Entanglement evolution of three-qubit states in a quantum-critical environment

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We investigate the entanglement dynamics of three-qubit states in a quantum-critical environment, which is an Ising model in a transverse field. By using negativity as entanglement measure, we find that the entanglement evolution depends not only on the system-environment couplings and the size of degrees of freedom of environment but also on the strength of transverse field and the symmetry of quantum states of concern. In particular, for the cases under study, our results imply that the entanglement decay can be enhanced by the quantum phase transition of the environment under weak coupling. Our analysis implies that the entanglement of any three-qubit quantum state will be completely destroyed by the decoherence under certain conditions, that we detail below.

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I. INTRODUCTION

Quantum entanglement plays an important role in quantum information processing (QIP),[1–3] and attracts much attention from physicists. Much progress concerning quantum entanglement both in theory and in experiment has been achieved.[4–7] As is known, the advantage of quantum computer over classical counterpart lies in the use of the quantum entanglement which has no classical analog. However, in the real world, QIP will always inevitably be affected by the decoherence[8] induced by the external environment. The evaluation of the extent to which the decoherence affects quantum entanglement is an interesting problem and much works have been done.[9–13] In most of the these studies, researchers usually considered the uncorrelated environment modeled by a reservoir consisting of harmonic oscillators or spin chains. However, particles of the environment always have interaction with each other. What is the effect of decoherence induced by a correlated environment on quantum states? To answer this question, researchers have made some works[14–16] where they analyzed the decay of Loschmidt echo, universal decoherence, and quantum bipartite entanglement. In particular, the authors in Ref. 16 analyzed the disentanglement of a bipartite system in which the two parties couple to the environment homogeneously. The effect of decoherence induced by a correlated environment on multipartite entanglement has not been investigated so far, and multipartite entanglement dynamics is an interesting problem.[17,18] So, in this paper, we will study the time evolution of three-qubit states in a critical environment and investigate the factors which determine the entanglement evolution and decay. In our model, we will not assume that the parties of system couple to the environment homogeneously to get a comprehensive analysis of decoherence induced by a critical environment on three-qubit states.

As is known, multipartite entanglement is crucial for practical QIP and much progress, either in theory or in experiment, has been made.[19–25] In these works, the measure of multipartite entanglement is a difficult problem which is far from being understood. Fortunately, based on the positive partial transpose criterion (PPT)[5] for separability, Zyczkowski *et al.*[26] proposed a quantity labeled as negativity by Vidal and Werner[27] as the entanglement measure to evaluate the quantum entanglement. Negativity can efficiently calculate multipartite entanglement of both pure and mixed quantum states. It should be noted that the PPT criterion is necessary and sufficient only for 2×2 and 2×3 quantum states, so negativity has limitation in measuring the quantum entanglement of multipartite quantum states with high dimensions. Due to its operational and calculation properties, we employ negativity to measure the entanglement of three-qubit quantum states as the minimal multipartite quantum states in this presentation. Our study of the entanglement evolution of three-qubit quantum states under a correlated environment may shed some light on the understanding of multipartite entanglement dynamics.

Our paper is arranged as follows. In Sec. II, we introduce the model of three-qubit system coupled to a transverse Ising model and give the analytical expression of the time evolution of density matrix of the system. In Sec. III, the time evolution of quantum entanglement and some discussions are given. We conclude our results in Sec. IV.

II. HAMILTONIAN EVOLUTION

In this paper, we consider a three-qubit system coupled to an environment consisting a transverse Ising model which exhibits a quantum phase transition. In this model, to get a general and comprehensive analysis of entanglement dynamics and decoherence, the parties of the system are assumed to couple to the correlated environment inhomogenously. The corresponding Hamiltonian reads

$$H=\sum_{i=-N_0}^{N_0}\sigma_i^x\sigma_{i+1}^x+\left(\omega+\sum_{j=1}^{3}g_js_j^z\right)\sum_{i=-N_0}^{N_0}\frac{\sigma_i^z}{2}, \tag{1}$$

where ω is the strength of the transverse field and $g_j(j=1,2,3)$ characterize the coupling constants between the environment and the three qubits $s_j(j=1,2,3)$. $\sigma_i^\alpha(\alpha=x,y,z)$ are the familiar Pauli operators representing the ith site in the lattice. Obviously, the total site of the spin chain is $N_t=2N_0+1$.

In order to get the time evolution of the density matrix of the system, we should follow the standard procedure of Hamiltonian diagonalization[28,29] by employing the Jordan-

Wigner transformation and the Fourier transformation to the momentum space. By using the following pseudospin operators $\sigma_{k\alpha}(\alpha=x,y,z)$:

$$\sigma_{kx} = \gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k}\gamma_k \quad (k=1,2,\ldots,N_0), \quad \sigma_{0z} = 2\gamma_0^\dagger\gamma_0 - 1,$$

$$\sigma_{ky} = -i\gamma_k^\dagger\gamma_{-k}^\dagger + i\gamma_{-k}\gamma_k, \quad \sigma_{kz} = \gamma_k^\dagger\gamma_k + \gamma_{-k}^\dagger\gamma_{-k} - 1, \quad (2)$$

where $\gamma_k^\dagger, \gamma_k\{k=0,1,2,\ldots\}$ denote the creation and annihilation operators of new fermions in the momentum space, we get the new Hamiltonian as

$$H = \sum_{k>0} e^{i(\theta_k/2)\sigma_{kx}}(E_k\sigma_{kz})e^{-i(\theta_k/2)\sigma_{kx}} + \left(-\frac{\Delta}{2}+1\right)\sigma_{0z}, \quad (3)$$

where $\Delta=\omega+\sum_{j=1}^{3} g_j s_j^z$ and parameters E_k, θ_k take the following expressions, respectively:

$$E_k = \sqrt{4+\Delta^2-4\Delta\cos(ka)},$$

$$\theta_k = \arctan\left[\frac{2\sin(ka)}{\Delta - 2\cos(ka)}\right]. \quad (4)$$

In Eq. (4), a denotes the lattice spacing and Δ is used as a c number.

