Electric field effects on motion of a charged particle through a saddle potential in a magnetic field

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The role of an electric field in electron transmission through a quantum point contact (QPC) in the presence of a magnetic field is analyzed here, modeling the QPC as a saddle potential. In this paper, we derive the relevant Green's function, including the effects of arbitrarily time-dependent electric and constant magnetic fields. The derivation is carried out using Schwinger's operator equation of motion approach. In the second part of the paper, we apply the Green's function to determine transmission of the electron guiding center through the QPC in constant electric and magnetic fields.

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I. INTRODUCTION

Recent years have seen growing interest in the development of smaller and faster semiconductor devices.[1] This has led to intensified study of nonlinear quantum transport for electrons in nanoscale systems in which point contacts have been appropriately modeled in terms of a saddle potential in a high magnetic field, with considerable success in explaining experimental data.[2–5] To investigate electron dynamics in such systems, we analyze the single-particle Schrödinger Green's function for a saddle potential with a time-dependent electric field and a perpendicular static magnetic field. The derivation is carried out by employing Schwinger's equation of motion technique, and the result facilitates the determination of the transmission coefficient for electrons across a point contact in the presence of electric and magnetic fields.

II. TWO DIMENSIONAL QUANTUM POINT CONTACT HAMILTONIAN WITH ELECTRIC AND MAGNETIC FIELDS

We model the quantum point contact (QPC) as a saddle potential of the form

$$V_{\mathrm{SP}}(x,y) = V_0 - \tfrac{1}{2}m\omega_x^2x^2 + \tfrac{1}{2}m\omega_y^2y^2, \tag{1}$$

where V_0 is the potential at the saddle point and the curvatures are expressed in terms of the frequencies ω_x and ω_y. The Hamiltonian for an electron in the saddle potential in the presence of a constant, uniform magnetic field, $\mathbf{B}=B\hat{\mathbf{z}}$, and a crossed external time-dependent electric field (uniform in space), $\mathbf{E}=E(t)\hat{\mathbf{x}}$, is given by ($\hbar \to 1$)

$$H = \frac{1}{2m}(\mathbf{p}+e\mathbf{A})^2 + V_{\mathrm{SP}}(x,y) + e\mathbf{x}\cdot\mathbf{E}(t), \tag{2}$$

where $\mathbf{A}=\frac{B}{2}(-y,x,0)$ is the vector potential in the symmetric gauge. This can be rewritten in the form

$$H = \frac{1}{2m}(p_x^2+p_y^2) + \frac{1}{8}m\omega_c^2(x^2+y^2) - \frac{1}{2}\omega_c(yp_x - xp_y) - \frac{1}{2}m(\omega_x^2x^2 - \omega_y^2y^2) + exE(t) + V_0, \tag{3}$$

where ω_c is the cyclotron frequency.

Following the sequence of transformations employed by Fertig and Halperin[6] (for details see Appendix A),

$$\begin{aligned} X &= \frac{1}{\sqrt{m\Omega}}(p_y \sin\phi + m\Omega x\cos\phi)e^{\theta_1},\\ P &= \frac{1}{\sqrt{m\Omega}}(p_x\cos\phi - m\Omega y\sin\phi)e^{-\theta_1},\\ s &= \frac{1}{\sqrt{m\Omega}}(m\Omega x\sin\phi - p_y\cos\phi)e^{-\theta_2},\\ p &= \frac{1}{\sqrt{m\Omega}}(m\Omega y\cos\phi + p_x\sin\phi)e^{\theta_2}, \end{aligned} \tag{4}$$

which obey the following commutation relations,

$$\begin{aligned} &[X,P]=[s,p]=i,\\ &[s,X]=[s,P]=[p,X]=[p,P]=0, \end{aligned} \tag{5}$$

with the following definitions for the various parameters,

$$\Omega^2 = (\omega_c/2)^2 + \Omega_-^2, \quad \Omega_-^2 = (\omega_y^2 - \omega_x^2)/2,$$

$$\Omega_+^2 = (\omega_x^2+\omega_y^2)/2, \quad \gamma = \Omega_+^2/(4\Omega),$$

$$\tan 2\phi = -\frac{\omega_c}{4\gamma}, \tag{6}$$

$$\tanh 2\theta_1 = \frac{\frac{1}{2}\Omega - \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}}{-\gamma},$$

$$\tanh 2\theta_2 = \frac{\gamma}{\frac{1}{2}\Omega + \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}},$$

we have

$$H = H_1 + H_2, \tag{7}$$

with $[H_1,H_2]=0$. Here, H_1 is given by

$$H_1 = E_1(P^2 - X^2) + F(t)X, \tag{8}$$

with

$$E_1 = \left[\gamma^2 - \left\{\frac{1}{2}\Omega - \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}\right\}^2\right]^{1/2} \tag{9}$$

and

$$F(t) = \frac{eE(t)}{\sqrt{m\Omega}} e^{-\theta_1} \cos\phi, \tag{10}$$

while H_2 is given by

$$H_2 = \tfrac{1}{2}E_2(p^2 + s^2) + G(t)s + V_0, \tag{11}$$

where

$$E_2 = 2\left[\left\{\frac{1}{2}\Omega + \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}\right\}^2 - \gamma^2\right]^{1/2} \tag{12}$$

and

$$G(t) = \frac{eE(t)}{\sqrt{m\Omega}} e^{\theta_2} \sin\phi. \tag{13}$$

It is important to note here that the two parts of the Hamiltonian obey the commutation relation $[H_1,H_2]=0$. The first part of the Hamiltonian, H_1, represents an electron in an inverted harmonic potential in the presence of an external time-dependent "electric field," $F(t)$, and describes its "guiding center" motion in terms of $X(t)$. The second part of the Hamiltonian, H_2, represents a one dimensional harmonic oscillator in a time-dependent electric field, $G(t)$, that describes the "cyclotron like" aspect of the motion of the electron.

