Fermi liquid identities for the infinite-U multichannel Anderson model

Eran Lebanon and P. Coleman
Center for Materials Theory, Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854-8019, USA
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We show how the electron-gas methods of Luttinger, Ward, and Nozières can be applied to an SU(N), multichannel generalization of the infinite-U Anderson impurity model within a Schwinger boson treatment. Working to all orders in a $1/N$ expansion, we show how the Friedel-Langreth relationship and the Yamada-Yosida-Yoshimori and the Shiba-Korringa relations can be derived under the assumption that the spinon and holon fields are gapped. One of the remarkable features of this treatment is that the Landau amplitudes depend on the exchange of low-energy virtual spinons and holons. We end the paper with a discussion on the extension of our approach to the lattice, where the spinon-holon gap is expected to close at a quantum critical point.

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I. INTRODUCTION

The standard model of interacting electronic systems is based on the theoretical framework established by Landau[1] in the 1950s, whereby the properties of interacting fluids of fermions are linked to noninteracting fermions by adiabatically turning on the interactions. This is the basis of the Landau Fermi liquid, and it leads to the concept of quasiparticles. It also provides the formal basis for the diagrammatic description of interacting Fermi fluids. In the early 1960s, Luttinger and Ward[2] showed how many aspects of the Landau Fermi liquid theory—such as the relationship between fermion density and Fermi-surface volume, the relationship between the linear specific heat capacity and the quasiparticle density of states—can be deduced from first-principles resummations of diagrammatic perturbation theory to infinite order.

Many new electronic materials discovered over the past two decades, such as the high-temperature superconductors, colossal magnetoresistance, and heavy electron materials, do not fit so naturally into the Landau scheme. In these materials, the interactions between the fermions become so large that they project out large tracts of the many-body Hilbert space. For example, in many narrow-band transition metal or f-electron materials, the charge fluctuations of the localized orbitals are restricted to certain valence states, and in local moment systems, the electrons in a localized moment are restricted to states of fixed definite occupancy. In these systems, the validity of the adiabatic approach is increasingly in question, and a more appropriate starting point is the one where the Coulomb interaction has been taken to infinity from the outset, explicitly removing certain states from the Hilbert space. For example, in the infinite-U Anderson model, states of double occupancy are excluded, whereas in the Kondo model, all charge fluctuations of the local moment are excluded, as illustrated in Fig. 1.

When cast as a field theory, infinite-U models become gauge theories in which the Hubbard operators that create and destroy the highly correlated electrons are written as composite products of "slave operators." In such an approach, the constraints on charge fluctuations become conserved local quantities associated with local gauge invariances and the basic fields entering into the Hamiltonian are "fractionalized" fields, which carry either spin or charge but not both. The gauge theory approach to strongly correlated electron systems poses a number of important technical and qualitative challenges. In particular, how does the Landau Fermi liquid emerge within a gauge theory of unbroken symmetry, and what happens to the spectrum of gauge particle excitations? This question assumes an increased importance in the context of recent speculations that fractional particles may become free at a quantum critical point.[3–6]

The past ten years have seen considerable progress in methods that combine the Luttinger-Ward approach with a new generation of large N expansions which use Schwinger bosons, rather than Abrikosov pseudofermions to represent the localized moments.[7–11] We now briefly review some of these recent developments. Cox and Ruckenstein[7] were the first to apply a large N approach to a multichannel Kondo single impurity Kondo model using a fermionic spin representation. Parcollet and Georges[8] developed the original formalism for the Schwinger boson large N approach to this model, employing a multichannel Kondo model, focusing their interest mainly on the overscreened case,[9] built upon this earlier work to develop a Luttinger-Ward approach, showing how it can be used to derive a Friedel sum rule for the single impurity model and a Luttinger sum rule for the

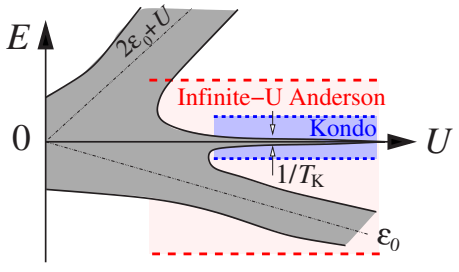


FIG. 1. (Color online) Schematic diagram illustrating the evolution of the impurity f-spectral function with the strength of the repulsive interaction U. Shaded areas denote the regions where the impurity f-spectral function is large. As U is increased, high-energy regions of the Hilbert space are projected out of the low-energy model, giving rise to the infinite-U Anderson model and Kondo model as effective low-energy theories. As long as adiabaticity is preserved, the spectral weight at the Fermi surface is preserved, no matter how large U becomes.

lattice. Assuming that the Fermi liquid state could be treated in a $1/N$ expansion. Rech *et al.*[10] showed how the fully screened Kondo impurity model could indeed be treated in the large N limit using the Schwinger boson method, demonstrating the emergence of Fermi liquid behavior. Lebanon *et al.*[11] extended these methods to the infinite-U Anderson model, showing how a Baym-Kadanoff approximation scheme can be developed from the $1/N$ expansion of the Luttinger-Ward functional using the leading-order term to develop a conserving approximation at finite N.

One of the key insights emerging from the large N solutions to the fully screened Kondo and Anderson impurity models is that the development of a Fermi liquid is accompanied by the formation of a gap Δ_g of order of the Kondo temperature T_K in the spectrum of the fractional particles.[10,11]

In this paper, we explore the general consequences of this observation for the infinite-U Anderson impurity model. In particular, we show that the assumption of a gap in the fractionalized particle spectrum enables us to extend the Luttinger-Ward approach beyond the leading orders considered so far to all orders in the $1/N$ expansion. In our approach, the expansion in powers of $1/N$ plays a completely analogous role to the expansion in powers of the interaction. The Luttinger-Ward approach was first applied to the finite-U Anderson model by Yamada and Yosida,[12] Shiba,[14] and Yoshimori,[13] who used conservation laws to develop a set of conserving identities that apply to all orders in the strength of the interaction U. Here, taking advantage of these new insights, we show how a parallel set of results can be obtained for the *infinite-U* Anderson model.

Many structural aspects of our work, including the interplay between virtual fractional excitations and the composite heavy quasiparticles, may enjoy a wider application to the Anderson and Kondo lattices. These are points that we return to in a discussion at the end of the paper.

II. SUMMARY OF KEY ASPECTS OF THE PAPER

The starting point for our work is the infinite-U Anderson model, which we formulate by representing the Hubbard operators as a product of slave fermion ("holon") and a Schwinger boson ("spinon"),[15,16] $X_{\alpha 0}=b_\alpha^\dagger \chi$, where $b_\alpha^\dagger$ creates a spinon with spin component α and $\chi^\dagger$ creates a charged holon. Since there is no natural perturbative expansion in terms of the parameters of the original model, we consider a family of infinite-U impurity Anderson models with SU(N) spin symmetry, employing a $1/N$ expansion around the large N limit. There are a number of special tricks that need to be carried out in order to realize such a large N expansion:

(a) The spin S of the moment, described by the number of Schwinger bosons $n_b=2S$, must be allowed to grow with N, so that $2S/N$ remains finite.

(b) More surprisingly, the number of electron-scattering channels K must also grow with N,[8] in order to screen the large moment S. The special case $2S=K$ describes the fully screened model.[10]

The introduction of K channels requires that we generalize the Hubbard operators as follows:[17]

$$X_{\alpha 0} \to X_{\alpha \nu} = b_\alpha^\dagger \chi_\nu, \tag{1}$$

where $\nu \in [1,K]$ is the channel index. $X_{\alpha\nu}$ creates a localized electron with spin component α and channel index ν.

We formulate these features into an infinite-U Anderson impurity model[11] as follows:

$$\mathcal{H} = \sum_{\vec{k}\nu\alpha} \epsilon_{\vec{k}} c_{\vec{k}\nu\alpha}^\dagger c_{\vec{k}\nu\alpha} + H_I. \tag{2}$$

Here, $c_{\vec{k}\nu\alpha}^\dagger$ creates a conduction electron with momentum $\vec{k}$, channel index $\nu \in [1,K]$, and spin index $\alpha \in [1,N]$. The term

$$H_I = \frac{V}{\sqrt{N}} \sum_{\vec{k}\nu\alpha} [c_{\vec{k}\nu\alpha}^\dagger \chi_\nu^\dagger b_\alpha + \text{H.c.}] + \epsilon_0 \sum_\alpha b_\alpha^\dagger b_\alpha$$

describes the interaction of the conduction electrons with a magnetic ion located at the origin. The $\sqrt{N}$ denominator in the hybridization ensures a well-defined large N limit. The energy of a singly occupied impurity ϵ_0 is taken to be negative. The conserved operator $Q=\Sigma_\alpha b_\alpha^\dagger b_\alpha + \Sigma_\nu \chi_\nu^\dagger \chi_\nu$ generalizes the no-double occupancy constraint of the infinite-U Anderson model by restricting the "valence" $n_b \leqslant Q$. Q is also the maximum size $S=Q/2$ of the local moment that can develop at each site. The condition $Q=K$ is required for perfect screening of the local moment.