With these analytical expressions, we can go straightforwardly to obtain the reduced density matrix of system via the following equation:

$$\rho_s(t) = \mathrm{Tr}_E[e^{-iHt}\rho_s(0)\otimes|\psi_E\rangle\langle\psi_E|e^{iHt}]. \quad (5)$$

Here, the initial state of the system is separable with the initial state of the environment, and the initial state of the environment is assumed as the vacuum state $|\psi_E\rangle=|0\rangle_{k=0}\otimes_{k\geqslant 0}|0\rangle_k|0\rangle_{-k}$, which satisfies the relation $\gamma_k|0\rangle_k=0$.

Let the initial state of the system be in a general form like

$$|\psi_s(0)\rangle = a_1|000\rangle + a_2|001\rangle + a_3|010\rangle + a_4|011\rangle + a_5|100\rangle + a_6|101\rangle + a_7|110\rangle + a_8|111\rangle, \quad (6)$$

where $a_1, a_2, \ldots, a_8$ are coefficients of the initial state $|\psi_s(0)\rangle$ and they satisfy the normalization relation $|a_1|^2+|a_2|^2+\cdots+|a_8|^2=1$. After a careful calculation, we obtain the time evolution of the system density matrix. Here, it should be pointed out that our calculation holds for mixed states,

$$\rho_s(t) = \begin{bmatrix} |a_1|^2 & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & M_{18} \\ M_{12}^* & |a_2|^2 & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} & M_{28} \\ M_{13}^* & M_{23}^* & |a_3|^2 & M_{34} & M_{35} & M_{36} & M_{37} & M_{38} \\ M_{14}^* & M_{24}^* & M_{34}^* & |a_4|^2 & M_{45} & M_{46} & M_{47} & M_{48} \\ M_{15}^* & M_{25}^* & M_{35}^* & M_{45}^* & |a_5|^2 & M_{56} & M_{57} & M_{58} \\ M_{16}^* & M_{26}^* & M_{36}^* & M_{46}^* & M_{56}^* & |a_6|^2 & M_{67} & M_{68} \\ M_{17}^* & M_{27}^* & M_{37}^* & M_{47}^* & M_{57}^* & M_{67}^* & |a_7|^2 & M_{78} \\ M_{18}^* & M_{28}^* & M_{38}^* & M_{48}^* & M_{58}^* & M_{68}^* & M_{78}^* & |a_8|^2 \end{bmatrix}, \quad (7)$$

where $M_{\alpha\beta}$, $\alpha,\beta=1,2,\ldots,8$ are elements of the reduced density matrix and $*$ denotes complex conjugation. Now, we write the matrix elements explicitly in the following equations:

$$M_{\alpha\beta} = a_\alpha a_\beta^* F_{\alpha\beta}, \quad \alpha,\beta=1,2,\ldots,8. \quad (8)$$

Due to symmetry, we need to know the moduli of $F_{\alpha\beta}$ in order to get the time evolution of entanglement of quantum states of concern by using negativity as the entanglement measure. It should be noted that once the all the moduli of $F_{\alpha\beta}$ are completely destroyed by decoherence, any state will become a separable one. The moduli of $F_{\alpha\beta}$ can be obtained in the following equation:

$$|F_{\alpha\beta}| = \prod_{k>0} F_k = \prod_{k>0}\{1-\sin^2(E_k^{\{\alpha\}}t)\sin^2(E_k^{\{\beta\}}t)\sin^2(\theta_k^{\{\alpha\}}-\theta_k^{\{\beta\}}) - [\sin(E_k^{\{\alpha\}}t)\cos(E_k^{\{\beta\}}t)\sin(\theta_k^{\{\alpha\}}) - \cos(E_k^{\{\alpha\}}t)\sin(E_k^{\{\beta\}}t)\sin(\theta_k^{\{\beta\}})]^2\}^{1/2}, \quad (9)$$

where $E_k^{\{\alpha\}}, \theta_k^{\{\alpha\}}$ can be calculated by replacing Δ with Δ_α in Eq. (4). Δ_α, $\alpha=1,\ldots,8$, take the following expressions:

$$\Delta_1 = \omega + \frac{g_1+g_2+g_3}{2}, \quad \Delta_2 = \omega + \frac{g_1+g_2-g_3}{2},$$

$$\Delta_3 = \omega + \frac{g_1-g_2+g_3}{2}, \quad \Delta_4 = \omega + \frac{g_1-g_2-g_3}{2},$$

$$\Delta_5 = \omega + \frac{-g_1+g_2+g_3}{2}, \quad \Delta_6 = \omega + \frac{-g_1+g_2-g_3}{2},$$

$$\Delta_7 = \omega + \frac{-g_1-g_2+g_3}{2}, \quad \Delta_8 = \omega + \frac{-g_1-g_2-g_3}{2}. \quad (10)$$

Let us make some discussion concerning the above results and define a quantity for entanglement decay when k_c is taken as a cutoff frequency similar to the procedure in Refs. 14 and 16,

$$E_c \equiv \prod_{k>0}^{k_c} F_k \geqslant |F_{\alpha\beta}|, \quad (11)$$

where E_c is defined to get a heuristic knowledge of the features of the effect of decoherence induced by the quantum-critical environment.

For small k, we get $E_k=|2-\Delta|$, $\sin^2(\theta_k^\alpha-\theta^\beta)=4(\Delta_\alpha-\Delta_\beta)^2(ka)^2/(2-\Delta_\alpha)^2(2-\Delta_\beta)^2$. As a result, if k_c is small enough, we obtain

$$\ln E_c \approx -2\Lambda(k_c)\Big\{[\sin(|2-\Delta_\alpha|t)\cos(|2-\Delta_\beta|t)|2-\Delta_\alpha|^{-1} - \sin(|2-\Delta_\beta|t)\cos(|2-\Delta_\alpha|t)|2-\Delta_\beta|^{-1}]^2 + \sin^2(|2-\Delta_\alpha|t)\sin^2(|2-\Delta_\beta|t)\frac{(\Delta_\alpha-\Delta_\beta)^2}{(2-\Delta_\alpha)^2(2-\Delta_\beta)^2}\Big\}, \quad (12)$$

where $\Lambda(k_c)=4\pi^2 N_c(N_c+1)(2N_c+1)/(6N_t^2)$ and N_c is the integer closest to $N_t k_c a/2\pi$.