III. ELECTRON GREEN'S FUNCTION IN A QUANTUM POINT CONTACT IN THE PRESENCE OF EXTERNAL FIELDS

In this section, we derive the Green's function for the problem at hand. The Green's function for the X motion (guiding center motion) is described by $H_1(t)$ and it may be constructed using Schwinger's approach as[7]

$$G_{H_1}(X,t;X',0) = K_{H_1}(X,X') \times \exp\left(-i\int_0^t \frac{\langle X(t')|H_1(t')|X'(0)\rangle}{\langle X(t')|X'(0)\rangle} dt'\right), \tag{14}$$

where $K_{H_1}(X,X')$ is independent of t and is determined by magnetic gauge considerations and the initial condition. The equations of motion for the operators X and P yield the following coupled equations:

$$\frac{dX}{dt} = -i[X,H_1] = 2E_1P \tag{15}$$

and

$$\frac{dP}{dt} = -i[P,H_1] = 2E_1X - F(t). \tag{16}$$

Combining the last two equations, we have

$$\frac{d^2X}{dt^2} - 4E_1^2X = -2E_1F(t), \tag{17}$$

which can be solved using the method of variation of parameters. The general solution is

$$X(t) = D_1\cosh(2E_1t) + D_2\sinh(2E_1t) + \cosh(2E_1t)\int_0^t F(t')\sinh(2E_1t')dt' - \sinh(2E_1t)\int_0^t F(t')\cosh(2E_1t')dt', \tag{18}$$

where D_1 and D_2 are arbitrary constants to be determined in terms of the initial values of $X(0)$ and $P(0)$. The form of $X(t)$ and $P(t)$ in terms of their initial values is given by

$$X(t) = X(0)\cosh(2E_1t) + P(0)\sinh(\omega t) - \int_0^t F(t')\sinh 2E_1(t-t')dt', \tag{19}$$

$$P(t) = \frac{1}{2E_1}\frac{dX}{dt} = X(0)\sinh(2E_1t) + P(0)\cosh(2E_1t) - \int_0^t F(t')\cosh 2E_1(t-t')dt'. \tag{20}$$

Eliminating $P(0)$, using Eq. (19), from Eq. (20) we obtain

$$P(t) = X(t)\coth(2E_1t) - \frac{1}{\sinh(2E_1t)}X(0) - \frac{1}{\sinh(2E_1t)}\int_0^t F(t')\sinh(2E_1t')dt'. \tag{21}$$

To determine the Green's function for H_1, we need to evaluate the matrix element $\langle X(t)|H_1(t)|X(0)\rangle$. In the process, we have to commute the $X(0)$ operator in P^2 to the right of $X(t)$. This is accomplished by using the commutation relation

$$[X(t),X(0)] = -i\sinh(2E_1 t). \tag{22}$$

Evaluation of Eq. (14) then yields

$$iG_{H_1}(X,t;X',0) = \frac{1}{\sqrt{2\pi i\sinh(2E_1t)}}\exp\Bigg[i\frac{1}{2}(X^2+X'^2)\coth(2E_1t) - iXX'\frac{1}{\sinh(2E_1t)} - \frac{i}{\sinh(2E_1t)}\int_0^t dt' F(t')\{X'\sinh[2E_1(t-t')] + X\sinh(2E_1t')\} - \frac{i}{\sinh(2E_1t)}\int_0^t dt'\int_0^{t'} dt'' F(t')\sinh[2E_1(t-t')]\sinh(2E_1t'')F(t'')\Bigg]. \tag{23}$$

Similarly, the equations of motion for the s motion are given by

$$\frac{ds}{dt} = -i[s,H_2] = E_2 p \tag{24}$$

and

$$\frac{dp}{dt} = -i[p,H_2] = -E_2 s - G(t), \tag{25}$$

leading to

$$\frac{d^2s}{dt^2} + E_2^2 s = -E_2 G(t). \tag{26}$$

Following the same technique used above for the X motion, the Green's function associated with the s motion (cyclotron center motion) is given by

$$iG_{H_2}(s,t;s',0) = \frac{1}{\sqrt{2\pi i\sin(E_2t)}}\exp\Bigg[i\frac{1}{2}(s^2+s'^2)\cot(E_2t) - iss'\frac{1}{\sin(E_2t)} - iV_0t - \frac{i}{\sin(E_2t)}\int_0^t dt' G(t') \times[s'\sin E_2(t-t') + s\sin(E_2t')] - \frac{i}{\sin(E_2t)}\int_0^t dt'\int_0^{t'} dt'' G(t') \times\sin[E_2(t-t')]\sin(E_2t'')G(t'')\Bigg]. \tag{27}$$

