{s\downarrow}$, associated with the second and third terms, give a well-known electrothermal effect, whereby a chemical potential gradient can drive an entropy current; they have units of a diffusion constant divided by temperature. The last two terms, which are new, have the same symmetry as corresponding terms in the number flux, which we now discuss. They imply that spin dynamics can drive an entropy current. We may call this spin pumping of an entropy current. $L_{s\vec{M}}$ has units of a diffusion constant divided by magnetization^2 $\times$temperature, and $L'_{s\vec{M}}$ has units of a diffusion constant divided by magnetization$\times$temperature. Note that when the fifth term is multiplied by $\partial_i T$ in Eq. (33) for R, unlike R the product is odd under $\mathcal{T}$; it thus must be canceled by another term (to be discussed below) or it must be zero.

Number fluxes $j_{\uparrow i}$ *and* $j_{\downarrow i}$. Each number flux, like the entropy flux, is a vector in real space whose reactive, or reversible, part is odd under time reversal $\mathcal{T}$. They take the forms [cf. Eq. (A14)]:

$$j_{\uparrow i} = -\frac{\sigma_\uparrow}{e^2}\partial_i\mu_\uparrow^* - L_{\uparrow s}\partial_i T - L_{\uparrow\downarrow}\partial_i\mu_\downarrow^* - L_{\uparrow M}(\vec{M} \times \partial_i\vec{M}) \cdot (\vec{M} \times \vec{H}) - L'_{\uparrow M}\partial_i\vec{M} \cdot (\vec{M} \times \vec{H}), \tag{A15}$$

$$j_{\downarrow i} = -\frac{\sigma_\downarrow}{e^2}\partial_i\mu_\downarrow^* - L_{\downarrow s}\partial_i T - L_{\downarrow\uparrow}\partial_i\mu_\uparrow^* - L_{\downarrow M}(\vec{M} \times \partial_i\vec{M}) \cdot (\vec{M} \times \vec{H}) - L'_{\downarrow M}\partial_i\vec{M} \cdot (\vec{M} \times \vec{H}). \tag{A16}$$

Each of these has five terms. The terms with unprimed coefficients are even under time reversal, signifying dissipation. The term with a primed coefficient is odd under time reversal, signifying no dissipation. $\sigma_{\uparrow\downarrow}$ gives the respective electrical conductivities. $L_{s\uparrow}$ and $L_{s\downarrow}$, associated with the second and third terms, give a well-known electrothermal effect, whereby a chemical potential gradient can drive an entropy current; they have units of a diffusion constant divided by temperature; the terms $L_{\downarrow\uparrow}$ and $L_{\uparrow\downarrow}$ have the same units. The last two terms in both Eqs. (A15) and (A16), which are new, have the same symmetry as corresponding terms in the number flux, which we now discuss. They imply that spin dynamics can drive a current. We call this spin-pumping of current. $L_{\uparrow\vec{M}}$ and $L_{\downarrow\vec{M}}$ have units of a diffusion constant divided by magnetization$^2\times$energy, and $L'_{\uparrow\vec{M}}$ and $L'_{\downarrow\vec{M}}$ have units of a diffusion constant divided by magnetization^2 $\times$energy. Note that when the fifth terms are respectively multiplied by $\partial_i n_\uparrow$ and $\partial_i n_\downarrow$ in Eq. (33) for R, unlike R the product is odd under $\mathcal{T}$, and thus these terms must be canceled by another cross term (discussed below) or they must be zero.

Nonequilibrium rotation rate $\vec{\Omega}$. The reactive, or reversible, part of $\vec{\Omega}$ is odd under $\mathcal{T}$. $\vec{\Omega}$ bears the brunt of the complexity of the Onsager coefficients. In detail, $\vec{\Omega}$ is given by

$$\vec{\Omega} = \lambda \hat{M} \times \vec{H} + L_{Ms}(\vec{M} \times \partial_i \vec{M})\partial_i T + L'_{Ms}\partial_i \vec{M}\partial_i T + L_{M\uparrow}(\vec{M} \times \partial_i \vec{M})\partial_i \mu^*_\uparrow + L'_{M\uparrow}\partial_i \vec{M}\partial_i \mu^*_\uparrow + L_{M\downarrow}(\vec{M} \times \partial_i \vec{M})\partial_i \mu^*_\downarrow + L'_{M\downarrow}\partial_i \vec{M}\partial_i \mu^*_\downarrow. \quad (A17)$$