The spinon and holon operators carry the conserved charge Q, and in order to discuss these fields, we need to consider the enlarged Fock space of general $Q \neq K$. The partition function for the general problem is given by

$$Z_Q = \text{Tr}[\mathcal{P}_Q e^{-\beta H}],$$

where the projection operator $\mathcal{P}_Q$ imposes the constraint $\hat{Q}=Q$. In doing so, we are really considering a whole family of Anderson and/or Kondo models which includes underscreened models, where $Q>K$ is larger than the number of screening channels, and the overscreened models, where the number of channels $K>Q$ exceeds the number of spinons per site.

We shall make the key assumption that the fully screened state where $Q=K$ develops an additional stability with respect to the over- and underscreened states where $Q \neq K$. This assumption is motivated by the discovery of a spinon-holon gap in the large N Schwinger boson solution of the one and two impurity Kondo models[10] and the leading Baym-Kadanoff approximation to the infinite-U Anderson model.[11] The structure of these solutions leads us to believe that the gap in the fractional particle excitation spectrum is robust in its extension to finite N. The emergence of a gap in the infinite N limit provides a vital infrared cuttoff to the fluctuations of the fractional excitations. This cutoff plays a dual role: it stabilizes a Fermi liquid with a large Fermi surface while, at the same time, providing a vital low-energy cutoff to the fluctuations of the fractional holon and spinon fields removing them from the low-energy Hilbert space. The presence of the gap guarantees that the $1/N$ expansion contains no nonperturbative infrared divergences, and this, in turn, guarantees its survival over a finite range of $1/N$ values.

Suppose E_0 is the energy of the ground state with $Q=K$ and $E_\pm$ be the ground-state energies of the state where Q

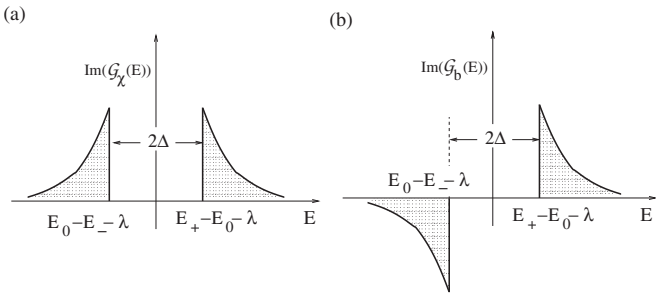


FIG. 2. The spectral gap in the holon and spinon spectra. Creation of a holon or spinon increases $Q \to K+1$, whereas the destruction of a holon or spinon decreases $Q \to K-1$.

$=K\pm 1$ is modified by one unit, then the "commensuration gap" associated with complete screening of the moment at site j_0 is

$$\Delta_g = \frac{1}{2}[E_+ + E_- - 2E_0]. \tag{3}$$

This gap is manifested as a gap in the spectral functions of the spinon and the holon, as illustrated in Fig 2. With a gap, we can calculate the low-temperature properties of the model in the grand-canonical ensemble where we remove the constraint and associate a chemical potential λ with the conserved charge Q,

$$Z_G[\lambda] \equiv e^{-\beta F[\lambda]} = \mathrm{Tr}[e^{-\beta[H-\lambda(Q-K)]}]. \tag{4}$$

Although the chemical potential λ only imposes $\hat{Q}=Q$ on the average, the existence of commensuration gap guarantees that there is a finite range of λ,

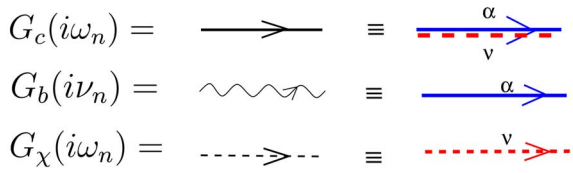
$$\lambda \in [E_0 - E_-, E_+ - E_0],$$

for which the constraint is imposed with exponential accuracy once $T \ll \Delta$.

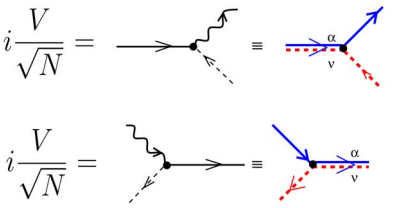
Our approach makes use of Coleman-Paul-Rech generalization of the Luttinger-Ward free energy functional to the gauge theory of interacting bosons and fermions,[10,18,19] which can be compactly written as

$$F = T\,\underline{\mathrm{Str}}[\ln(-\mathcal{G}^{-1}) + \Sigma\mathcal{G}] + Y[\mathcal{G}], \tag{5}$$

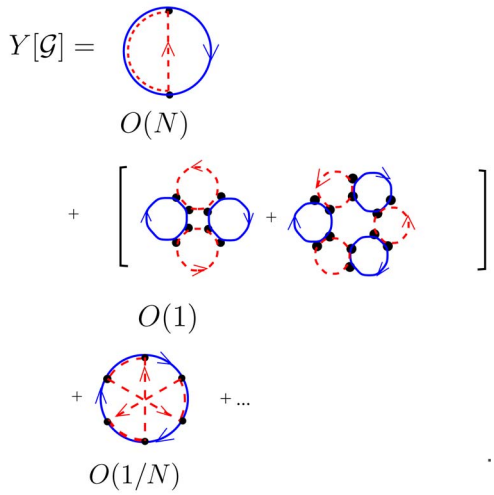
where $\underline{\mathrm{Str}}[A]=\underline{\mathrm{Tr}}[A_B]-\underline{\mathrm{Tr}}[A_F]$ denotes the supertrace of a matrix containing both bosonic (B) and fermionic (F) components (where the underline notation is used to denote a sum over internal frequencies and a trace over the internal quantum numbers of the matrix). $\mathcal{G}=(\mathcal{G}_0^{-1}-\Sigma)^{-1}$ is the matrix describing the fully dressed Green's function of all elementary particles and fields entering the Lagrangian, including the slave particles, where Σ is the self-energy matrix and $\mathcal{G}_0$ the bare propagator of the fields. Diagrammatically, the propagators for the Anderson model are denoted as follows:



In the above expressions, we display two alternative notations for the propagators. In the second column, we adopt a traditional notation using continuous, wavy, and dashed lines to represent the conduction, spinon, and holon propagators, respectively. In the third column, we display the alternative "railway track" notation that we use in cases where it is necessary to clearly show the flow of spin and charge in the respective propagators. At the interaction vertices



the spin and charge of the electron fields divide up amongst the holon and spinon fields. In the Feynman diagrams, each vertex is associated with a factor $iV/\sqrt{N}$. The generating functional $Y[\mathcal{G}]$ may be written as the sum of all closed-loop two-particle irreducible skeleton Feynman diagrams. These can be ordered in a $1/N$ expansion as follows:



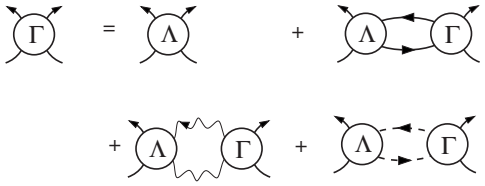
(6)

Each blue loop over spin gives a factor of N, whereas each red loop over charge gives a factor of $K=kN$.

The functional $Y[\mathcal{G}]$ is the generating functional for the self-energies of the fields, so that variation of $Y[\mathcal{G}]$ with respect to the full Green's functions $\mathcal{G}_\zeta$ of the conduction, spinon, and holon fields, $\Sigma_\zeta=-\eta_\zeta\beta\delta Y/\delta G_\zeta$, self-consistently determines the self-energies Σ_ζ of these fields,[9,19] where $\eta_\zeta=+1$ (-1) for bosons (fermions). We refer to $Y[\mathcal{G}]$ as the Luttinger-Ward functional. Approximations to this functional are the basis of Kadanoff-Baym conserving many-body approaches.[20]

The key difference between our approach and the classic Luttinger-Ward approach to the interacting electron gas is the appearance of the holon and spinon fields. In the Luttinger-Ward approach, the $Y[\mathcal{G}]$ is a generating functional for the self-energies of the fields. The second derivative of the generating functional with respect to the Green's functions generates the off-shell scattering vertex function between the fields $\Lambda = \beta^2 \delta^2 Y[\mathcal{G}]/\delta \mathcal{G}^2$. Nozières and Luttinger[21] demonstrated that the multiple forward scattering generated by this vertex gives rise to the on-shell Fermi liquid amplitudes between the electron quasiparticles. The Landau amplitudes are determined as solutions of the Dyson equation represented by the following Feynman diagrams:

This feature is preserved in our approach, but the off-shell scattering vertex also involves virtual holon or spinon pairs, as shown below:



In this way, scattering off the virtual fractionalized excitations determines the scattering amplitudes of low-energy Fermi liquid. Remarkably, despite these effects, all of the key conservation properties of the Landau Fermi liquid are preserved, as long as the fractionalized excitations remain gapped.