Since $[H_1,H_2]=0$, the Green's function for the full Hamiltonian $H=H_1+H_2$ can be written as (Appendix B)

$$G_H(X,s,t;X',s',0) = iG_{H_1}(X,t;X',0)G_{H_2}(s,t;s',0). \tag{28}$$

IV. ELECTRON TRANSMISSION THROUGH A QPC

We will analyze electron tunneling and/or scattering through a QPC in crossed fields for the case of a constant electric field. In conjunction with the Hamiltonian, Eq. (7), we consider a general product state (in a representation where X and s are diagonal) as $\chi(X,s)=\phi(X)\psi_n(s)$. Here, $\psi_n(s)$ is taken to satisfy a harmonic-oscillator-like equation,

$$H_2\psi_n(s') = \varepsilon_2^{(n)}\psi_n(s') = \left[\left(n+\frac{1}{2}\right)E_2 - \frac{G^2}{2E_2} + V_0\right]\psi_n(s'), \tag{29}$$

where $s'=s+G/E_2$, and G is given from Eq. (13) as $G=\frac{eE}{\sqrt{m\Omega}}e^{\theta_2}\sin\phi$. The X part of the product state, $\phi(X)$, satisfies

$$H_1\phi(X) = (\mathcal{E} - \varepsilon_2^{(n)})\phi(X), \tag{30}$$

with $\mathcal{E}$ the total energy corresponding to the state $\chi(X,s)$ for the system described by H.

The time development of an arbitrary state of the system, $\Psi(X,s;t)$, arising from an initial state, $\Psi(X',s';0)$, is given by (Appendix B)

$$\Psi(X,s;t) = i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dX'ds' G_H(X,s,t;X',s',0)\Psi(X',s';0), \tag{31}$$

where $G_H(X,s,t;X',s',0)$ is the Green's function derived above. Specifically, if we choose the initial state as a product state, $\Psi(X',s';0)=\Psi_1(X',0)\Psi_2(s',0)$, the time development is described by Eq. (28) as

$$\Psi(X,s;t) = i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dX'ds'G_H(X,s,t;X',s',0)\Psi(X',s';0)$$
$$= \left[\int_{-\infty}^{\infty} dX' iG_{H_1}(X,t;X',0)\Psi_1(X',0)\right]$$
$$\times\left[\int_{-\infty}^{\infty} ds' iG_{H_2}(s,t;s',0)\Psi_2(s',0)\right], \quad (32)$$

which indicates that the two parts of such a product state propagate independently. Following Fertig and Halperin,[6] we choose the s'-dependent initial state to lie in the nth oscillator level $\psi_n(s)$ of H_2. Then, $\psi_n(s';t)\sim e^{-i\varepsilon_2^{(n)}t}$ has constant modulus for all t and the transmission coefficient of the two dimensional problem is reduced to the one dimensional transmission coefficient associated with the guiding center X motion. Hence, from Eq. (32) we need only to evaluate

$$\Phi_{out}(X,t) = \int_{-\infty}^{\infty} dX' iG_{H_1}(X,t;X',0)\Phi_{in}(X',0), \quad (33)$$

where $\Psi_1(X',0)\rightarrow\Phi_{in}$ $[\Psi(X,s;t)\rightarrow\Phi_{out}]$ is the incoming (outgoing) state. In the case of constant electric field, the Green's function of Eq. (23) for the guiding center motion reduces to

$$iG_{H_1}(X,t;X',0) = \sqrt{\frac{b}{2\pi i}}\exp\Big[i\tfrac{1}{2}c(X^2+X'^2) - ibXX'$$
$$- if^2E_1t - if(c-b)(X+X'-f)\Big], \quad (34)$$

where $f=F/2E_1$, $b=1/\sinh(2E_1t)$, and $c=\coth(2E_1t)$.

In regard to the initial "in" state, $\phi(X,0)$ is formed from the even, $\phi_e(X)$, and odd, $\phi_o(X)$, solutions of Eq. (30) for the guiding center motion in a constant electric field,

$$H_1\phi_{e,o}(X) = [E_1(P^2-X^2)+FX]\phi_{e,o}(X) = (\mathcal{E}-\varepsilon_2)\phi_{e,o}(X), \quad (35)$$

where, from Eq. (10), F is given as $F=\frac{eE}{\sqrt{m\Omega}}e^{-\theta_1}\cos\phi$. Completing the square, Eq. (35) can be written as

$$\left(\frac{d^2}{dX'^2}+X'^2+\varepsilon\right)\phi_{e,o}(X') = 0, \quad (36)$$

where $X'=X-f$ and

$$\varepsilon = \frac{E_G-V_0}{E_1} + \frac{G^2}{2E_1E_2} - \left(\frac{F}{2E_1}\right)^2. \quad (37)$$