There are seven terms. The terms with unprimed coefficients are even under $\mathcal{T}$, and thus are dissipative. The terms with primed coefficients are odd under $\mathcal{T}$, and thus are not dissipative. The first term gives Landau-Lifshitz damping. (We believe that any theory based on irreversible thermodynamics that has an energy term varying as $-\vec{H}M \cdot d\hat{M}$ will give a Landau-Lifshitz damping term, since then the corresponding thermodynamic force is $\hat{M} \times \vec{H}$.) The three terms proportional to $\vec{M} \times \partial_i \vec{M}$ are even under $\mathcal{T}$, so they are dissipative. They correspond to spin torque by electric current and by entropy current. Note that there are two types of spin torque, corresponding to the two types of spins. Both of these correspond to what has been called adiabatic spin torque. The term driven by entropy current is new. The three terms proportional to $\partial_i \vec{M}$ are odd under $\mathcal{T}$, so they are nondissipative. They, too, correspond to spin torque by electric current and by entropy current. Again, there are two types of spin torque, corresponding to the two types of spins. Both of these correspond to what has been called adiabatic spin torque. The term driven by entropy current is new.

Rate of heat production R. The rate of entropy production is strictly even under time reversal. We now rewrite Eq. (33) in light of Eq. (A10), which eliminates the divergence term, and in light of the equations for the various thermodynamic fluxes and sources. We then have

$$R = \alpha(\mu^*_\uparrow - \mu^*_\uparrow)^2 + \frac{\kappa}{T}(\partial_i T)^2 + \frac{\sigma_\uparrow}{e^2}(\partial_i \mu^*_\uparrow)^2 + \frac{\sigma_\downarrow}{e^2}(\partial_i \mu^*_\downarrow)^2 + \frac{\lambda}{M}(\vec{M} \times \vec{H})^2 + (\partial_i T)(\partial_i \mu^*_\uparrow)(L_{s\uparrow} + L_{\uparrow s}) + (\partial_i T)(\partial_i \mu^*_\downarrow)(L_{s\downarrow} + L_{\uparrow s}) + (\partial_i \mu^*_\uparrow)(\partial_i \mu^*_\downarrow)(L_{\uparrow\downarrow} + L_{\downarrow\uparrow}) + (\vec{M} \times \partial_i \vec{M}) \cdot (\vec{M} \times \vec{H})[(L_{sM} + L_{Ms})\partial_i T + (L_{\uparrow M} + L_{M\uparrow})\partial_i \mu^*_\uparrow + (L_{\downarrow M} + L_{M\downarrow})\partial_i \mu^*_\downarrow] + (\partial_i \vec{M}) \cdot (\vec{M} \times \vec{H})[(L'_{sM} + L'_{Ms})\partial_i T + (L'_{\uparrow M} + L'_{M\uparrow})\partial_i \mu^*_\uparrow + (L'_{\downarrow M} + L'_{M\downarrow})\partial_i \mu^*_\downarrow]. \quad (A18)$$

To ensure that R is invariant under time reversal, the nondissipative cross terms (which are odd under $\mathcal{T}$) must be eliminated. This is done by imposing the conditions

$$L'_{Ms} = -L'_{sM}, \quad L'_{M\uparrow} = -L'_{\uparrow M}, \quad L'_{M\downarrow} = -L'_{\downarrow M}. \quad (A19)$$

In addition, in order to satisfy the Onsager relations for dissipative cross terms (which are even under $\mathcal{T}$), we must impose the conditions

$$L_{\uparrow s} = L_{s\uparrow}, \quad L_{\downarrow s} = L_{s\downarrow}, \quad L_{\uparrow\downarrow} = L_{\downarrow\uparrow}, \quad L_{Ms} = L_{sM}, \quad L_{M\uparrow} = L_{\uparrow M}, \quad L_{M\downarrow} = L_{\downarrow M}. \quad (A20)$$

We then have

$$R = \alpha(\mu^*_\uparrow - \mu^*_\uparrow)^2 + \frac{\kappa}{T}(\partial_i T)^2 + \frac{\sigma_\uparrow}{e^2}(\partial_i \mu^*_\uparrow)^2 + \frac{\sigma_\downarrow}{e^2}(\partial_i \mu^*_\downarrow)^2 + \frac{\lambda}{M}(\vec{M} \times \vec{H})^2 + 2L_{\uparrow\downarrow}(\partial_i \mu^*_w)(\partial_i \mu^*_\downarrow) + 2L_{s\uparrow}(\partial_i T)(\partial_i \mu^*_\uparrow) + 2L_{s\downarrow}(\partial_i T)(\partial_i \mu^*_\downarrow) + 2(\vec{M} \times \partial_i \vec{M}) \cdot (\vec{M} \times \vec{H})[L_{sM}(\partial_i T) + L_{\uparrow M}(\partial_i \mu^*_\uparrow) + L_{\downarrow M}(\partial_i \mu^*_\downarrow)]. \quad (A21)$$

The first five terms are diagonal in the fluxes and source. They are always positive. The other six terms, involving off-diagonal products of the thermodynamic forces, can have either sign, dependent on the relative directions of these gradients. The constraint of positive definiteness for R then imposes limits on the magnitudes of the off-diagonal coefficients, which we do not enumerate.