In the context of impurity models, the Landau Fermi liquid becomes Nozières' "local" Fermi liquid,[22,23] in which the energies of each electronic quasiparticles are expressed in terms of the elastic scattering phase shift $\delta_{\nu\alpha}$, where ν is the channel index and α the spin index. If $\Delta\epsilon$ is the spacing of states in the continuum, then the energy of each electron in the continuum is shifted down by an amount $-\left(\frac{\delta_{\nu\alpha\mu}}{\pi}\right)\Delta\epsilon$. The Fermi liquid theory is then defined in terms of the dependence of the phase shifts on the quasiparticle occupancies,

$$\delta_{\nu\alpha}[\epsilon,\{n_{k\alpha\nu}\}] = \frac{\pi}{N} + \epsilon\delta' + \sum_{k',\nu',\alpha'} \phi_{\nu\alpha,\nu'\alpha'}\delta n_{k\nu'\alpha'}, \quad (7)$$

where the impurity Landau parameter

$$\phi_{\nu\alpha,\nu'\alpha'} = \langle \nu,\alpha;\nu'\alpha'|\hat{\phi}|\nu,\alpha;\nu'\alpha\rangle \quad (8)$$

is the expectation value of the interaction energy between quasiparticles of flavor and/or spin (ν,α) and (ν',α').

By comparing the free energy of the infinite-U Anderson model obtained in the Luttinger-Ward approach, we are able to diagrammatically identify each of the terms in the above expansion. In particular, the conduction electron phase shift is given by

$$\delta_{\nu\alpha}(\epsilon) = \mathrm{Im}\ \ln(1 - i\pi\rho\Sigma_c(\epsilon - i0^+)),$$

where $\Sigma_c(\epsilon)$ is the conduction electron self-energy, ρ the density of conduction electron states at the Fermi surface, and 0^+ is a small positive infinitesimal. We are also able to identify the Landau parameter with the on-shell forward scattering amplitude between quasiparticles

$$\phi_{\nu\alpha,\nu'\alpha'} = \Gamma_{\nu\alpha,\nu'\alpha'}(\omega = 0, \omega' = 0).$$

As in the finite-U Anderson model, each of these quantities enjoys expression entirely in terms of conduction electron states.

The ability to link the quasiparticle physics with the gauge theory diagrammatics enables us to derive the key conservation relationships for the infinite-U Anderson impurity model, working to all orders in the $1/N$ expansion. These relationships survive in the leading Kadanoff-Baym[11] approximation that defines the large N limit, and we can illustrate them explicitly in practical calculations. The key conservation laws that appear from this approach are the Friedel sum rule,[24–26]

$$\delta_c = \frac{\pi}{N}\frac{K - n_\chi}{K} + O\left(\frac{T_K}{ND}\right), \quad (9)$$

the Langreth relation,[26]

$$\mathrm{Im}\ t\ (\epsilon - i0^+)|_{\epsilon=0} = \sin^2\ \delta_c/\pi\rho, \quad (10)$$

where ρ is the density of states of the electron fluid, the Yamada-Yosida-Yoshimori relationship[12,13] between the spin, charge, and channel susceptibilities and the linear specific heat coefficient,

$$NK\frac{\gamma}{\gamma^0} = K\frac{N^2-1}{N+K}\frac{\chi_s}{\chi_s^0} + N\frac{K^2-1}{K+N}\frac{\chi_f}{\chi_f^0} + \frac{\chi_c}{\chi_c^0}, \quad (11)$$

and the Korringa-Shiba relationship[14] between the dynamical spin susceptibility and the impurity susceptibility, $\chi''(\omega)/\omega|_{\omega=0} = (N\pi/K)(\chi/N)^2$.

The structure of the paper is as follows. In Sec. III, we set up various formal preliminaries. In Sec. IV, we recapitulate the relationship between conservation laws and Ward identities, using them to rederive a general set of Friedel sum rules. In Sec. V, we show how these can be used to derive the Langreth[26] relation for the scattering t matrix and a general set of Friedel sum rules. In Sec. VI, we introduce the off-shell interaction vertices amongst the electrons and the fractional particles. Using the Friedel relations, we express the charge, spin, and flavor susceptibilities in terms of these vertices. In Sec. VII, we use our gap hypothesis to derive an expression for the low-temperature free energy and use this to relate the specific heat with the spin, charge, and flavor susceptibilities. In Sec. VIII, we identify the Nozières interaction parameters[22,23] with the on-shell interaction vertices of the gapless electrons and use our results to derive an identity between the linear specific heat coefficient and a linear combination of the spin, charge, and flavor susceptibilities. Our result here is a generalization of the Yamada-Yosida-Yoshimori[12,13] identities for the finite-U

Anderson model. In Sec. IX, we derive the Korringa-Shiba relationship[14] between the uniform susceptibilities and the power spectrum at low energies. Finally, in Sec. X, we discuss the extension of these ideas to a lattice environment.

III. PRELIMINARIES

Our starting point is the infinite-U Anderson model,

$$\mathcal{H} = \sum_{\vec{k}\nu\alpha} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\nu\alpha} c_{\vec{k}\nu\alpha} + \frac{V}{\sqrt{N}} \sum_{\vec{k}\nu\alpha} [c^{\dagger}_{\vec{k}\nu\alpha} \chi^{\dagger}_{\nu} b_{\alpha} + \text{H.c.}] + \epsilon_0 \sum_{\alpha} b^{\dagger}_{\alpha} b_{\alpha} - \lambda(Q - 2S), \tag{12}$$

where the spin components α are taken to run from $-j$ to $+j$, where $N=2j+1$, and the flavor components ν are taken to run from $-f$ to $+f$, where $K=2f+1$. The bare conduction electron, spinon, and holon Green's functions are given by

$$\mathcal{G}^{(0)}_{c\vec{k},\vec{k}}(z) = \frac{1}{z - \epsilon_{\vec{k}}} \delta_{\vec{k},\vec{k}},$$

$$\mathcal{G}^{(0)}_{b}(z) = \frac{1}{z - (\epsilon_0 - \lambda)},$$

$$\mathcal{G}^{(0)}_{\chi}(z) = \frac{1}{z + \lambda}, \tag{13}$$

where we use the symbol z to denote the analytic extension of the Matsubara frequencies into the complex plane. In an impurity model, it is generally convenient to trace the conduction electron propagator over its momentum indices to obtain the local propagator. The bare local conduction propagator is given by

$$\mathcal{G}^{(0)}_{c}(z) = \sum_{\vec{k}} \frac{1}{z - \epsilon_{\vec{k}}} \approx \mp i\pi\rho \quad \text{for } z = \omega \pm i0^{+}, \tag{14}$$

where the expression on the right is the large bandwidth limit of the local propagator, choosing the negative sign for the retarded propagator (z above the real axis) and the positive sign for the advanced propagator (z below the real axis).

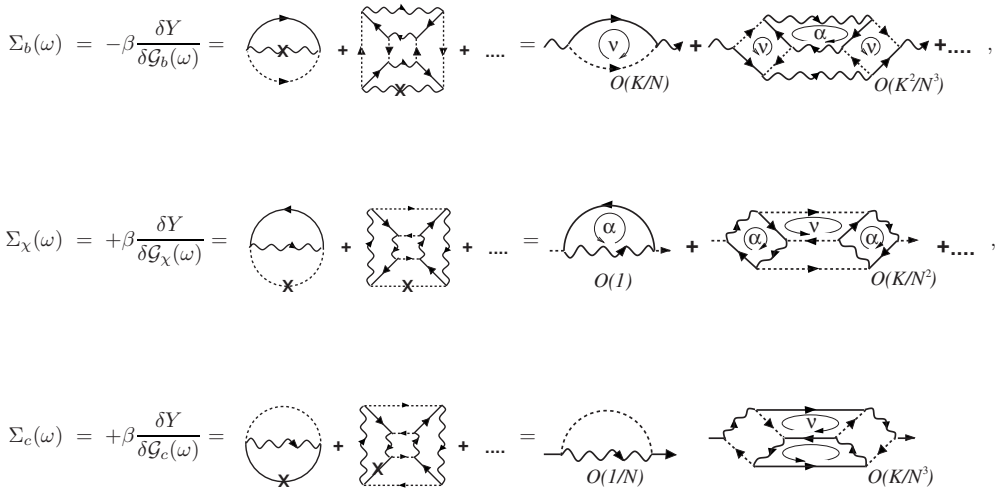
When interactions are turned on, each of the fields acquires a self-energy correction in the full propagators,

$$\mathcal{G}_c(z) = \{[\mathcal{G}^{(0)}_c(z)]^{-1} - \Sigma_c(z)\}^{-1},$$

$$\mathcal{G}_b(z) = [z - (\epsilon_0 - \lambda) - \Sigma_b(z)]^{-1},$$

$$\mathcal{G}_{\chi}(z) = [z + \lambda - \Sigma_{\chi}(z)]^{-1}. \tag{15}$$

Each of these self-energies is obtained by differentiating the Luttinger-Ward functional, shown in Eq. (6) with respect to the Green's functions,



(16)

where the cross indicates the line which is eliminated by the functional differential. Although our discussion today will focus on keeping all orders in this expansion, it is interesting to briefly reflect on the leading terms in the diagrammatic expansion. The leading-order contributions to the self-energies are derived from the virtual separation of the electron into spinon-holon pairs. These terms are already sufficient to develop a conserving approximation with all the key features of the infinite order resummation. The second-order terms in this expansion contain the subleading effects of electron-electron scattering.

