Striped phases in two-dimensional dipole systems

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We prove that a system of discrete two-dimensional (2D) in-plane dipoles with four possible orientations, interacting via a three-dimensional (3D) dipole-dipole interaction plus a nearest neighbor ferromagnetic term, has periodic striped ground states. As the strength of the ferromagnetic term is increased, the size of the stripes in the ground state increases, becoming infinite, i.e., giving a ferromagnetic ground state, when the ferromagnetic interaction exceeds a certain critical value. We also give a rigorous proof of the *reorientation transition* in the ground state of a 2D system of discrete dipoles with six possible orientations, interacting via a 3D dipole-dipole interaction plus a nearest neighbor antiferromagnetic term. As the strength of the antiferromagnetic term is increased, the ground state flips from being striped and in plane to being staggered and out of plane. An example of a rotator model with a sinusoidal ground state is also discussed.

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I. INTRODUCTION

Recent advances in film growth techniques and in the experimental control of spin-spin interactions have revived interest in the low temperature physics of thin films.[1–12] These quasi-two-dimensional (2D) systems show a wide range of ordering effects including formation of striped states, reorientation transitions, bubble formation in strong magnetic fields, etc.[13–15] The origins of these phenomena are, in many cases, traced to competition between short-ranged exchange (ferromagnetic) interactions, favoring a homogeneous ordered state, and the long-ranged dipole-dipole interaction, which opposes such ordering on the scale of the whole sample. The present theoretical understanding of these phenomena is based on a combination of variational methods and a variety of approximations, e.g., mean-field and spin-wave theory.[14,16–19] The comparison between the predictions of these approximate methods and the results of Monte Carlo simulations are often difficult because of the slow relaxation dynamics associated with the long-range nature of the dipole-dipole interactions.[12,16] It would clearly be desirable to have more rigorous results about the spontaneous formation of such patterns. In a previous paper,[20] we began to investigate these questions by means of a spin-block reflection-positivity method which, combined with *a priori* estimates on the Peierls' contours, allowed us to do the following:

(i) Describe the zero temperature phase diagram of a one-dimensional (1D) Ising model with nearest neighbor ferromagnetic and long-range, reflection positive, antiferromagnetic interactions. These include power-law-type interactions, such as dipolarlike interactions. We proved, in particular, the existence of a sequence of phase transitions between periodic states with longer and longer periods as the nearest neighbor ferromagnetic exchange strength J was increased.

(ii) Derive upper and lower bounds on the ground state energy of a class of 2D Ising models with similar competing interactions, which agreed within exponentially small terms in J with the energy of the striped state.

In this paper, we prove for some models of dipole systems with discrete orientations, on a 2D lattice, that their ground states *display a periodic striped order*. As in the 1D case, the stripe size increases with the strength of the nearest neighbor exchange interaction, becoming infinite, i.e., giving a ferromagnetic ground state, when the ferromagnetic interaction exceeds a certain critical value. The proof is again based on a combination of reflection-positivity and Peierls' estimates and on an exact reduction of our 2D model to the 1D Ising model studied in Ref. 20. The analysis takes explicit account of the tensorial nature of the three-dimensional (3D) dipole-dipole interaction that, in the *absence* of any short-range exchange, tends to produce order in the form of polarized columns (or rows) of aligned spins, with alternating polarization. For dipoles oriented along four possible directions at each site, the ground state shown in Fig. 1 is fourfold degenerate (spin reversal and 90° rotations) and these are the *only* ground states (see Ref. 16 for a more detailed description of pure dipole states and Ref. 21 for some proofs).

The order described in Fig. 1 is induced by the fact that two dipoles with the same orientation attract if their axes are parallel to their relative position vector, and they repel if both their axes are perpendicular to their relative position vector: in other words, the dipole-dipole interaction, although overall antiferromagnetic (in the sense that it prefers total spin equal to zero) is, roughly speaking, ferromagnetic (FM) in one direction and antiferromagnetic in the orthogonal direction. It is then possible to show that, in the presence of an additional nearest neighbor ferromagnetic exchange, the FM order will persist in one direction. In the orthogonal direction, the column-column interaction can then be effectively described in terms of a 1D model, which can be treated by the methods of Ref. 20.

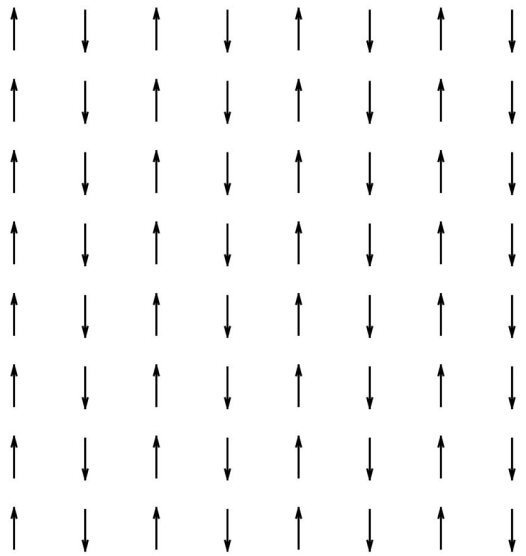

FIG. 1. A ground state of the pure dipole-dipole system in 2D.

A similar method allows us to investigate the zero temperature phase diagram of the 2D discrete dipole model in the presence of a nearest neighbor *antiferromagnetic* exchange, even in the case that the dipoles are allowed to orient along six different directions (four in plane and two out of plane). In this case, we prove the so-called *reorientation transition*,[16,22] consisting in a flip from an in-plane ground state, like the one depicted in Fig. 1, to an out-of-plane staggered state, as the strength of the antiferromagnetic exchange interaction is increased.

The effects of the interplay between the tensorial dipole-dipole interaction and a short-range exchange interaction is still far from being understood in general systems of continuous (Heisenberg-like) spins or in general anisotropic models with different assumptions on the allowed spin orientations and/or on the short-range exchange interactions. It is worth remarking that in these classes of models, the naive procedure of minimizing the Fourier transform of the pair interaction not only is wrong (because it neglects the local constraints coming from the requirement that the norm of the spin vector at each site is equal to 1) but generally leads to very bad estimates on the ground state energy and on the size of the characteristic zero temperature patterns. However, it is interesting to note that, for an $O(n)$ spin model, $n \geqslant 2$, with a *scalar* interaction whose Fourier transform admits a nontrivial minimum at $\mathbf{k}_0 \neq \mathbf{0}$, it is relatively simple to show the existence of "soft" striped, i.e., sinusoidal 1D spin wave, ground states. This was shown via a very general argument by Nussinov[23] for classical $O(n)$ spins. This general result is independent of the dimension and of any reflection positivity of the interactions. It is applied here to the concrete example of rotators in two dimensions interacting with a nearest neighbor ferromagnetic interaction and a weak long-range scalar $1/r^3$ interaction, i.e., a scalar $O(n=2)$ model with competing long-range interactions, imitating the decay properties of the real dipole-dipole potential. The fact that the solution to this scalar isotropic problem is easy adds further motivation to the study of the harder and more exciting case of real dipole systems. (Note that even the existence of the thermodynamic limit in an external field, which is expected to be shape dependent, is unproven for a 3D dipole system).

The rest of this paper is organized as follows. In Sec. II, we introduce the 2D spin model with discrete orientations and state our results about its zero temperature phase diagram. This includes the existence of striped order (in the case of a short-range ferromagnetic exchange) and the existence of a reorientation transition from an in-plane to an out-of-plane ordered state (in the case of an antiferromagnetic exchange). In Sec. III, we discuss an example of a 2D rotator model with long-range scalar interactions whose ground states are given by sinusoidal 1D spin waves ("soft stripes"). In Secs. IV–VII, we present the details of the proof for the 2D spin model with discrete orientations, both for the case of ferromagnetic and of antiferromagnetic exchange. In Sec. VIII, we summarize some aspects of the conjectured positive-temperature behavior of dipole systems with competing interactions. The rigorous analysis of the phase diagram at positive temperatures is presently beyond our reach.

Some results and proofs in the present paper rely on those of the previous paper.[20] We have discovered a minor technical error in the proofs of Theorems 1 and 2 of Ref. 20, which was caused by overlooking several exponentially small terms of the form e^{-cL} which came from the use of periodic boundary conditions on a ring of length L. These errors can be repaired, but we found that everything can be done more easily and clearly using open boundary conditions on the line rather than periodic boundary conditions on the circle. This improved methodology, which might be independently interesting for future work, is given here in the Appendix.

II. TWO-DIMENSIONAL DISCRETE DIPOLES: THE MODEL AND THE MAIN RESULTS

In this section, we introduce the 2D model with discrete orientations and state the main results on the structure of its zero temperature phase diagram. We shall first discuss the stripe formation phenomenon in the presence of a nearest neighbor ferromagnetic interaction and then the reorientation transition phenomenon in the presence of an antiferromagnetic exchange interaction.

A. Ferromagnetic case

Let $\Lambda \subset \mathbb{Z}^2$ be a simple cubic 2D torus of side $2L$ and let $\vec{S}_{\mathbf{x}}$, $\mathbf{x} \in \Lambda$, be an in-plane unit vector with components $\{S^i_{\mathbf{x}}\}_{i=1,2}$. We shall assume that $\vec{S}_{\mathbf{x}}$ can only be oriented along the two coordinate directions of Λ [i.e., that $S^1_{\mathbf{x}}$ and $S^2_{\mathbf{x}}$ can take values $\{-1,0,1\}$, with $(S^1_{\mathbf{x}})^2+(S^2_{\mathbf{x}})^2=1$]. We shall denote by Ω_Λ the corresponding spin configuration space and, for later convenience, we shall define $\Omega^V_\Lambda=\{\vec{S}_\Lambda \in \Omega_\Lambda : S^1_{\mathbf{x}}=0, \forall \mathbf{x} \in \Lambda\}$ and $\Omega^H_\Lambda=\{\vec{S}_\Lambda \in \Omega_\Lambda : S^2_{\mathbf{x}}=0, \forall \mathbf{x} \in \Lambda\}$ to be the subspaces of vertical and horizontal spin configurations. The Hamiltonian is of the form

$$H = \sum_{i,j=1}^{2} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} S^i_{\mathbf{x}} W_{ij}(\mathbf{x}-\mathbf{y}) S^j_{\mathbf{y}} - \sum_{\langle \mathbf{x},\mathbf{y}\rangle \in \Lambda} \{J\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}} + \lambda[(\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}})^2 - 1]\}, \quad (2.1)$$

where, denoting the Yukawa potential by $Y_\varepsilon(\mathbf{x}) = e^{-\varepsilon|\mathbf{x}|}|\mathbf{x}|^{-1}$, the interaction matrix $W(\mathbf{x})$ is of the dipole form given by

$$W_{ij}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} (-\partial_i\partial_j) Y_\varepsilon(\mathbf{x}+2\mathbf{n}L), \quad \mathbf{x}\neq\mathbf{0}, \quad (2.2)$$

and $W_{ij}(\mathbf{0}) = \Sigma_{\mathbf{n}\neq\mathbf{0}}(-\partial_i\partial_j) Y_\varepsilon(2\mathbf{n}L)$. The second sum in Eq. (2.1) runs over pairs of nearest neighbor sites in Λ and the constants J and λ will be assumed non-negative. The λ term is inserted in H to discourage neighboring spins from having orthogonal polarizations. It has the effect of encouraging stripes, but this term alone cannot create stripes. Without the J term, the ground state would be as in Fig. 1.

One of the main result of this section concerns the zero temperature phase diagram of model (2.1) with $\varepsilon=0$, that is, in the case that the long-range dipole-dipole interaction is the "real" dipolar one. It is summarized in the following theorem.