3. Derived quantities

We now turn to the derived quantities j^n_i, $Q_{\alpha i}$, and N_α.

Number current density j^n_i. By Eqs. (23), (A15), and (A16), we have

$$j^n_i = -\partial_i T(L_{\uparrow s} + L_{\downarrow s}) - \partial_i \mu^*_\uparrow\left(\frac{\sigma_\uparrow}{e^2} + L_{\downarrow\uparrow}\right) - \partial_i \mu^*_\downarrow\left(\frac{\sigma_\downarrow}{e^2} + L_{\uparrow\downarrow}\right) - (\vec{M} \times \partial_i \vec{M}) \cdot (\vec{M} \times \vec{H})(L_{\downarrow M} + L_{\uparrow M}) - \partial_i \vec{M} \cdot (\vec{M} \times \vec{H})(L'_{\downarrow M} + L'_{\uparrow M}). \quad (A22)$$

Magnetization flux $\vec{Q}_i$. This quantity is the sum of ten terms, which we obtain from Eqs. (29), (A15), and (A16). We have

$$\vec{Q}_i = \gamma(\hbar/2)\hat{M}\left[\partial_i T(L_{\uparrow s} - L_{\downarrow s}) + \partial_i \mu^*_\uparrow\left(\frac{\sigma_\uparrow}{e^2} - L_{\downarrow\uparrow}\right) - \partial_i \mu^*_\downarrow\left(\frac{\sigma_\downarrow}{e^2} - L_{\uparrow\downarrow}\right) - (\vec{M}\times\partial_i\vec{M})\cdot(\vec{M}\times\vec{H})(L_{\downarrow M} - L_{\uparrow M}) - \partial_i\vec{M}\cdot(\vec{M}\times\vec{H})(L'_{\downarrow M} - L'_{\uparrow M})\right]. \tag{A23}$$

The terms involving gradients of the electrochemical potentials are diffusive in nature.

Magnetization source $\vec{N}$. This quantity is the sum of 18 terms, which we obtain from Eqs. (30) and (A15)–(A17). We have

$$\vec{N} = \gamma(\hbar/2)\partial_i\hat{M}\left[\partial_i T(L_{\uparrow s} - L_{\downarrow s}) + \partial_i \mu^*_\uparrow\left(\frac{\sigma_\uparrow}{e^2} - L_{\downarrow\uparrow}\right) - \partial_i \mu^*_\downarrow\left(\frac{\sigma_\downarrow}{e^2} - L_{\uparrow\downarrow}\right) - (\vec{M}\times\partial_i\vec{M})\cdot(\vec{M}\times\vec{H})(L_{\downarrow M} - L_{\uparrow M}) - \partial_i\vec{M}\cdot(\vec{M}\times\vec{H})(L'_{\downarrow M} - L'_{\uparrow M})\right] - (\gamma\hbar)\hat{M}(-1)\Gamma(\mu^*_\uparrow - \mu^*_\downarrow) - \vec{M}\times\left[\lambda\hat{M}\times\vec{H} + L_{Ms}(\vec{M}\times\partial_i\vec{M})\partial_i T + L'_{Ms}\partial_i\vec{M}\partial_i T + L_{M\uparrow}(\vec{M}\times\partial_i\vec{M})\partial_i\mu^*_\uparrow + L'_{M\uparrow}\partial_i\vec{M}\partial_i\mu^*_\uparrow + L_{M\downarrow}(\vec{M}\times\partial_i\vec{M})\partial_i\mu^*_\downarrow + L'_{M\downarrow}\partial_i\vec{M}\partial_i\mu^*_\downarrow\right]. \tag{A24}$$

To end this section, we note that, by Eq. (31), in the absence of temperature and chemical potential gradients, the non-Larmor spin torque has a transverse part given only by the Landau-Lifshitz term. This is in contrast to the case of the generic conducting magnet, where two new coefficients can contribute to the damping, in a manner that depends on the magnetic texture. For the two-band magnet, there is no such dependence.

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