IV. FRIEDEL SUM RULES AND LUTTINGER-WARD IDENTITIES

The Fermi liquid description of this model centers around the relationship between the conduction electron phase shift

TABLE I. The diagonal components of the conserved charge, spin, and flavor operators q^C, q^S, and q^F, resolved into their conduction (c), spinon (b), and holon (χ) components.

	Conserved quantity		
Particle	q^C	q^S	q^F
c	1	$\tilde{\alpha}$	$\tilde{\nu}$
b	0	$\tilde{\alpha}$	0
χ	−1	0	$-\tilde{\nu}$

and the thermodynamics and the charge, spin, and flavor susceptibilities. We begin with a derivation of the Friedel sum rule. This part of the paper closely follows earlier work on the Kondo model.

In addition to the local conserved charge $Q=n_b+n_\chi$, there are three globally conserved physical quantities, charge (Q^C), (Table I) magnetization ($M=Q^S$), and flavor ($F=Q^F$), associated with this model, which we write as

$$\hat{Q}^C = \sum_{\vec{k}\nu\alpha} c^\dagger_{\vec{k}\nu\alpha} c_{\vec{k}\nu\alpha} - \sum_\nu \chi^\dagger_\nu \chi_\nu,$$

$$\hat{Q}^S = \sum_{\vec{k}\nu\alpha} \tilde{\alpha} c^\dagger_{\vec{k}\nu\alpha} c_{\vec{k}\nu\alpha} + \sum_\alpha \tilde{\alpha} b^\dagger_\alpha b_\alpha = \hat{M},$$

$$\hat{Q}^F = \sum_{\vec{k}\nu\alpha} \tilde{\nu} c^\dagger_{\vec{k}\nu\alpha} c_{\vec{k}\nu\alpha} - \sum_\nu \tilde{\nu} \chi^\dagger_\nu \chi_\nu, \tag{17}$$

respectively, where $\tilde{\alpha}=\mathrm{sgn}(\alpha)$ is 1 for "up spins" and −1 for "down spins," and $\tilde{\nu}=\mathrm{sgn}(\nu)$ is 1 for flavors $\nu>0$ and −1 for flavor index $\nu<0$. Each of these conserved quantities is associated with a corresponding external field, which we introduce into the Hamiltonian by writing

$$H = \mathcal{H} - \Delta\mu \hat{Q}^C - B\hat{M} - B^F \hat{Q}^F,$$

$$= \mathcal{H} - \sum_{A=C,S,F} B^A \hat{Q}^A, \tag{18}$$

where we introduce the notation $(B^C, B^S, B^F) \equiv (\Delta\mu, B, B^F)$. We shall write each of these conserved quantities using the shorthand

$$\hat{Q}^A = \psi^\dagger \hat{q}^A \psi, \tag{19}$$

where $\psi^\dagger \equiv (c^\dagger_{\vec{k}}, b^\dagger, \chi^\dagger)$ denotes the complete spinor of electron, spinon, and holon fields. We can write the expectation value of these operators in terms of the triplet of electron, spinon, and holon Green's functions,

$$\mathcal{G} = \{\mathcal{G}_{c\vec{k}\vec{k}}, \mathcal{G}_b, \mathcal{G}_\chi\}, \tag{20}$$

as

$$\langle \hat{Q}^A \rangle = \mathrm{Tr}\{\psi^\dagger \hat{q}^A \psi \varrho\} = -T\, \underline{\mathrm{Str}}[\hat{q}^A \mathcal{G}], \tag{21}$$

where ϱ is the thermal density matrix and the supertrace denotes a trace over each particle species (with a minus sign for fermions) and their quantum numbers, and the underline beneath the supertrace denotes a summation over Matsubara frequencies,

$$\underline{\mathrm{Str}}[\hat{q}^A \mathcal{G}] = \sum_{i\alpha_n} \mathrm{Str}[\hat{q}^A \mathcal{G}(i\alpha_n)]$$

$$= \sum_{i\nu_n} \mathrm{Tr}[q^A \mathcal{G}_B(i\nu_n)] - \sum_{i\omega_n} \mathrm{Tr}[q^A \mathcal{G}_F(i\omega_n)].$$

An alternative, and convenient way to formulate the conserved quantities at absolute zero is to replace q^A by its more general form, $q^A \to q^A \gamma(\omega)$, where

$$\gamma(\omega) = \frac{\partial \mathcal{G}_0^{-1}(\omega)}{\partial \omega}, \tag{22}$$

so that

$$\langle \hat{Q}^A \rangle = -T\, \underline{\mathrm{Str}}[\hat{q}^A \gamma \mathcal{G}]. \tag{23}$$

This is a particularly useful device in the single impurity model where we work with the local conduction electron propagator. In this case, the charges do not couple trivially in the local propagator and we need to introduce frequency dependent vertices with the external fields. To see this, note that the bare local propagator becomes

$$\mathcal{G}_c^0(z,B) = \sum_{\mathbf{k}} [(z-\epsilon_{\mathbf{k}}) + q_c^A B^A]^{-1}$$

$$= \mathcal{G}_c^0(z) - [\mathcal{G}_c^0(z)]^2 [B^A q_c^A \gamma(z)] + O(B^2),$$

where

$$\gamma_c(z) = \frac{\partial [\mathcal{G}_c^0(z)]^{-1}}{\partial z} = \frac{\sum_{\vec{k}} \left(\frac{1}{z-\epsilon_{\mathbf{k}}}\right)^2}{[\mathcal{G}_c^0(z)]^2} \tag{24}$$

is identified as the frequency dependent vertex function. The conduction electron vertex $\gamma_c(\omega)$ vanishes in the wideband limit $\gamma_c \to 0$. This is the basis of the famous "Anderson-Clogston" compensation theorem,[27,28] according to which the conduction band polarization of charge, spin, or flavor degrees of freedom in the ground state is

$$\langle Q_c^A \rangle = \mathrm{Tr}[q_c^A \gamma G_c] = O(T_K/D),$$

where D is the electron bandwidth and T_K is the Kondo temperature.

Ward identities are properties of the zero temperature self-energies, Green's functions, and scattering vertices that result from the conservation laws. A key step in making the transition from finite to zero temperature is the replacement of Matsubara sums at finite temperature by continuous integrals along the imaginary axis at low temperature,

$$T\sum_{\alpha_n} A(i\alpha_n) \overset{T\to 0}{=} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} A(i\alpha) \equiv \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} A(\omega). \tag{25}$$

Here, α_n represents the Bose-Matsubara frequencies $\nu_n = 2n\pi k_B T$ along bosonic propagator lines, or the fermion Matsubara frequencies $\omega_n = (2n+1)\pi k_B T$ along fermion lines. Our ability to make this replacement presumes the ex-

istence of well-defined scales associated with the excitation spectrum. This is a situation that may not be satisfied at a quantum critical point, or an overscreened Kondo model where the spinon and holon propagators may exhibit E/T scaling, with temperature as the relevant excitation scale. However, away from a quantum critical point, the conduction electron lines are characterized by a well-defined Fermi energy and the existence of a well-defined gap in the spinon and holon spectrum guarantees that the continuous version of the Matsubara sums is valid for these lines too. Provided that the replacement [Eq. (25)] is allowed, the zero temperature conserved quantities may be written in the form

$$\langle \hat{Q}^A \rangle = -\int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \mathrm{Str}\{\hat{q}^A \gamma(\omega)\mathcal{G}(\omega)\}. \tag{26}$$

The starting point for the development of Ward identities is the Luttinger-Ward functional. The conserved charges are conserved at each vertex in the diagrams contributing to the generating function Y. For the quantity Q^A, the charge q_ζ^A is associated with each propagator line $\mathcal{G}_\zeta$ inside the functional. When we shift the frequency of each propagator by a constant differential times the corresponding charge,

$$\mathcal{G}_\zeta(\omega) \to \mathcal{G}_\zeta(\omega + \delta\omega_\zeta^A), \quad \delta\omega_\zeta^A = \delta\omega q_\zeta^A, \tag{27}$$

the generating functional is unchanged. Since the derivative of the generating functional with respect to the Green's functions is the self-energy, this leads to the Ward identity

$$\frac{\delta Y[\mathcal{G}]}{\delta \omega^A} = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \mathrm{Str}\left\{\Sigma \frac{d\mathcal{G}}{d\omega}\hat{q}^A\right\} = 0, \tag{28}$$

and integration by parts then yields

$$\int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \mathrm{Str}\left\{\hat{q}^A \frac{d\Sigma}{d\omega}\mathcal{G}\right\} = 0. \tag{29}$$