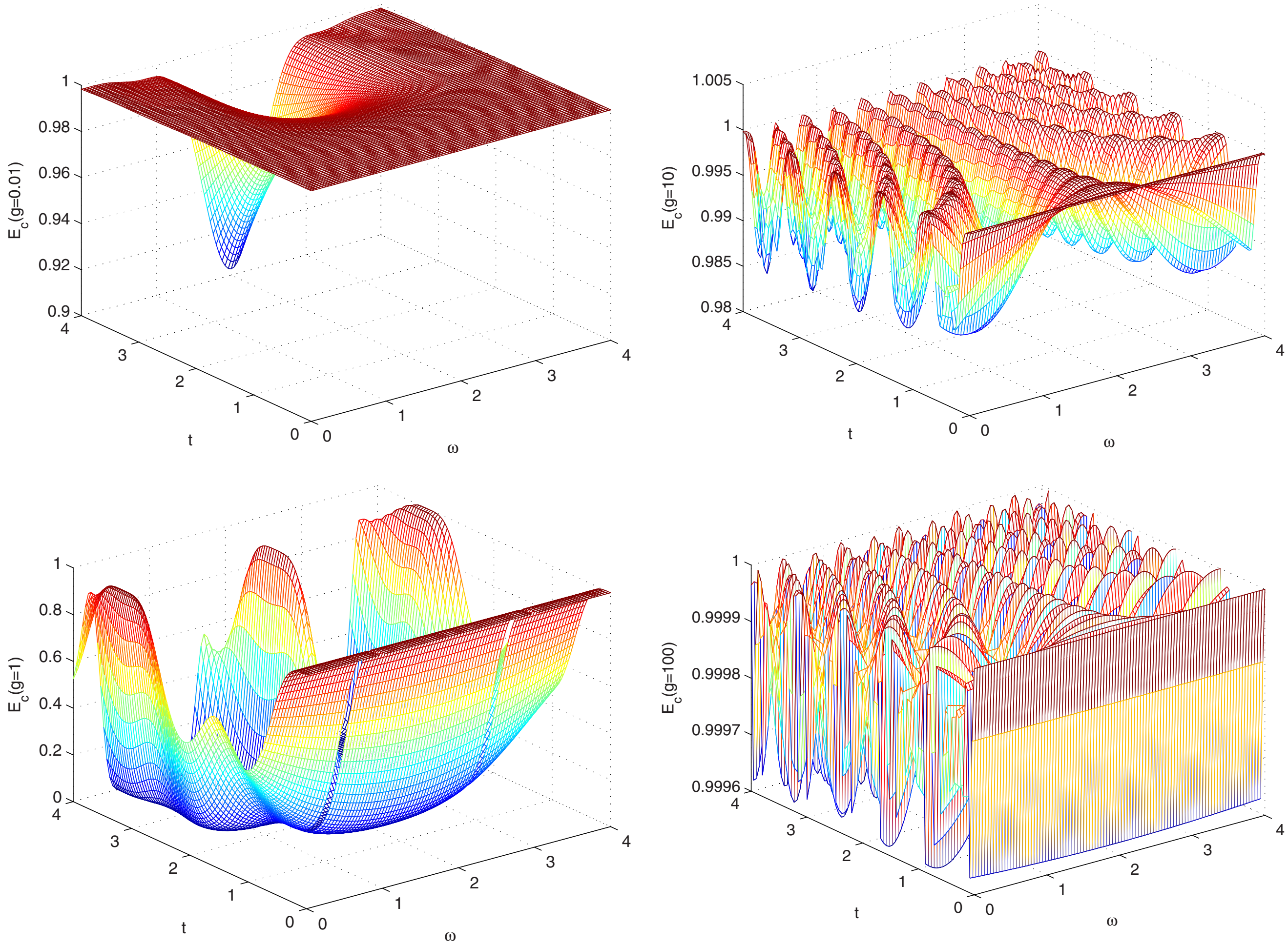

FIG. 1. (Color online) Decoherence factor E_c versus time t and ω with a critical value of 2 is demonstrated for $g=0.01,1,10,100$ when the momentum k takes small values, where $g\equiv(g_1+g_2+g_3)/2$ and $\Delta_\alpha,\Delta_\beta$ take expressions of Δ_1,Δ_8, respectively.

In the illustrations of Fig. 1, we have assumed that the constant $\Lambda(k_c)$ take a value of 0.5. It should be noted that the above demonstrations of quantity E_c in Fig. 1 has employed the fact that $\Delta_\alpha=\Delta_1$, $\Delta_\beta=\Delta_8$. The numerical results implies that when $g=(g_1+g_2+g_3)/2$ takes a small value, as shown in Fig. 1, the decay of E_c is enhanced by the quantum phase transition of the environment. However, when g takes a large value such as $g=1$, the decay of E_c is complicated and no enhancement of decay of E_c has been found as some recurrences appear near the critical line. When $g=10$, the quantity of E_c shows a small oscillation. When $g=100$, the quantity of E_c is almost stable and close to 1. Numerical calculation implies that the decay of E_c will be enhanced by the quantum phase transition of the environment when g takes a small value such as $g=0.01$. Similarly, for the cases, namely, (a) $\Delta_\alpha=\Delta_2$, $\Delta_\beta=\Delta_7$, (b) $\Delta_\alpha=\Delta_3$, $\Delta_\beta=\Delta_6$, (c) $\Delta_\alpha=\Delta_4$, $\Delta_\beta=\Delta_5$, the decay of the corresponding quantity of E_c will be enhanced by the quantum phase transition of the environment when $(g_1+g_2-g_3)/2$, $(g_1-g_2+g_3)/2$, and $(g_1-g_2-g_3)/2$ take small values, respectively. Even though the calculation of the decoherence factor has employed some approximations and thus the result cannot give an exact description of the effect of the quantum-critical environment, the analysis will be helpful in obtaining a heuristic knowledge of decoherence from the critical environment.

