Coulomb gas on the Keldysh contour: Anderson-Yuval-Hamann representation of the nonequilibrium two-level system

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(Received 16 May 2007; revised manuscript received 7 July 2007; published 27 August 2007)

The nonequilibrium tunneling center model of a localized electronic level coupled to a fluctuating two-state system and to two electronic reservoirs is solved via an Anderson-Yuval-Hamann mapping onto a plasma of alternating positive and negative charges time ordered along the two "Keldysh" contours needed to describe nonequilibrium physics. The interaction between charges depends both on whether their time separation is small or large compared to a dephasing scale defined in terms of the chemical potential difference between the electronic reservoirs and on whether their time separation is larger or smaller than a decoherence scale defined in terms of the current flowing from one reservoir to another. A renormalization group transformation appropriate to the nonequilibrium problem is defined. An important feature is the presence in the model of a new coupling, essentially the decoherence rate, which acquires an additive renormalization similar to that acquired by the energy in equilibrium problems. The method is used to study interplay between the dephasing-induced formation of independent resonances tied to the two chemical potentials and the decoherence which cuts off the scaling and leads effectively to classical long-time behavior. We determine the effect of departures from equilibrium on the localization-delocalization phase transition.

DOI: 10.1103/PhysRevB.76.085342 PACS number(s): 73.23.−b, 05.30.−d, 71.10.−w, 71.38.−k

I. INTRODUCTION

Understanding the nonequilibrium behavior of interacting quantum mechanical systems is one of the important open issues in condensed matter physics, with applications in nanoscience,[1] the study of cold atoms in optical lattices,[2] nonlinear spectroscopies,[3] and transport at quantum critical points.[4] One may distinguish three classes of nonequilibrium situations: response of a system initially in an equilibrium state to a strong transient pulse, time evolution of a system from a particular initial condition, and the steady state behavior of a driven system. In this paper, we shall be concerned with one of the simplest examples of the third class of problems: a quantum mechanical system with only a few degrees of freedom, coupled to two reservoirs with which particles and energy may be exchanged, and with the nonequilibrium drive arising from a difference, $\Delta\mu$, in chemical potential μ between the reservoirs. This model is of experimental relevance in the context of single molecule devices[5] and of quantum dots[6] and is important as a paradigm problem for the development of techniques and insights. Two crucial issues in nonequilibrium physics are dephasing and decoherence. In the model studied here dephasing arises because the wave functions in the two reservoirs evolve in time at rates which differ by $\Delta\mu$, whereas decoherence arises from the flow of energy and particles across the system. An important issue in nonequilibrium physics is to develop methods which allow these effects to be systematically analyzed.

On the formal level, investigation of equilibrium systems is based on the partition function. Powerful techniques, most importantly the renormalization group method, enable one to eliminate putatively unimportant degrees of freedom and derive an effective theory governing the low energy behavior of interest. In equilibrium problems the renormalization group has been implemented in two intimately related ways: by considering changes in the self-energies, vertex functions, and correlation functions in diagrammatic calculations, and by working directly with the partition function, generating an effective action describing only the degrees of freedom of interest.

Most applications of renormalization group ideas to nonequilibrium problems have been based on the first approach: a diagrammatics is constructed using the Keldysh technique and then the flow of vertices, self-energies, and response functions under changes in cutoff is studied. In pioneering work, Rosch *et al.*[7,8] constructed a nonequilibrium scaling theory for the Kondo problem by identifying logarithms in perturbative calculations. Measurable quantities were computed; scaling equations for the coupling constants of the Hamiltonian were inferred from logarithmic dependences of observables on the upper frequency cutoff. In a very recent paper, Borda *et al.*[9] used similar techniques to study the nonequilibrium behavior of the tunneling center model by perturbation theory in the dot-lead coupling. Paaske *et al.* provided further insight[10,11] into the physically crucial issue of decoherence rates, showing that voltage-induced decoherence enters differently into different observables, so that the analogy between temperature and decoherence is not precise. In a more recent set of approaches, a transformation is performed on the Hamiltonian itself.[12,13]

While these approaches have established a number of basic results and concepts, the development of the subject remains incomplete. In equilibrium, defining a renormalization group transformation directly on the free energy provided important insights into the meaning of the transformations

and the formal structure of the theory. A similar analysis out of equilibrium should lead to valuable insights including a clearer understanding of decoherence and dissipation and a more precise definition of the charges in the renormalization group equations and the ability to map one problem onto another. In this paper, we therefore examine a simple model, the tunneling center problem, from the effective action point of view. The "tunneling center" model is perhaps the simplest example of a wide class of quantum impurity models such as the Kondo problem or the spin-boson problem, which involve a finite number of local degrees of freedom coupled to reservoirs. The equilibrium behavior of these models is very rich, involving nontrivial correlated states, dynamically generated energy scales, and quantum phase transitions.[14] The equilibrium physics revolves around a competition between formation of a quantum coherent state of the local degrees of freedom and the decoherence associated with coupling to the reservoir. We wish to characterize the effect of departures from equilibrium on this physics.

Effective actions for nonequilibrium problems have been discussed by several authors; for a review, see, e.g., Ref. 15 Here, we define and analyze a renormalization group transformation directly on the effective action. Our approach is somewhat similar to previous work of König and co-workers,[16,17] who defined a renormalization group transformation directly on the equation of motion for the density matrix. These authors treated the dot-lead coupling via (self-consistently resummed) perturbation theory. We study a simpler model in which we are able to treat the dot-lead coupling exactly. Our formalism allows us to deal with decoherence and orthogonality physics on the same footing. We show that the dephasing scale $\Delta\mu$ defines a crossover beyond which both the physics and the formalism change. We find that the effective theory valid for energy scales lower than $\Delta\mu$ is characterized by a richer structure of charges than is the corresponding equilibrium theory. It also involves, as a new parameter, a decoherence rate. This arises physically from the flow of current through the system. It enters the theory as an additional parameter, which is subject to additive renormalization (as is the total energy in usual renormalization group treatments). This is discussed in more detail in Sec. III.

The rest of the paper is organized as follows. In Sec. II, we define the model we consider and obtain the effective action. In Sec. III, we derive the relevant scaling equation, and in Sec. IV we present the solution and its physical content, among other things displaying the different physics of decoherence and dephasing. Section V is a summary and conclusion, which outlines implications for other problems.

II. MODEL AND DERIVATION OF THE COULOMB GAS

There are many realizations of the tunneling center model.[18] We consider a two-state system, which we represent in spin notation, linearly coupled to the density of electrons in an N-fold degenerate electronic level (creation operator $d_\alpha^\dagger$ with $\alpha=1\ldots N$), which is itself hybridized with two (L and R) leads characterized by free-fermion statistics with possibly different chemical potentials, i.e., $\Delta\mu=\mu_L-\mu_R\neq 0$. We consider mainly temperature $T=0$. We assume that the coupling to the leads preserves the "pseudospin" index α. The Hamiltonian may then be written as ($S_{z,x}=\frac{1}{2}\sigma_{x,z}$ is a spin matrix)

$$H = H_{loc} + H_{bath}, \tag{1}$$

$$H_{loc} = S_z B + \Delta_T S_x + \lambda D S_z \sum_{\alpha=1\cdots N} d_\alpha^\dagger d_\alpha, \tag{2}$$

$$H_{bath} = \sum_{a=L,R,\alpha=1\cdots N} \int d\epsilon \frac{\epsilon}{D} c_{\epsilon a\alpha}^\dagger c_{\epsilon a\alpha} + \sqrt{\frac{1}{\pi}} \int d\epsilon \sum_{a=L,R,\alpha=1\cdots N} (\cos\theta_a c_{\epsilon a\alpha}^\dagger d_\alpha + \text{H.c.}), \tag{3}$$

where $0\leqslant\theta_L=\frac{\pi}{2}-\theta_R\leqslant\frac{\pi}{2}$ and we have absorbed the mean hybridization strength into the variable ε. The density of states per pseudospin α in the leads is D^{-1}. For the two-level system, we choose the spin basis, which diagonalizes the coupling to the d electrons, and parametrize this coupling by a dimensionless variable λ and the lead density of states. The "magnetic field" B is the level splitting of the two-level system and the parameter Δ_T gives the tunneling between the states. Note that if $B=0$, the Hamiltonian has a particle-hole symmetry, so its energies are invariant under combined operations of changing the state of the two-level system from "up" to "down" and changing particles to holes ($d\leftrightarrow d^\dagger$); this simplifies the algebra without changing the conclusions.

The model requires an upper cutoff ξ^{-1} for the energy integrals. It is most convenient to assume that in each lead, the cutoff is symmetrical about the chemical potential in that lead, i.e., in lead a the energy integrals run from $\mu_a-\xi^{-1}$ to $\mu_a+\xi^{-1}$ (see Fig. 1). We assume that in the starting model the cutoff energies are much larger than the chemical potential difference or than the level splitting parameter B.

Crucial parameters of the model are the nonequilibrium phase shifts introduced in Ref. 19 and defined by

$$\delta_a = \arctan\left[\frac{\lambda\cos^2\theta_a}{1-\frac{\lambda^2}{4}-i\,\text{sgn}(\mu_a-\mu_{\bar{a}})\lambda\sin^2\theta_a}\right]. \tag{4}$$

Note that the phase shifts are not independent variables, but are related to each other as $\delta_{eq}=\delta_L+\delta_R=\arctan\lambda/(1-\lambda^2/4)$. Also note that while δ_{eq} is just the difference of the equilibrium phase shifts $\pm\tan^{-1}\frac{\lambda}{2}$ associated with the two states $S_z=\pm\frac{1}{2}$, $\delta_{L,R}$ are not the differences of the nonequilibrium phase shifts associated with the two states.