Equations (26) and (29) sum up to an integral over a full differential; rotating the integration contour around the negative real axis, we get the Friedel sum rules

$$\langle \hat{Q}^A \rangle = -\frac{1}{\pi} \mathrm{Im}\ \mathrm{Str}\{\hat{q}^A \ln[-\mathcal{G}^{-1}(-i0^+)]\}. \tag{30}$$

In impurity problems, we are interested in change in these quantities that results from coupling the impurity to its environment, which is given by

$$\begin{aligned}\Delta Q^A &= \langle \hat{Q}^A \rangle - \langle \hat{Q}^A \rangle_0 \\ &= -\frac{1}{\pi} \mathrm{Im}\ \mathrm{Str}\{\hat{q}^A \ln[\mathcal{G}^{-1}/\mathcal{G}_0^{-1}]\}\Big|_{\omega=-i0^+} = \\ &\quad -\frac{1}{\pi} \mathrm{Im}\ \mathrm{Str}\{\hat{q}^A \ln[1-\mathcal{G}_0(\omega)\Sigma(\omega)]\}\Big|_{\omega=-i0^+}.\end{aligned} \tag{31}$$

The quantities on the right-hand side are the "phase shifts" of the electrons, spinons, and holons. If we identify

$$\delta_\zeta = \mathrm{Im}\{\ln[1-\mathcal{G}_{0\zeta}(\omega)\Sigma_\zeta(\omega)]\}|_{\omega=-i0^+} \quad (\zeta = c,b,\chi), \tag{32}$$

then we can write

$$\Delta Q^A = \sum q_{c\nu\alpha}^A \frac{\delta_{\nu\alpha}^c}{\pi} + \sum q_{\chi\nu}^A \frac{\delta_\nu^\chi}{\pi} - \sum q_{b\alpha}^A \frac{\delta_{\nu\alpha}^b}{\pi}.$$

At first sight, this is very different from the expression we would expect for a Fermi liquid, where there are no holon or spinon contributions. However, as long as the spinons and holons are gapped, their corresponding phase shifts are either π or 0 and do not vary with the external fields. General arguments lead us to believe that in the impurity model, the spinon phase shifts are zero, while the χ phase shifts are π. To see this, consider the Friedel sum rule for the conserved charge $Q=n_b+n_\chi$, for which $(q_c,q_b,q_\chi)=(0,1,1)$. In this case,

$$\Delta Q = 2S = K = -N\frac{\delta_b}{\pi} + K\frac{\delta_\chi}{\pi}. \tag{33}$$

Provided all the phase shifts are positive definite, then this quantity can never exceed K and the maximum value is only attained if $\delta_b=0$ and $\delta_\chi=\pi$. If the spinon phase shifts are zero, then for a perfectly screened impurity, with $K=2S$, it follows that the holon phase shifts equal π and the spinon phase shifts vanish as long as the spinon-holon gap is preserved.

The general sum rules then become

$$\Delta Q^C = \sum_{\nu\alpha} \frac{\delta_{c\nu\alpha}}{\pi} - \sum_\nu \frac{\delta_{\chi\nu}}{\pi} = \sum_{\nu\alpha} \frac{\delta_{c\nu\alpha}}{\pi} - K,$$

$$\Delta Q^A = \sum_{\nu\alpha} q_{c\nu\alpha}^A \frac{\delta_{c\nu\alpha}}{\pi} \quad (A = S,F), \tag{34}$$

where the conduction electron phase shift is

$$\delta_{c\nu\alpha} = \mathrm{Im}\ \ln[1-\mathcal{G}_c^0\Sigma_{c\nu\alpha}]|_{\omega=-i0^+}. \tag{35}$$

The χ fermion contribution only enters into the charge Friedel sum rule since $\Sigma_\nu q_{\chi\nu}^C=0$ and $\Sigma_\nu q_{\chi\nu}^{S,F}=0$. The first sum rule expresses the conventional Friedel sum rule. In general, the impurity charge ΔQ^C contains a conduction electron and a holon contribution,

$$\Delta Q^C = \Delta Q_e^C - \langle n_\chi \rangle. \tag{36}$$

According to the Anderson-Clogston theorem,[27,28] the first term is of order T_K/D, where D is the bandwidth, and vanishes in the infinite bandwidth limit. The remaining term is simply the charge associated with the empty f states at the impurity, $\Delta Q^C=-\langle n_\chi \rangle$. Using the phase shift expression for ΔQ^C in Eq. (34), the Friedel sum rule can then be rewritten as

$$\sum_{\nu\alpha} \frac{\delta_{c\nu\alpha}}{\pi} = K - \langle n_\chi \rangle. \tag{37}$$

In the case where there is no magnetic or flavor polarization, all phase shifts are equal, $\delta_{c\nu\alpha}=\delta_c$ and $\Sigma_{\nu\alpha}\delta_{c\nu\alpha}=NK\delta_c$, so that the Friedel sum rule becomes

$$\delta_c = \frac{\pi}{N}\left(1 - \frac{\langle n_\chi \rangle}{K}\right). \tag{38}$$

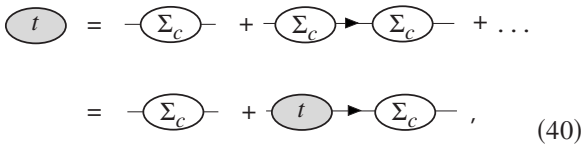
V. LANGRETH SUM RULE AND THE KONDO RESONANCE

The emergence of the Kondo resonance is one of the signature features of the Kondo effect. The "Langreth sum rule"[26] links the Friedel sum rule to the formation of the Kondo resonance by providing a rigorous constraint on the size of the spectral function associated with the localized state.

The t matrix of the conduction electrons is determined by the spectral function of the electrons localized in the magnetic moment,

$$t(i\omega_n) = -\frac{V^2}{N}\int_0^\beta \langle T X^\dagger_{\alpha\nu}(\tau) X_{\alpha\nu}(0)\rangle e^{i\omega_n \tau} d\tau. \tag{39}$$

We shall assume here that there is no magnetic or flavor polarization, so that the t matrix is identical for all ν and α. The t matrix is determined diagrammatically by the process of repeated scattering off the localized state and is given by



(40)

or

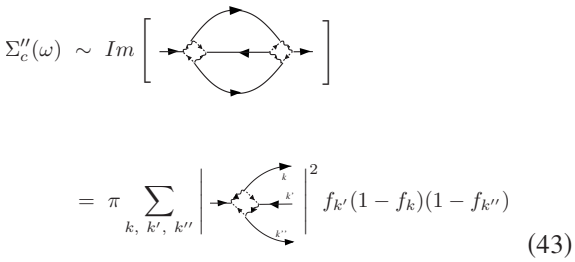
$$t(z) = \Sigma_c(z) + \Sigma_c(z) G_c^{(0)}(z)\Sigma_c(z) + \cdots = \frac{\Sigma_c(z)}{1 - G_c^0(z)\Sigma_c(z)}. \tag{41}$$

From Eq. (35), we see that the denominator in the t matrix is also the argument of the scattering phase shift. In the limit of a broad bandwidth, we can replace $G_c^{(0)}(\omega+i\delta)=-i\pi\rho$, so that the phase shift and t matrix become

$$t(\omega + i\epsilon) = \frac{\Sigma_c(\omega + i\epsilon)}{1 + i\pi\rho\Sigma_c(\omega + i\epsilon)},$$

$$\delta^c = \mathrm{Im}\ \ln[1 + i\pi\rho\Sigma_c]|_{\omega=-i0^+}. \tag{42}$$

If we examine the decay processes associated with the conduction electron self-energy, we see that there are two types of process. In one type, the electron decays into holon and spinon combinations, but since these are gapped excitations, they produce no contribution to the inelastic electron decay processes at the Fermi energy. The only remaining decay processes involve the production of electron-hole pairs, such as



(43)

where the internal loop of gapped fractional quasiparticles does not contribute to the low-energy decay processes. These are the well-known decay processes of the conventional Fermi liquid, producing an inelastic decay rate that grows quadratically with energy and temperature, $\Sigma_c''(\omega)\sim[\omega^2+(\pi T)^2]$. At zero temperature and frequency, the conduction electron self-energy is thus entirely real, $\Sigma_c''(0)=0$. So from Eq. (42), $\tan\delta_c=\pi\rho\Sigma_c(0)$, and the on-shell t matrix is given by

$$t(0 + i\epsilon) = \frac{1}{\pi\rho}\frac{\tan\delta_c}{(1 + i\tan\delta_c)} = \frac{\sin\delta_c}{\pi\rho}e^{-i\delta_c}, \tag{44}$$

and the imaginary part of the t matrix is then

$$\mathrm{Im}\ t(0 + i\epsilon) = \frac{\sin^2\delta_c}{\pi\rho}. \tag{45}$$

The existence of this quantity as an adiabatic invariant is well known in the finite-U Anderson model, and its appearance in the limit that $U\to\infty$ is an important test of these methods.