In the following content, we will carry out numerical simulation of the exact expression of Eq. (9) to investigate the entanglement dynamics of three-qubit states.

III. ENTANGLEMENT DECAY

In this section, firstly, we introduce the operational and calculation entanglement measures. Secondly, we analyze the entanglement dynamics of three-qubit states under decoherence. Finally, a general discussion of entanglement decay in a quantum-critical environment is given.

Due to good operational and calculation properties, negativity has been used to measure the multipartite entanglement of quantum states with high dimensions. For a given quantum state with density matrix ρ, the negativity of ρ is defined by

$$N(\rho)=\frac{\|\rho^{T_i}\|-1}{2}, \tag{13}$$

where $\|\rho^{T_i}\|$ is the sum of the absolute values of the eigenvalues of ρ^{T_i}, and ρ^{T_i} denotes the partial transpose of density

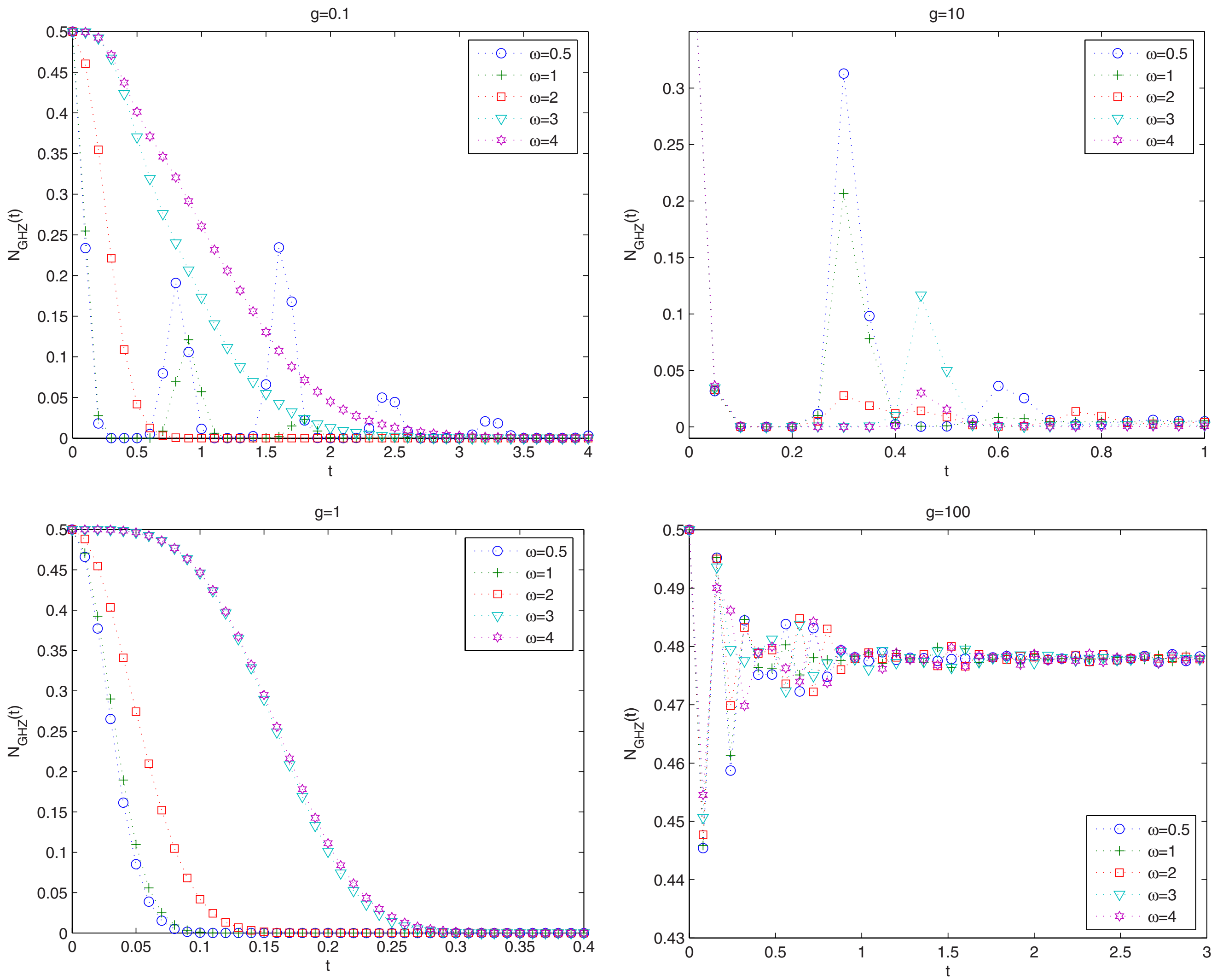

FIG. 2. (Color online) Negativity versus time t is plotted for different kinds of coupling: $g=0.1,1,10,100$ when ω as the strength of the transverse field of the environment takes different values, where $g\equiv(g_1+g_2+g_3)/2$ and $N_t=300$.

matrix ρ with respect to party i. As is known, the partial transpose of a density matrix does not change the trace of a density matrix, so we can easily find that the negativity of ρ equals the sum of absolute values of the negative eigenvalues of ρ^{T_i}. It was proved that negativity is a local operation and classical communication entanglement monotone.[27]

Negativity can be applied to multipartite quantum state. For example, a tripartite quantum state ρ_{ABC} can be splitted into three bipartitions $AB-C, BC-A, AC-B$. Every bipartition gives a negativity, so there are three negativities $N_{AB-C}, N_{BC-A}, N_{AC-B}$ to measure the quantum correlation between one group with two parties and the other group with one party. We also consider the residual entanglement of a reduced density matrix, which can be obtained by tracing one party off the density matrix ρ_{ABC}, and three reduced density matrices appear. For the density matrix of ρ_{ABC}, label the reduced density matrices by $\rho_{AB}^r, \rho_{BC}^r, \rho_{AC}^r$, and we can analyze the residual entanglement with negativities $N_{A-B}, N_{B-C}, N_{A-C}$, respectively.