The behavior of the two-level system is specified by the reduced density matrix, given at time t in terms of an initial condition $\rho(0)$ at time $t=0$ by

$$\hat{\rho}_S = \text{Tr}_{el}[e^{-iHt}\rho(0)e^{iHt}]. \tag{5}$$

Here, Tr_{el} indicates a trace over all of the electronic degrees of freedom. We follow Anderson *et al.*[20] and expand $\hat{\rho}_S$ perturbatively in the "spin-flip" amplitude Δ_T. The spin-flip

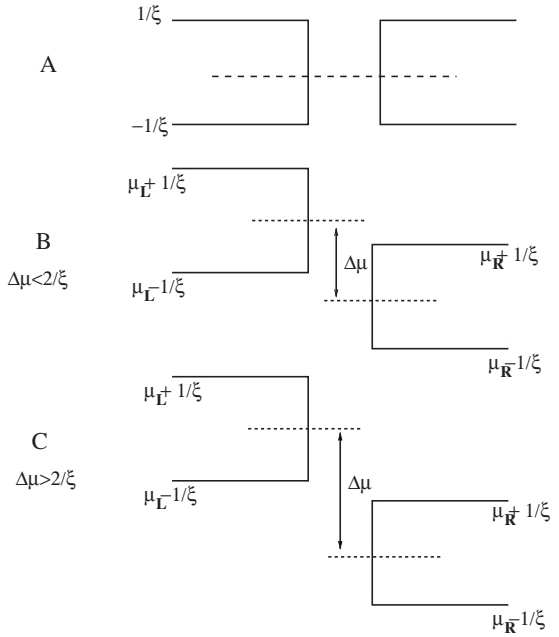

FIG. 1. Sketch of energy levels and cutoffs; band energy cutoffs chosen for each lead to be symmetrical about Fermi energy in that lead. (A) (Upper panel) chemical potential difference $\Delta\mu=\mu_L-\mu_R=0$. (B) (Middle panel) Chemical potential difference $\Delta\mu\neq 0$ but less than cutoff $2\xi^{-1}$. (C) (Lower panel) Chemical potential difference $\Delta\mu$ greater than cutoff $2\xi^{-1}$; no dissipative processes possible.

events are viewed as particles with "fugacity" $\ln[\xi\Delta_T]$ and interaction determined by the trace over electrons. The new features are the need for two time contours and a dependence of the interaction on the chemical potential difference $\Delta\mu$.

A term in the expansion of the density matrix consists of n_- spin-flip events at times running from 0_- to t along the time-ordered contour, followed by n_+ antitime ordered from t to 0_+. Some examples are shown on the left side of Fig. 2: the top panel has $n_+=4$, $n_-=0$, whereas in the two lower panels $n_+=n_-=2$. Labeling the times in this "Keldysh order" (time ordered on the $-$ contour and antitime ordered on the $+$ contour), we have for the diagonal components of the density matrix ($\sigma=\pm 1$ below represents the state of the impurity spin),

$$\langle\sigma|\hat{\rho}_S(t)|\sigma\rangle = \sum_{n_-+n_+=even} (-i)^{n_-}i^{n_+}\langle\sigma|\tau_x^{n_-}\rho_{S0}\tau_x^{n_+}|\sigma\rangle \times\left[\int_0^t \frac{dt_{n_-}}{\xi}\int_0^{t_{n_-}}\frac{dt_{n_--1}}{\xi}\cdots\int_0^{t_2}\frac{dt_1}{\xi}\right] \times\left[\int_0^t \frac{dt_{n_-+1}}{\xi}\int_0^{t_{n_-+1}}\frac{dt_{n_-+2}}{\xi}\cdots\int_0^{t_{n_-+n_+-1}}\frac{dt_{n_++n_-}}{\xi}\right] \times(\Delta_T\xi)^{n_-+n_+}\exp[I_0(\{t_k\})]. \quad (6)$$

The interaction I_0 is obtained by evaluating the Tr_{el}, in other words by solving the Keldysh problem of electrons in the time-dependent potential specified by the spin flips. In principle, this is a multiparticle interaction depending on all of the times $\{t_k\}$. In the equilibrium problem, an essentially complete analytical solution exists,[20] showing that the interaction is pairwise, with a logarithmic time dependence and coefficients given by the product of the sign of the charges (i.e., whether the spin flips from up to down) and changes in scattering phase shifts. In the nonequilibrium case, general analytical expressions are not known. The available evidence, including solutions at times short and long compared to $\Delta\mu$, perturbative calculations,[22] and numerics,[23] suggests the following structure, which is only slightly more involved than in equilibrium.

To specify the structure, it is convenient to collapse the two-contour problem onto a single time axis by defining classical $\left(V_{cl}=\frac{\lambda D}{2}\{S_z(t_-)+S_z(t_+)\}\right)$ and quantum fields $\left(V_q=\frac{\lambda D}{2}\{S_z(t_-)-S_z(t_+)\}\right)$. We then have a four state system, in which either $V_q=0$ and $V_{cl}=\pm\lambda D/2$ or $V_{cl}=0$ and $V_q=\pm\lambda D/2$. Transitions between these states are instantons, which we label by an integer $n=\pm 1$ giving the sign of the change in the quantum field (this is just the usual Coulomb gas charge) and an integer $q=\pm 1$ corresponding to the sign

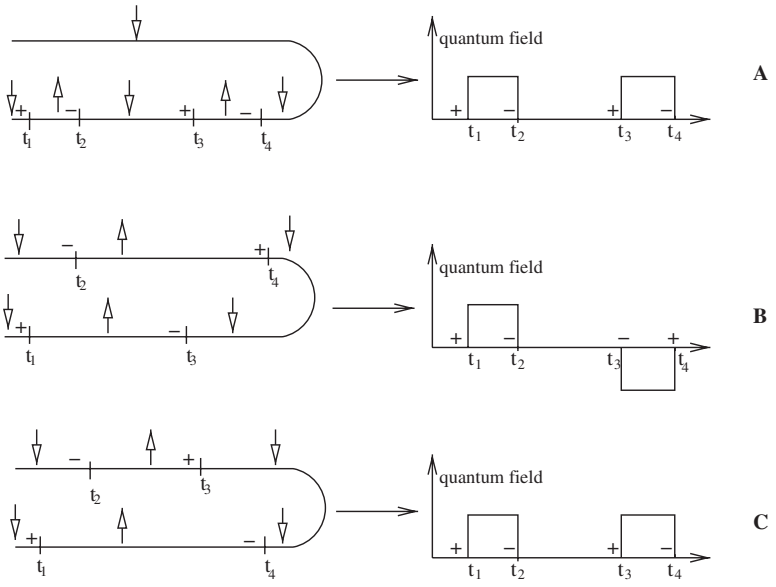

FIG. 2. Examples of spin-flip events in the Keldysh two axis representation (left side) and in the single axis representation employing quantum and classical fields (right side). The charges n_i are indicated on the figure. In the top and bottom examples, the fields $q=+1$ for all instantons. In the middle example, $q=+1$ for the left hand pair of instantons and $=-1$ for the right hand pair. Although the first and the third examples have the same quantum field configuration, the different positions of the times on the Keldysh contour lead to different phase factors.

of the quantum field in the region where it is nonzero. The right hand panels of Fig. 2 show some examples.

There are in principle 16 pairwise interactions between the four kinds of instantons, but calculations reveal a simpler structure (see Appendix A). We find that the interaction between instantons at times t_i and t_j is pairwise, as in equilibrium. As in equilibrium, the sign of the interaction between instantons i and j is determined in the usual way by the product of the charges $n_i n_j$, while the magnitude is determined by the product of the scattering phase shifts. However, in the general nonequilibrium case, the phase shifts are complex and take value δ_i if $q_i=1$ and δ_i^* if $q_i=-1$ (see Appendix A). Finally, with each interaction is a phase factor. The interaction has a logarithmic time dependence, given approximately by $\ln \frac{ia_{ij}|t_i-t_j|+\xi}{\xi}$, with phase factor $a_{ij}=\pm 1$ determined by the *Keldysh* time ordering of t_i and t_j, so that in the upper panel of Fig. 2 $a_{23}=1$ whereas in the middle panel $a_{23}=-1$. We find it convenient to combine the phase factors into an overall phase Φ. The result is

$$\begin{aligned}\langle\sigma|\hat{\rho}_S(t)|\sigma\rangle = & \sum_{n_-+n_+=even} (-i)^{n_-} i^{n_+} \langle\sigma|\tau_x^{n_-}\rho_{S0}\tau_x^{n_+}|\sigma\rangle \\ & \times \int_0^t \frac{dt_{n_-+n_+}}{\xi} \int_0^{t_{n_-+n_+-1}} \\ & \times \ldots \int_0^{t_2} dt_1 (\Delta_T \xi)^{n_-+n_+} \sum_{n_i q_i v_i}{}' \\ & \times \exp\left[\sum_{j<i} n_i n_j C_0\left(q_i, q_j, \frac{t_i-t_j}{\xi}, \Delta\mu\xi\right) + i\Phi\right].\end{aligned} \tag{7}$$

The prime symbol on the sum above is to keep track of constraints, such as two charges of the same sign cannot appear more than twice in sequence, and $\Sigma_i n_i=0$ for an expansion involving the diagonal component of the density matrix.