THEOREM 1. *Let* $\varepsilon=0$ *and* $0\leqslant J<J_0(0)$, *where*

$$J_0(0) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} dk \frac{\pi^2 m^2}{\sqrt{4\pi^2 m^2 + k^2}} \left(\sinh \frac{\sqrt{4\pi^2 m^2 + k^2}}{2} \right)^{-2}. \quad (2.3)$$

There exists $\lambda_0(J)$ *such that, if* $\lambda\geqslant\lambda_0(J)$, *then the specific ground state energy of Eq. (2.1) in the thermodynamic limit is given by:*

$$\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} E_0(\Lambda) = \min_{h\in\mathbb{Z}^+} e(h), \quad (2.4)$$

where $e(h) \overset{def}{=} \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} E^{(h)}_{per}(\Lambda)$ *and* $E^{(h)}_{per}(\Lambda)$ *is the energy of a periodic configuration with either vertical or horizontal stripes all of size h and alternate magnetization. If the side of* Λ *is divisible by the optimal period [i.e., by* $2h^*(J)$, *with* $h^*(J)$ *the minimizer of the right-hand side (rhs) of Eq. (2.4)], then the* only *ground states are the periodic configuration with either vertical or horizontal stripes all of size* $h^*(J)$ *and alternate magnetization.*

If $\lambda=+\infty$, *i.e., if we consider model (2.2) restricted to* $\Omega^V_\Lambda \cup \Omega^H_\Lambda$, *then the same conclusions are valid for all* $J\geqslant 0$.

Remarks.

(1) For any $J<J_0(0)$, the ground state has nontrivial stripes of finite size, that is, the minimizer $h^*(J)$ defined in the theorem is finite, and $h^*(J)$ diverges logarithmically as J tends to $J_0(0)$. In the hard core ($\lambda=+\infty$) case, Theorem 1 implies that for any $J\geqslant J_0(0)$, we have $h^*(J)=+\infty$ and the ground state is ferromagnetic.

(2) The constant $\lambda_0(J)$ introduced in the theorem is proportional to $h^*(J)$, i.e., it is of the form $\lambda_0(J)=Ch^*(J)$, with C independent of J. In particular, $\lambda_0(J)$ diverges at $J=J_0(0)$. It is not clear whether this divergence is an artifact of our proof and whether one should expect the same result to be valid for smaller values of λ (in particular, for $\lambda=0$). For $\lambda=0$, we tried to look for states with all four possible orientations and energy smaller than the one given by Eq. (2.4), but we did not succeed. It could very well be that the same result is actually valid for any $\lambda\geqslant 0$, but unfortunately, we do not know how to prove or disprove it.

In the case that $\varepsilon>0$, we can extend the previous result to all values of $J\geqslant 0$ and finite λ. The result is summarized in Theorem 2.

THEOREM 2. *The conclusions of Theorem 1 are also valid if* $\varepsilon>0$, $J\geqslant 0$, *and* $\lambda\geqslant\max\{C_\varepsilon - J, 0\}$, *where*

$$C_\varepsilon = \text{const} \begin{cases} |\log\varepsilon| & \text{if } \varepsilon\leqslant 1/2 \\ \varepsilon e^{-\varepsilon} & \text{if } \varepsilon > 1/2. \end{cases} \quad (2.5)$$

Remarks.

(1) Similarly to the $\varepsilon=0$ case, the optimal stripe size $h^*(J)$ is finite if and only if $J<J_0(\varepsilon)$, where

$$J_0(\varepsilon) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} dk \frac{\pi^2 m^2}{\alpha(m,k,\varepsilon)} \left[\sinh \frac{\alpha(m,k,\varepsilon)}{2} \right]^{-2}, \quad (2.6)$$

with

$$\alpha(m,k,\varepsilon) = \sqrt{4\pi^2 m^2 + k^2 + \varepsilon^2}. \quad (2.7)$$

The constant in the rhs of Eq. (2.5) is chosen such that C_ε is larger than $J_0(\varepsilon)$, for all $\varepsilon>0$. Note that the conclusions of Theorems 1 and 2 are also valid under the assumptions $\varepsilon>0$, $J\geqslant 0$, and $\lambda\geqslant$ const $h^*(J)$.

(2) For small ε, the bound on the value of λ above which we get the striped state is proportional to $|\log\varepsilon|$ and, in particular, diverges at $\varepsilon=0$. Also in this case, it is not clear whether this divergence is just an artifact of our proof and whether one should expect the same result to be valid for smaller values of λ.

In summary, we prove that the ground state of Eq. (2.1) is a periodic array of stripes if the λ term is large enough. More precisely, in the dipole case, $\varepsilon=0$, if λ is larger than a constant proportional to the period of the $\lambda=+\infty$ ground state; in the Yukawa case, $\varepsilon>0$, if λ is larger than a constant depending on the Yukawa mass (and diverging logarithmically as it goes to zero). The proof of Theorems 1 and 2 in the $\lambda=+\infty$ case, described in Sec. IV below, is based on an exact reduction to an effective 1D model, to be treated by the methods of Ref. 20: as a byproduct of the proof, we also get explicit bounds for the energies of excited states. The extension to finite λ, both in the $\varepsilon=0$ and $\varepsilon>0$ cases, is based on a Peierls' estimate on the energy of a droplet of horizontal spins surrounded by vertical spins. The proof of Peierls' estimate is very simple under the assumptions of Theorem 2, that is, in the presence of exponential decay with mass ε and of a large penalty λ (large nonuniformly in the mass ε); this proof, together with the necessary definitions of Peierls' contours and droplets, is described in Sec. V. Extending the result to the case $\varepsilon=0$ and λ finite (i.e., the case considered in Theorem 1) is highly nontrivial: in fact, since the long-range potential decays as the third power of the distance, the naive dimensional estimate on the dipole energy of a droplet of size ℓ decreases as $-\ell\log\ell$. So, in order to exclude the

presence of droplets of spins with the "wrong" orientation, we have to use cancellations, that is, we have to prove the presence of screening in the ground state. The proof is given in Sec. VI.

B. Antiferromagnetic case

Let us now consider the case that the $\vec{S}_{\mathbf{x}}$ have six possible orientations, four in plane and two out of plane, i.e., $S^1_{\mathbf{x}}$, $S^2_{\mathbf{x}}$, and $S^3_{\mathbf{x}}$ can take values $\{-1,0,1\}$, with $(S^1_{\mathbf{x}})^2+(S^2_{\mathbf{x}})^2+(S^3_{\mathbf{x}})^2=1$. Their interaction is given by the Hamiltonian [Eq. (2.1)], with $\mathbf{x}$ still located at the sites of the 2D torus Λ considered above, with $J\leqslant 0$ and $\varepsilon=0$.

THEOREM 3. *There exists an absolute constant* $\lambda_0\geqslant 0$ *such that if* $\lambda\geqslant\max\{\lambda_0-|J|,0\}$, *then the specific ground state energy is given by*

$$\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}E_0(\Lambda)=\min\{e_1,e_0-2|J|\},\tag{2.8}$$

with e_1 *the specific energy of the planar antiferromagnetic state described in Fig. 1 and* $e_0-2|J|$ *the specific energy of the out-of-plane staggered state. One has* $e_0>e_1$. *For* $|J|<(e_0-e_1)/2$ *and* Λ *large enough, the planar antiferromagnetic states (i.e., the one described in Fig. 1 plus those obtained from it by translation and/or by a 90° rotation) are the* only *ground states. For* $|J|>(e_0-e_1)/2$ *and* Λ *large enough, the out-of-plane Néel states are the* only *ground states. For* $|J|=(e_0-e_1)/2$ *and* Λ *large enough, the out-of-plane Néel state and the in-plane state described in Fig. 1 are the* only *ground states.*

Theorem 3 is proven in Sec. VII. Its proof goes along the lines of the Proof of Theorem 2. In other words, we first consider the restricted problem with the dipoles all constrained to be parallel (either in or out of plane) and determine the ground states using a reflection-positivity argument. Then, we prove that for λ large enough, the picture does not change, using Peierls' argument. Note that now the critical value λ_0 is an absolute constant, independent of J. The theorem can be extended to cover the $\varepsilon>0$ case, in which case λ_0 is also independent of ε. This is due to the fact that the reference states we now need to consider (i.e., the ground states of the restricted problem) are always antiferromagnetic with period 2, and this induces a strong screening which effectively corresponds to a faster decay of interactions, at least as the inverse distance to the power 4.

III. $O(n)$ SPINS WITH SCALAR INTERACTIONS

In this section, we describe an example of a spin system with dipole-type interactions that displays periodic striped ground states. The construction of the ground state, based on an observation about $O(n)$ spin models with *scalar* interactions, was originally given by Nussinov.[23] This shows that striped states are, very naturally, the ground states for a large class of $O(n)$ models with $n\geqslant 2$ and competing interactions. For this class of models, the local constraint on the norm of the spin at each site is not strong enough to invalidate the naive procedure for determining the ground states based on minimization of the Fourier transform of the interaction.

Let us consider an $O(n)$ model, $n\geqslant 2$, on a simple cubic d-dimensional torus $\Lambda\subset\mathbb{Z}^d$ of side L with $d\geqslant 1$ and Hamiltonian

$$H=\sum_{\mathbf{x},\mathbf{y}}J(\mathbf{x}-\mathbf{y})\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}},\tag{3.1}$$

where $J(\mathbf{x})$ is a function of the distance $|\mathbf{x}|$ only and the $\vec{S}_{\mathbf{x}}$ are n-dimensional unit vectors, $n\geqslant 2$. The Hamiltonian [Eq. (3.1)] can be rewritten in Fourier space as follows:

$$H=\sum_{i=1}^{n}\sum_{\mathbf{k}\in\mathcal{D}_L}\hat{J}(\mathbf{k})\hat{S}^i_{\mathbf{k}}\cdot\hat{S}^i_{-\mathbf{k}},\tag{3.2}$$

where $\mathcal{D}_L=\{\mathbf{k}=2\pi L^{-1}\mathbf{n},\mathbf{n}\in[0,L)^d\cap\mathbb{Z}^d\}$ and

$$\hat{S}^i_{\mathbf{k}}=\frac{1}{\sqrt{|\Lambda|}}\sum_{\mathbf{x}}S^i_{\mathbf{x}}e^{i\mathbf{x}\cdot\mathbf{k}},\quad \hat{J}(\mathbf{k})=\sum_{\mathbf{x}}J(\mathbf{x})e^{i\mathbf{x}\cdot\mathbf{k}}.\tag{3.3}$$

PROPOSITION 1 (Ref. 23). *If* $\mathbf{k}_0$ *is a minimizer for* $\hat{J}(\mathbf{k})$ *and* $g\in O(n)$ *is an* $n\times n$ *orthogonal matrix, then all states of the form*

$$\vec{S}^{g,\mathbf{k}_0}_{\mathbf{x}}=g\vec{S}^{\mathbf{k}_0}_{\mathbf{x}}\quad\text{with}\ \vec{S}^{\mathbf{k}_0}_{\mathbf{x}}=[\cos(\mathbf{k}_0\cdot\mathbf{x}),\sin(\mathbf{k}_0\cdot\mathbf{x}),0,\ldots,0]\tag{3.4}$$

are ground states of models (3.2) and (3.3).

Proof. By Eq. (3.2) and the normalization condition $\Sigma_{i,\mathbf{k}}|\hat{S}^i_{\mathbf{k}}|^2=|\Lambda|$, we see that the ground state energy cannot be smaller than $|\Lambda|\min_{\mathbf{k}}\hat{J}(\mathbf{k})$; on the other hand, the states [Eq. (3.4)] all have energy equal to $|\Lambda|\min_{\mathbf{k}}\hat{J}(\mathbf{k})$ and they satisfy the normalization $|\vec{S}_{\mathbf{x}}|^2=1$, so this proves that they are ground states. ■

Note that if $\mathbf{k}_0\neq 0$, the ground states $\{\vec{S}^{g,\mathbf{k}_0}_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ are "striped," in the sense that they are 1D sinusoidal spin waves in the direction $\mathbf{k}_0/|\mathbf{k}_0|$ with wavelength $2\pi/|\mathbf{k}_0|$; we call these *soft stripes*. Under additional conditions on the geometry of the set of minima of $\hat{J}(\mathbf{k})$, one could actually prove that such states are the only possible ground states, but we will not investigate this question in the greatest possible generality.[24] Instead, as an illustration, we will discuss an explicit example where these ideas can be used to infer that all ground states are soft stripes. A similar analysis can be performed to show that the previous remark also applies to the case of a 3D $O(n)$ model, $n\geqslant 2$, with spins interacting via a short-range ferromagnetic interaction plus a positive long-range Coulombic $1/r$ interaction, i.e., to the class of models considered in Ref. 25.