VI. STATIC SUSCEPTIBILITIES

The susceptibilities of the impurity contribution to the conserved charges can now be obtained by differentiating the above expressions with respect to the corresponding fields, $\chi^A = d\langle\hat{Q}^A\rangle_{\mathrm{imp}}/dB^A$, at zero field $B^A=0$. Since the spinon and holon are gapped, their phase shift is pinned to the constant values of 0 and π, and therefore the only contribution to the susceptibilities comes from derivatives of the conduction electron phase shift $\delta=\mathrm{Im}\ \ln[1-G_c^0\Sigma_c]$:

$$\chi_A = \frac{1}{\pi}\sum_{\alpha\nu} q^A_{c\alpha\nu}\frac{d\delta_{\alpha\nu}}{dB^A}\bigg|_{B^A=0}. \tag{46}$$

Explicitly, the derivatives of the phase shift are given by

$$\frac{d\delta}{dB^A} = -\mathrm{Im}\ \mathcal{G}_c\frac{d\Sigma_c}{dB^A} - \mathrm{Im}(\mathcal{G}_c + \mathcal{G}_c^0)\gamma_c\hat{q}_c^A\bigg|_{-i0^+}, \tag{47}$$

and in the wide bandwidth limit wherein γ_c vanishes,

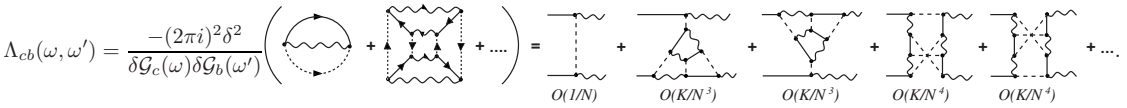
$$\frac{d\delta}{dB^A} = -\mathrm{Im}\ \mathcal{G}_c\frac{d\Sigma_c}{dB^A}\bigg|_{-i0^+}. \tag{48}$$

To calculate the derivatives of the self-energies, it is

useful to introduce the bare and full off-shell interaction vert*ices. The bare interaction vertex Λ is defined as the functional Hessian of the Luttinger-Ward functional,

$$\Lambda_{aa'}(\omega,\omega') = (2\pi i)^2 \eta_a \eta_{a'} \frac{\delta^2 Y[\mathcal{G}]}{\delta \mathcal{G}_a(\omega)\delta \mathcal{G}_{a'}(\omega')} = 2\pi i \eta_{a'} \frac{\delta \Sigma_a(\omega)}{\delta \mathcal{G}_{a'}(\omega')}, \tag{49}$$

where a and a' list the particle species c, b, and χ and their corresponding indices α and/or ν, while $(\eta_b,\eta_c,\eta_\chi)=(1,-1,-1)$ grade the anticommuting character of the Fermi fields. The $2\pi i$ factors come to compensate for the integration measure and replace the $1/T$ factors in the zero temperature limit when the discrete Matsubara frequencies are replaced by the continuous imaginary variable ω. For example, the interaction vertex between an electron and a spinon is



The derivatives of the self-energies with respect to the fields B^A are expressed in terms of the bare interaction vertex by

$$\frac{d\Sigma_a(\omega)}{dB^A} = -\sum_{a'} \int \frac{d\omega'}{2\pi i} \Lambda_{aa'}(\omega,\omega')\eta_{a'}\mathcal{G}^2_{a'}(\omega') \times \left\{ \gamma_{a'}(\omega')\hat{q}^A_{a'} - \frac{d\Sigma_{a'}(\omega')}{dB^A} \right\}. \tag{50}$$

Adopting a heuristic matrix notation, we may write this as

$$\frac{d\Sigma}{dB^A} = -\Lambda \eta \mathcal{G}^2 \left(\gamma q^A - \frac{d\Sigma}{dB^A} \right), \tag{51}$$

which implies that

$$(1-\Lambda\eta\mathcal{G}^2)\frac{d\Sigma}{dB^A} = -\Lambda\eta\mathcal{G}^2(\gamma q^A), \tag{52}$$

or

$$\frac{d\Sigma}{dB^A} = -\Gamma\eta\mathcal{G}^2(\gamma q^A), \tag{53}$$

where

$$\Gamma = (1-\Lambda\eta\mathcal{G}^2)^{-1}\Lambda \tag{54}$$

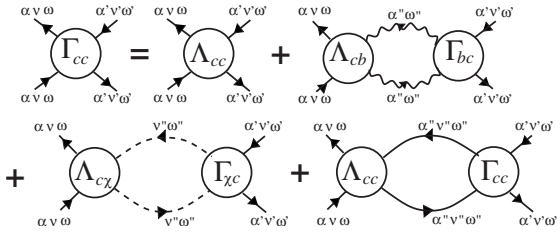
is the solution to the Dyson equation

$$\Gamma = \Lambda + \Lambda\eta\mathcal{G}^2\Gamma.$$

Restoring the indices, we then have

$$\Gamma_{aa'}(\omega,\omega') = \sum_{a''} \int \frac{d\omega''}{2\pi i} \Lambda_{aa''}(\omega,\omega'')\{2\pi i \delta_{a'',a'}\delta(\omega''-\omega') + \eta_{a''}\mathcal{G}^2_{a''}(\omega'')\Gamma_{a''a'}(\omega'',\omega')\}.$$

It is instructive to represent this equation diagrammatically. For example, the vertex with incoming and outgoing electrons is Γ_{cc}, represented as



The self-energy derivatives are then reduced to

$$\frac{d\Sigma_a(\omega)}{dB^A} = -\sum_{a'} \int \frac{d\omega'}{2\pi i} \Gamma_{aa'}(\omega,\omega')\eta_{a'}\mathcal{G}^2_{a'}(\omega')\gamma_{a'}(\omega')q^A_{a'}. \tag{55}$$

Unlike the Fermi liquid in the original works of Luttinger and Ward, the on-shell interaction vertex which represents the Fermi liquid is normalized by the appearance of intermediate spinon and holon states.

Incorporating the expression for the self-energy derivatives within Eqs. (46) and (48), we can write the susceptibilities as

$$\chi^A = \frac{\mathrm{Im}\, G_c}{\pi} \int \frac{d\omega}{2\pi i} \sum_{\alpha\nu a} q^A_{c\alpha\nu}\Gamma_{c\alpha\nu,a}(0,\omega)\mathcal{G}^2_a(\omega)\eta_a q^A_a \gamma_a. \tag{56}$$

Since we work in the wideband limit and γ_c vanishes, the summation over $\gamma_a\hat{q}_a$ gives nonvanishing contribution only for one particle species,

$$\chi^S = \frac{\mathrm{Im}\, G_c}{\pi} K \sum_{\alpha\alpha'} \tilde{\alpha}\tilde{\alpha}' \int \frac{d\omega}{2\pi i} \Gamma_{c\alpha,b\alpha'}(0,\omega)\mathcal{G}^2_b(\omega), \tag{57}$$

$$\chi^F = \frac{\mathrm{Im}\, G_c}{\pi} N \sum_{\nu\nu'} \tilde{\nu}\tilde{\nu}' \int \frac{d\omega}{2\pi i} \Gamma_{c\nu,\chi\nu'}(0,\omega)\mathcal{G}^2_\chi(\omega), \tag{58}$$

$$\chi^C = \frac{\mathrm{Im}\, G_c}{\pi} N \sum_{\nu\nu'} \int \frac{d\omega}{2\pi i} \Gamma_{c\nu,\chi\nu'}(0,\omega) \mathcal{G}_\chi^2(\omega). \tag{59}$$

VII. LOW-TEMPERATURE FREE ENERGY

For the recapitulation of the hard argument of Luttinger-Ward, if we use the zero temperature Green's function $\tilde{\mathcal{G}} \equiv \mathcal{G}|_{T=0}$, the leading change in the free energy at finite temperatures, $\Delta F = F(T) - F(T=0)$, can be computed solely from the finite temperature trace of the logarithm of the *zero* temperature Green's function,

$$\Delta F = [T\, \underline{\mathrm{Str}}\{\ln(-\tilde{\mathcal{G}}^{-1})\}]_{T=0}^{T}. \tag{60}$$

This is a quite nontrivial result, and for clarity, here we repeat the original Luttinger-Ward derivation in the context of the current work. At small finite temperature, we have two things to keep track of in the free energy; the discreteness of the frequency summations and the temperature dependence of the Green's functions. However, since the free energy is stationary at low temperatures with respect to small changes in $\mathcal{G}$, to calculate the low-temperature corrections to the free energy, we may use the zero temperature Green's function $\tilde{\mathcal{G}} \equiv \mathcal{G}|_{T=0}$ in the Luttinger-Ward expression

$$F[T \sim 0] = T\, \underline{\mathrm{Str}}[\ln(-\tilde{\mathcal{G}}^{-1}) + \tilde{\Sigma}\tilde{\mathcal{G}}] + Y[\tilde{\mathcal{G}}, T], \tag{61}$$

where $(\mathcal{G}_0^{-1} - \tilde{\mathcal{G}}^{-1}) = \tilde{\Sigma}$ is the zero temperature self-energy.