The long-time[19] and short-time[20,21] limits of the interaction function C_0 are known. At short times $t<1/\Delta\mu$,

$$C_0(\Delta\mu t \ll 1) = N\left(\frac{\delta_L+\delta_R}{\pi}\right)^2 \ln\left(\frac{|t|}{\xi}\right), \tag{8}$$

independent of quantum fields. At long times $t>\frac{1}{\Delta\mu}$, we have

$$\begin{aligned} & C_0(q=+,q=+,\Delta\mu t>1;\Delta\mu\xi\ll 1)(t) \\ & = \{C_0(q=-,q=-)\}^* \\ & = N\left(\frac{\delta_L^2}{\pi^2}+\frac{\delta_R^2}{\pi^2}\right)\ln(|\Delta\mu t|)+\Gamma_{neq}|t| \\ & \quad + N\left(\frac{\delta_L+\delta_R}{\pi}\right)^2\left[\ln\frac{1}{\Delta\mu\xi}\right], \end{aligned} \tag{9}$$

$$\begin{aligned} & C_0(q=+,q=-,\Delta\mu t>1;\Delta\mu\xi\ll 1)(t) \\ & = C_0(q=-,q=+) \\ & = N\left(\frac{|\delta_L|^2}{\pi^2}+\frac{|\delta_R|^2}{\pi^2}\right)\ln(|\Delta\mu t|)+\Gamma_{neq}|t| \\ & \quad + N\left(\frac{\delta_L+\delta_R}{\pi}\right)^2\left[\ln\frac{1}{\Delta\mu\xi}\right]. \end{aligned} \tag{10}$$

From Eqs. (9) and (10), note that the dynamics is characterized by an exponential time decay, which we use to define the *decoherence rate* Γ_{neq} which plays a fundamental role in the subsequent analysis:

$$\begin{aligned}\Gamma_{neq} &= \gamma_{neq}\Delta\mu = N\Delta\mu\frac{|\delta_L''-\delta_R''|}{2\pi} \\ &= N\Delta\mu\xi\left(\frac{1}{\pi}\lambda^2\cos^2\theta_L\sin^2\theta_L+\cdots\right).\end{aligned} \tag{11}$$

The physics expressed by the interaction C_0 in the $\Delta\mu\xi \ll 1$ limit is as follows: for times less than the dephasing scale $t_{dephasing}=(\Delta\mu)^{-1}$, one has the equilibrium result. The two-level system interacts with one coherent combinations of the two leads ($\bar{c}^\dagger=\cos\theta_L c_L^\dagger+\sin\theta_R c_R^\dagger$); the other combination decouples. The coupling leads to the usual power law interaction with exponents given by the coherent phase shift $\delta_{eq}=\delta_L+\delta_R$. Note that δ_{eq} is independent of quantum fields. For times longer than the dephasing scale, one has a richer structure. The model is effectively a two channel model with separate couplings to left and right leads. The interaction between instantons acquires a dependence on the quantum field. At times longer than the decoherence scale Γ_{neq}^{-1}, the interaction is cut off altogether.

These limiting forms suggest the following decomposition of the interaction:

$$\begin{aligned} C_0(q,q',t-t',\Delta\mu,\xi) = & Q_0(q,q')h_0\left(\frac{t-t'}{\xi}\right) \\ & + Q_M(q,q')h_M\left(\frac{t-t'}{\xi},\Delta\mu(t-t')\right) \\ & + \gamma_{neq}h_{neq}\left(\frac{t-t'}{\xi},\Delta\mu(t-t')\right), \end{aligned} \tag{12}$$

where the "charges" or phase shifts $Q_{0,M}$ obey the property

$$Q_{0,M}(q,q') = [Q_{0,M}(-q,-q')]^*, \tag{13}$$

$$Q_{0,M}(q,-q) = Q_{0,M}(-q,q) = > \mathrm{Im}[Q_{0,M}(q,-q)] = 0. \tag{14}$$

The form of $h_{0,M,neq}$ depends on the cutoff scheme, but for $t/\xi \gg 1$ and $\Delta\mu\xi \ll 1$ (see Appendix B),

$$h_0(t/\xi) = \ln(|t|/\xi), \tag{15}$$

$$h_M(t/\xi,\Delta\mu t) = h_0(t/\xi), \quad \Delta\mu t \ll 1, \tag{16}$$

$$= \ln\frac{1}{\Delta\mu\xi}, \quad \Delta\mu t \gg 1, \tag{17}$$

$$h_{neq}(t/\xi,\Delta\mu t) = \frac{2}{\pi}(\Delta\mu t)^2, \quad \Delta\mu t \ll 1, \tag{18}$$

$$= |\Delta\mu t|, \quad \Delta\mu t \gg 1, \tag{19}$$

implying that for a model with $\Delta\mu\xi \ll 1$,

$$Q_0(+,+) = N\left(\frac{\delta_L^2}{\pi^2} + \frac{\delta_R^2}{\pi^2}\right), \tag{20}$$

$$Q_M(+,+) = \frac{2N}{\pi^2}\delta_L\delta_R, \tag{21}$$

$$Q_0(+,-) = N\left(\frac{\delta_L\delta_L^*}{\pi^2} + \frac{\delta_R\delta_R^*}{\pi^2}\right), \tag{22}$$

$$Q_M(+,-) = N\frac{\delta_L\delta_R^* + \delta_R\delta_L^*}{\pi^2}. \tag{23}$$

The coefficient γ_{neq} is independent of the quantum fields and is given by

$$\gamma_{neq} = N\frac{|\delta_L'' - \delta_R''|}{2\pi}. \tag{24}$$

The term proportional to Q_0 in Eq. (12) represents the effect of processes in which an electron emerges from one lead and is scattered back into the same lead; it is independent of $\Delta\mu$. The term proportional to Q_M represents the effect of processes in which an electron is transferred from one lead to another; it depends on $\Delta\mu$. Finally, as will be seen, the last term expresses the decoherence; it vanishes if the coupling is only to one lead.

It is also interesting to consider the expression for C_0 in a model in which $\Delta\mu\xi > 1$. This situation arises after rescaling. In this case, the regime $\Delta\mu t < 1$ is not defined, while for all times $t > \xi^{-1}$ we find

$$C_0(q=+,q=+,\Delta\mu t > 1) = Q_0\left[\ln\frac{|t|}{\xi}\right], \tag{25}$$

with (for the model with $\Delta\mu\xi > 1$) $Q_M = 0$ and

$$Q_0 \to Q_0' = \left(\frac{\tan^{-1}\dfrac{\lambda\cos^2\theta_L}{1-\dfrac{\lambda^2}{4}}}{\pi}\right)^2 + \left(\frac{\tan^{-1}\dfrac{\lambda\cos^2\theta_R}{1-\dfrac{\lambda^2}{4}}}{\pi}\right)^2. \tag{26}$$

In this limit there is no decay because the theory has no real process which allows nonconservation of energy (see Fig. 1, case C).

Comparison of Eqs. (9) and (25) reveals an important point. If we apply the usual renormalization process of reducing bandwidth, we must pass from the model which gives rise to Eq. (9) and contains decoherence to the model which gives rise to Eq. (25). No decoherence processes exist in this latter model, but the renormalization maps one model onto another model with the same physical content. We therefore conclude that one consequence of renormalization must be the generation of a decay rate, which appears as an extra parameter, additional to what is directly computed from the small bandwidth model.

III. DERIVATION OF SCALING EQUATIONS

In the formulation given in Eq. (7), the nonequilibrium two-level system is seen to be a function of the dimensionless parameters $\Delta_T\xi$, $\Delta\mu\xi$, and $\delta_{L,R}$. In this section, we construct a renormalization group analysis by following the usual procedure of reducing the energy cutoff, i.e., increasing the time cutoff from ξ to $\xi' = (1+\Lambda)\xi$, integrating out the degrees of freedom in the eliminated interval and determining the consequences for the remaining degrees of freedom. These effects were considered for the equilibrium problem in Ref. 20. Reducing the energy cutoff (increasing the time cutoff) leads to changes arising from the ξ dependence of C_0. Rescaling leads to the simple "engineering dimension" changes $(\Delta_T, \Delta\mu) \to (\Delta_T, \Delta\mu)(\xi/\xi') \approx (\Delta_T, \Delta\mu)(1-\Lambda)$. Finally, some kink-antikink pairs fall within the time interval between ξ and $\xi(1+\Lambda)$ and must therefore be removed from the renormalized theory. The procedure for the nonequilibrium problem is similar to the equilibrium problem, in that in the starting formulation the minimum separation in time between tunneling events is ξ. In the rescaled theory, spin-flip events separated by time intervals less than ξ' cannot explicitly appear, but their presence will lead to a renormalization of the interaction between the processes which do explicitly appear in the theory. For small $\Delta_T\xi\Lambda$, the only sequence of close tunneling events which appears with any probability are the "close pairs" shown in Fig. 3. A close pair may lie between the two spin-flip events that occur on the same Keldysh axis (example pairs A and B in Fig. 3 that lie between spin-flip events t_1, t_2) or may lie between spin-flip events on opposite Keldysh axes (examples C and D that lie between spin-flip events t_2, t_4). Physically, a close pair corresponds to a dipole and leads to a screening of the interaction between other spin flip events that may lie on the same Keldysh axis or on different Keldysh axes. Note that we always consider close pairs that lie on the same axis. A close pair with one member on each contour cannot be considered as a dipole; its removal would change the spin state at the final time t or initial time $t=0$.

Mathematically, the calculation is easiest to perform in the collapsed single axis representation in terms of classical and quantum fields. As an example consider Fig. 4. The interaction between the charge at t_i and the charge at t_k can get renormalized by integrating out eight kinds of close pairs, four of which are nearest neighbors to charge at t_i, while the other four are nearest neighbors to the charge at t_k. We show the four close pairs that are nearest neighbors to t_i in Fig. 4. Close pair I can occur in two ways which have the same quantum field configuration, but different classical field configuration.