EXAMPLE. *Let* $\Lambda\subset\mathbb{Z}^2$ *be a simple cubic 2D torus of side L and let H be defined as*

$$H=-J\sum_{\langle\mathbf{x},\mathbf{y}\rangle}\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}}+\varepsilon\sum_{\mathbf{m}\in\mathbb{Z}^2}\sum_{\substack{\mathbf{x},\mathbf{y}\in\Lambda\\ \mathbf{x}\neq\mathbf{y}}}\frac{\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}+\mathbf{m}L|^3},\tag{3.5}$$

where the first sum ranges over nearest neighbor sites of Λ *and the* $\vec{S}_{\mathbf{x}}$*'s are 2D unit vectors. Then, there exists a constant* $c>0$ *such that if* $\varepsilon\leqslant cJ$ *and L is large enough, all the ground*

states of H are soft striped states.

Remark. This result, as well as the general remark above, depends crucially on the fact that the interaction is scalar and *isotropic*, i.e., for each pair $(\mathbf{x},\mathbf{y})$, the coupling depends only on $\vec{S}_{\mathbf{x}}\cdot\vec{S}_{\mathbf{y}}$. In the case of anisotropic interactions, the ground state may look very different and will be in general very difficult to identify. As an illustrative example, take the case of soft scalar spins, i.e., scalar spins $S^1_{\mathbf{x}}$ constrained to satisfy $|S^1_{\mathbf{x}}|\leqslant 1$, interacting via a pair potential $J(\mathbf{x})$ as in Eq. (3.1), with the further property that $J(\mathbf{0})=0$. This example can be viewed as an extreme anisotropic case, where all couplings between the ith components of the spins, $i\neq 1$, have been switched off. In this case, the ground state will be of the Ising type; this can be shown in the following way. The Hamiltonian is linear in $S^1_{\mathbf{x}}$, for each $\mathbf{x}\in\Lambda$, and it has to be minimized under the constraint that $|S^1_{\mathbf{x}}|\leqslant 1$, for all $\mathbf{x}\in\Lambda$. By the aforementioned linearity, the minimum will be attained at the boundary, that is, $S^1_{\mathbf{x}}=\pm 1$, for all $\mathbf{x}\in\Lambda$. The problem then reduces to determining the ground state of an Ising model with both long-range antiferromagnetic and nearest neighbor ferromagnetic interactions, that is, a very difficult and, in many respects, open problem, see Refs. 19 and 20. For intermediate values of the anisotropy, the problem is even more difficult and it is unclear whether the transition from a sinusoidal spin-wave state to an Ising-like state as the couplings between the ith components of the spins, $i\neq 1$, are decreased will be a sharp transition or rather a continuous one.

Proof of the Example. The Hamiltonian [Eq. (3.5)], up to an overall constant, can be rewritten as in Eq. (3.2), with $\hat{J}(\mathbf{k})=2J\Sigma_i(1-\cos k_i)+\varepsilon\Sigma_{\mathbf{x}\neq\mathbf{0}}e^{i\mathbf{kx}}|\mathbf{x}|^{-3}$. Note that $\Sigma_{\mathbf{x}\neq\mathbf{0}}e^{i\mathbf{kx}}|\mathbf{x}|^{-3}$ can be conveniently rewritten as $\hat{g}(\mathbf{k})-1+\Sigma_{\mathbf{p}\in 2\pi\mathbb{Z}^2}\int d\mathbf{x}e^{i(\mathbf{p}+\mathbf{k})\mathbf{x}}(\mathbf{x}^2+1)^{-3/2}$, where $\hat{g}(\mathbf{k})$ is the Fourier transform of $g(\mathbf{x})=(1-\delta_{\mathbf{x}})[|\mathbf{x}|^{-3}-(\mathbf{x}^2+1)^{-3/2}]$ [note that $g(\mathbf{x})$ goes to zero as $|\mathbf{x}|^{-5}$ as $|\mathbf{x}|\to\infty$]. Using the fact that $\int d\mathbf{x}e^{i(\mathbf{p}+\mathbf{k})\mathbf{x}}(\mathbf{x}^2+1)^{-3/2}=2\pi e^{-|\mathbf{p}+\mathbf{k}|}$,[26] we find that

$$\hat{J}(\mathbf{k})=2J\sum_{i=1}^{2}(1-\cos k_i)+2\pi\varepsilon\sum_{\mathbf{p}\neq\mathbf{0}}e^{-|\mathbf{p}+\mathbf{k}|}+\varepsilon\hat{g}(\mathbf{k})-\varepsilon, \tag{3.6}$$

where $\hat{g}(\mathbf{k})$ is twice differentiable and even in $\mathbf{k}$. An elementary study of the minima of Eq. (3.6) shows that for ε/J small, they are located within $O(\varepsilon/J)^2$ from the points $(\pm\pi\varepsilon/J,0)$, $(0,\pm\pi\varepsilon/J)$. Moreover, the minima have the following property (that we shall call *nondegeneracy*): there are no two distinct (unordered) pairs of minimizing vectors $(\mathbf{p},\mathbf{q})$, $(\mathbf{p}',\mathbf{q}')$, such that $\mathbf{p}+\mathbf{q}=\mathbf{p}'+\mathbf{q}'$; this is because the minima all lie on a surface of strictly positive curvature. This implies that all ground states are striped, that is, they are all of the form of Eq. (3.4), with stripes that are all (almost) horizontal or vertical. In fact, one can show that no state obtained as a superposition of different minimizing modes can be a ground state: this is proven by using the normalization $|\vec{S}_{\mathbf{x}}|^2=1$ and the nondegeneracy condition. For a similar discussion, see Ref. 23.

IV. TWO-DIMENSIONAL DISCRETE DIPOLES: THE $\lambda=+\infty$ CASE

As mentioned in Sec. II, we shall first prove Theorems 1–3 in the case $\lambda=+\infty$, that is, in the case that the spins are either all vertical or all horizontal or, possibly, all out of plane. Then, we will show that, if λ is large enough, the spins in the ground state configurations will, in fact, be either all vertical or all horizontal. We will do this first for the ferromagnetic plus Yukawa case (Sec. V), then for the ferromagnetic plus dipole case (Sec. VI), and finally for the antiferromagnetic plus dipole case (Sec. VII).

A. Ferromagnetic case

Let us assume here that ε, $J\geqslant 0$ in Eqs. (2.1) and (2.2), and the spins are oriented along four possible directions, as described in Sec. II A. Let us denote by H_∞ the Hamiltonian [Eq. (2.1)] with the hard core interaction corresponding to $\lambda=+\infty$. By definition, H_∞ is the restriction of Eq. (2.1) to $\Omega^V_\Lambda\cup\Omega^H_\Lambda$, i.e., to the space of configurations with spins either all vertical or all horizontal. Without loss of generality, let us consider the case that all spins are vertical, that is, $\vec{S}_{\mathbf{x}}=(0,\sigma_{\mathbf{x}})$, $\sigma_{\mathbf{x}}=\pm 1$, $\forall\mathbf{x}\in\Lambda$. In the following, we shall alternatively refer to the spins as "up or down" spins or "plus and minus" spins, with interchangeable meaning. The key remark is that in this case, the system is reflection positive with respect to ferromagnetic reflections in horizontal lines. The relevant reflection symmetry is defined as follows. Let π be a *pair* of horizontal lines midway between two lattice rows which bisect the torus Λ of side $2L$ into two pieces Λ_+ and Λ_- of equal size. Let r denote the reflection of sites with respect to π. Clearly, $r\Lambda_-=\Lambda_+$. We define

$$\theta\sigma_{\mathbf{x}}=\sigma_{r\mathbf{x}}. \tag{4.1}$$

For any function $F(\{\sigma_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda})$, we shall define the reflected function θF as $\theta F(\{\sigma_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda})=[F(\{\theta\sigma_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda})]^*$. Note that if F_+ depends only on the spins in Λ_+, then $F_-=\theta F_+$ will depend only on the spins in Λ_-. We shall say that H_∞ is reflection positive (RP) with respect to reflections in horizontal planes if it can be written in the form $H_\infty=H_++\theta H_+-\int C_+(x)\theta C_+(x)d\rho(x)$ for a positive measure $d\rho(x)$, with $H_+,C_+(x)$ depending only on the spins in Λ_+. In our case, this representation can be achieved by defining H_+ as the interaction of the spins in Λ_+ among themselves, $H_-=\theta H_+$ as the interaction of the spins in Λ_+ among themselves, and by suitably rewriting the interaction term $\Sigma_{\mathbf{x}\in\Lambda_+}\Sigma_{\mathbf{y}\in\Lambda_-}\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}[W_{22}(\mathbf{x}-\mathbf{y})-J\delta_{|\mathbf{x}-\mathbf{y}|,1}]$, in the following way.

Let $x_2>0$ and let us rewrite

$$Y_\varepsilon(\mathbf{x})=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi}{\mathbf{k}^2+\varepsilon^2}e^{i\mathbf{kx}}=\frac{1}{2\pi}\int\frac{d\mathbf{k}_\perp}{\sqrt{\mathbf{k}_\perp^2+\varepsilon^2}}e^{ik_1x_1}e^{-|x_2|\sqrt{\mathbf{k}_\perp^2+\varepsilon^2}}, \tag{4.2}$$

where in the last expression $\mathbf{k}_\perp=(k_1,k_3)$. If $x_2>y_2$ and $\partial_2=\frac{\partial}{\partial x_2}$,

$$\sum_{\substack{\mathbf{x}\in\Lambda_+\\ \mathbf{y}\in\Lambda_-}} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} W_{22}(\mathbf{x}-\mathbf{y}) = -\sum_{\mathbf{n}\in\mathbb{Z}^2}\sum_{\substack{\mathbf{x}\in\Lambda_+\\ \mathbf{y}\in\Lambda_-}} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\partial_2^2 Y_\varepsilon(\mathbf{x}-\mathbf{y}+2\mathbf{n}L)$$

$$= -\sum_{\mathbf{n}\in\mathbb{Z}^2}\frac{1}{2\pi}\int d\mathbf{k}_\perp \sqrt{\mathbf{k}_\perp^2+\varepsilon^2}$$

$$\times \sum_{\substack{\mathbf{x}\in\Lambda_+\\ \mathbf{y}\in\Lambda_-}} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} e^{ik_1(x_1-y_1+2n_1L)} e^{-|x_2-y_2+2n_2L|\sqrt{\mathbf{k}_\perp^2+\varepsilon^2}} \tag{4.3}$$

that has clearly the correct structure $-\int C(x)\theta C(x)d\rho(x)$. Note that the exchange interaction between Λ_+ and Λ_- admits a representation of the form $-J\Sigma_i\sigma_i\theta\sigma_i$, with σ_i the spins in Λ_+ at distance 1 from Λ_-. This concludes the proof that H_∞ is RP with respect to reflections in horizontal planes.