Here, $E_G=\mathcal{E}-(n+1/2)\,E_2$ is the energy associated with the guiding center motion of the electron and $\mathcal{E}$ is total energy. The solutions for $\phi_{e,o}$ can be expressed in terms of the confluent hypergeometric function ${}_1F_1$ (Ref. 8) as

$$\phi_e(X') = e^{-iX'^2/2}\,{}_1F_1\left(\frac{1}{4}+\frac{i\varepsilon}{4};\frac{1}{2};iX'^2\right) \quad (38)$$

and

$$\phi_o(X') = X'e^{-iX'^2/2}\,{}_1F_1\left(\frac{3}{4}+\frac{i\varepsilon}{4};\frac{3}{2};iX'^2\right). \quad (39)$$

The initial in state $\Phi_{in}(X,0)$ is taken as a linear combination of these functions,

$$\Phi_{in}(X,0) = A\phi_e(X') + B\phi_e(X') = A\phi_e(X-f) + B\phi_e(X-f) \quad (40)$$

or

$$\Phi_{in}(X,0) = Ae^{-i(X-f)^2/2}\,{}_1F_1\big(\alpha;\tfrac{1}{2};i(X-f)^2\big)$$
$$+ B(X-f)e^{-i(X-f)^2/2}\,{}_1F_1\big(\alpha+\tfrac{1}{2};\tfrac{3}{2};i(X-f)^2\big), \quad (41)$$

where the A and B coefficients are determined such that $\Phi_{in}(X,0)$ reduces asymptotically as $e^{-i(X-f)^2/2}\rightarrow e^{-iX^2/2}$. Accordingly, A and B satisfy $\frac{A}{B}=-\frac{\Gamma(\alpha^*)}{2\Gamma(\alpha^*+\frac{1}{2})}e^{i\pi/46}$ and $\alpha=\frac{1}{4}+i\frac{\varepsilon}{4}$, where $\Gamma(x)$ denotes the gamma function. Substituting the Green's function, Eq. (34), and the form of the incoming state, Eq. (41), into Eq. (33), we obtain

$$\Phi_{out}(X,t) = \varphi_1(X,t) + \varphi_2(X,t), \quad (42)$$

where

$$\varphi_1(X,t) = A\sqrt{\frac{b}{2\pi i}}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}$$
$$\times\int_{-\infty}^{\infty} e^{i(c-1)\xi^2/2}e^{-ib(X-f)\xi}\,{}_1F_1\big(\alpha;\tfrac{1}{2};i\xi^2\big)d\xi \quad (43)$$

and

$$\varphi_2(X,t) = B\sqrt{\frac{b}{2\pi i}}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}$$
$$\times\int_{-\infty}^{\infty} e^{i(c-1)\xi^2/2}e^{-ib(X-f)\xi}\,{}_1F_1\big(\alpha+\tfrac{1}{2};\tfrac{3}{2};i\xi^2\big)d\xi. \quad (44)$$

The integrals involved in the time development of $\Phi_{out}(X,t)$ are performed in Appendix C, with the result

$$\frac{\Phi_{out}(X,t)}{B} = \left[\frac{e^{-i\pi/2}}{\Gamma^*(1-\alpha)} - \frac{\Gamma(\alpha^*)}{\Gamma(\alpha)\Gamma\big(\alpha^*+\frac{1}{2}\big)}\right]\frac{\sqrt{\pi}}{2}(X-f)^{2\alpha-1}e^{i\pi\alpha/2}e^{-i\varepsilon E_1t}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}e^{-ib^2(X-f)^2/4}, \quad (45)$$

which is to be compared with the asymptotic form of the incoming wave function of Eq. (41) (Appendix C),

$$\frac{\Phi_{\rm in}(X,0)}{B} = -\left[\frac{1}{\Gamma\left(\alpha^* + \frac{1}{2}\right)} + \frac{1}{\Gamma(1-\alpha)}\right]\frac{\sqrt{\pi}}{2} \times |X-f|^{-2\alpha} e^{-i\pi(\alpha-3/4)/2} e^{-i(X-f)^2/2}. \quad (46)$$

The transmission coefficient for the problem at hand can be written as

$$T = \lim_{X\to\infty} \frac{|\Phi_{\rm out}|^2}{|\Phi_{\rm in}|^2}. \quad (47)$$

The squared modulus of the outgoing wave function is given by

$$\left|\frac{\Phi_{\rm out}(X,t)}{B}\right|^2 = \frac{\pi}{4}\frac{e^{-\pi\varepsilon/4}}{(X-f)}\left|\frac{e^{-i\pi/2}}{\Gamma(1-\alpha^*)} - \frac{\Gamma(\alpha^*)}{\Gamma(\alpha)\Gamma\left(\alpha^* + \frac{1}{2}\right)}\right|^2, \quad (48)$$

and the corresponding expression for the incoming wave function is

$$\left|\frac{\Phi_{\rm in}(X,0)}{B}\right|^2 = \frac{\pi e^{-\pi\varepsilon/4}}{(X-f)\left|\Gamma\left(\alpha^* + \frac{1}{2}\right)\right|^2}. \quad (49)$$