The change in the density matrix on integrating out the four close pairs is ($\bar{\Delta}_T = i\Delta_T\xi$)

$$\rho^{2n+2} = \rho^{2n}[1 + \bar{\Delta}_T^2(2\times \mathrm{I} + \mathrm{II} + \mathrm{III})]. \tag{27}$$

Summing over all possible positions of the relative separation ϵ between the two charges of the close pair, and their center of mass position τ one obtains

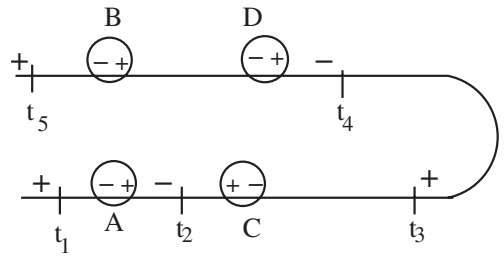

FIG. 3. Scheme for integrating out close pairs.

$$\mathrm{I} = \int_{t_{i-1}}^{t_i} \frac{d\tau}{\xi} \int_{\xi}^{\xi(1+\Lambda)} \frac{d\epsilon}{\xi} \exp\Bigg[- C_0(+,+,\epsilon) + \sum_k n_i n_k \Bigg(- C_0\Big(+,q_k,\Big|t_k - \tau - \frac{\epsilon}{2}\Big|\Big) + C_0\Big(+,q_k\Big|t_k - \tau + \frac{\epsilon}{2}\Big|\Big)\Bigg)\Bigg], \tag{28}$$

$$\mathrm{II} = \int_{t_i}^{t_{i+1}} \frac{d\tau}{\xi} \int_{\xi}^{\xi(1+\Lambda)} \frac{d\epsilon}{\xi} \exp\Bigg[- C_0(+,+,\epsilon) + \sum_k n_i n_k \Bigg(C_0\Big(+,q_k,\Big|t_k - \tau - \frac{\epsilon}{2}\Big|\Big) - C_0\Big(+,q_k\Big|t_k - \tau + \frac{\epsilon}{2}\Big|\Big)\Bigg)\Bigg], \tag{29}$$

$$\mathrm{III} = \int_{t_i}^{t_{i+1}} \frac{d\tau}{\xi} \int_{\xi}^{\xi(1+\Lambda)} \frac{d\epsilon}{\xi} \exp\Bigg[- C_0(-,-,\epsilon) + \sum_k n_i n_k \Bigg(- C_0\Big(-,q_k,\Big|t_k - \tau - \frac{\epsilon}{2}\Big|\Big) + C_0\Big(-,q_k\Big|t_k - \tau + \frac{\epsilon}{2}\Big|\Big)\Bigg)\Bigg]. \tag{30}$$

In the above three expressions, the first term in the argument of the exponent represents the self-interaction of the close pair, and the second term represents the interaction of the close pair with all other charges. The latter may be Taylor expanded in ϵ. The term I on Taylor expansion takes the form

$$\mathrm{I} \sim \int_{t_{i-1}}^{t_i} \frac{d\tau}{\xi} \int_{\xi}^{\xi(1+\Lambda)} \frac{d\epsilon}{\xi} e^{-C_0(+,+\epsilon)} \Bigg(1 - \epsilon \sum_k n_i n_k \frac{\partial C_0(|t_k - \tau|)}{\partial \tau} \Bigg). \tag{31}$$

The integrals over τ and ϵ can easily be performed. The result is of $O(\Lambda)$, which implies that the change to the density matrix in Eq. (27) is $O(\bar{\Delta}_T^2 \Lambda) \ll 1$, and therefore can be reexponentiated. Finally, integrating out the four close pairs of Fig. 4 leads to the following renormalization of the density matrix;

$$\rho^{2n+2} = \rho^{2n} e^{\bar{\Delta}_T^2 e^{-C_0(\xi)} \Lambda(-2t_{i-1}+2t_{i+1})/\xi} e^{\Lambda \bar{\Delta}_T^2 e^{-C_0(\xi)} \Sigma_k n_i n_k [-2C_0(+,q_k,|t_k-t_i|)+2C_0(+,q_k,|t_k-t_{i-1}|)+C_0(+,q_k,|t_k-t_{i+1}|)-C_0(+,q_k,|t_k-t_i|)-C_0(-,q_k,|t_{i+1}-t_k|)+C_0(-,q_k,|t_k-t_i|)]}. \tag{32}$$

Note that the term $e^{-C_0(\xi)}$ is from the self-interaction of the close pair. Since this term arises at short times $t \sim \xi$, it has no explicit quantum field dependence, and therefore the quantum field label has been dropped.

Repeating the above computation for all possible positions of close pairs, one finds that the initial factor of $e^{\bar{\Delta}_T^2 e^{-C_0(\xi)} \Lambda(2t_{i+1}-2t_{i-1})/\xi}$ cancels among each other, while the function $C_0(q_i, q_k, |t_k - t_i|)$ is renormalized in the following way:

$$C_0(q_i,q_k,|t_k-t_i|) \to C_0(q_i,q_k,|t_k-t_i|) - \Lambda \bar{\Delta}_T^2 [6C_0(q_i,q_k,|t_k-t_i|) - C_0(-q_i,q_k,|t_k-t_i|) - C_0(q_i,-q_k,|t_k-t_i|)] e^{-C_0(t/\xi=1)}. \tag{33}$$

Note that $C_0(t/\xi=1)=0$ if $\Delta\mu\xi \ll 1$ but $=\Gamma_{neq}\xi$ if $\Delta\mu\xi > 1$.

The second effect arises from the explicit dependence of the interaction C_0 on the cutoff ξ. We write, for infinitesimal Λ,

$$C_0(t,\xi,\Delta\mu) = C_0\Big(\frac{t}{\xi'}, \Lambda\Delta\mu t\Big) + \delta C_0, \tag{34}$$

with

$$\delta C_0 = \frac{dC_0\Big(\frac{t}{\xi}, \Delta\mu t\Big)}{d\ln\xi}. \tag{35}$$

Adding Eqs. (33) and (34) leads to the fundamental scaling equation,

$$\frac{dC_0(q,q',t)}{d\ln\xi} = -\bar{\Delta}_T^2 e^{-C_0(1)} [6C_0(q,q',t) - C_0(-q,q',t) - C_0(q,-q',t)] + \frac{dC_0(q,q',t/\xi,\Delta\mu\xi)}{d\ln\xi}. \tag{36}$$

The second term in Eq. (36) may be computed from the fundamental Eq. (A13). While a precise general expression is not known, the limits are established. For $\Delta\mu\xi \ll 1$, Eqs. (9) and (10) show that $\frac{dC_0}{d\ln\xi}$ is independent of time, with coeffi-

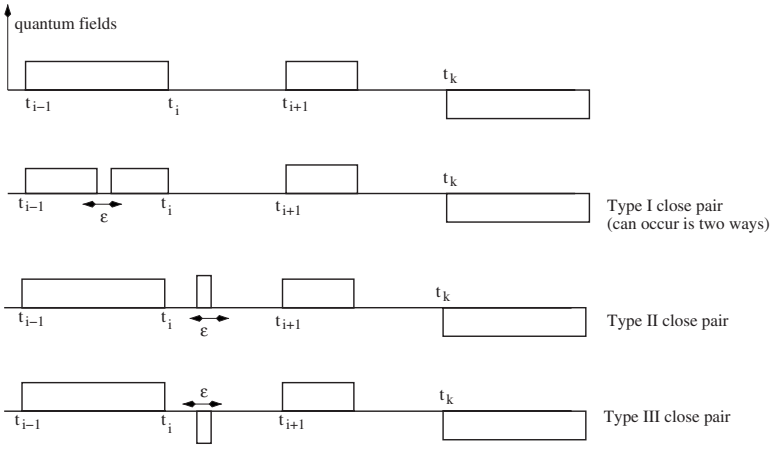

FIG. 4. Close pairs that are nearest neighbors to charge at time t_i.

cient Q_0+Q_M independent of quantum fields. This is just the equilibrium scaling. For $\Delta\mu\xi \gg 1$, Eq. (25) shows that $\frac{dC_0}{d\ln\xi}$ is again independent of time, with some coefficient Q_0 derived from scaling and not, in general, given by Eq. (26). The regime $\Delta\mu\xi \sim 1$ requires a more careful treatment. Differentiation of the perturbative results derived in Appendix B yields two contributions: a function $g_M = \frac{dh_M}{d\ln\xi}$ which expresses the dephasing crossover by "turning off" the Q_M contribution to C_0 as $\Delta\mu\xi$ increases through unity, and an additional contribution proportional to t for $\Delta\mu t > 1$ which expresses decoherence by cutting off the interaction between instantons. The decoherence term is characterized by a coefficient $g_{neq} = Lt_{t\to\infty}\left[\gamma_{neq}\frac{1}{t}\frac{dh_{neq}}{d\ln\xi}\right]$. Figure 5 shows g_{neq} calculated from Appendix B within perturbation theory to $\mathcal{O}(\lambda^2)$ and for hard and soft cutoffs. (Note that the hard cutoff model gives rise to oscillations which we have neglected.)

In summary, our scaling theory must keep track of the changes in the function C_0 as the cutoff is changed, but consideration of the short- and long-time limits shows that the model has effectively six coupling constants: "Coulomb gas charges" $Q_0(+,+), Q_0(+,-)$ expressing the part of logarithmic interaction between tunneling events involving the same lead, Coulomb gas charges $Q_M(+,+), Q_M(+,-)$ expressing the part of the logarithmic interaction which depends on coherence between leads, a tunneling amplitude Δ_{Tq} (which acquire a quantum field dependence labeled by q), and the decoherence rate Γ_{neq}. These are renormalized according to

$$\frac{d\bar{\Delta}_{T\pm}}{d\ln\xi} = \bar{\Delta}_{T\pm}\left[1 - \frac{1}{2}[Q_0(\pm,\pm) + Q_M(\pm,\pm)g_M(\Delta\mu\xi)]\right], \tag{37}$$

$$\frac{d\Gamma_{neq}(+,+)}{d\ln\xi} = -4\left[\Gamma_{neq}(+,+)\bar{\Delta}_{T+}^2 + \frac{1}{2}(\Gamma_{neq}(+,+)\bar{\Delta}_{T+}^2 - \Gamma_{neq}(+,-)\bar{\Delta}_{T-}^2)\right]e^{-C_0(1)} + g_{neq}(\Delta\mu\xi), \tag{38}$$

$$\frac{d\Gamma_{neq}(-,-)}{d\ln\xi} = -4\left[\Gamma_{neq}(-,-)\bar{\Delta}_{T-}^2 + \frac{1}{2}(\Gamma_{neq}(-,-)\bar{\Delta}_{T-}^2 - \Gamma_{neq}(-,+)\bar{\Delta}_{T+}^2)\right]e^{-C_0(1)} + g_{neq}(\Delta\mu\xi), \tag{39}$$

$$\frac{d\Gamma_{neq}(+,-)}{d\ln\xi} = -[3\Gamma_{neq}(+,-)\bar{\Delta}_{T-}^2 + 3\Gamma_{neq}(-,+)\bar{\Delta}_{T+}^2 - \Gamma_{neq}(+,+)\bar{\Delta}_{T+}^2 - \Gamma_{neq}(-,-)\bar{\Delta}_{T-}^2]e^{-C_0(1)} + g_{neq}(\Delta\mu\xi), \tag{40}$$

$$\frac{dQ_{0,M}(+,+)}{d\ln\xi} = -4\left[Q_{0,M}(+,+)\bar{\Delta}_{T+}^2 + \frac{1}{2}(Q_{0,M}(+,+)\bar{\Delta}_{T+}^2 - Q_{0,M}(+,-)\bar{\Delta}_{T-}^2)\right]e^{-C_0(1)}, \tag{41}$$