In practice, we are interested in some explicit examples. Here, firstly, we consider the entanglement dynamics of two types of initial pure states, the GHZ state and the W state with expressions $|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ and $|W\rangle=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$, which are known to bear incompatible multipartite correlations, in the sense that they cannot be transformed into each other by local operations and the classical communication.[30] Secondly, we discuss the entanglement evolution of the Werner-like state[31] as an example of a mixed state under the environment.

Case 1. Let the initial state of the system be a three-qubit GHZ state. $|\psi_s(0)\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$. Here, $|0\rangle$ and $|1\rangle$ are the eigenstates of s_j^z with corresponding eigenvalues of $1/2$ and $-1/2$, respectively. From Eq. (7), we obtain the time evolution of entanglement of the GHZ state,

$$N_{AB-C}(t)=\frac{1}{2}|F_{18}(t)|. \tag{14}$$

For the symmetry of the GHZ state, the other two negativities of N_{AC-B}, N_{BC-A} take the same expression as N_{AB-C}. Tracing one party off the GHZ state, we find that the reduced

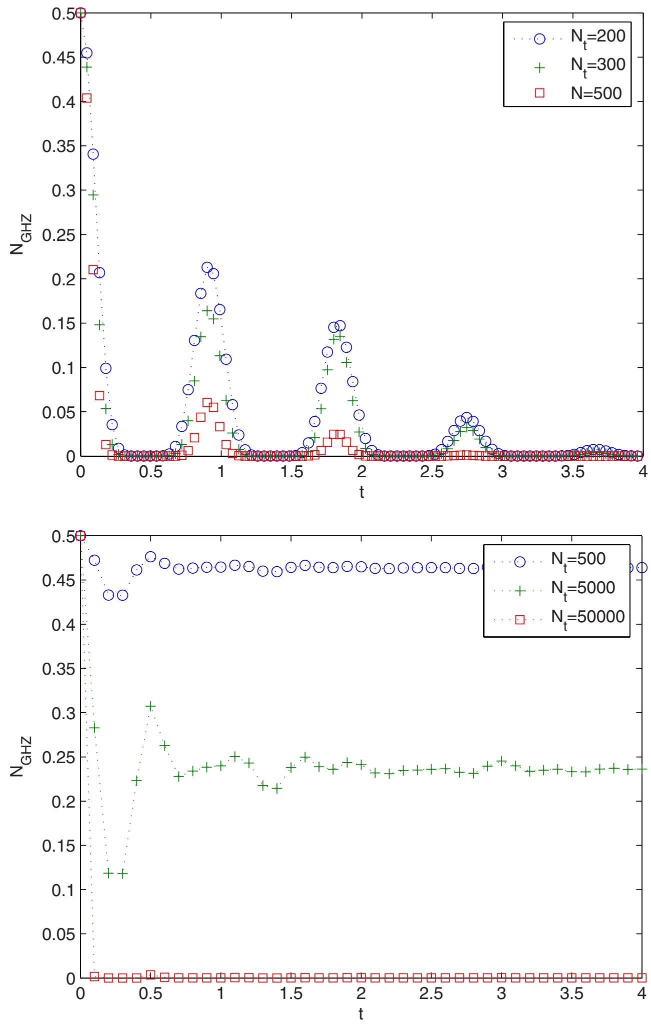

FIG. 3. (Color online) Negativity of the GHZ state versus time t is plotted under different sizes of the degree freedom of the environment, where $g \equiv (g_1+g_2+g_3)/2=0.1, 100$ for the left and the right subfigures, $\omega=1$.

density matrix is separable, so the residual entanglement of $N_{A-C}, N_{B-C}, N_{A-B}$ is zero.

To examine the effect of environment on the quantum entanglement of the GHZ state, we numerically calculate the exact expression of Eq. (9) and plot the entanglement evolution under different conditions in Fig. 2. Here and in the following numerical simulation, we have used the fact that the lattice spacing a takes the value of $2\pi/N_t$. From Figs. 2 and 3, we can find that the entanglement evolution of the GHZ state depends not only on the parameter of ω, the coupling constants g_j $(j=1,2,3)$, but also on N_t, which is the size of the degrees of freedom of the environment. Under weak coupling $(g=0.1)$, there are revivals of entanglement when $\omega<2$ seen from the cases of $\omega=0.5, 1$, while the entanglement will vanish monotonously with time when $\omega \geqslant 2$, and the decay of entanglement is enhanced by the quantum phase transition at which $\omega=2$. For the case $g=1$ in Fig. 2, we can find that the entanglement will not be revived anymore, but will vanish monotonously to zero with time going long. However, when the coupling strength takes the expression of $g=10$, some revivals occur in the time evolution of entanglement of the GHZ state; only when the time is long enough can the entanglement vanish completely. Here, even though the fluctuations for $\omega=2$ are smaller than the other cases in short time, yet, near $t=0.8$, the fluctuations for $\omega=2$ are bigger among all the cases. Compared to the cases $g=0.1$ and $g=1$, one loses compelling evidence of any strong enhancement of entanglement decay due to the phase transition. Taking a look at the final subfigure in Fig. 2 where $g=100$, we find that the entanglement of the GHZ state fluctuates initially and tends to a constant near 0.478 when the time is long enough. This result is not surprising because the size of degrees of freedom of the environment is assumed to be finite. In fact, a practical environment has infinite degrees of freedom and can vanish the quantum entanglement completely. The right subfigure in Fig. 3 can give a naive argument for such a claim, where we plot the entanglement evolution of the GHZ state for the case $g=100$ under the environment with different sizes of the degrees of freedom. Such a result is consistent with our forehead analysis of the evolution of decoherence factor E_c. It should be pointed out that we numerically calculated the decoherence factor $|F_{18}|$, which involves the case $\Delta_\alpha=\Delta_1$, $\Delta_\beta=\Delta_8$ with corresponding parameter $g=(g_1+g_2+g_3)/2$, and similar results should also be applied to the decoherence factors $|F_{27}|$, $|F_{36}|$, and $|F_{45}|$ with corresponding parameters $(g_1+g_2-g_3)/2$, $(g_1-g_2+g_3)/2$, and $(g_1-g_2-g_3)/2$, respectively. We now discuss the decoherence factor $|F_{27}|$. Under weak coupling, when $(g_1+g_2-g_3)/2$ takes small values, the decay of decoherence factor $|F_{27}|$ will be enhanced. While $(g_1+g_2-g_3)/2$ takes large values, the evolution of decoherence factor $|F_{27}|$ will take a similar behavior to that of $|F_{18}|$. Similar conclusion can also be applied to the decoherence factors $|F_{36}|$, $|F_{45}|$.