Forming the transmission coefficient, we have

$$T = \frac{1}{4}\left|\frac{e^{-i\pi/2}}{\Gamma(1-\alpha^*)} - \frac{\Gamma(\alpha^*)}{\Gamma(\alpha)\Gamma\left(\alpha^* + \frac{1}{2}\right)}\right|^2 \left|\Gamma\left(\alpha^* + \frac{1}{2}\right)\right|^2, \quad (50)$$

which can be reduced further using the identity [Ref. 8, p. 256, Eq. (6.1.32)]

$$\Gamma\left(\tfrac{1}{4} + iy\right)\Gamma\left(\tfrac{3}{4} - iy\right) = \frac{\sqrt{2}\pi}{\cosh \pi y + i \sinh \pi y}.$$

The resulting expression for the transmission coefficient is

$$T = \frac{1}{1 + e^{-\pi\varepsilon}}, \quad (51)$$

where

$$\varepsilon = \frac{E_G - V_0}{E_1} + \frac{G^2}{2E_1E_2} - \left(\frac{F}{2E_1}\right)^2, \quad (52)$$

with

$$E_G = \mathcal{E} - (n + 1/2)E_2.$$

V. CONCLUSIONS

In summary, we have determined the coefficient for electron transmission through a quantum point contact in the presence of crossed electric and magnetic fields, with the incoming electron state characterized as an appropriate linear combination of states of the "guiding center" part of the Hamiltonian, constructed to reduce properly asymptotically. Our interest has been focused on exploring the role of the electric field $\mathbf{E}$ in the transmission coefficient analyzed previously[6] with the magnetic field alone. Our result, $T=(1+e^{-\pi\varepsilon})^{-1}$, involves the electric field through ε [Eq. (52)], which may be rewritten as

$$\varepsilon = \frac{\mathcal{E} - (n+1/2)E_2 - V_0}{E_1} + \frac{1}{2E_1E_2}\left(\frac{eE}{\sqrt{m\Omega}} e^{\theta_2}\sin\phi\right)^2 - \left(\frac{eE}{2E_1\sqrt{m\Omega}} e^{-\theta_1}\cos\phi\right)^2, \quad (53)$$

where Ω, θ_1, θ_2, ϕ, E_1, and E_2 are given in Eqs. (6), (9), and (12) and have no dependence on the electric field. As written, Eqs. (53) and (52) are valid for $\omega_c > \sqrt{2(\omega_y^2 - \omega_x^2)}$. (Failing this, a simple reconsideration of $|\Phi_{\rm out}|^2/|\Phi_{\rm in}|^2$ will provide a real expression for T.)

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APPENDIX A: TRANSFORMED HAMILTONIAN

In this appendix, we provide a step by step procedure of transforming the Hamiltonian of Eq. (3). To this end, we introduce the operators

$$a_1 = \sqrt{\frac{m\Omega}{2}}x + \frac{i}{\sqrt{2m\Omega}}p_x, \quad a_1^\dagger = \sqrt{\frac{m\Omega}{2}}x - \frac{i}{\sqrt{2m\Omega}}p_x,$$

$$a_2 = \sqrt{\frac{m\Omega}{2}}y + \frac{i}{\sqrt{2m\Omega}}p_y, \quad a_2^\dagger = \sqrt{\frac{m\Omega}{2}}y - \frac{i}{\sqrt{2m\Omega}}p_y, \quad (A1)$$

where

$$\Omega^2 = \left(\frac{\omega_c}{2}\right)^2 + \Omega_-^2,$$

$$\Omega_-^2 = \frac{1}{2}(\omega_y^2 - \omega_x^2), \quad (A2)$$

$$\Omega_+^2 = \frac{1}{2}(\omega_x^2 + \omega_y^2).$$

In terms of these operators, (x,p_x) and (y,p_y) are expressed as

$$x = \frac{1}{\sqrt{2m\Omega}}(a_1^\dagger + a_1), \quad p_x = i\sqrt{\frac{m\Omega}{2}}(a_1^\dagger - a_1),$$

$$y = \frac{1}{\sqrt{2m\Omega}}(a_2^\dagger + a_2), \quad p_y = i\sqrt{\frac{m\Omega}{2}}(a_2^\dagger - a_2), \quad (A3)$$

and the Hamiltonian, Eq. (3), can now be written as

$$H = \Omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) - \frac{i\omega_c}{2}(a_1^\dagger a_2 - a_2^\dagger a_1) + \gamma[(a_2^\dagger + a_2)^2 - (a_1^\dagger + a_1)^2] + \frac{eE(t)}{\sqrt{2m\Omega}}(a_1^\dagger + a_1) + V_0. \tag{A4}$$

Introducing the Bogoliubov transformation,

$$a_1 = ib_1 \cos\phi + b_2 \sin\phi,$$
$$a_2 = -b_1 \sin\phi - ib_2 \cos\phi, \tag{A5}$$

with

$$\tan\phi = -\frac{\omega_c}{4\gamma}, \tag{A6}$$

and $\gamma = \Omega_+^2/(4\Omega)$, the b and $b^\dagger$ operators satisfy canonical commutation relations

$$[b_1, b_1^\dagger] = [b_2, b_2^\dagger] = 1$$

and

$$[b_1, b_2] = [b_1, b_2^\dagger] = 0.$$

The Hamiltonian in terms of the b and $b^\dagger$ operators has the form

$$H = \Omega(b_1^\dagger b_1 + b_2^\dagger b_2 + 1) + \gamma[(b_1^2 + b_1^{\dagger 2}) - (b_2^2 + b_2^{\dagger 2})] + 2\sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}(b_2^\dagger b_2 - b_1^\dagger b_1) + \frac{eE(t)}{\sqrt{2m\Omega}}[(b_2^\dagger + b_2)\sin\phi - i(b_1^\dagger - b_1)\cos\phi] + V_0. \tag{A7}$$