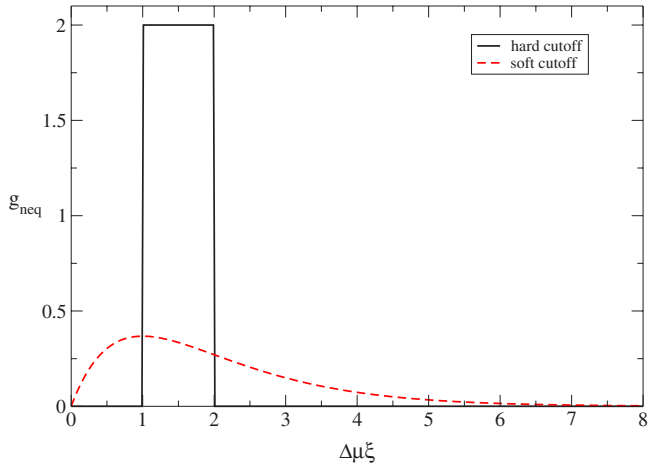

FIG. 5. (Color online) Crossover function g_{neq} (in units of $\frac{\pi}{2}Q_M\Delta\mu$) describing additive renormalization of decay constant Γ_{neq}, for hard (solid line) and soft (dashed line) cutoffs.

$$\frac{dQ_{0,M}(+,-)}{d\ln\xi} = -[3Q_{0,M}(+,-)\bar{\Delta}_{T-}^2 + 3Q_{0,M}(-,+)\bar{\Delta}_{T+}^2 - Q_{0,M}(+,+)\bar{\Delta}_{T+}^2 - Q_{0,M}(-,-)\bar{\Delta}_{T-}^2]e^{-C_0(1)}, \tag{42}$$

$$\frac{dQ_{0,M}(-,-)}{d\ln\xi} = -4\left[Q_{0,M}(-,-)\bar{\Delta}_{T-}^2 + \frac{1}{2}(Q_{0,M}(-,-)\bar{\Delta}_{T-}^2 - Q_{0,M}(-,+)\bar{\Delta}_{T+}^2)\right]e^{-C_0(1)}. \tag{43}$$

Note that in these expressions, $C_0(1)=\Gamma_{neq}\xi$ for $\Delta\mu\xi \gg 1$ and =0 otherwise. The meaning of Eqs. (38)–(40) is that as the renormalized chemical potential passes through the scale $\Delta\mu\xi=1$ an additive contribution to Γ_{neq} is generated.

IV. SOLUTION OF THE SCALING EQUATIONS

We now discuss the solution of the scaling equations. Notice from Eqs. (20)–(23) that in equilibrium ($\Delta\mu=0$), the charges $Q_{0,M}$ are independent of the quantum fields. Moreover, out of equilibrium, any explicit dependence on the quantum fields appears to $O(\lambda^3)$ for asymmetric couplings and to $O(\lambda^4)$ for symmetric couplings. We will present results that are valid to $O(\lambda^2)$, and hence ignore the explicit quantum field dependence of the charges. In this limit, as we shall show, the dominant effect of voltage is due to the decoherence term.

The physically relevant starting point for the solution of the scaling equations is one where the chemical potential difference is small compared to the cutoff scale. In this limit, $C_0(1), g_{neq} \approx 0$, and $g_M(\Delta\mu\xi) \approx 1$; thus, the effective Coulomb gas charge $Q=Q_0+Q_M$ (physically, the level couples to the coherent combination of the leads). The scaling equations thus become

$$\frac{d\bar{\Delta}_T}{d\ln\xi} = \bar{\Delta}_T\left[1-\frac{Q}{2}\right], \tag{44}$$

$$\frac{dQ}{d\ln\xi} = -4Q\bar{\Delta}_T^2, \tag{45}$$

$$\frac{d\Gamma_{neq}}{d\ln\xi} = -4\Gamma_{neq}\bar{\Delta}_T^2, \tag{46}$$

$$\frac{dQ_M}{d\ln\xi} = -4Q_M\bar{\Delta}_T^2. \tag{47}$$

Equations (44) and (45) are the usual equilibrium scaling equations and are solved as usual; from the solution, the behavior of Q_M and Γ_{neq} is computed. The equations leave the combination $I=\bar{\Delta}_T^2-\frac{1}{2}\left(\frac{Q-2}{2}-\ln(Q/2)\right)$ invariant. If $I<0$ then the model scales toward $\Delta_T=0$ (i.e., is localized) while if $I>0$ then it is on the delocalized side of the phase boundary and Δ_T increases. We discuss the two cases separately.

If the initial conditions are such that the model is on the localized side of the equilibrium phase diagram, then scaling proceeds until the cutoff crosses through the dephasing scale $\Delta\mu\xi=1$. Beyond this point, changes occur. First, the leads decohere $g_M \to 0$, so the term proportional to Q_M drops out of the scaling equations and the charge becomes Q_0. Depending on the sign of Q_M (i.e., the relative sign of δ_R and δ_L), this may either make the system more localized or more delocalized. Second, and of greater significance, the decoherence rate Γ_{neq} acquires a positive additive term, arising from the function g_{neq} in the scaling equation. Third, the effective coupling becomes $(\bar{\Delta}_T^*)^2 \equiv \bar{\Delta}_T^2 e^{-C_0(1)} \approx \bar{\Delta}_T^2 e^{-\Gamma_{neq}\xi}$. Thus, the theory at the scale $\xi > (\Delta\mu)^{-1}$ is characterized by a fugacity $\ln \Delta_T^*\xi$, a decoherence rate Γ_{neq}, a charge Q_0, and by the scaling equations

$$\frac{d\bar{\Delta}_T}{d\ln\xi} = \bar{\Delta}_T\left[1-\frac{Q_0}{2}\right], \tag{48}$$

$$\frac{d\Gamma_{neq}}{d\ln\xi} = -4\Gamma_{neq}\bar{\Delta}_T^2 e^{-C_0(1)}, \tag{49}$$

$$\frac{dQ_0}{d\ln\xi} = -4Q_0\bar{\Delta}_T^2 e^{-C_0(1)}. \tag{50}$$

We see that scaling proceeds in the usual way until the *nonequilibrium scale* $\xi=1/\Gamma_{neq}$ is reached; beyond this point, the factor $e^{-C_0(1)}$ cuts off the scaling and we are left with a perturbative theory.

A particularly important special case occurs if $I=0$. In this case, the equilibrium fixed point is $\Delta_T=0$, $Q=2$, and if the initial value of $\Delta\mu$ is sufficiently small, this fixed point is approached very closely, so that at the scale $\xi=1/\Delta\mu$

$$\bar{\Delta}_T(\xi) = \frac{\bar{\Delta}_{T0}}{1+2\bar{\Delta}_{T0}\ln\frac{1}{\Delta\mu\xi_0}} \approx \frac{1}{2\ln\frac{1}{\xi_0\Delta\mu}}, \tag{51}$$

$$Q \approx 2, \tag{52}$$

$$Q_M \to Q_M^*. \tag{53}$$

Scaling through the crossover region then drives $Q_M \to Q_M^*$, changes the basic charge to $Q_0=2-Q_M^*$, generates a $\Gamma_{neq}=\frac{\pi}{2}|Q_M^*|\Delta\mu$, and does not change Δ_T significantly. Scaling then proceeds until $\xi \to 1/\Gamma_{neq} \sim 2/(\pi|Q_M^*|\Delta\mu)$. We therefore see that the dephasing crossover typically shifts the system away from the critical point, and that decoherence then cuts off the scaling. The decoherence cutoff occurs very rapidly, unless $Q_M^* \ll 1$, meaning that one of the leads is much more weakly coupled than the other one. In this case, a significant nonequilibrium scaling regime can exist.

If the initial condition is on the delocalized side of the equilibrium phase diagram, then again we distinguish two cases, according to whether or not the model flows to strong coupling before $\Delta\mu\xi=1$ or not. In the latter case, the treatment outlined above applies. In the former case, the Kondo or coherence scale is larger than the dissipation rate and a different treatment, beyond the scope of this paper, is needed.

V. CONCLUSIONS

In this paper, we have expressed the nonequilibrium tunneling center model in terms of a Coulomb gas defined on the Keldysh contour. The nonequilibrium problem has a richer structure than the corresponding equilibrium problem. In particular, the effective low energy theory is shown to be a Coulomb gas characterized by two parameters (charge and sign of quantum field). Crucial ingredients of the resulting theory are the dephasing arising because the wave functions in the two leads precess at rates differing by the chemical potential difference and a decoherence arising from the dissipative processes again allowed when the model is driven out of equilibrium. We showed explicitly that the decoherence effects cut off the power law interaction between instantons which is found at $T=0$ in equilibrium.

Further, we generalized the standard equilibrium scaling theory of the model to the nonequilibrium case. We found that scaling through the dephasing crossover $\Delta\mu\xi\sim 1$ generates an additive renormalization to the decoherence rate. From this, we conclude that the decoherence rate is a fundamental parameter of the nonequilibrium theory, which must be explicitly considered in a renormalization process. We further showed explicitly how the decoherence cuts off the renormalization group flow.