In order to examine the effect of the size of the degrees of freedom of environment on the quantum entanglement, we numerically calculated the entanglement evolution under different sizes and found that the entanglement vanishes at sufficient time takes long enough for the case $g=0.1$ seen in the left subfigure of Fig. 3. During the process of entanglement evolution, there are some revivals of entanglement. With regards to the revivals, the numerical results imply that the larger the size is, the smaller the revivals of entanglement is. For the case $g=100$, we can find that the entanglement will decrease to a stable value for a finite size of the environment; the larger the size of the degrees of freedom of the environment is, the smaller the stable value is. A practical environment has an infinite size of degrees of freedom, so it can destroy the quantum entanglement completely even for large g, which is a finite coupling constant. In fact, a true quantum phase transition will not happen in a finite-sized environment but in an infinite-sized environment. However, it is difficult to carry out numerical simulation of the effect from an infinite-sized environment on quantum states. Therefore, we can say that our study can shed some light on the decoherence induced by a quantum-critical environment on quantum states to some extent.

Case 2. W state is one important quantum state like the GHZ state in QIP. Here, we consider that the initial state of the system under the environment is the W state $|\psi_s(0)\rangle$

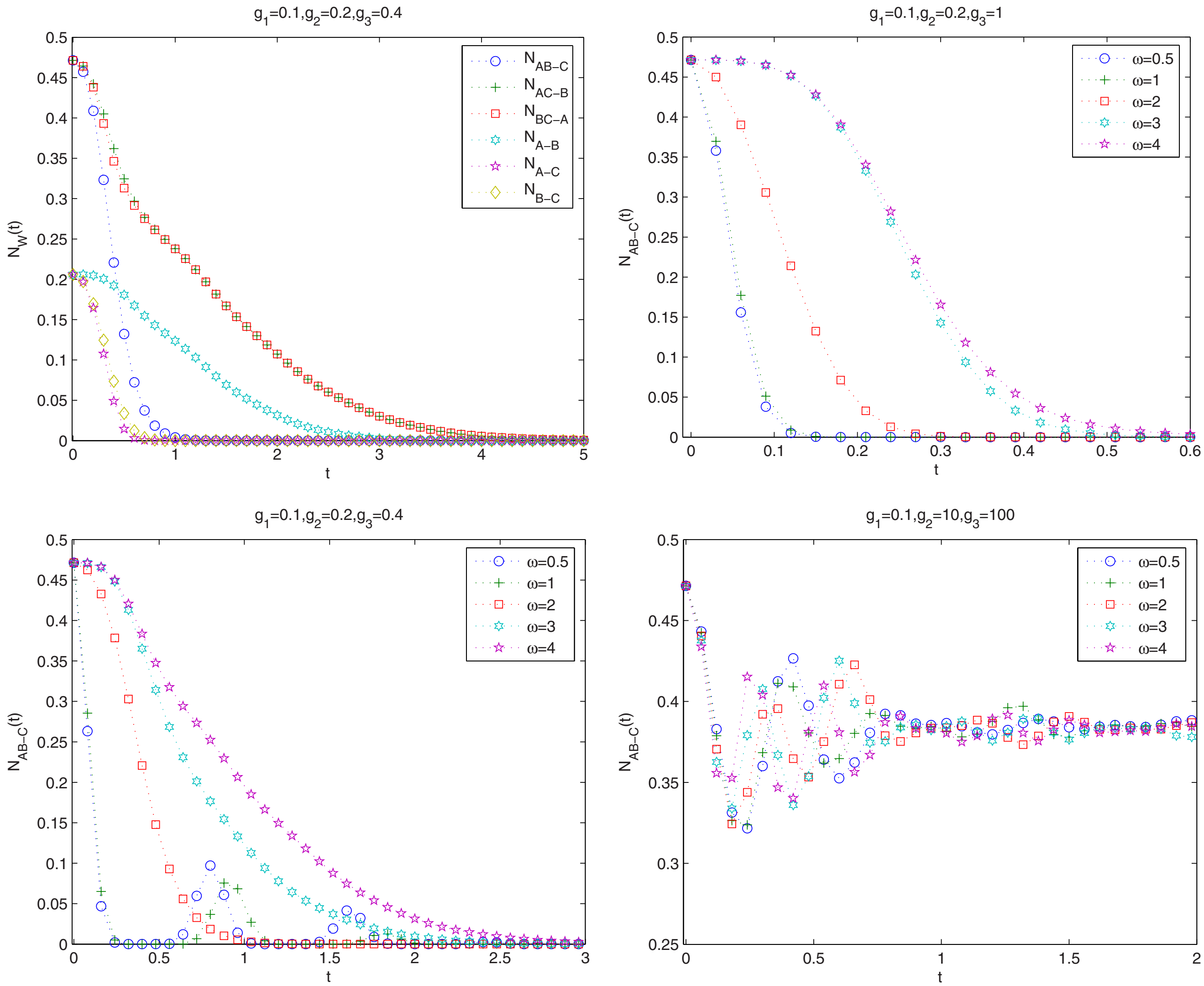

FIG. 4. (Color online) Negativities of the W state under a fixed-transverse magnetic field $\omega=2$ and the quantity of N_{AB-C} with different kinds of coupling are plotted, where $N_t=300$.