Let $\mathcal{R}_i$ be the ith row of Λ and, given a spin configuration $\underline{\sigma}_\Lambda=\{\sigma_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$, let $\mathcal{S}_i(\underline{\sigma}_\Lambda)$ be the spin configuration $\underline{\sigma}_\Lambda$ =restricted to $\mathcal{R}_i$. By reflection positivity and, more specifically, by the chessboard estimate (see Theorem 4.1 in Ref. 27), the energy of $\underline{\sigma}_\Lambda$ can be bounded below as

$$H_\infty(\underline{\sigma}_\Lambda) \geqslant \frac{1}{2L}\sum_{i=1}^{2L} H_\infty[\{\mathcal{S}_i(\underline{\sigma}_\Lambda),\ldots,\mathcal{S}_i(\underline{\sigma}_\Lambda)\}], \tag{4.4}$$

where $\{\mathcal{S}_i(\underline{\sigma}_\Lambda),\ldots,\mathcal{S}_i(\underline{\sigma}_\Lambda)\}$ is the spin configuration in Λ obtained by repeating in every row the same spin configuration $\mathcal{S}_i(\underline{\sigma}_\Lambda)$. Then, the problem of minimizing H_∞ among all possible spin configurations is reduced to the problem of minimizing H_∞ among the spin configuration in which all spins in a given column have the same direction. Let us consider one such configuration. If column x_1 has all spins pointing up, then we will label it by $\sigma_{x_1}=+$, and if it has all spins pointing down, we will label it by $\sigma_{x_1}=-$. The energy of the configuration labeled by $\{\sigma_x\}_{x=1,\ldots,2L}$ is

$$E = 2L\left[w_0 - J + \sum_{x\neq y=1}^{2L}\sigma_x w_{\uparrow\uparrow}(x-y)\sigma_y - J\sum_{x=1}^{2L}\sigma_x\sigma_{x+1}\right], \tag{4.5}$$

with $w_0=\Sigma_{n\geqslant 1}W_{22}(n\hat{e}_2)$ and

$$w_{\uparrow\uparrow}(x-y) = \sum_{\mathbf{n}\in\mathbb{Z}^2} W_{22}[(x-y+2Ln_1)\hat{e}_1 + n_2\hat{e}_2]$$

$$= \frac{1}{2\pi}\sum_{\mathbf{n}\in\mathbb{Z}^2}\int \frac{dk_2dk_3}{\sqrt{k_2^2+k_3^2+\varepsilon^2}}$$

$$\times k_2^2 e^{ik_2n_2} e^{-|x-y+2Ln_1|\sqrt{k_2^2+k_3^2+\varepsilon^2}}, \tag{4.6}$$

where $\hat{e}_1,\hat{e}_2$ are the two coordinate unit vectors and in the second line we used the representation [Eqs. (4.2) and (4.3)]. Performing the summations over n_1,n_2, we get

$$w_{\uparrow\uparrow}(x-y) = \sum_{m\in\mathbb{Z}}\int dk \frac{4\pi^2m^2}{\alpha(m,k,\varepsilon)}\frac{e^{-|x-y|\alpha(m,k,\varepsilon)}+e^{-(2L-|x-y|)\alpha(m,k,\varepsilon)}}{1-e^{-2L\alpha(m,k,\varepsilon)}}, \tag{4.7}$$

where $\alpha(m,k,\varepsilon)$ was defined in Eq. (2.7). This potential decays exponentially at large distances and is reflection positive with respect to the antiferromagnetic reflections. The ground states of this one-dimensional model can be studied by the same methods of Ref. 20. As mentioned in the Introduction, the proofs of Theorems 1 and 2 in Ref. 20 actually contain a minor technical error, which was caused by overlooking exponentially small terms which came from periodic boundary conditions. While fixing the error, we discovered that the same results can be proven by a simpler and more elegant method, which is presented here in the Appendix. The result is that the ground states for the one-dimensional system described by the effective Hamiltonian [Eq. (4.5)] consist of periodic arrays of blocks of alternating sign, all of size h, with h the positive integer minimizing the function

$$e(h) = w_1 - 2J + \frac{2J}{h} - \frac{2}{h}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty} dk \frac{4\pi^2m^2}{\alpha(m,k,\varepsilon)}$$

$$\times \frac{e^{-\alpha(m,k,\varepsilon)}}{(1-e^{-\alpha(m,k,\varepsilon)})^2}\tanh\frac{h\alpha(m,k,\varepsilon)}{2}, \tag{4.8}$$

where $w_1=w_0+\Sigma_{n\geqslant 1}w_{\uparrow\uparrow}(n)$. Note that $e(h)$ is the specific energy of periodic configurations with blocks all of the same size and alternating sign. An elementary study of the behavior of $e(h)$ shows that if $J\geqslant J_0$, with J_0 defined as in Eq. (2.6), then the ground state is ferromagnetic. If $0\leqslant J<J_0$, then the ground state is nontrivial and h is the integer part of the solution to

$$J_0 - J = 2\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty} dk$$

$$\times \frac{4\pi^2m^2e^{-(h+1)\alpha(m,k,\varepsilon)}[1+e^{-h\alpha(m,k,\varepsilon)}+h\alpha(m,k,\varepsilon)]}{\alpha(m,k,\varepsilon)(1-e^{-\alpha(m,k,\varepsilon)})^2(1+e^{-h\alpha(m,k,\varepsilon)})^2}. \tag{4.9}$$

In terms of the original dipole system, this shows that for $J\geqslant J_0$, the ground state is ferromagnetic, while for $0\leqslant J<J_0$, the ground state is striped, with stripes either all horizontal or all vertical. The stripes have alternating orientation and their thickness varies from 1 to ∞ as J is increased from 0 to J_0.

The results in Ref. 20 and in the Appendix also imply a bound on the energy of any given 2D state $\underline{\sigma}_\Lambda$ *different* from the striped ground state $\underline{\sigma}^*_\Lambda$. From now on, let us assume for simplicity that the minimization problem $\min_{h\in\mathbb{Z}^+}e(h)$ is solved by a unique h^* with the property that $e(h^*)=\min_{h\in\mathbb{Z}^+}e(h)=\min_{h\in\mathbb{R}^+}e(h)$. Note that this is *not* a generic property: in general, $\min_{h\in\mathbb{Z}^+}e(h)\neq\min_{h\in\mathbb{R}^+}e(h)$ and for some special values of J, it could even happen that the minimization problem on $\mathbb{Z}^+$ is solved by two consecutive values h^*,h^*+1. However, the general case can be treated in a way completely analogous to the one described below, at the

price of slightly more cumbersome notation[28]. So let us assume that $e(h^*)=\min_{h\in\mathbb{Z}^+}e(h)=\min_{h\in\mathbb{R}^+}e(h)$ and let $c_J=\frac{1}{2}\min_{h\neq h^*}[e(h)-e(h^*)]$. Let us think $\underline{\sigma}_\Lambda$ as a collection of 1D configurations corresponding to the configurations $\mathcal{S}_i(\underline{\sigma}_\Lambda)$ on different rows (here, $i=1,\ldots,2L$ is the index labeling different rows). Moreover, let us think each $\mathcal{S}_i(\underline{\sigma}_\Lambda)$ as a collection of blocks B of size h_B and alternate magnetization. Equation (23) in Ref. 20 implies that the energy of $\underline{\sigma}_\Lambda$ can be bounded from below as

$$H_\infty(\underline{\sigma}_\Lambda)\geqslant|\Lambda|e(h^*)+\sum_{B:h_B\neq h^*}2c_Jh_B. \tag{4.10}$$

So for any block of wrong size h_B, we pay a penalty c_Jh_B, and for any block of optimal size h^*, we do not pay *a priori* any penalty. We can actually improve the estimate above in the case that a block of optimal size on a given row is next to a nonoptimal block of size h_B on the same row: in this case, we can use the proof of Eq. (26) in Ref. 20 to infer that for each such pair, we pay a penalty $2d_J(h^*+h_B)$, with $d_J=\frac{1}{4}\min_{h\neq h^*}[e(\{h,h^*\})-e(h^*)]$, and $e(\{h,h^*\})$ is the specific energy of a periodic configuration with blocks of sizes $(\ldots,h,h,h^*,h^*,h,h,h^*,h^*,\ldots)$. Combining Eq. (4.10) with the refinement we just discussed, we find

$$H_\infty(\underline{\sigma}_\Lambda)\geqslant|\Lambda|e(h^*)+\sum_{B:h_B\neq h^*}c_Jh_B+\sum_{\substack{\langle B_1,B_2\rangle:\\ h_{B_1}\neq h_{B_2}=h^*}}d_J(h_{B_1}+h_{B_2}), \tag{4.11}$$

where the second sum runs over pairs of nearest neighbor horizontal blocks, such that one of the two blocks in the pair is of optimal length. A straightforward computation, based on the explicit expression [Eq. (4.8)] of the specific energy, allows one to check that $c_J,d_J\geqslant\kappa e^{-\alpha h^*}$, for suitable constants $\alpha,\kappa>0$. This concludes the proof of Theorems 1 and 2 for the $\lambda=+\infty$ case.

B. Antiferromagnetic case

Let us now assume that $\varepsilon\geqslant0$, $J\leqslant0$ and that the dipoles can have six possible orientations, as described in Sec. II B. In the $\lambda=+\infty$ case, the spins can only be oriented all parallel to each other, either out of plane or vertical in plane or horizontal in plane.

If the spins are all oriented out of plane, then the Hamiltonian reduces to a long-range antiferromagnetic Ising Hamiltonian of the form

$$H^3_\infty=\sum_{\mathbf{x},\mathbf{y}\in\Lambda}S^3_{\mathbf{x}}W_{33}(\mathbf{x}-\mathbf{y})S^3_{\mathbf{y}}+|J|\sum_{\langle\mathbf{x},\mathbf{y}\rangle\in\Lambda}S^3_{\mathbf{x}}S^3_{\mathbf{y}}, \tag{4.12}$$

with $S^3_{\mathbf{x}}=\pm1$. As a consequence of the analysis in Ref. 29 (see Proof of Theorem 5.1), the ground state of the Hamiltonian [Eq. (4.12)] is the usual period-2 staggered state, for any value of $|J|\geqslant0$. Its ground state energy is $e_0-2|J|$, with

$$e_0=w_0^{AF}-\sum_{m\in\mathbb{Z}+1/2}\int dk\frac{k^2}{\alpha(m,k,\varepsilon)}\frac{e^{-\alpha(m,k,\varepsilon)}}{1+e^{-\alpha(m,k,\varepsilon)}}, \tag{4.13}$$

where $w_0^{AF}=\Sigma_{n\geqslant1}(-1)^nW_{11}(n\hat{e}_2)$ and $\alpha(m,k,\varepsilon)$ was defined in Eq. (2.7). In order to prove Theorem 3 in the $\lambda=+\infty$ case, we need to compare this specific energy with the specific energy of the best possible in-plane spin configurations.