An additional Bogoliubov transformation of the form

$$\begin{pmatrix} b_i \\ b_i^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\theta_i & \sinh\theta_i \\ \sinh\theta_i & \cosh\theta_i \end{pmatrix}\begin{pmatrix} c_i \\ c_i^\dagger \end{pmatrix}, \tag{A8}$$

with $i=1$, 2, reduces the Hamiltonian, Eq. (A7), to

$$H = E_1(c_1^2 + c_1^{\dagger 2}) + E_2\left(c_2^\dagger c_2 + \frac{1}{2}\right) + \frac{eE(t)}{\sqrt{2m\Omega}}[(c_2^\dagger + c_2)e^{\theta_2}\sin\phi - i(c_1^\dagger - c_1)e^{-\theta_1}\cos\phi] + V_0. \tag{A9}$$

Here, the following definitions have been introduced:

$$E_1 = \left[\gamma^2 - \left\{\frac{1}{2}\Omega - \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}\right\}^2\right]^{1/2}, \tag{A10}$$

$$E_2 = 2\left[\left\{\frac{1}{2}\Omega + \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}\right\}^2 - \gamma^2\right]^{1/2}, \tag{A11}$$

and

$$\tanh 2\theta_1 = \frac{\frac{1}{2}\Omega - \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}}{-\gamma}, \tag{A12}$$

$$\tanh 2\theta_2 = \frac{\gamma}{\frac{1}{2}\Omega + \sqrt{\gamma^2 + \left(\frac{\omega_c}{4}\right)^2}}. \tag{A13}$$

From Eq. (A8), it follows that $[c_i, c_j^\dagger] = \delta_{ij}$, with all other commutators vanishing.

Finally, we introduce the canonically paired operators

$$X = \frac{1}{i\sqrt{2}}(c_1^\dagger - c_1),$$
$$P = \frac{1}{\sqrt{2}}(c_1^\dagger + c_1), \tag{A14}$$

with $[X, P] = i$, and

$$s = \frac{1}{\sqrt{2}}(c_2 + c_2^\dagger),$$
$$p = \frac{1}{i\sqrt{2}}(c_2 - c_2^\dagger), \tag{A15}$$

with $[s, p] = i$, and $[s, X] = [s, P] = [p, X] = [p, P] = 0$. The Hamiltonian, Eq. (A9), can now be written in the form

$$H = E_1(P^2 - X^2) + \frac{1}{2}E_2(p^2 + s^2) + \frac{eE(t)}{\sqrt{m\Omega}}(se^{\theta_2}\sin\phi + Xe^{-\theta_1}\cos\phi) + V_0, \tag{A16}$$

which can be recast as the sum of two commuting parts

$$H = H_1 + H_2, \tag{A17}$$

with

$$H_1 = E_1(P^2 - X^2) + F(t)X \tag{A18}$$

and

$$H_2 = \tfrac{1}{2}E_2(p^2 + s^2) + G(t)s + V_0, \tag{A19}$$

where

$$F(t) = \frac{eE(t)}{\sqrt{m\Omega}}e^{-\theta_1}\cos\phi, \tag{A20}$$

$$G(t) = \frac{eE(t)}{\sqrt{m\Omega}}e^{\theta_2}\sin\phi. \tag{A21}$$

APPENDIX B: GREEN'S FUNCTION FOR A SEPARABLE HAMILTONIAN

In this appendix, we derive the form of the Green's function for a system whose time-independent Hamiltonian H

can be written as the sum of two commuting parts, i.e., $H=H_1+H_2$ with $[H_1,H_2]=0$. The time evolution of a general state Ψ in the case where $H\neq H(t)$, from some initial time t' to a later time t, is given by

$$|\Psi(t)\rangle = e^{-iH(t-t')/\hbar}|\Psi(t')\rangle. \tag{B1}$$

Since $[H_1,H_2]=0$, we have $e^{-iHt/\hbar}=e^{-iH_1t/\hbar}e^{-iH_2t/\hbar}$. Moreover, using the eigenstates of the individual Hamiltonians H_1 and H_2,

$$H_1|\phi_n\rangle = \varepsilon_1^{(n)}|\phi_n\rangle, \qquad H_2|\psi_m\rangle = \varepsilon_2^{(m)}|\psi_m\rangle, \tag{B2}$$

the eigenstates of H can be written as $|\chi_{n,m}\rangle=|\phi_n\rangle|\psi_m\rangle$. Introducing the joint product space identity operator

$$\sum_{n,m}|\phi_n\rangle|\psi_m\rangle\langle\psi_m|\langle\phi_n| = I, \tag{B3}$$

and writing the configuration space kets as $|\mathbf{x}\rangle=|\mathbf{x}_1\rangle|\mathbf{x}_2\rangle$, with $|\mathbf{x}_i\rangle$ the space associated with H_i, we obtain