$=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$. Similarly, we get the negativities of the W state in the following expression:

$$N_{AB-C}=\frac{1}{3}\sqrt{|F_{23}|^2+|F_{25}|^2},\quad N_{A-B}=\frac{1}{6}(\sqrt{4|F_{35}|^2+1}-1),$$

$$N_{AC-B}=\frac{1}{3}\sqrt{|F_{23}|^2+|F_{35}|^2},\quad N_{A-C}=\frac{1}{6}(\sqrt{4|F_{25}|^2+1}-1),$$

$$N_{BC-A}=\frac{1}{3}\sqrt{|F_{25}|^2+|F_{35}|^2},\quad N_{B-C}=\frac{1}{6}(\sqrt{4|F_{23}|^2+1}-1). \tag{15}$$

Here, we make some comparison of the entanglement evolution of the GHZ state and the W state. Firstly, the entanglement evolution of the W state is different from that of the GHZ state in that the difference in coupling constants of g_j can be reflected by the time evolution of the negativities of the W state from Fig. 4. Secondly, the residual entanglement of the W state is not zero but takes a value; this fact is familiar. Thirdly, when all the coupling constants take the same value, we find that the W state will not perceive the presence of the environment. In this sense, the W state is called as a decoherence-free quantum state and can be used to design noiseless quantum codes.[32–34] Such a result can also be applied to the W class states which read $|W\rangle=w_1|011\rangle+w_1|101\rangle+w_3|110\rangle$, where w_1,w_2,w_3 are complex coefficients. From the above analysis, we can say that the W state is more robust that the GHZ state in the context here.

To examine the effect of quantum phase transition on the entanglement of the W state, we numerically calculate the quantity of N_{AB-C} under different couplings, either weak or strong. Because of symmetry, the other quantities of entanglement of the W state are omitted here. From Fig. 4, we can find that the behavior of time evolution of the W state is similar to that of the GHZ state under different couplings, either weak or strong. Under weak coupling such as $g_1=0.1$, $g_2=0.2$, $g_3=0.4$, the decay of quantity N_{AB-C} is enhanced by the quantum phase transition at which $\omega=2$. Under strong coupling, as shown in the final subfigure of Fig. 4, we do not find the enhancement of entanglement decay. Especially, numerical calculation implies that the entanglement of quantity N_{AB-C} tends to a constant near 0.38 under strong coupling shown in the final subfigure of Fig. 4. Though we just single out one quantity N_{AB-C} to demonstrate the effect of environment on the quantum entanglement evolution, the results can be applied to other quantities of the W state because of symmetry.

Case 3. Beyond the pure quantum states, mixed states should also be discussed. Here, we employ a Werner-like state to investigate the effect of a critical quantum environment on mixed states. Werner-like state reads

$$\rho_{Werner} = \frac{pI_{8\times 8}}{8} + (1-p)|\psi\rangle\langle\psi|, \tag{16}$$

where p is a parameter characterizing the extent to which the white noise exists in state $|\psi\rangle$, and it ranges from 0 to 1. Operator $I_{8\times 8}$ is an identity matrix with rank of 8, and $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Due to decoherence, the time evolution of the Werner-like state takes the following expression:

$$N_{AB-C} = \frac{1-p}{2}|F_{18}| - \frac{p}{8}. \tag{17}$$

Equation (17) holds under the condition that $\frac{1-p}{2}|F_{18}| > \frac{p}{8}$; otherwise, negativity N_{AB-C} takes a zero value. Due to a high symmetry, the other negativities of N_{AC-B}, N_{BC-A} take the same expression with N_{AB-C}. The entanglement evolution of the Werner-like state is illustrated in Fig. 5. From Fig. 5, we find that the more noise applied to state $|\psi\rangle$ is, the shorter the time for entanglement persists. As one observation of Fig. 5, for the case of $p=0$, the Werner-like state is reduced to the GHZ state, and the time for complete disentanglement is infinite; however, for the cases of $p\neq 0$ including cases of $p=0.1,0.3,0.5,0.7$, the entanglement of the Werner-like state vanishes completely due to the decoherence induced by the environment. Usually, researchers call this phenomenon of finite-time disentanglement as "sudden entanglement death" firstly observed by Yu and Eberly in the study of spontaneous emission.[35] To illustrate the effect of different couplings on the quantum entanglement of the Werner-like state when the transverse magnetic field takes the critical value of $\omega=2$, we plot the right subfigure in Fig. 5 where the cases $g=0.1,1,10,100$ are plotted. When the coupling takes a large value such as $g=1$, the entanglement vanishes much faster. Here, the result does not conflict with the conclusion of the enhancement of entanglement decay for the GHZ and W states under weak coupling. When the coupling $g=10$, the time evolution of entanglement shows an oscillating behavior. If the coupling takes a value large enough as $g=100$, we find that the entanglement will be reduced quickly to a constant near 0.418. Additionally, the entanglement evolution of the Werner-like state is similar to that of the GHZ state due to the similar expression, so the effect of quantum phase transition of environment on the entanglement of the Werner-like state can be obtained directly from the analysis of the GHZ state.