So let us consider the case that the spins are all oriented in plane; we can assume without loss of generality that they are all horizontal. [The choice of horizontal rather than vertical spins is made here in order to keep the definition of the pair π of reflection planes the same as in the previous subsection, see the lines preceding Eq. (4.1).] In this case, we define the variables $\sigma_{\mathbf{x}}$ in such a way that $\vec{S}_{\mathbf{x}}=(\sigma_{\mathbf{x}},0)$. Now, the relevant reflection in horizontal planes, replacing Eq. (4.1), is $\theta\sigma_{\mathbf{x}}=-\sigma_{r\mathbf{x}}$ (note the minus sign). By the chessboard estimate, we reduce to an expression analog to Eq. (4.5), given by

$$E=2L\left[w_0^{AF}-|J|+\sum_{x\neq y=1}^{2L}\sigma_xw(x-y)\sigma_y+|J|\sum_{x=1}^{2L}\sigma_x\sigma_{x+1}\right], \tag{4.14}$$

with σ_x now representing the direction (right or left) of the spin in $(x,0)$, w_0^{AF} the constant defined after Eq. (4.13), and

$$w(x-y)=-\sum_{m\in\mathbb{Z}+1/2}\int_{-\infty}^{+\infty}dk\alpha(m,k,\varepsilon)\times\frac{e^{-|x-y|\alpha(m,k,\varepsilon)}+e^{-(2L-|x-y|)\alpha(m,k,\varepsilon)}}{1-e^{-2L\alpha(m,k,\varepsilon)}}. \tag{4.15}$$

For $|J|=0$, the ground state of Eq. (4.14) is ferromagnetic (that is, in terms of the original 2D dipole model, the ground state is given by Fig. 1—rotated by 90°). If $|J|$ is increased, then, using the same methods of Ref. 20, one can prove that the ground state has a sequence of transitions from the ferromagnetic state to periodic states of *antiferromagnetic* blocks of size h with alternating staggered polarization (of period h or $2h$, depending whether h is odd or even). As an illustration, the states with $h=2,3,4$ (with periods 4,3,8, respectively) are given by $(\cdots+--++-\cdots)$, $(\cdots+-++-++-+\cdots)$, and $(\cdots+-+--+-++-+-\cdots)$. For any value of $|J|$, the optimal period $h^*(|J|)$ is obtained as usual by minimizing over h the specific energy $e^{AF}(h)$ of the states with AF blocks all of size h and alternating staggered magnetization. Here, $e^{AF}(h)$ is given by

$$e^{AF}(h)=w_1^{AF}-2|J|+\frac{2|J|}{h}-\frac{2}{h}\sum_{m\in\mathbb{Z}+1/2}\int_{-\infty}^{+\infty}dk\alpha(m,k,\varepsilon)\times\frac{e^{-\alpha(m,k,\varepsilon)}}{(1+e^{-\alpha(m,k,\varepsilon)})^2}\frac{1-(-e^{-\alpha(m,k,\varepsilon)})^h}{1+(-e^{-\alpha(m,k,\varepsilon)})^h}, \tag{4.16}$$

with

$$w_1^{AF} = w_0^{AF} + \sum_{m \in \mathbb{Z}+1/2} \int_{-\infty}^{\infty} dk \alpha(m,k,\varepsilon) \frac{e^{-\alpha(m,k,\varepsilon)}}{1+e^{-\alpha(m,k,\varepsilon)}}.$$

It is straightforward to check that the $h^*(|J|)$ minimizing $e^{AF}(h)$ for a given value of $|J|$ is equal to $+\infty$ as soon as $|J|$ is larger than a critical value $|J_1| = \Sigma_m \int dk \alpha e^{-\alpha}(1+e^{-\alpha})^{-2}$. This means that the minimal energy in-plane state is the staggered antiferromagnet for $|J| \geqslant |J_1|$. If we define $e_h^{dip} = e^{AF}(h) + 2|J|(1-h^{-1})$, we have $e_1^{dip} < e_h^{dip}$, for all $h > 1$, and moreover $e_1^{dip} < e_0$, with e_0 defined in Eq. (4.13). For $h > 1$, the critical value of $|J|$ for which $e^{AF}(1) = e^{AF}(h)$ is given by $2|J_c^h| = (e_h^{dip} - e_1^{dip})/(1-h^{-1})$. Similarly, the critical value of $|J|$ for which $e^{AF}(1)$ is equal to the specific energy of the out-of-plane staggered state is given by $|J_c^0| = (e_0 - e_1^{dip})/2$.

It is now clear that in order to prove Eq. (2.8) in the $\lambda = +\infty$ case, it is enough to show that $|J_c^h| > |J_c^0|$, for all $h > 1$. This simply follows from a computation, whose details we omit. The uniqueness property stated in Theorem 3 follows along the same lines as in the ferromagnetic case. This concludes the proof of Theorem 3 in the $\lambda = +\infty$ case.

V. FERROMAGNETIC CASE: FINITE λ AND POSITIVE MASS

In this section, we prove Theorem 2. We prove it by showing that if $\varepsilon > 0$ and λ is larger than $\max\{C_\varepsilon - J, 0\}$, see Eq. (2.5), then in the ground state of Eq. (2.1) the spins must be either all horizontal or all vertical. Note that if this is the case, then Theorem 2 simply follows from the discussion of the previous section. Let us recall that spins now can assume four possible directions (those parallel to the two in-plane coordinate axis). We shall denote by $\vec{S}_X$ spin configurations in $X \subset \mathbb{Z}^2$ (in particular, we shall denote by $\vec{S}_\Lambda$ spin configurations in Λ) and by $\vec{S}^H$ and $\vec{S}^V$ two (arbitrarily chosen) $\lambda = +\infty$ infinite volume ground state configurations with horizontal and vertical stripes, respectively.

We need to introduce some definitions. As in the basic Peierls construction, we introduce the definitions of *contours* and *droplets*. Given any configuration $\vec{S}_\Lambda$, we define $\Delta_V = \Delta_V(\vec{S}_\Lambda)$ to be the set of sites at which the spins are vertical. We draw around each $\mathbf{x} \in \Delta_V$ the four sides of the unit square centered at $\mathbf{x}$ and suppress the faces which occur twice: we obtain in this way a *closed polygon* $\Gamma(\Delta_V)$ which can be thought as the boundary of Δ_V. Each face of $\Gamma(\Delta_V)$ separates a point $\mathbf{x} \in \Delta_V$ from a point $\mathbf{y} \in \Delta_V$. Along a vertex of $\Gamma(\Delta_V)$, there can be either two or four lines meeting. In the case of four lines, we deform slightly the polygon, "chopping off" the vertex from the cubes containing a horizontal spin. When this is done, $\Gamma(\Delta_V)$ splits into disconnected polygons $\gamma_1, \ldots, \gamma_r$, which we shall call *contours*. Note that, because of the choice of periodic boundary conditions, all contours are closed but can possibly wind around the torus Λ. The definition of contours naturally induces a notion of connectedness for the spins in Δ_V: given that $\mathbf{x}, \mathbf{x}' \in \Delta_V$, we say that $\mathbf{x}$ and $\mathbf{x}'$ are connected if and only if there exists a sequence $(\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n = \mathbf{x}')$ such that $\mathbf{x}_m, \mathbf{x}_{m+1}$, $m = 0, \ldots, n-1$, are nearest neighbors and none of the bonds $(\mathbf{x}_m, \mathbf{x}_{m+1})$ crosses $\Gamma(\Delta_V)$. The maximal connected components δ_i of Δ_V will be called V-droplets and the set of V-droplets of Δ_V will be denoted by $\mathcal{D}_V(\Delta_V) = \{\delta_1, \ldots, \delta_s\}$, or simply $\mathcal{D}_V$. Note that the boundaries $\Gamma(\delta_i)$ of the V-droplets $\delta_i \in \mathcal{D}_V$ are all distinct subsets of $\Gamma(\Delta_V)$ with the property: $\cup_{i=1}^s \Gamma(\delta_i) = \Gamma(\Delta_V)$. Similarly, we can introduce the notion of H-droplets and of the set $\mathcal{D}_H(\Delta_H)$. A *droplet* will be either a V-droplet or an H-droplet. The set of droplets will be denoted by $\mathcal{D} = \mathcal{D}_H \cup \mathcal{D}_V$.

The same kind of construction allows one to define FM-droplets as the maximal connected regions of spins with the same orientation. We shall call FM-contours the boundaries of FM-droplets.

Given the previous definitions, we can now state and prove the main results of this section.

LEMMA 1 (Peierls' estimate—massive case—small J). *If* $\lambda \geqslant J + \bar{C}_\varepsilon$, *with*

$$\bar{C}_\varepsilon = \begin{cases} c_1 |\log \varepsilon| & \text{if } \varepsilon \leqslant 1/2 \\ c_2 \varepsilon e^{-\varepsilon} & \text{if } \varepsilon > 1/2, \end{cases}$$

for two suitable constants $c_1, c_2 > 0$ *[chosen in such a way that, in particular,* $\bar{C}_\varepsilon \geqslant J_0$, *with* J_0 *the constant in Eq. (2.6)], then the following is true. Let* $\vec{S}_\Lambda$ *be a spin configuration in* Λ *and let* $\delta \in \mathcal{D}$ *be one of its droplets. If* δ *is a V-droplet (H-droplet), the spin configuration* $\vec{T}_\Lambda$ *coinciding with* $\vec{S}_\Lambda$ *on* δ^c *and with* $\vec{S}^H$ $(\vec{S}^V)$ *on* δ *satisfies* $H(\vec{S}_\Lambda) - H(\vec{T}_\Lambda) > 0$.

LEMMA 2 (Peierls' estimate—massive case—large J). *Let* $\bar{C}_\varepsilon$ *the same as in Lemma 1 and* $\lambda \geqslant 0$. *Then, the energy of any spin configuration associated with the set* Γ *of FM-contours can be bounded below by* $E_{FM}(\Lambda) + (J - \bar{C}_\varepsilon)\Sigma_{\gamma \in \Gamma} |\gamma|$, *where* $E_{FM}(\Lambda)$ *is the energy of the ferromagnetic state and* $|\gamma|$ *is the length of the contour* γ.

Proof of Theorem 2. A consequence of Lemma 1 is that for $\lambda \geqslant J + \bar{C}_\varepsilon$, the set Δ_V is either empty or the whole Λ, i.e., there are no contours in the ground state. This, together with the discussion in the previous section, implies the result of Theorem 2 in the case $\lambda \geqslant J + \bar{C}_\varepsilon$. Moreover, a consequence of Lemma 2 is that for $J > \bar{C}_\varepsilon$ and $\lambda \geqslant 0$, the ground state has no FM-contours, i.e., it is ferromagnetic. Since $\bar{C}_\varepsilon \geqslant J_0$, this, in particular, means that under the same conditions, the conclusions of Theorem 2 are valid (see Remark 1 after Theorem 2). Theorem 2 follows by the combination of these two results, choosing $C_\varepsilon > 3\bar{C}_\varepsilon$.

Proof of Lemma 1 and 2. Let $\gamma = \Gamma(\delta)$ and note that $H(\vec{S}_\Lambda) - H(\vec{T}_\Lambda)$ is given bounded below by $(\lambda - J)|\gamma|$ plus the difference between the self-energies of $\vec{S}_\delta$ and $\vec{T}_\delta$ plus the difference between the inside-outside interactions of the spins in δ with the spins in δ^c. By the exponential decay of the potential and the fact that $\vec{T}_\Lambda$ coincides on δ with the $\lambda = +\infty$ ground state, the first difference is bounded below by a positive constant minus, possibly, a term of size const$|\gamma|$; similarly, the second difference is bounded above and below by const$|\gamma|$. A computation shows that these two constants can be bounded above by $c_1 \log(1/\varepsilon)$, for small ε, and by

$c_2\varepsilon e^{-\varepsilon}$, for large ε, where c_1, c_2 are two suitable constants. So Lemma 1 is proven. The proof of Lemma 2 goes exactly along the same lines: the energy of a state with a nontrivial set of FM-contours, compared to the energy of the FM state, is given by $\Sigma_{\gamma\in\Gamma}J|\gamma|$ plus the inside-outside energy associated with any FM-droplet. The latter is bounded below by $-\text{const}\,\Sigma_{\gamma\in\Gamma}|\gamma|$, with the constant bounded as discussed above.

VI. FERROMAGNETIC CASE: FINITE λ AND ZERO MASS

In this section, we want to discuss how to generalize Peierls' estimate of Lemma 1 to the case $\varepsilon=0$. In this section, we shall only consider the case $0\leqslant J<J_0$, with J_0 defined as in Eq. (2.6). The main result of this section is a generalization of Lemma 1 to the massless case. Its proof requires the use of the screening properties of the $\lambda=+\infty$ ground state (related to its striped nature, in particular, to the fact that its total polarization vanishes). In order to describe the result, we also need to introduce the notion of *simple droplet*: we shall say that a droplet δ is simple if either it is simply connected or it winds around the torus and its complement is connected. One crucial property of simple droplets δ we shall need is that the number of sites in δ at a fixed lattice distance d from δ^c is bounded above by $|\Gamma(\delta)|$. Note also that any collection of droplets $\mathcal{D}$ associated with some spin state $\vec{S}_\Lambda$ contains at least one simple droplet.