$$\begin{aligned} G_H(\mathbf{x},t;\mathbf{x}',t') &= -i\langle\mathbf{x}|e^{-iH(t-t')/\hbar}|\mathbf{x}'\rangle = -i\sum_{n,m}\sum_{n',m'}\langle\mathbf{x}_2|\langle\mathbf{x}_1|(|\phi_n\rangle|\psi_m\rangle)(\langle\psi_m|\langle\phi_n|e^{-iH(t-t')/\hbar}|\phi_{n'}\rangle|\psi_{m'}\rangle)(\langle\psi_{m'}|\langle\phi_{n'}|)|\mathbf{x}_1'\rangle|\mathbf{x}_2'\rangle \\ &= -i\sum_{n,m}\sum_{n',m'}\langle\mathbf{x}_1|\phi_n\rangle\langle\mathbf{x}_2|\psi_m\rangle(\langle\phi_n|e^{-iH_1(t-t')/\hbar}|\phi_{n'}\rangle\langle\psi_m|e^{-iH_2(t-t')/\hbar}|\psi_{m'}\rangle)\langle\psi_{m'}|\mathbf{x}_2'\rangle\langle\phi_{n'}|\mathbf{x}_1'\rangle \\ &= -i\sum_{n,n'}\langle\mathbf{x}_1|\phi_n\rangle\langle\phi_n|e^{-iH_1(t-t')/\hbar}|\phi_{n'}\rangle\langle\phi_{n'}|\mathbf{x}_1'\rangle\sum_{m,m'}\langle\mathbf{x}_2|\psi_m\rangle\langle\psi_m|e^{-iH_2(t-t')/\hbar}|\psi_{m'}\rangle\langle\psi_{m'}|\mathbf{x}_2'\rangle, \end{aligned} \tag{B4}$$

whence

$$\begin{aligned} G_H(\mathbf{x},t;\mathbf{x}',t') &= -i\sum_{n,n'}\langle\mathbf{x}_1|\phi_n\rangle e^{-i\varepsilon_1^{(n)}(t-t')/\hbar}\delta_{n,n'}\langle\phi_{n'}|\mathbf{x}_1'\rangle\sum_{m,m'}\langle\mathbf{x}_2|\psi_m\rangle e^{-i\varepsilon_2^{(m)}(t-t')/\hbar}\delta_{m,m'}\langle\psi_{m'}|\mathbf{x}_2'\rangle \\ &= -i\sum_n\langle\mathbf{x}_1|\phi_n\rangle e^{-i\varepsilon_1^{(n)}(t-t')/\hbar}\langle\phi_n|\mathbf{x}_1'\rangle\sum_m\langle\mathbf{x}_2|\psi_m\rangle e^{-i\varepsilon_2^{(m)}(t-t')/\hbar}\langle\psi_m|\mathbf{x}_2'\rangle \\ &= -i\langle\mathbf{x}_1|e^{-iH_1(t-t')/\hbar}|\mathbf{x}_1'\rangle\langle\mathbf{x}_2|e^{-iH_2(t-t')/\hbar}|\mathbf{x}_2'\rangle. \end{aligned} \tag{B5}$$

Alternatively expressed,

$$\begin{aligned} G_H(\mathbf{x},t;\mathbf{x}',t') &= -i[iG_{H_1}(\mathbf{x}_1,t;\mathbf{x}_1',t')][iG_{H_2}(\mathbf{x}_2,t;\mathbf{x}_2',t')] \\ &= iG_{H_1}(\mathbf{x}_1,t;\mathbf{x}_1',t')G_{H_2}(\mathbf{x}_2,t;\mathbf{x}_2',t'). \end{aligned} \tag{B6}$$

Translating this to the problem considered in this work, we have

$$G_H(X,s,t;X',s',t') = iG_{H_1}(X,t;X',t')G_{H_2}(s,t;s',t'), \tag{B7}$$

and the time evolution of a general product state $[\Psi(X',s';t')=\Psi_1(X',t')\Psi_2(s',t')]$ is given by

$$\begin{aligned} \Psi(X,s;t) &= i\int dX'ds'G_H(X,s,t;X',s',t')\Psi(X',s';t') \\ &= \int dX' iG_{H_1}(X,t;X',t')\Psi_1(X',t') \\ &\quad\times\int ds' iG_{H_2}(s,t;s',t')\Psi_2(s',t'). \end{aligned} \tag{B8}$$