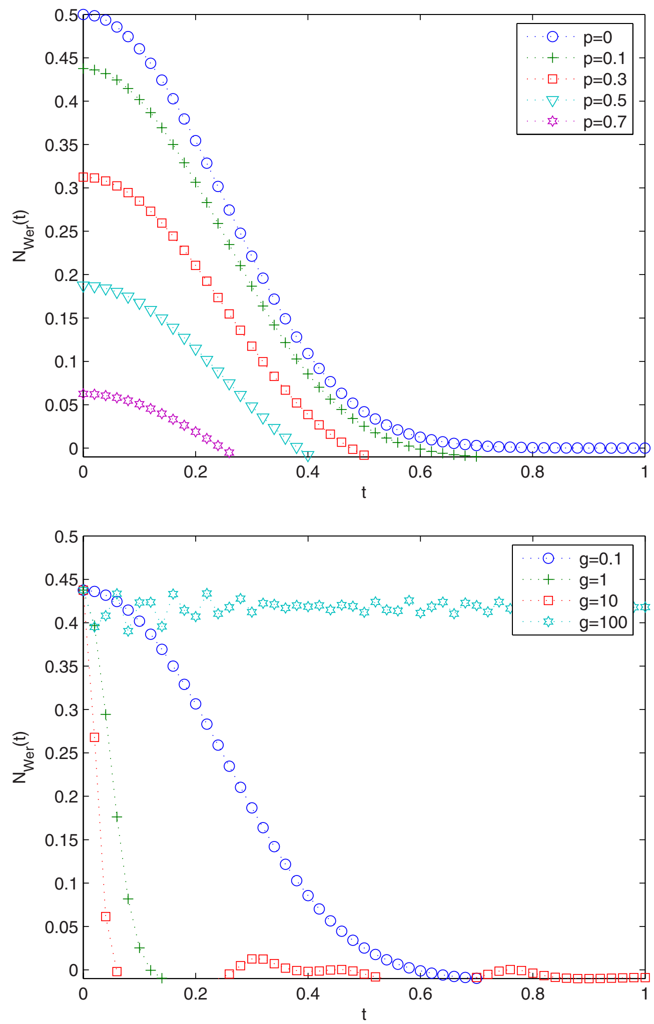

FIG. 5. (Color online) Negativities of the Werner-like state versus time t are plotted for (1) different values of p and a fixed value of $g\equiv(g_1+g_2+g_3)/2=0.1$ and (2) different values of g and a fixed value of $p=0.1$, where $N_t=300$, $\omega=2$.

With the above three cases, we analyze the effect of a quantum-critical environment on the entanglement of three-qubit states. From the above cases, we find that the entanglement dynamics depends not only on the system-environment coupling and the size of degrees of freedom of the environment but also on the strength of transverse magnetic field and the symmetry of the state of concern. Furthermore, the numerical results imply that the quantum phase transition of the environment plays a positive role in enhancing the decay of entanglement for these states when the system couples to the environment weakly. Does such a conclusion apply to all the entangled states? We cannot answer this question here because such a question is difficult to prove. In principle, the entanglement evolution of any state of a three-qubit system can be discussed as well. Here, for convenience, the other examples are omitted.

IV. DISCUSSION AND CONCLUSION

Getting a further insight into the decoherence factor $F_{\alpha\beta}(\alpha,\beta=1,2,\ldots,8)$ which determines the effect of environment on quantum states of concern, we can find that $F_{\alpha\beta}=1$ if $\Delta_\alpha=\Delta_\beta$, and thus the quantum states will remain unchanged during the interaction process with the environment. In particular, when the coupling constants of $g_j(j=1,2,3)$ take the same value of g, we find a linear decoherence-free space[32–34] S_{LDF} consisting the following basis: $|000\rangle\langle000|$, $|001\rangle\langle001|$, $|010\rangle\langle010|$, $|011\rangle\langle011|$, $|100\rangle\langle100|$, $|101\rangle\langle101|$, $|110\rangle\langle110|$, $|111\rangle\langle111|$, $|001\rangle\langle010|$, $|001\rangle\langle100|$, $|010\rangle\langle001|$, $|010\rangle\langle100|$, $|011\rangle\langle101|$, $|011\rangle\langle110|$, $|100\rangle\langle001|$, $|100\rangle\langle010|$ $|101\rangle\langle011|$, $|101\rangle\langle110|$, $|110\rangle\langle011|$, $|110\rangle\langle101|$. Any state that can be expanded by the basis of S_{LDF} will remain under decoherence when all the coupling constants take the same value; in contrast, any state that cannot be expanded by the basis of S_{LDF} will lose coherence. When $\Delta_\alpha\neq\Delta_\beta$ $(\alpha\neq\beta)$, $F_{\alpha\beta}$ will be a value less than 1 and thus the coherence of quantum states will change. Especially, when $F_{\alpha\beta}(\alpha\beta)$ decays exponentially with time, then any density matrix will evolve into the final form like $\rho(t\to\infty)=\Sigma_\alpha c_\lambda|\lambda\rangle\langle\lambda|$ which is a separable quantum density matrix, where c_λ are real and non-negative coefficients and $|\lambda\rangle$ are the orthogonal basis of quantum states.

To conclude, we have investigated the entanglement dynamics of three-qubit quantum states in a correlated environment. Our results imply that the entanglement evolution depends not only on the strength of the transverse field, the coupling constants $g_j(j=1,2,3)$, but also on N_t as the size of degrees of freedom of the environment. Due to different symmetries, the entanglement evolution of different quantum states shows different behaviors. Specifically, we employ the GHZ state, the W state, and the Werner-like state to investigate their entanglement evolution with negativity as the entanglement measure. Our results from the cases imply that quantum entanglement decay is enhanced by the quantum phase transition of environment under weak coupling, while under strong coupling, the entanglement evolution is complicated with the order of magnitude of the coupling strength change. Specially, numerical calculation shows that the entanglement tends to be a constant when the coupling strength takes a large enough value. Furthermore, we have investigated the effect of the size of degrees of freedom of the environment on quantum entanglement and found that the size affects the quantum entanglement too. In short time, the larger the size is, the smaller the revival of entanglement is. Finally, we have made a general discussion of the effect of decoherence on the quantum entanglement. When all the coupling constants take the same value, a linear decoherence-free quantum space is identified; however, when all the couplings satisfy some certain relations and the relation of $\Delta_\alpha\neq\Delta_\beta$ holds for all the cases of $\alpha\neq\beta$, then any entangled quantum state will become a separable one. In a word, our results will contribute to a clear understanding of entanglement dynamics of three-qubit quantum states in a critical environment and can shed some light on the multipartite entanglement dynamics in a correlated environment.

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