We are now ready to state the main result of this section.

LEMMA 3 (Peierls' estimate—massless case). *Let* $\varepsilon=0$. *If* $0\leqslant J<J_0$ *and* $\lambda\geqslant\text{const}\,h^*(J)$, *for a suitable constant and with* $h^*(J)$ *the minimizer of the rhs of Eq. (2.4), then the following is true. Let* $\vec{S}_\Lambda$ *be a spin configuration in* Λ *and let* $\delta\in\mathcal{D}$ *be one of its droplets. If* δ *is a simple V-droplet (H-droplet), then the spin configuration* $\vec{T}_\Lambda$ *coinciding with* $\vec{S}_\Lambda$ *on* δ^c *and with* $\vec{S}^H$ ($\vec{S}^V$) *on* δ *satisfies* $H(\vec{S}_\Lambda)-H(\vec{T}_\Lambda)>0$.

Proof of Theorem 1. An immediate consequence of Lemma 3 is that in the ground state, $|\mathcal{D}|=1$. In fact, if by contradiction the ground state configuration $\vec{S}_\Lambda$ had $|\mathcal{D}|>1$, then it would be possible to reduce the energy by changing $\vec{S}_\Lambda$ into $\vec{T}_\Lambda$ in the way described above (note that we are using that, as remarked above, any droplet configuration $\mathcal{D}$ always contains at least one simple droplet). Then, all spins are either horizontal or vertical and this, together with the discussion of Sec. IV, implies the result stated in Theorem 1.

Proof of Lemma 3. With no loss of generality, we assume that δ is a *V*-droplet. We rewrite $\vec{S}_\Lambda$ in the form $\vec{S}_\Lambda=\vec{S}_\delta\cup\vec{S}_{\delta^c}$, where the spins in $\vec{S}_\delta$ are all vertical and, in particular, they consitute a maximally connected component of vertical spins. We rewrite

$$H(\vec{S}_\Lambda)-H(\vec{T}_\Lambda)=H_0(\vec{S}_\delta)-H_0(\vec{S}^H_\delta)+H_1(\vec{S}_\delta|\vec{S}_{\delta^c})-H_1(\vec{S}^H_\delta|\vec{S}_{\delta^c}), \tag{6.1}$$

where $H_0(\vec{S}_\delta)$ is the internal energy of the spins in δ and $H_1(\vec{S}^H_\delta|\vec{S}_{\delta^c})$ is the inside-outside interaction between the spins in δ and those in δ^c.

Let us consider the auxiliary configuration $\vec{D}_\Lambda=\vec{S}_\delta\cup\vec{S}^V_{\delta^c}$ coinciding with $\vec{S}_\Lambda$ inside δ and coinciding with the ($\lambda=+\infty$) vertical ground state $\vec{S}^V_\Lambda$ outside δ. Equation (6.1) can be rewritten as

$$\begin{aligned}H(\vec{S}_\Lambda)-H(\vec{T}_\Lambda)=&[H(\vec{D}_\Lambda)-H(\vec{S}^V_\Lambda)]+H_1(\vec{S}_\delta|\vec{S}_{\delta^c})\\&+[H_0(\vec{S}^V_\delta)-H_0(\vec{S}^H_\delta)-H_1(\vec{S}^H_\delta|\vec{S}_{\delta^c})\\&+H_1(\vec{S}_\delta|\vec{S}^V_{\delta^c})-H_1(\vec{S}^V_\delta|\vec{S}^V_{\delta^c})].\end{aligned} \tag{6.2}$$

We shall now separetely estimate the contributions from the three square brackets in the rhs.

By Eq. (4.11) and defining $\gamma\equiv\Gamma(\delta)$, the first term can be bounded from below by

$$\begin{aligned}H(\vec{D}_\Lambda)-H(\vec{S}^V_\Lambda)\geqslant&-2J|\gamma|+\sum_{B:h_B\neq h^*}\kappa e^{-\alpha h^*}h_B\\&+\sum_{\substack{\langle B_1,B_2\rangle:\\ h_{B_1}\neq h_{B_2}=h^*}}\kappa e^{-\alpha h^*}(h_{B_1}+h_{B_2}),\end{aligned} \tag{6.3}$$

where the notation is the same as in Eq. (4.11) and we used that $c_J,d_J\geqslant\kappa e^{-\alpha h^*}$, see lines following Eq. (4.11). The second term in Eq. (6.2) can be bounded below as

$$\begin{aligned}H_1(\vec{S}_\delta|\vec{S}_{\delta^c})\geqslant&\lambda|\gamma|-\sum_{B:h_B\neq h^*}|H_{dip}(B_\delta|\vec{S}_{\delta^c})|\\&-\sum_{\substack{\langle B_1,B_2\rangle:\\ h_{B_2}=h^*}}|H_{dip}(\{B_1,B_2\}_\delta|\vec{S}_{\delta^c})|,\end{aligned} \tag{6.4}$$

where $H_{dip}(B_\delta|\vec{S}_{\delta^c})$ is the dipole-dipole interaction energy between the spins in $B\cap\delta$ and $\vec{S}_{\delta^c}$, and similarly $H_{dip}(\{B_1,B_2\}_\delta|\vec{S}_{\delta^c})$ is the interaction energy between the spins in $\{B_1\cup B_2\}\cap\delta$ and $\vec{S}_{\delta^c}$. Note that now in the second sum, also pairs of blocks with both blocks of optimal size are included. The rhs of Eq. (6.4) can be bounded below by

$$\begin{aligned}&\lambda|\gamma|-\text{const}\left[\sum_{\mathbf{x}\in B:h_B\neq h^*}\frac{1}{z_{\mathbf{x}}}+\sum_{\substack{\mathbf{x}\in B_1\cup B_2:\\ h_{B_1}\neq h_{B_2}=h^*}}\frac{1}{z_{\mathbf{x}}}\right]\\&-\sum_{\substack{\langle B_1,B_2\rangle:\\ h_{B_1}=h_{B_2}=h^*}}|H_{dip}(\{B_1,B_2\}_\delta|\vec{S}_{\delta^c})|,\end{aligned} \tag{6.5}$$

where $z_{\mathbf{x}}\geqslant 1$ is the lattice distance between $\mathbf{x}$ and δ^c. In Eq. (6.5), we are only left with the summation over pairs of optimal blocks. Some of these pairs can be at a distance from δ^c smaller than $h^*(J)$, and the contribution from all such pairs is bounded below by $-\text{const}\,h^*(J)|\gamma|$. Now, note that a pair of nearest neighbor optimal blocks has vanishing polarization, so we can bound the contribution of any optimal pair $\{B_1,B_2\}$ at a distance from δ^c larger than $h^*(J)$ from below by

$$-\text{const}\, h^*(J) \sum_{\substack{\mathbf{x}\in B_1\cup B_2:\\ h_{B_1}=h_{B_2}=h^*}} \frac{1}{z_{\mathbf{x}}^2}$$

(note that the $\frac{1}{z_{\mathbf{x}}}$ has been now replaced by $\frac{1}{z_{\mathbf{x}}^2}$). The total contribution due to pairs of optimal blocks can then be bounded below as

$$-\text{const}\, h^*(J)|\gamma| - \text{const}\, h^*(J) \sum_{d\geq 1} \sum_{\substack{\mathbf{x}\in\delta:\\ z_{\mathbf{x}}=d}} \frac{1}{d^2} \geq -\text{const}\, h^*(J)|\gamma|,$$

where the last constant is in general different from those at left-hand side and we used that the number of sites at a distance d from δ^c is bounded above by $|\gamma|$ (because δ is a simple droplet). Using this result in Eq. (6.5), we find that, for a suitable $C>0$,

$$H_1(\vec{S}_\delta|\vec{S}_{\delta^c}) \geq [\lambda - Ch^*(J)]|\gamma| - C\left[\sum_{\mathbf{x}\in B:h_B\neq h^*} \frac{1}{z_{\mathbf{x}}} + \sum_{\substack{\mathbf{x}\in B_1\cup B_2:\\ h_{B_1}\neq h_{B_2}=h^*}} \frac{1}{z_{\mathbf{x}}}\right]. \tag{6.6}$$

Similarly, the contribution from the last square bracket in Eq. (6.2) is bounded below by $-Ch^*(J)|\gamma|$. The conclusion is that

$$H(\vec{S}_\Lambda) - H(\vec{T}_\Lambda) \geq (\lambda - C_1 h^*)|\gamma| + \sum_{\mathbf{x}\in B:h_B\neq h^*} \left(\kappa e^{-\alpha h^*} - \frac{C_2}{z_{\mathbf{x}}}\right) + \sum_{\substack{\mathbf{x}\in B_1\cup B_2:\\ h_{B_1}\neq h_{B_2}=h^*}} \left(\kappa e^{-\alpha h^*} - \frac{C_2}{z_{\mathbf{x}}}\right), \tag{6.7}$$

for suitable constants $C_1, C_2>0$. To the purpose of a bound from below, we can throw away all terms with $z_{\mathbf{x}} \geq C_2 e^{\alpha h^*}/\kappa$ in the two sums. We are then left with

$$H(\vec{S}_\Lambda) - H(\vec{T}_\Lambda) \geq (\lambda - C_1 h^*)|\gamma| - 3C_2 \sum_{\substack{\mathbf{x}\in\delta:\\ \kappa z_{\mathbf{x}}\leq C_2 e^{\alpha h^*}}} \frac{1}{z_{\mathbf{x}}}. \tag{6.8}$$

Since δ is simple, the number of points at a distance d from δ^c is at most $|\gamma|$. Then, the last summation is bounded below by $-\text{const}|\gamma|\log(C_2 e^{\alpha h^*}/\kappa)$ and the proof of the lemma is concluded.

VII. ANTIFERROMAGNETIC CASE

In this section, we conclude the proof of Theorem 3. We already proved it in the $\lambda=+\infty$ case, see Sec. IV B. So, we now simply need to show that if either (i) $|J|$ is larger than some absolute constant κ_1 and $\lambda=0$ or (ii) $|J|\leq\kappa_1$ and λ larger than some absolute constant κ_2, then the system always prefers to have the spins all oriented either out of plane or in plane horizontally or vertically. If this is the case, then Theorem 3 follows, with $\lambda_0=\kappa_1+\kappa_2$.

The easiest case to handle that we shall treat first is the case of $|J|$ large and $\lambda=0$. In this case, both the in-plane and the out-of-plane minimal energy states display staggered antiferromagnetic order. In analogy with the FM-contours defined in Sec. V, we can introduce the notion of AF-contours, obtained as union of dual bonds separating nearest neighbor spins which are either identical or orthogonal. By construction, the AF-contours separate maximally connected regions of spins displaying staggered antiferromagnetic order (AF-droplets). Let $\mathcal{D}_i$ ($\mathcal{D}_o$) be the set of in-plane (out-of-plane) AF-droplets and let $e_{ip}=e_\infty^{dip}-2|J|$ and $e_{op}=e_0-2|J|$ be the in-plane and out-of-plane minimal energies (we recall that e_∞^{dip} and e_0 were defined in Sec. IV B). As discussed in Sec. IV B, we have $e_{ip}>e_{op}$. Given a spin configuration $\vec{S}_\Lambda$ whose set of droplets is $\mathcal{D}_i\cup\mathcal{D}_o$, its energy can be bounded below by

$$\sum_{\delta\in\mathcal{D}_i}\left[e_{ip}|\delta| + \left(\frac{|J|}{2} - \text{const}\right)|\Gamma(\delta)|\right] + \sum_{\delta\in\mathcal{D}_o}\left[e_{op}|\delta| + \left(\frac{|J|}{2} - \text{const}\right)|\Gamma(\delta)|\right], \tag{7.1}$$

where the two (absolute) constants take into account the dipole interaction energy of the spins inside the droplet with the spins outside (note that this inside-outside interaction is simply proportional to the length of the contour because of the screening effect associated with the antiferromagnetic phase). By Eq. (7.1), we see that if $|J|$ is larger than an absolute constant κ_1, then the unique ground state is the out-of-plane staggered antiferromagnetic state.