Small finite temperatures introduce a discreteness into the frequency summations that produces corrections of order T^2 in the free energy. To keep track of these changes, we must write

$$T \sum_{i\alpha_n} A[i\alpha_n] = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} [1 + \epsilon(\omega)] A(\omega),$$

where

$$\epsilon(\omega) = \left(2\pi i T \sum_n \delta(i\omega + \alpha_n) - 1 \right) \tag{62}$$

keeps track of the discreteness. Luttinger and Ward made the insightful observation that at low temperatures, one can regard $\epsilon(\omega)$ to be infinitesimal, so that inside the Luttinger-Ward functional, the leading-order effect of finite temperature is taken into account by making a variation $\tilde{\mathcal{G}} \to \tilde{\mathcal{G}} + \delta\mathcal{G}$, where

$$\delta\mathcal{G} = + \epsilon\tilde{\mathcal{G}},$$

in other words,

$$Y[\tilde{\mathcal{G}}, T] - \tilde{Y}[\tilde{\mathcal{G}}] = \int \frac{d\omega}{2\pi i} \mathrm{Tr}\left[2\pi i \frac{\delta\tilde{Y}}{\delta\tilde{\mathcal{G}}(\omega)} \epsilon(\omega) \tilde{\mathcal{G}}(\omega) \right], \tag{63}$$

where $\tilde{Y} = Y[\tilde{\mathcal{G}}, 0]$.

It is instructive to briefly digress to see how this replacement works in the leading-order contribution to the Luttinger-Ward functional, given by

$$Y_1[\mathcal{G}, T] = KN \left(\frac{V}{\sqrt{N}} \right)^2 T^2 \sum_{\omega_n, \nu_r} \mathcal{G}_c(\omega_n) \mathcal{G}_\chi(i\nu_r - i\omega_n) \mathcal{G}_b(i\nu_r). \tag{64}$$

So we can write

$$Y_1[\tilde{G}, T] = KV^2 \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi i)^3} 2\pi i \delta(i\omega_1 + i\omega_2 - i\omega_3) \times [1 + \epsilon(\omega_1)][1 + \epsilon(\omega_2)][1 + \epsilon(\omega_3)] \tilde{\mathcal{G}}_c(\omega_1) \tilde{\mathcal{G}}_\chi(\omega_2) \tilde{\mathcal{G}}_b(\omega_3). \tag{65}$$

To leading order in ϵ, we have

$$\begin{aligned} \delta Y_1 &= KV^2 \int \frac{d\omega_1 d\omega_2 d\omega_3}{(2\pi i)^3} 2\pi i \delta(i\omega_1 + i\omega_2 - i\omega_3) \\ &\quad \times [\epsilon(\omega_1) + \epsilon(\omega_2) + \epsilon(\omega_3)] \tilde{\mathcal{G}}_c(\omega_1) \tilde{\mathcal{G}}_\chi(\omega_2) \tilde{\mathcal{G}}_b(\omega_3) \\ &= \int \frac{d\omega}{2\pi i} \left[NK 2\pi i \frac{\delta\tilde{Y}_1}{\delta\tilde{\mathcal{G}}_c(\omega)} \epsilon(\omega) \tilde{\mathcal{G}}_c(\omega) \right. \\ &\quad \left. + K 2\pi i \frac{\delta\tilde{Y}_1}{\delta\tilde{\mathcal{G}}_\chi(\omega)} \epsilon(\omega) \tilde{\mathcal{G}}_\chi(\omega) + N 2\pi i \frac{\delta\tilde{Y}_1}{\delta\tilde{\mathcal{G}}_b(\omega)} \epsilon(\omega) \tilde{\mathcal{G}}_b(\omega) \right] \\ &= \int \frac{d\omega}{2\pi i} \mathrm{Tr}\left[2\pi i \frac{\delta\tilde{Y}_1}{\delta\tilde{\mathcal{G}}(\omega)} \epsilon(\omega) \tilde{\mathcal{G}}(\omega) \right], \end{aligned}$$

where the factors of N, NK, and K arise because the functional derivative refers to the variation of Y_1 with respect to a given spin and/or flavor leg of the propagator. The same series of manipulations can be carried out on diagrams of arbitrary order.

Returning from the digression, let us now use the relation

$$\delta Y = -\, \underline{\mathrm{Str}}[\Sigma\, \delta\mathcal{G}] \overset{T\to 0}{=} -\int \frac{d\omega}{2\pi i} \mathrm{Str}[\Sigma(\omega)\, \delta\mathcal{G}(\omega)] \tag{66}$$

to identify

$$2\pi i \frac{\delta\tilde{Y}}{\delta\tilde{\mathcal{G}}(\omega)} = -\, \eta\tilde{\Sigma}(\omega). \tag{67}$$

With this identification, we can rewrite Eq. (63) as

$$\Delta Y = -\int \frac{d\omega}{2\pi i} \mathrm{Str}[\tilde{\Sigma}(\omega) \epsilon(\omega) \tilde{\mathcal{G}}(\omega)] = -\, T[\underline{\mathrm{Str}}\{\tilde{\Sigma}\tilde{G}\}]_{T=0}^{T}. \tag{68}$$

We see that the finite temperature change in the generating functional correction term exactly cancels the change from the second self-energy term in the free energy functional, so that for small finite temperatures,

$$\begin{aligned} \Delta F &= [T\, \underline{\mathrm{Str}}\{\ln(-\tilde{\mathcal{G}}^{-1}) + \tilde{\Sigma}\tilde{\mathcal{G}}\}]_{T=0}^{T} + \Delta Y \\ &= [T\, \underline{\mathrm{Str}}\{\ln(-\tilde{\mathcal{G}}^{-1})\}]_{T=0}^{T}, \end{aligned} \tag{69}$$

as promised.

Now, in principle, this expression involves finite temperature contributions from all fields, including the fractionalized excitations. If we carry out the Matsubara summations, we see that

$$\Delta F = -\frac{T^2\pi^2}{3}\frac{1}{2\pi}\frac{d}{d\omega}[\mathrm{Str\ Im\ ln}(-\tilde{\mathcal{G}}^{-1})]_{\omega=-i0^+}, \quad (70)$$

and the density of states of each field contributes to the specific heat coefficient $\gamma=-d^2F/dT^2$. It is this observation that primarily motivates our gap hypothesis, for with a gap in the holon-spinon spectrum, the only remaining contribution to the linear specific heat is derived from the conduction electrons,

$$\gamma = \frac{2\pi^2}{3}\frac{d}{d\omega}\frac{1}{2\pi}\left[\sum_{\alpha\nu}\mathrm{Im\ ln}\left(\frac{\tilde{\mathcal{G}}_c^{-1}}{(\tilde{\mathcal{G}}_c^0)^{-1}}\right)\right]_{\omega=-i0^+}. \quad (71)$$

Note that as was previously done for susceptibilities, we subtract the effect of the empty conduction band to get the impurity contribution to the specific heat coefficient. We have removed the tilde sign $\tilde{\mathcal{G}}\to\mathcal{G}$ as from now on, we restrict our derivation to the Green's function at zero temperature.

In the single impurity model, the low-energy thermodynamics can then be entirely expressed in terms of the conduction electron phase shift

$$\delta_c = \mathrm{Im\ ln}[1-\mathcal{G}_c^0\Sigma_c], \quad (72)$$

in terms of which

$$\gamma = \frac{\pi^2}{3}\sum_{\alpha\nu}\frac{1}{\pi}\left.\frac{d\delta_c}{d\omega}\right|_{\omega=-i0^+}. \quad (73)$$

The linear coefficient of the specific heat is then

$$\gamma = -\frac{\pi^2}{3}\sum_{\alpha\nu}\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{G}_c\left.\frac{d\Sigma_c}{d\omega}\right|_{-i0^+}, \quad (74)$$

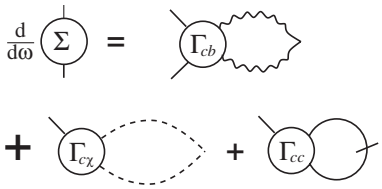
and we see that the renormalization of the density of states is entirely encoded in the frequency dependence of the conduction electron self-energy at the Fermi surface.