A few words on the generalization of this approach to other quantum impurity models, such as the nonequilibrium Kondo model.[10] The key difference between the model studied here and the nonequilibrium Kondo model is the term $\Delta_T S_x$ in Eq. (2) responsible for spin-flip processes. The analog of $\Delta_T S_x$ for the Kondo model is $(J_{LL}+J_{RR})(S_+ s_- + \text{H.c.})$, where S is the impurity spin, while $\vec{s}=\Sigma_{\sigma\sigma' kk'}\psi^\dagger_{k\sigma}\vec{\tau}\psi_{k'\sigma'}$ are the electron spins which have been written as the following linear combination of the two leads $\psi_{k\sigma}=\frac{\sqrt{J_{LL}}}{\sqrt{J_{LL}+J_{RR}}}c_{k\sigma L}+\frac{\sqrt{J_{RR}}}{\sqrt{J_{LL}+J_{RR}}}c_{k\sigma R}$. Thus in the Anderson-Yuval-Hamann procedure applied to the Kondo model, the (n_-+n_+)th order expansion in the spin-flip amplitude involves the computation of

$$\begin{aligned}&\mathrm{Tr}_{el}[Te^{-i\int_0^t dt'\tilde{V}_\uparrow(t')}s_-(t_{n_-})\cdots s_+(t_1)\\&\quad\times\rho^\uparrow_{S0}\tilde{T}e^{+i\int_0^t dt'\tilde{V}_\uparrow(t')}s_-(t_{n_-+n_+})\cdots s_+(t_{n_-+1})]\\&=e^{C_0(t_1,t_2,\ldots,t_{n_-+n_+})}L(t_1,t_2,\ldots,t_{n_-+n_+}),\end{aligned}\tag{54}$$

rather than the quantity needed for the model studied in this paper

$$\mathrm{Tr}_{el}[Te^{-i\int_0^t dt'\tilde{V}_\uparrow}\rho^\uparrow_{S0}\tilde{T}e^{+i\int_0^t dt'\tilde{V}_\uparrow}]=e^{C_0(t_1,t_2,\ldots,t_{n_-+n_+})}.\tag{55}$$

In equilibrium, the quantity L in Eq. (54) (referred to in the literature as the open line contribution)[21] acquires the same structure as that of the closed loop part e^{C_0}, namely, that of a Cauchy determinant. Thus, Eqs. (54) and (55) and therefore the Kondo model and the tunneling center model may be related by a simple redefinition of the phase shifts. Out of equilibrium, this analysis breaks down because the dephasing between the leads occuring at $\Delta\mu t>1$ gives rise to a two channel structure similar to that analyzed by Fabrizio *et al.*[24] and Vladár *et al.*[25] A direct numerical evaluation or a mapping to an explicit two channel Kondo model (with decoherence) could be employed.

This work suggests several generalizations. First, the long-time exponential cutoff suggests that a numerical estimation of the perturbation series may be possible. Second, the key issue in seeing a wide nonequilibrium scaling range is to get the decoherence time to be very large compared to the dephasing time. This does not occur naturally in the simple two lead models we have studied. A search for models, involving for example three leads, where this separation of scales occurs more naturally, may be of interest.

Wide classes of models have been studied in equilibrium by Hubbard-Stratonovich methods, in which the partition function is expressed as a sum over configurations of auxiliary fields. In the strong coupling limit of many quantum impurity models, a small number of auxiliary field configurations are relevant and the physics is controlled by tunneling between them.[26] Generalizing this analysis to the nonequilibrium situation is an important open problem,[22] for which the methods developed here may be useful. A useful first step might be a comparison to the Bethe-ansatz solvable interacting resonant level model.[27]

ACKNOWLEDGMENTS

This work was supported by NSF-DMF 0431350.

APPENDIX A: DERIVATION OF EQ. (6)

We start from the Hamiltonian in Eq. (1) which we write as a sum of two parts:

$$H=H_0+H_1,\tag{A1}$$

$$\begin{aligned}H_0&=S_zB+\lambda DS_z\sum_{\alpha=1\cdots N}d^\dagger_\alpha d_\alpha+\sum_{a=L,R,\alpha=1\cdots N}\int d\epsilon\frac{\epsilon}{D}c^\dagger_{\epsilon a\alpha}c_{\epsilon a\alpha}\\&\quad+\sqrt{\frac{1}{\pi}}\sum_{a=L,R,\alpha=1\cdots N}\int_{\mu_a-1/\xi}^{\mu_a+1/\xi}d\epsilon(\cos\theta_a c^\dagger_{\epsilon a\alpha}d_\alpha+\text{H.c.})\end{aligned}$$

$$H_1=\Delta_T S_x.\tag{A2}$$

When $\Delta_T=0$, the Hamiltonian is exactly solvable and represents a noninteracting resonant level hybridized with the leads. The Anderson-Yuval-Hamann approach involves a perturbative expansion in Δ_T, treating H_0 exactly. This procedure was originally carried out for the partition function; we apply it here to the time-dependent density matrix $\rho(t)$ determined from an initial condition $\rho_0(t)$ via

$$\rho(t)=e^{-iHt}\rho_0 e^{iHt}.\tag{A3}$$

The reduced density matrix for the impurity spin is defined as

$$\rho_S(t)=\mathrm{Tr}_{el}\,\rho(t)\tag{A4}$$

and is a 2×2 matrix whose diagonal elements give the probability of the spin S to be up or down, while the off diagonal elements contain information about phase coherence. Rewriting

$$e^{-iHt} = e^{-iH_0 t} T e^{-i\int_0^t dt V(t)}, \tag{A5}$$

$$V(t) = e^{iH_0 t} H_1 e^{-iH_0 t} = \Delta_T [e^{iH_0 t} S_x e^{-iH_0 t}] = \Delta_T S_x(t), \tag{A6}$$

a perturbative expansion in $V(t)$ of Eq. (A4) may be carried out, yielding

$$\hat{\rho}_S(t) = \sum_{n_-,n_+} (-i)^{n_-} i^{n_+} (\Delta_T \xi)^{n_- + n_+} \times \left[\int_0^t \frac{dt_{n_-}}{\xi} \int_0^{t_{n_-}} \frac{dt_{n_- -1}}{\xi} \cdots \int_0^{t_2} \frac{dt_1}{\xi} \right] \times \left[\int_0^t \frac{dt_{n_- +1}}{\xi} \int_0^{t_{n_- +1}} \frac{dt_{n_- +2}}{\xi} \cdots \int_0^{t_{n_- + n_+ -1}} \frac{dt_{n_+ + n_-}}{\xi} \right]$$

$$\times \mathrm{Tr}_{el}[S_x(t_{n_-}) \cdots S_x(t_2) S_x(t_1) \rho_0 S_x(t_{n_- + n_+}) \cdots S_x(t_{n_- +1})]. \tag{A7}$$

We assume the initial density matrix

$$\rho_0 = \begin{pmatrix} \rho_{0S}^{\uparrow} \rho_{leads}^{\uparrow} & 0 \\ 0 & \rho_{0S}^{\downarrow} \rho_{leads}^{\downarrow} \end{pmatrix}, \tag{A8}$$

where $\uparrow,\downarrow$ represent the direction of the local impurity spin, while $\rho_{leads}^{\uparrow/\downarrow}$ represents the steady state distribution of the electrons when the local spin is oriented along $\uparrow/\downarrow$. The effect of the spin-flip term would be to modify the diagonal components of ρ_S from its initial value and also to introduce off diagonal terms. The perturbative expansion for the diagonal component of ρ_S is (note $H_0^{\uparrow/\downarrow}$ appearing below implies H_0 corresponding to $S_z = \uparrow/\downarrow$),

$$\langle\uparrow|\rho_S|\uparrow\rangle = \sum_{n_-,n_+=0,2,4\ldots} (-i)^{n_-} i^{n_+} (\Delta_T \xi)^{n_- + n_+} \left[\int_0^t \frac{dt_{n_-}}{\xi} \int_0^{t_{n_-}} \frac{dt_{n_- -1}}{\xi} \cdots \int_0^{t_2} \frac{dt_1}{\xi} \right] \left[\int_0^t \frac{dt_{n_- +1}}{\xi} \int_0^{t_{n_- +1}} \frac{dt_{n_- +2}}{\xi} \cdots \int_0^{t_{n_- + n_+ -1}} \frac{dt_{n_+ + n_-}}{\xi} \right]$$

$$\mathrm{Tr}_{el}[e^{+iH_0^{\downarrow} t_{n-}} e^{-iH_0^{\uparrow} t_{n-}} \cdots e^{+iH_0^{\downarrow} t_1} e^{-iH_0^{\uparrow} t_1} \rho_{S0}^{\uparrow} e^{+iH_0^{\uparrow} t_{n_- + n_+}} e^{-iH_0^{\downarrow} t_{n_- + n_+}} \cdots e^{+iH_0^{\uparrow} t_{n_- +1}} e^{-iH_0^{\downarrow} t_{n_- +1}}]$$

$$+ \sum_{n_-,n_+=1,3,5,\ldots} (-i)^{n_-} i^{n_+} (\Delta_T \xi)^{n_- + n_+} \left[\int_0^t \frac{dt_{n_-}}{\xi} \int_0^{t_{n_-}} \frac{dt_{n_- -1}}{\xi} \cdots \int_0^{t_2} \frac{dt_1}{\xi} \right] \left[\int_0^t \frac{dt_{n_- +1}}{\xi} \int_0^{t_{n_- +1}} \frac{dt_{n_- +2}}{\xi} \cdots \int_0^{t_{n_- + n_+ -1}} \frac{dt_{n_+ + n_-}}{\xi} \right]$$

$$\mathrm{Tr}_{el}[e^{+iH_0^{\uparrow} t_{n-}} e^{-iH_0^{\downarrow} t_{n-}} \cdots e^{+iH_0^{\uparrow} t_1} e^{-iH_0^{\downarrow} t_1} \rho_{S0}^{\downarrow} e^{+iH_0^{\downarrow} t_{n_- + n_+}} e^{-iH_0^{\uparrow} t_{n_- + n_+}} \cdots e^{+iH_0^{\downarrow} t_{n_- +1}} e^{-iH_0^{\uparrow} t_{n_- +1}}]. \tag{A9}$$