Let us now turn to the case $|J|\leq\kappa_1$ and λ is larger than some absolute constant κ_2, to be determined below. Given any spin configuration $\vec{S}_\Lambda$, let us define H-droplets and V-droplets as in Sec. V and let us also introduce the notion of O-droplets as the maximal connected regions of out-of-plane spins (the set of O-droplets will be denoted by $\mathcal{D}_O$). Since the dipole interaction between an in-plane and an out-of-plane spin is zero, the energy $H(\vec{S}_\Lambda)$ can be rewritten as

$$H(\vec{S}_\Lambda) = \left[\frac{\lambda}{2}\sum_{\delta\in\mathcal{D}_H\cup\mathcal{D}_V}|\Gamma(\delta)| + H_0(\mathcal{D}_H\cup\mathcal{D}_V)\right] + \left[\frac{\lambda}{2}\sum_{\delta\in\mathcal{D}_O}|\Gamma(\delta)| + H_0(\mathcal{D}_O)\right] \equiv (\text{I}) + (\text{II}), \tag{7.2}$$

where $H_0(\mathcal{D}_H\cup\mathcal{D}_V)$ $[H_0(\mathcal{D}_O)]$ is the interaction energy among the in-plane (out-of-plane) spins, corresponding to the Hamiltonian [Eq. (2.1)] with $J\leq 0$ and $\lambda=0$. Using the chessboard estimate, (II) can be easily bounded below by $(\lambda/2)\Sigma_{\delta\in\mathcal{D}_O}|\Gamma(\delta)|+\Sigma_{\delta\in\mathcal{D}_O}(e_0-2|J|)|\delta|$. In order to bound (I), we follow the same strategy of Sec. VI, but we first need to fill the regions occupied by the O-droplets by an auxiliary configuration of in-plane spins. To this purpose, we define the auxiliary spin configuration $\vec{T}_\Lambda$, which coincides with $\vec{S}_\Lambda$ on $\mathcal{D}_H\cup\mathcal{D}_V$, and with $\vec{S}_\Lambda^H$ on $\mathcal{D}_O$. We have

$$(\mathrm{I}) \geqslant H(\vec{\underline{T}}_\Lambda) - \sum_{\delta \in \mathcal{D}_O} [e^* |\delta| + \mathrm{const} |\Gamma(\delta)|], \qquad (7.3)$$

where e^* is the specific energy of $\vec{\underline{S}}_\Lambda^H$ [in terms of the notation introduced in Sec. IV B, $e^* = e^{AF}(h^*)$ and $h^* \equiv h^*(|J|)$]. Now, $H(\vec{\underline{T}}_\Lambda)$ can be bounded below by the method discussed in Sec. VI. However, now, the role of "blocks of the wrong size" is played by "elementary defects," i.e., sequences of three contiguous identical spins. Note, in fact, that none of the minimal energy states discussed in Sec. IV B contains such subconfiguration of spins and a straightforward computation shows that the state obtained from an elementary defect by repeated reflections has specific energy larger than $e^{AF}(h)$, for all $h \geqslant 1$. This makes the present discussion much simpler than the discussion of Sec. VI. The result is that, as long as λ is larger than an absolute constant, we have that $H(\vec{\underline{T}}_\Lambda) \geqslant |\Lambda| e^*$. Combining this bound with Eqs. (7.2) and (7.3), we finally conclude that

$$H(\vec{\underline{S}}_\Lambda) \geqslant (e_0 - 2|J|) \sum_{\delta \in \mathcal{D}_O} |\delta| + e^* \sum_{\delta \in \mathcal{D}_H \cup \mathcal{D}_V} |\delta|, \qquad (7.4)$$

and this concludes the proof of Eq. (2.8) and of Theorem 3.

VIII. CONCLUDING REMARKS

We rigorously proved the existence of periodic striped order with periods of length $h(J)$ in the ground states of a 2D system of dipoles with restricted orientations, with 3D dipole-dipole long-range interactions competing with a nearest neighbor *ferromagnetic* exchange interaction of strength J. We also considered such system with an *antiferromagnetic* exchange $|J|$, in which case we proved the existence of a reorientation transition from an in-plane to an out-of-plane ordered ground state, as $|J|$ increased. Finally, we gave an example of soft striped order in the form of 1D sinusoidal spin wave.[23]

Unfortunately, even the ground states of more realistic models used to describe thin films are still far from being solved exactly. On the basis of variational arguments and approximations, one expects spontaneous formation of mesoscopic stripes in anisotropic systems of out-of-plane spins interacting via the dipole-dipole (or Coulomb) long-range interaction and a short-range FM exchange, like those considered in Refs. 18, 19, 25, 30, and 31. It is also expected that in the presence of a sufficiently strong uniform magnetic field, oriented perpendicular to the plane, the ground state should exhibit periodic order in the form of bubbles, i.e., domains of spins parallel to the external field of quasicircular shape.[32] Striped or bubble patterns are also expected on the basis of an effective (mean field) free energy functional in 2D electron gases,[10,11,33] Langmuir monolayers, and liquid crystals.[34]

Even less is known rigorously for positive temperatures. In particular, it is unclear, even on a heuristic level, whether the expected striped or bubbled order for discrete spins should have strict long-range order (LRO) or rather quasi-long-range order (QLRO) characterized by order on short scales and a power law decay of the order parameter correlation functions. One of the few rigorous results, which we are aware of, about the positive-temperature behavior of this class of systems is the recent proof by Biskup *et al.* of the absence of ferromagnetism in d-dimensional Ising models with long-range repulsive interactions, decaying as $1/r^p$, $d < p \leqslant d+1$, interactions.[35]

The models discussed here are related to a class of systems with Kac potentials [i.e., long-range potentials of the form $\gamma^d v(\gamma r)$] considered by Lebowitz and Penrose in Ref. 36. There, they computed the free energy density of a system of particles interacting via a short-range interaction, favoring phase segregation, and a long-range non-negative definite Kac potential. They showed that in the limit $\gamma \to 0$, there is no phase transition in the thermodynamic sense, even though the pair distribution function has the form characterizing a phase transition, at least over length scales much smaller than γ^{-1}. They concluded that the repulsive Kac potential causes the distinct phases of a normal first-order phase transition to break into droplets, or froth, of characteristic length large compared to the range of the short-range potential and small compared to γ^{-1}. We do not know the scale of these domains or even whether they destroy the first-order transition in 2D and 3D when γ is small but finite.

In 1D, we can show, using the method of Ref. 20, for $H = -J\Sigma \sigma_i \sigma_{i+1} + \Sigma_{i,j} \gamma \exp\{-\gamma |i-j|\}$, that for $\gamma \to 0$, the ground states are periodic with period proportional to $\gamma^{-1}(J\gamma)^{1/3}$, i.e., on the macroscopic scale γ^{-1}, the period goes as $(J\gamma)^{1/3}$. This is very reminiscent of what happens to the minimizers of continuum energy functionals used to model microphase separation of diblock copolymers and many other physical systems.[37–42]

For Heisenberg spins with long-range interactions, the Hohenberg-Mermin-Wagner argument is not applicable and it could very well be that the long-range tails stabilize phases against thermal fluctuations, even in two dimensions. This issue has been discussed in some detail for the case of rotators in two dimensions, interacting via a 3D pure dipole-dipole interaction:[22,43] this is a case where linear spin-wave theory predicts the nonexistence of LRO, while nonlinear corrections and renormalized spin-wave theory seem to suggest that LRO survives at positive temperatures.

For scalar fields describing the local magnetization or electron density in an effective free energy functional theory, the general expectation is that in the presence of an anisotropy term (possibly induced by the underlying crystalline structure), QLRO should survive at positive temperatures. On the contrary, even QLRO should generally be destroyed by thermal fluctuations in the case of isotropic interactions.[12] The issue is rather subtle; however, analyses based on a Hartree approximation would generically predict the presence of a first-order phase transition from a high temperature disordered phase to a low temperature striped phase.[44] If both predictions are correct, this class of models would exhibit a rather peculiar first-order phase transition from a disordered state to a locally ordered state without strict LRO. The issue has been investigated by Jamei *et al.*[10,11,33] in the context of 2D electron gases and by Tarjus *et al.*[12] in the context of the frustration-based appoach to the glass transition in supercooled liquids and structural glasses. Jamei *et al.* argued that for 2D electron gases, in the presence of Coulomb inter-

actions, first-order transitions (as those predicted by Brazovskii) are not allowed and a sequence of transition between different mesoscopic patterned states should generically appear instead. On the contrary, as discussed in Ref. 12, numerical simulations of frustrated spin models with Coulomb interaction predict a finite jump in the energy at a critical line separating the paramagnetic state from a locally order striped state.

Refinements of the mean field or variational arguments seem very difficult: the natural effective continuum theories describing systems whose interaction has a Fourier transform with a nontrivial minimum in k space are, at least naively, nonrenormalizable.[45,46] It would be very interesting to provide convincing arguments for the patterned states to be (globally) stable against the presence of thermal or quantum fluctuations, as well as against "generic" perturbations in the form of long- or short-range interactions.

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APPENDIX: A REVISED PROOF OF PERIODIC ORDER IN REFLECTION-POSITIVE ONE-DIMENSIONAL ISING SYSTEMS

We mentioned in the Introduction that the proofs of Theorems 1 and 2 in Ref. 20 contain a technical error, which was caused by overlooking some exponentially small terms that came from the choice of periodic boundary conditions. In this appendix, we show how to repair this error. We exploit a different method, which is simpler than the one used in Sec. III of Ref. 20 and which does not make use of periodic but rather of open boundary conditions, as will be discussed below. The proof given in this appendix also implies the results discussed in Sec. IV.

First of all, let us point out the mistake in Ref. 20: a term

$$\sum_{1\leqslant i<j\leqslant M/2}(-1)^{i-j}(1-e^{-\alpha h_i})(1-e^{-\alpha h_j})e^{-2\alpha N}\prod_{i\leqslant k\leqslant j}e^{\alpha h_k} \tag{A1}$$

is missing in the definition of $H_R(\alpha,\underline{h}_R)$ in (I.9), i.e., in Eq. (9) of Ref. 20. Taking into account this term, one can check that, whenever the total length $N_L=\Sigma_{M/2<i\leqslant 0}h_i$ of the blocks on the left is different from the total length $N_R=\Sigma_{1\leqslant i\leqslant M/2}h_i$ of the blocks on the right, the energy functions $H(\alpha,(\underline{h}_L,\hat{\theta}\underline{h}_L))$ and $H(\alpha,(\hat{\theta}\underline{h}_R,\underline{h}_R))$ obtained after the first reflection, see Eq. (11) of Ref. 20, are not periodic. For this reason, it does not seem possible to repeatedly reflect, as one must do in order to obtain the checkerboard estimate, see Ref. 20, Eq. (12).