APPENDIX C: TRANSMISSION COEFFICIENT

In this appendix, we derive the form of the outgoing wave and use it to obtain the transmission coefficient. Considering the evaluation of Eq. (42), there are two parts. We rewrite the first part, Eq. (43), using the integral representation of the confluent hypergeometric function [Ref. 8, p. 505, Eq. (13.2.1)],

$${}_1F_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xu}u^{a-1}(1-u)^{c-a-1}du, \tag{C1}$$

so that $\varphi_1(X,t)$ is given by

$$\begin{aligned} \varphi_1(X,t) &= A\sqrt{\frac{b}{2\pi i}}\frac{\Gamma(1/2)}{|\Gamma(\alpha)|^2}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}\int_0^1 du\, u^{\alpha-1} \\ &\quad\times(1-u)^{-(\alpha+1/2)}\int_{-\infty}^{\infty}d\xi\, e^{i[u+(c-1)/2]\xi^2/2}e^{-ib(X-f)\xi}, \end{aligned} \tag{C2}$$

where we used the Γ function relation $\Gamma^*(z)=\Gamma(z^*)$ [Ref. 8, p. 256, Eq. (6.1.23)]. The ξ integration can be performed using Ref. 9, Eq. (3.896.2) (p. 480), resulting in

$$\varphi_1(X,t) = A\left(\frac{b}{2}\right)^{1/2}\frac{\Gamma(1/2)}{|\Gamma(\alpha)|^2}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)} \times\int_0^1 du\frac{u^{\alpha-1}(1-u)^{-(\alpha+1/2)}}{\left(u+\frac{c-1}{2}\right)^{1/2}}\exp\left[-i\frac{b^2(X-f)^2}{4\left(u+\frac{c-1}{2}\right)}\right]. \quad \text{(C3)}$$

In the large t limit, $c-1=\coth(2E_1t)-1\to 0$, or $c\to 1$, and the subsequent substitution $u=1/(1+s)$ reduces the u integral to

$$\int_0^1 du\frac{u^{\alpha-1}(1-u)^{-(\alpha+1/2)}}{\left(u+\frac{c-1}{2}\right)^{1/2}}\exp\left[-i\frac{b^2(X-f)^2}{4\left(u+\frac{c-1}{2}\right)}\right] = e^{-ib^2(X-f)^2}\int_0^\infty ds s^{-\alpha-1/2}e^{-ib^2(X-f)^2s/4}. \quad \text{(C4)}$$

The s integral can be evaluated using Ref. 9, Eq. (3.381.5) (p. 318), resulting in

$$\int_0^\infty ds s^{-\alpha-1/2}e^{-ib^2(X-f)^2s/4} = \Gamma\left(\frac{1}{2}-\alpha\right) \times\left[\frac{b(X-f)}{2}\right]^{2\alpha-1}e^{-i\pi(1/2-\alpha)/2}. \quad \text{(C5)}$$

Consequently, Eq. (C3) can be written as

$$\varphi_1(X,t) = A\left(\frac{b}{2}\right)^{1/2}\frac{\Gamma(1/2)}{|\Gamma(\alpha)|^2}e^{-i(-\alpha+1/2)\pi/2}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}e^{-ib^2(X-f)^2}, \quad \text{(C6)}$$

and in the limit of large t $\left[b/2=\frac{1}{2}\operatorname{csch}(2E_1t)\to e^{-2E_1t}\right]$ it reduces to

$$\varphi_1(X,t) = A\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{|\Gamma(\alpha)|^2}(X-f)^{2\alpha-1}e^{-i\varepsilon E_1t}e^{i\pi(\alpha-1/2)/2}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}e^{-ib^2(X-f)^2/4}. \quad \text{(C7)}$$

A similar procedure applied to $\varphi_2(X,t)$ results in

$$\varphi_2(X,t) = B\frac{\sqrt{\pi}}{2}\frac{\Gamma(1-\alpha)}{|\Gamma(1-\alpha)|^2}(X-f)^{2\alpha-1}e^{-i\varepsilon E_1t}e^{i\pi(\alpha-1)/2}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}e^{-ib^2(X-f)^2/4}. \quad \text{(C8)}$$

Combining Eqs. (C7) and (C8), we form the outgoing wave function $\Phi_{\text{out}}(X,t)$ as

$$\Phi_{\text{out}}(X,t) = \varphi_1(X,t) + \varphi_2(X,t).$$

To simplify this expression, we use the relation between the coefficients A and B mandated by the required asymptotic behavior of $\Phi_{\text{in}}(X,0)$,

$$\frac{A}{B} = -e^{i\pi/4}\frac{\Gamma(\alpha^*)}{2\Gamma\left(\alpha^*+\frac{1}{2}\right)},$$

and obtain

$$\frac{\Phi_{\text{out}}(X,t)}{B} = \left[\frac{e^{-i\pi/2}}{\Gamma^*(1-\alpha)} - \frac{\Gamma(\alpha^*)}{\Gamma(\alpha)\Gamma\left(\alpha^*+\frac{1}{2}\right)}\right]\frac{\sqrt{\pi}}{2}(X-f)^{2\alpha-1}e^{i\pi\alpha/2}e^{-i\varepsilon E_1t}e^{i(cX^2/2-E_1f^2t)}e^{-icf(X-f/2)}e^{-ib^2(X-f)^2/4}. \quad \text{(C9)}$$

We also write the incoming wave function $\Phi_{\text{in}}(X,0)$, Eq. (41), for large values of $|X-f|$ as

$$\frac{\Phi_{\text{in}}(X,0)}{B} = -\left[\frac{1}{\Gamma\left(\alpha^*+\frac{1}{2}\right)} + \frac{1}{\Gamma(1-\alpha)}\right]\frac{\sqrt{\pi}}{2}|X-f|^{-2\alpha}e^{-i\pi(\alpha-3/4)/2}e^{-i(X-f)^2/2}. \quad \text{(C10)}$$

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