The above may be written as

$$\langle\uparrow|\rho_S|\uparrow\rangle = \sum_{n_-,n_+=0,2,4\ldots} (-i)^{n_-} i^{n_+} (\Delta_T \xi)^{n_- + n_+} \left[\int_0^t \frac{dt_{n_-}}{\xi} \int_0^{t_{n_-}} \frac{dt_{n_- -1}}{\xi} \cdots \int_0^{t_2} \frac{dt_1}{\xi} \right] \left[\int_0^t \frac{dt_{n_- +1}}{\xi} \int_0^{t_{n_- +1}} \frac{dt_{n_- +2}}{\xi} \cdots \int_0^{t_{n_- + n_+ -1}} \frac{dt_{n_+ + n_-}}{\xi} \right]$$

$$\mathrm{Tr}_{el}[T e^{-i\int_0^t dt' \tilde{V}_{\uparrow}(t')} \rho_{S0}^{\uparrow} \tilde{T} e^{+i\int_0^t dt' \tilde{V}_{\uparrow}(t')}]$$

$$+ \sum_{n_-,n_+=1,3,5,\ldots} (-i)^{n_-} i^{n_+} (\Delta_T \xi)^{n_- + n_+} \left[\int_0^t \frac{dt_{n_-}}{\xi} \int_0^{t_{n_-}} \frac{dt_{n_- -1}}{\xi} \cdots \int_0^{t_2} \frac{dt_1}{\xi} \right] \left[\int_0^t \frac{dt_{n_- +1}}{\xi} \int_0^{t_{n_- +1}} \frac{dt_{n_- +2}}{\xi} \cdots \int_0^{t_{n_- + n_+ -1}} \frac{dt_{n_+ + n_-}}{\xi} \right]$$

$$\mathrm{Tr}_{el}[T e^{-i\int_0^t dt' \tilde{V}_{\downarrow}(t')} \rho_{S0}^{\downarrow} \tilde{T} e^{+i\int_0^t dt' \tilde{V}_{\downarrow}(t')}], \tag{A10}$$

where

$$\tilde{V}_{\uparrow/\downarrow}(t) = \lambda D \left[S_z(t) + \frac{(-/+)1}{2} \right] \sum_{\alpha=1\cdots N} d_\alpha^\dagger(t) d_\alpha(t), \tag{A11}$$

$S_z(t)$ in Eq. (A11) is the expectation value of the operator S_z and switches between $(+/-)\frac{1}{2}$ at times $t_1, t_2 \ldots t_{n_- + n_+}$.

The first term in Eq. (A10) ($n_\pm = 0,2,4\cdots$) represents "outscattering," the second term ($n_\pm = 1,3,5\cdots$) represents "in scattering." To evaluate the Tr_{el} in the outscattering terms we write the lead states in the basis of scattering states appropriate to the static potential $(\lambda D/2)\Sigma_{\alpha=1\cdots N} d_\alpha^\dagger(t) d_\alpha(t)$. The potential $\tilde{V}_{\uparrow}(t)$ then alternates between the values $-\lambda D \Sigma_{\alpha=1\cdots N} d_\alpha^\dagger(t) d_\alpha(t)$ and 0. Similarly to evaluate the Tr_{el} in the in-scattering term, we write the lead states in the basis of scattering states appropriate to the static potential $(-\lambda D/2)\Sigma_{\alpha=1\cdots N} d_\alpha^\dagger(t) d_\alpha(t)$ so $\tilde{V}_{\downarrow}$ alternates between $\lambda D \Sigma_{\alpha=1\cdots N} d_\alpha^\dagger(t) d_\alpha(t)$ and 0. The density of states of the two scattering problems is identical.

Since the leads are noninteracting electrons, with nonequilibrium imposed by $\mu_a \neq \mu_b$, the evaluation of the Tr_{el} for a given configuration of spin flips reduces to a problem of single particle quantum mechanics in a time-dependent potential. We briefly outline the solution based on the nonequilibrium linked cluster theorem,[22] which implies

$$\mathrm{Tr}_{el}[T e^{-i\int_0^t dt' \tilde{V}_{\uparrow}(t')} \rho_{S0}^{\uparrow} \tilde{T} e^{+i\int_0^t dt' \tilde{V}_{\uparrow}(t')}] = e^{-C_0(t)}, \tag{A12}$$

where

$$C_0(t) = \sum_\alpha \int_0^1 \frac{dg}{g} \int_0^t dt_1 \, \mathrm{Tr}[\phi_{g,q}(t_1)\{G_\alpha^{Kg}(t_1,t_1) + G_\alpha^{Zg}(t_1,t_1)\}], \tag{A13}$$

where for in scattering, $G_\alpha^{K,Zg}(t_1,t_1)$ are Green's functions appropriate to the classical field $\phi_{cl}=\frac{\lambda D}{2}(S_z(t_+)+S_z(t_-)+1)$ and the quantum field $\phi_{g,q}=\frac{g\lambda D}{2}(S_z(t_-)-S_z(t_+))$. They obey the Dyson equation

$$\hat{G}^g = \hat{g} + \hat{g}(\phi_{cl}1 + \phi_{g,q}\tau_x)\hat{G}^g, \tag{A14}$$

where

$$\hat{G}^g = \begin{pmatrix} G_g^R & G_g^K \\ G_g^Z & G_g^A \end{pmatrix}, \quad \hat{g} = \begin{pmatrix} g^R & g^K \\ 0 & g^A \end{pmatrix}.$$

Here $g^{R,A,K}$ are the standard retarded, advanced, and Keldysh Green's functions of H, Eq. (1), with $S_z=-1/2$. Note that $g^{R,A}$ are short ranged in time and may be approximated as delta functions

It is convenient to recast Eq. (A14) as

$$\hat{G}^g = \hat{\bar{g}} + \hat{\bar{g}}\phi_{g,q}\tau_x\hat{G}^g, \tag{A15}$$

with $\hat{\bar{g}}=(1-\phi_{cl}\hat{g})^{-1}\hat{g}$. Explicitly, we have

$$\bar{g}^{R,A}(t-t') \approx -\frac{(\phi_{cl}(t)-\lambda D/2) \pm iD}{(\phi_{cl}(t)-\lambda D/2)^2 + D^2}\delta(t-t'), \tag{A16}$$

$$\bar{g}^K(t,t') = \bar{g}^R h - h\bar{g}^A = -\frac{2D(\cos^2\theta_L e^{-i\mu_L(t-t')} + \cos^2\theta_R e^{-i\mu_R(t-t')})}{(iD+\lambda D/2-\phi_{cl}(t))(-iD+\lambda D/2-\phi_{cl}(t'))}P\frac{1}{t-t'}, \tag{A17}$$

with $h\sim 1/(t-t')$ the usual distribution function. Note that for times t,t' such that $\phi_q\neq 0$, $\phi_{cl}=\lambda D/2$.

Rearranging Eq. (A15) explicitly, we find that G^K and G^Z obey the equations

$$G^Z = \frac{\bar{g}^A\phi_{gq}}{1-\bar{g}^R\phi_{gq}\bar{g}^A\phi_{gq}}[\bar{g}^R + \bar{g}^K(\bar{g}^A)^{-1}G^Z], \tag{A18}$$

$$G^K = \frac{1}{1-\bar{g}^R\phi_{gq}\bar{g}^A\phi_{gq}}[\bar{g}^K + \bar{g}^R\phi_{gq}\bar{g}^A + \bar{g}_K\phi_{gq}G^K]. \tag{A19}$$

Equations (A18) and (A19) are singular integral equations. Noting that $G^Z(t,t')\neq 0$ only when $\phi_q(t)\neq 0$ and that $\bar{g}^{R,A}$ are effectively delta functions, we see that the long-time behaviors of G^K and G^Z are the same. In equilibrium, we may set $\mu_{L,R}=0$ and write Eq. (A19) explicitly for t,t' separated widely in time. The important term is the last one, which is

$$\int' dt_1 \frac{1}{t-t_1}\frac{\lambda}{1-\frac{\lambda^2}{4}}G^K(t_1,t') \tag{A20}$$

where the prime denotes an integration only over those times t_1 for which $\phi_q(t_1)\neq 0$. From the standard properties of singular integral equations,[20,21] we identify the coefficient as the tangent of the phase shift, obtaining

$$\tan\delta_{eq} = \frac{\lambda}{1-\frac{\lambda^2}{4}} \tag{A21}$$

and recovering the usual equilibrium Coulomb gas.

In the nonequilibrium long-time limit, we follow Ng[19] and write $G^K=e^{-i\mu_L(t-t')}G_L^K+e^{-i\mu_R(t-t')}G_R^K$. We substitute this expression into Eq. (A19), write separate equations for G_L^K and G_R^K, use Eq. (A17), and note that for $\Delta\mu(t-t')\gg 1$ the cross term between the $e^{i\mu_L}$ term in g^K and the $e^{i\mu_R}$ term in G_R^K gives an effective delta function contribution to the equation for G^K. This leads to a singular term of the form of Eq. (A20) but with the phase shift replaced by

$$\tan\delta_{a=L,R}(t) = \frac{2D\phi_{gq}(t)\cos^2\theta_a}{D^2-\phi_{gq}^2(t) - i\,\mathrm{sgn}(\mu_L-\mu_R)\sin^2\theta_a 2\phi_{gq}(t)D}. \tag{A22}$$

Note from Eq. (A22) that $\delta_a(-\phi_{gq})=-\delta_a^*(\phi_{gq})$.