Let us now show how to correct the proofs of Theorems 1 and 2 in Ref. 20. Let us denote Eq. (x) of Ref. 20 by (I.x) and let H_N be the periodic Hamiltonian in (I.1),

$$H_N(\underline{\sigma})=-J\sum_{i=-N+1}^{N}\sigma_i\sigma_{i+1}+\sum_{-N+1\leqslant i<j\leqslant N}\sigma_iJ_p(j-i)\sigma_j,$$

$$J_p(j-i)=\sum_{n\in\mathbb{Z}^d}\frac{1}{|i-j+2nN|^p}, \tag{A2}$$

where $\sigma_{N+1}\equiv\sigma_{-N+1}$ and $p>1$. As discussed in Ref. 20, Theorems 1 and 2 are consequences of the *chessboard estimate*, used in Ref. 20 and stating that

$$H(h_1,\ldots,h_n)\geqslant\sum_{i=1}^{n}h_ie(h_i),\quad\sum_{i=1}^{n}h_i=N, \tag{A3}$$

where $H(h_1,\ldots,h_n)$ is the energy $H_N(\underline{\sigma})$ of a spin configuration whose corresponding block configuration is $\underline{h}=\{h_1,\ldots,h_n\}$ and $e(h)$ is the energy per site of the infinite system with a periodic configuration of blocks all of the same size and alternating sign (let us recall that a block is a maximal sequence of spins all of the same sign). In the following, we shall also need to introduce the analog of $H(\underline{h})$ with *open boundary conditions*, to be denoted by $H^0(\underline{h})$. Since the long-range potential is summable, we have

$$H(\underline{h})=\lim_{m\to\infty}\frac{1}{m}H^0(\overbrace{\underline{h},\ldots,\underline{h}}^{m\text{ times}}), \tag{A4}$$

and we find that the chessboard estimate [Eq. (A3)] is a consequence of the following:

CHESSBOARD ESTIMATE WITH OPEN BOUNDARY CONDITIONS. *Given a finite sequence of blocks* $\underline{A}_i=\{h_1^i,\ldots,h_{m_i}^i\}$, *let us denote by* $\theta\underline{A}_i=\{h_{m_i}^i,\ldots,h_1^i\}$ *the reflection of* $\underline{A}_i$ *(with the sign of the spins in the reflected blocks being opposite to what they were originally). By* $e(\underline{A}_i)$, *we denote the infinite volume energy per site of the configuration* $(\ldots,\underline{A}_i,\theta\underline{A}_i,\underline{A}_i,\theta\underline{A}_i,\ldots)$ *and let* $a_i=\Sigma_{j=1}^{m_i}h_j^i<\infty$. *Then, for any collection* $\underline{A}_0,\ldots,\underline{A}_{n+1}$, *with* $n\geqslant 1$ *and* $\Sigma_{i=0}^{n+1}a_i=N$, *we have*

$$H^0(\underline{A}_0,\underline{A}_1,\ldots,\underline{A}_n,\underline{A}_{n+1})\geqslant(a_0+a_{n+1})e_0+\sum_{i=1}^{n}a_ne(\underline{A}_n), \tag{A5}$$

where e_0 *is the infinite volume specific ground state energy.*

Proof. Let us first note that, for any sequence of blocks $\underline{A}$, $e(\underline{A})\geqslant ae_0$, where $a=\Sigma_{h\in\underline{A}}h$. This can be proven as follows. Denote by $\underline{\sigma}_a$ the spin configuration corresponding to $\underline{A}$, so that $H^0(\underline{A})=H_a^0(\underline{\sigma}_a)$. If $\omega=\pm$, we have $H_{2a}^0(\underline{\sigma}_a,\omega\underline{\sigma}_a)=2H_a^0(\underline{\sigma}_a)+\omega E_{int}(\underline{\sigma}_a;\underline{\sigma}_a)$, where $(\underline{\sigma}_a,\omega\underline{\sigma}_a)$ is the spin configuration of length $2a$ obtained by juxtaposing $\underline{\sigma}_a$ and $\omega\underline{\sigma}_a$, and $\omega E_{int}(\underline{\sigma}_a;\underline{\sigma}_a)$ is the interaction energy between the two halves. For one of the two choices $\omega=\pm$, this interaction energy is nonpositive, so that $2H_a^0(\underline{\sigma}_a)\geqslant\min_{\underline{\sigma}}H_{2a}^0(\underline{\sigma})$. Iterating, we find

$$H_a^0(\underline{\sigma}_a) \geqslant \lim_{m\to\infty} 2^{-m} \min_{\underline{\sigma}} H_{2^m a}^0(\underline{\sigma}) = ae_0, \tag{A6}$$

which is the desired estimate.

Next, let us recall that reflection positivity of the long-range potential $1/r^p$ implies the following basic estimate.

LEMMA A.1. *Given two finite sequences of blocks* $\underline{A}_- = \{h_{-M+1}, \ldots, h_0\}$ *and* $\underline{A}_+ = \{h_1, \ldots, h_N\}$, *with* $M, N \geqslant 1$, *let* $\theta\underline{A}_- = \{h_0, \ldots, h_{-M+1}\}$ *and* $\theta\underline{A}_+ = \{h_N, \ldots, h_1\}$ *be their reflections. Then, we have*

$$H^0(\underline{A}_-, \underline{A}_+) \geqslant \frac{1}{2} H^0(\theta\underline{A}_+, \underline{A}_+) + \frac{1}{2} H^0(\underline{A}_-, \theta\underline{A}_-). \tag{A7}$$

Now, we are ready to prove Eq. (A5). We proceed by induction. If $n=1$ in Eq. (A5), then by reflection positivity, i.e., by Lemma A.1, we have

$$H^0(\underline{A}_0, \underline{A}_1, \underline{A}_2) \geqslant \frac{1}{2} H^0(\theta\underline{A}_2, \underline{A}_2) + \frac{1}{2} H^0(\underline{A}_0, \underline{A}_1, \theta\underline{A}_1, \theta\underline{A}_0). \tag{A8}$$

By Eq. (A6), the first term in the rhs can be bounded from below by $a_2 e_0$. The second term can be bounded by a second reflection,

$$\frac{1}{2} H^0(\underline{A}_0, \underline{A}_1, \theta\underline{A}_1, \theta\underline{A}_0) \geqslant \frac{1}{4} H^0(\theta\underline{A}_0, \underline{A}_0) + \frac{1}{4} H^0[\underline{A}_0, (\underline{A}_1)^{\otimes 4}, \theta\underline{A}_0], \tag{A9}$$

where by definition $(\underline{A}_1)^{\otimes 4} = (\underline{A}_1, \theta\underline{A}_1, \underline{A}_1, \theta\underline{A}_1)$. By Eq. (A6), the first term in the rhs of Eq. (A9) can be bounded from below by $a_0 e_0/2$, so we end up with

$$H^0(\underline{A}_0, \underline{A}_1, \underline{A}_2) \geqslant a_2 e_0 + \frac{1}{2} a_0 e_0 + \frac{1}{4} H^0[\underline{A}_0, (\underline{A}_1)^{\otimes 4}, \theta\underline{A}_0]. \tag{A10}$$

Iterating, we find

$$H^0(\underline{A}_0, \underline{A}_1, \underline{A}_2) \geqslant a_2 e_0 + a_0 e_0 \sum_{n\geqslant 1} 2^{-n} + \lim_{n\to\infty} 2^{-n} H^0[\underline{A}_0, (\underline{A}_1)^{\otimes 2^n}, \theta\underline{A}_0]. \tag{A11}$$

Note that the last term is equal to $a_1 e(\underline{A}_1)$, so the desired bound is proven for $n=1$.

Let us now assume by induction that the bound is valid for all $1 \leqslant k \leqslant n-1$, $n \geqslant 2$, and let us prove it for $k=n$. There are two cases.

(a) $n=2p$ for some $p \geqslant 1$. If we reflect once, we get:

$$H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_{2p}, \underline{A}_{2p+1}) \geqslant \frac{1}{2} H^0(\theta\underline{A}_{2p+1}, \ldots, \theta\underline{A}_{p+1}, \underline{A}_{p+1}, \ldots, \underline{A}_{2p+1}) + \frac{1}{2} H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_p, \theta\underline{A}_p, \ldots, \theta\underline{A}_1, \theta\underline{A}_0). \tag{A12}$$

If we now regard $\underline{A}'_{p+1} \equiv (\theta\underline{A}_{p+1}, \underline{A}_{p+1})$ and $\underline{A}'_p \equiv (\underline{A}_p, \theta\underline{A}_p)$ as two new sequences of blocks, the two terms in the rhs of Eq. (A12) can be regarded as two terms with $n=2p-1$ and, by the induction assumption, they satisfy the bounds

$$H^0(\theta\underline{A}_{2p+1}, \ldots, \underline{A}'_{p+1}, \ldots, \underline{A}_{2p+1}) \geqslant 2a_{2p+1} e_0 + 2 \sum_{i=p+1}^{2p} a_i e(\underline{A}_i), \tag{A13}$$

$$H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}'_p, \ldots, \theta\underline{A}_1, \theta\underline{A}_0) \geqslant 2a_0 e_0 + 2 \sum_{i=1}^{p} a_i e(\underline{A}_i), \tag{A14}$$

where we used that, by construction, $a'_p = 2a_p$, $a'_{p+1} = 2a_{p+1}$, $e(\underline{A}'_p) = e(\underline{A}_p)$, and $e(\underline{A}'_{p+1}) = e(\underline{A}_{p+1})$. Therefore, the desired bound is proven.

(b) $n=2p+1$ for some $p \geqslant 1$. If we reflect once, we get

$$H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_{2p+1}, \underline{A}_{2p+2}) \geqslant \frac{1}{2} H^0(\theta\underline{A}_{2p+2}, \ldots, \theta\underline{A}_{p+2}, \underline{A}_{p+2}, \ldots, \underline{A}_{2p+2}) + \frac{1}{2} H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_{p+1}, \theta\underline{A}_{p+1}, \ldots, \theta\underline{A}_1, \theta\underline{A}_0). \tag{A15}$$

The first term in the rhs corresponds to $n=2p$ so by the induction hypothesis, it is bounded below by $a_{2p+2} e_0 + \sum_{i=p+2}^{2p+1} a_1 e(\underline{A}_i)$. As regards the second term, using reflection positivity again, we can bound it from below by

$$\frac{1}{4} H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_p, \theta\underline{A}_p, \ldots, \theta\underline{A}_1, \theta\underline{A}_0) + \frac{1}{4} H^0(\underline{A}_0, \underline{A}_1, \ldots, \underline{A}_p, (\underline{A}_{p+1})^{\otimes 4}, \theta\underline{A}_p, \ldots, \theta\underline{A}_1, \theta\underline{A}_0). \tag{A16}$$

By the induction hypothesis, the first term is bounded below by $a_0 e_0/2 + (1/2) \sum_{i=1}^{p} a_i e(\underline{A}_i)$, and the second can be bounded using reflection positivity again. Iterating, we find

$$H^0(\underline{A}_0,\underline{A}_1,\ldots,\underline{A}_{2p+1},\underline{A}_{2p+2}) \geqslant a_{2p+2}e_0 + \sum_{i=p+2}^{2p+1} a_i e(\underline{A}_i) + \left(\sum_{n\geqslant 1} 2^{-n}\right)\left[a_0 e_0 + \sum_{i=1}^{p} a_i e(\underline{A}_i)\right] + \lim_{n\to\infty} 2^{-n} H^0[\underline{A}_0,\underline{A}_1,\ldots,\underline{A}_p,(\underline{A}_{p+1})^{\otimes 2^n},\theta\underline{A}_p,\ldots,\theta\underline{A}_1,\theta\underline{A}_0]. \quad \text{(A17)}$$

Note that the last term is equal to $a_{p+1}e(\underline{A}_{p+1})$, so Eq. (A17) is the desired bound. This concludes the proof of the chessboard estimate with open boundary conditions and, as mentioned above, of Eq. (A3) and of Theorems 1 and 2 in Ref. 20.

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[24] We thank the anonymous referee for the following comment: "It may be noted that in addition to the simple striped ground states of the form of Eq. (3.4), there may generally exist other, more sophisticated, variants which are not simple soft stripes. That is, for $n>3$ spin systems, and for $\hat{J}(\mathbf{k})$ that display multiple minima, $\pm\mathbf{k}_0$, $\pm\mathbf{k}_1$,… (the appearance of such multiple minima is often dictated by the presence of point group lattice symmetries), many more ground states generally appear. In such cases, for general wave numbers $(\mathbf{k}_0,\mathbf{k}_1,\ldots)$, the ground states generally correspond to spirals (of wave numbers $\mathbf{k}_0$, $\mathbf{k}_1$,…) in independent planes of the n-dimensional spin space."

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