We briefly discuss the structure of the solution in the long-time nonequilibrium limit. From Eq. (A13), we are eventually interested in the equal time limit of the Green's functions, which just as in equilibrium have a divergent contribution due to the long-time approximation made in deriving them. This issue can be resolved by solving for $G_{K,Z}$ by assuming that the Green's functions adiabatically follow the time-dependent potential. The nondivergent part of $G_{K,Z}$ leads to the logarithmic interaction between the charges, which has the following form:

$$C_0^{\ln} = -\int_0^1 \frac{dg}{g}\int dt \sum_{a=L,R}\left[\frac{\phi_{gq}}{\pi}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](t)\frac{\partial \ln X_a(t)}{\partial t}, \tag{A23}$$

where

$$X_{a=L,R}(t) = \exp\left[\frac{P}{2\pi i}\int\frac{dt''}{t-t''}\ln\frac{1-i\tan\delta_a(t'')}{1+\tan\delta_a(t'')}\right]. \tag{A24}$$

Using Eq. (A24), one may rewrite $\ln X_a(t) = -\frac{1}{\pi}P\int dt''\frac{\delta_a(t'')}{(t-t'')}$; further integrating Eq. (A23) by parts, one finds

$$C_0^{\ln} = -\frac{1}{\pi^2}P\int_0^1\frac{dg}{g}\int dt dt'' \sum_{a=L,R}\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](t)\frac{1}{t''-t}\frac{\delta_a(t'')}{dt''}. \tag{A25}$$

Our model allows for a sequence of quantum fields which alternates between 0 and $\pm\frac{g\lambda D}{2}$, and therefore may be written as

$$\phi_{gq}(t) = \frac{g\lambda D}{2} \sum_{i=1\cdots} q_i[\theta(t-t_{2i-1}) - \theta(t-t_{2i})], \quad \text{(A26)}$$

i denoting times at which the quantum field changes, while q_i denoting the sign of the quantum field in the region where it is nonzero. From Eq. (A22), it follows that

$$\frac{\delta_a(t)}{dt} = \sum_{i=1\cdots} \delta_a(q_i)[\delta(t-t_{2i-1}) - \delta(t-t_{2i})]. \quad \text{(A27)}$$

In a similar manner,

$$\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](t) = \sum_{i=1\cdots}\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](q_i)[\theta(t-t_{2i-1}) - \theta(t-t_{2i})]. \quad \text{(A28)}$$

Substituting Eqs. (A27) and (A28) in Eq. (A25), we find

$$C^{\ln} = -\frac{1}{\pi^2}P\int_0^1 \frac{dg}{g}\int dt \sum_{a=L,R;j}\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](q_j)[\theta(t-t_{2j-1}) - \theta(t-t_{2j})]\sum_i \delta_a(q_i)\left[\frac{1}{t_{2i-1}-t} - \frac{1}{t_{2i}-t}\right]. \quad \text{(A29)}$$

The integral over time t give rise to the logarithmic interaction between charges. Moreover, the coefficient of the logarithmic interaction between charges at t_i and t_j depend on the quantum fields $q_{i,j}$. In particular, if $q_i=q_j=q$, the coupling constant integral yields a coefficient

$$2\sum_{a=L,R}\int_0^1 \frac{dg}{g}\delta_a(q)\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](q) = \sum_{a=L,R}\delta_a^2. \quad \text{(A30)}$$

On the other hand, if $q_i=-q_j=q$, the coefficient of the logarithm interaction between charges is

$$\int_0^1 \frac{dg}{g}\sum_{a=L,R}\left(\delta_a^*(q)\left[\phi_{gq}\frac{\partial\delta_a}{\partial\phi_{gq}}\right](q) + \delta_a(q)\left[\phi_g\frac{\partial\delta_a}{\partial\phi_{gq}}\right]^*(q)\right) = \sum_{a=L,R}\delta_a\delta_a^* \quad \text{(A31)}$$

The overall $\pm$ signs before the coefficients of the logarithmic interaction essentially keep track of whether the spin has flipped up or down and may be used to define the Coulomb gas charge n_i.

The discussion so far is valid for $\Delta\mu t \gg 1$. In order to obtain an expression for C_0 for arbitrary $\Delta\mu t$, we solve the Dyson equation perturbatively and obtain expressions for C_0 correct to second order in the scattering potential λ. This is outlined in Appendix B. The expression obtained interpolates the exact analytic expressions for $\Delta\mu t \ll 1$ and $\Delta\mu t \gg 1$.

APPENDIX B: PERTURBATIVE EVALUATION OF C_0 FOR A SYMMETRIC AND HARD CUTOFF

Let us turn to the evaluation of the time evolution operator at $O(\Delta_T^2)$ (which corresponds to a single instanton in the quantum field),

$$e^{-C_0(t)} = \mathrm{Tr}_{el}[Te^{-i\int_0^t dt'\tilde{V}_\uparrow(t')}\rho_{S0}^\uparrow], \quad \text{(B1)}$$

with $\tilde{V}_\uparrow(t')$ defined in Eq. (A11).

By using the convenient representation $d_\alpha(t) = \frac{1}{\sqrt{\pi}}\sum_{k,a=L,R}\cos\theta_a c_{k\alpha a}(t)$, the expression for C_0 at $O(\lambda^2)$ is

$$C_0(t) = \frac{N\lambda^2}{\pi^2}\sum_{\alpha\beta=L,R}\cos^2\theta_\alpha\cos^2\theta_\beta\int_0^t dt_1\int_0^{t_1} dt_2\int_{-\xi^{-1}+\mu_\alpha}^{\mu_\alpha} d\epsilon_1 \times\int_{\mu_\beta}^{\xi^{-1}+\mu_\beta} d\epsilon_2\langle c^\dagger_{\epsilon_1\alpha}(t_1)c_{\epsilon_1\alpha}(t_2)\rangle\langle c_{\epsilon_2\beta}(t_1)c^\dagger_{\epsilon_2\beta}(t_2)\rangle, \quad \text{(B2)}$$

where $0\leqslant\theta_L=\frac{\pi}{2}-\theta_R\leqslant\frac{\pi}{2}$. The limits of integration for $\epsilon_{1,2}$ correspond to band edges that are abrupt and symmetrically located with respect to the chemical potentials (see Fig. 1). At zero temperatures,

$$\begin{aligned} C_0(t) &= \frac{N\lambda^2}{\pi^2}\sum_{\alpha\beta=L,R}\cos^2\theta_\alpha\cos^2\theta_\beta\int_0^t dt_1\int_0^{t_1} dt_2\int_{-\xi^{-1}+\mu_\alpha}^{\mu_\alpha} d\epsilon_1 \times\int_{\mu_\beta}^{\xi^{-1}+\mu_\beta} d\epsilon_2 e^{-i(\epsilon_1-\epsilon_2)(t_2-t_1)} \\ &= -\frac{N\lambda^2}{\pi^2}\sum_{\alpha\beta=L,R}\cos^2\theta_\alpha\cos^2\theta_\beta \times\int_0^t dt_1\int_0^{t_1} dt_2 e^{-i(\mu_\beta-\mu_\alpha)(t_1-t_2)}\frac{[1-e^{-i(t_1-t_2)/\xi}]^2}{(t_1-t_2)^2}. \end{aligned} \quad \text{(B3)}$$

It is also of interest to define a soft cutoff model with density of states $\rho_{L,R}(\epsilon)\sim e^{-|\epsilon-\mu_{L,R}|\xi}$. This gives Eq. (B3) but with $\frac{[1-e^{-i(t_1-t_2)/\xi}]}{t_1-t_2}\to\frac{1}{t_1-t_2-i\xi}$.

Writing the above as a symmetric and antisymmetric combination of the time arguments t_1, t_2, one obtains

$$C_0 = \mathrm{Re}[C_0] + \mathrm{Im}[C_0], \quad \text{(B4)}$$

where $\mathrm{Re}[C_0]$ is given by the symmetric combination of t_1, t_2, which after a straightforward evaluation of time integrals leads to

$$\begin{aligned} \mathrm{Re}[C_0](t,\Delta\mu,\xi) &= N\lambda^2\left(\frac{\cos^4\theta_L}{\pi^2}+\frac{\cos^4\theta_R}{\pi^2}\right)[F_0(2t/\xi)-2F_0(t/\xi)] \\ &+ 2N\lambda^2\frac{\cos^2\theta_L\cos^2\theta_R}{\pi^2}\left[F_0(\Delta\mu t)+\frac{F_0(2t/\xi+\Delta\mu t)+F_0(2t/\xi-\Delta\mu t)}{2}-F_0(t/\xi+\Delta\mu t)-F_0(t/\xi-\Delta\mu t)\right] \\ &+ 2N\lambda^2\frac{\cos^2\theta_L\cos^2\theta_R}{\pi^2}\left[F_1(\Delta\mu t)+\frac{F_1(2t/\xi+\Delta\mu t)+F_1(2t/\xi-\Delta\mu t)}{2}-F_1(t/\xi+\Delta\mu t)-F_1(t/\xi-\Delta\mu t)\right], \end{aligned}$$

with

$$F_0(x) = -\ln|x| - \gamma + \mathrm{Ci}(x) + (\cos x - 1), \tag{B5}$$

$$F_1(x) = x\mathrm{Si}(x), \tag{B6}$$

where $\mathrm{Ci}(z) = \gamma + \ln z + \int_0^z dt \frac{\cos t - 1}{t}$. Note $F_0(x \ll 1) = -x^2/2$, $F_1(x \ll 1) = x^2$, $F_1(x \gg 1) = \frac{\pi}{2}x$.

The antisymmetric combination of t_1, t_2 leads to

$$\mathrm{Im}[C_0(t)] \propto it \tag{B7}$$

and represents unimportant energy renormalization that vanishes for the particle-hole symmetric case.

Identifying the coefficients above with perturbative expressions for the appropriate phase shifts defined in Eq. (4), and defining the functions

$$h_0(t/\xi) = [F_0(2t/\xi) - 2F_0(t/\xi)], \tag{B8}$$

$$h_M(t/\xi, \Delta\mu t) = \left[F_0(\Delta\mu t) + \frac{F_0(2t/\xi + \Delta\mu t) + F_0(2t/\xi - \Delta\mu t)}{2} - F_0(t/\xi + \Delta\mu t) - F_0(t/\xi - \Delta\mu t) \right], \tag{B9}$$

$$h_{neq}(t/\xi, \Delta\mu t) = \frac{2}{\pi}\left[F_1(\Delta\mu t) + \frac{F_1(2t/\xi + \Delta\mu t) + F_1(2t/\xi - \Delta\mu t)}{2} - F_1(t/\xi + \Delta\mu t) - F_1(t/\xi - \Delta\mu t) \right], \tag{B10}$$

one obtains Eq. (12).

To obtain the scaling function g_{neq} giving rise to the additive renormalization of Γ_{neq}, we differentiate Eq. (B3) in its soft cutoff analog with respect to $\ln \xi$ and (to extract the long-time behavior) t. The resulting integral may easily be evaluated. We plot the real part in Fig. 5.

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