The derivative of the self-energies with respect to the frequencies can be calculated in a similar way to the derivatives with respect to the fields B^A. The analogous expression is

$$\frac{d\Sigma_a(\omega)}{d\omega} = -\sum_{a'}\int\frac{d\omega'}{2\pi i}\Gamma_{aa'}(\omega,\omega')\,\eta_{a'}\mathcal{G}_{a'}^2(\omega')\,\gamma_{a'}(\omega') + \sum_{a'}\Gamma_{aa'}(\omega,0)\,\eta_{a'}\left.\frac{\mathrm{Im}\,\mathcal{G}_{a'}}{\pi}\right|_{-i0^+}. \quad (75)$$

The additional second term results from the discontinuous jump across the real axis: $\delta\mathcal{G}=-2i\,\mathrm{Im}\,\mathcal{G}(0-i\eta)$. In the first term, the contribution from the local conduction electrons to the summation vanishes in the band limit. On the other hand, in the second term, only the conduction electrons contribute since the holon and boson are gapped and their spectral function vanishes at the Fermi level. Explicitly,

$$\left.\frac{d\Sigma_c^{\alpha\nu}}{d\omega}\right|_{-i0^+} = -\int\frac{d\omega}{2\pi i}\left\{\sum_{\alpha'}\Gamma_{c\alpha,b\alpha'}(0,\omega)\mathcal{G}_b^2(\omega) - \sum_{\nu'}\Gamma_{c\nu,\chi\nu'}(0,\omega)\mathcal{G}_\chi^2(\omega)\right\} - \left.\frac{\mathrm{Im}\,\mathcal{G}_c}{\pi}\right|_{-i0^+}\sum_{\alpha'\nu'}\Gamma_{c\alpha\nu,c\alpha'\nu'}(0,0), \quad (76)$$



The next step in our derivation is to compute the Ward identities that relate the frequency and field dependences of the electron self-energies to the electron Landau parameters.

The derivation of the Friedel sum rules in Sec. II relied on the Ward identities of Eq. (29), which were derived by shifting the frequencies in diagrams of the Luttinger-Ward functional $Y[G]$ on by a shift $\delta\omega_a^A=\delta\omega q_a^A$ proportional to the conserved charge Q^A carried by the corresponding particle: $G_c(\omega)\to G_c(\omega+\delta\omega_c^A)$. In the Luttinger-Ward diagrams, such shifts correspond to shifts along closed loops. Since the interaction vertex conserves the charges Q^A, it is also possible to think of similar shifts in the self-energy diagrams.[12,13,29,30] The interesting case is when the external legs of the self-energy carry the relevant charge $q_a^A\neq 0$; in this case, in addition to the closed loops, also the external frequency is shifted and the self-energy change can be related to the frequency derivative of the self-energy,

$$\frac{\delta\Sigma_a}{\delta\omega^A} = \frac{1}{q_a^A}\frac{d\Sigma_a}{d\omega}. \quad (77)$$

These identities introduce, in addition to Eq. (75), alternative equations for the frequency derivative of the self-energies,

$$\frac{1}{q_a^A}\frac{d\Sigma_a(\omega)}{d\omega} = -\sum_{a'}\int\frac{d\omega'}{2\pi i}\Gamma_{aa'}(\omega,\omega')\,\eta_{a'}\mathcal{G}_{a'}^2(\omega')\,\gamma_{a'}q_{a'}^A + \sum_{a'}\Gamma_{aa'}(\omega,0)q_{a'}^A\,\eta_{a'}\left.\frac{\mathrm{Im}\,\mathcal{G}_{a'}}{\pi}\right|_{-i0^+}. \quad (78)$$

The conduction electrons carry the three charges $q_c^A\neq 0$, and it is possible to plug these derivatives into the expression for the specific heat coefficient of Eq. (74), which gives

$$\frac{\gamma}{\gamma_0} = \frac{1}{NK}\left\{\int\frac{d\omega}{2\pi i}\sum_{\alpha\nu a}q_{c\alpha\nu}^A\Gamma_{c\alpha\nu,a}(0,\omega)\mathcal{G}_a^2(\omega)\,\eta_a\hat{q}_a^A\gamma_a - \sum_{\alpha\nu a}[\hat{q}_c^A]_{\alpha\nu}\Gamma_{c\alpha\nu,a}(0,0)q_a^A\eta_a\frac{\mathrm{Im}\,\mathcal{G}_a}{\pi}\right\}, \quad (79)$$

with $\gamma_0=(NK\pi^2/3)[(\mathrm{Im}\,G_c)/\pi]$ the specific heat coefficient of the bare conduction band. Rewriting the latter equation,

we can relate the specific heat coefficient to the various susceptibilities,

$$\frac{\gamma}{\gamma_0} = \frac{\chi^A}{\chi_0^A} - \frac{1}{NK}\sum_{\alpha\nu a} \hat{q}^A_{c\alpha\nu}\Gamma_{c\alpha\nu,a}(0,0)q^A_a \eta_a \frac{\mathrm{Im}\,\mathcal{G}_a}{\pi}, \quad (80)$$

where $\chi_0^A = (NK\,\mathrm{Im}\,\mathcal{G}_c)/\pi$ are the bare Pauli susceptibilities. The latter expression can be simplified as a set of Ward identities relating the specific heat to the susceptibilities and the on-shell Fermi liquid interaction vertex,

$$\begin{aligned}\frac{\gamma}{\gamma_0} &= \frac{\chi^S}{\chi_0^S} + \frac{1}{NK}\frac{\mathrm{Im}\,\mathcal{G}_c}{\pi}\sum_{\alpha\nu\alpha'\nu'} \tilde{\alpha}\tilde{\alpha}'\Gamma_{c\alpha\nu,c\alpha'\nu'}(0,0) \\ &= \frac{\chi^F}{\chi_0^F} + \frac{1}{NK}\frac{\mathrm{Im}\,\mathcal{G}_c}{\pi}\sum_{\alpha\alpha'\nu\nu'} \tilde{\nu}\tilde{\nu}'\Gamma_{c\alpha\nu,c\alpha'\nu'}(0,0) \\ &= \frac{\chi^C}{\chi_0^C} + \frac{1}{NK}\frac{\mathrm{Im}\,\mathcal{G}_c}{\pi}\sum_{\alpha\alpha'\nu\nu'} \Gamma_{c\alpha\nu,c\alpha'\nu'}(0,0). \end{aligned} \quad (81)$$

In deriving these identities, we assumed gapped spinon and holon spectra which restrict the summation over a in Eq. (80) to the conduction electrons. As discussed in the following section, it is possible to directly relate the on-shell conduction electron interaction vertex $\Gamma_{c\alpha\nu,c\alpha'\nu'}$ to the Fermi liquid amplitudes $\phi_{\alpha\nu,\alpha'\nu'}$ by comparing the latter Ward identities to the Nozières Fermi liquid expressions. Note that it is possible to derive three additional spectral sum rules for the off-shell interaction vertices by equating Eq. (76) with Eq. (78).

VIII. YAMADA-YOSIDA RELATIONS

As was mentioned in the Introduction, the phase shift in a Nozières-type local Fermi liquid can be written as a function of the energy and the occupation,

$$\delta_{\alpha\nu}[\omega,\{n_{k'\alpha',\nu'}\}] = \delta_0 + \delta'\omega + \sum_{k'\alpha'\nu'} \phi_{\alpha\nu,\alpha'\nu'}\delta n_{k'\alpha'\nu'}, \quad (82)$$

where δ' is the energy derivative of phase shift and the impurity Landau parameter

$$\phi_{\alpha\nu,\alpha'\nu'} = \langle \alpha\nu,\alpha'\nu'|\hat{\phi}|\alpha\nu\alpha'\nu'\rangle \quad (83)$$

is the expectation value of the interaction between quasiparticles. The density of the quasiparticles is $\rho_0 + \delta'/\pi$, where ρ_0 is the bare density of states, which implies that the impurity contribution to the specific heat coefficient is given by $\gamma = \gamma_0\delta'/\pi\rho_0$. For a local Fermi liquid, the impurity Landau parameter can be parametrized in terms of two numbers ϕ_1 and ϕ_2 through

$$\phi_{\alpha\nu,\alpha'\nu'} = \phi_1(1-\delta_{\alpha\alpha'}\delta_{\nu\nu'}) + \phi_2(\delta_{\nu\nu'} - \delta_{\alpha\alpha'}). \quad (84)$$

The response of the conserved charges Q^A to their fields B^A can be represented in terms of the phase shifts through

$$\chi^A = \frac{1}{\pi}\sum_{\alpha\nu} q^A_{c\alpha\nu}\frac{\Delta\delta_{\alpha\nu}}{B^A}. \quad (85)$$

The resulting susceptibilities read

$$\begin{aligned}\frac{\chi^S}{\chi_0^S} - \frac{\gamma}{\gamma_0} &= \sum_{\alpha\alpha'\nu\nu'} \tilde{\alpha}\tilde{\alpha}'\phi_{\alpha\nu,\alpha'\nu'} = -\phi_1 - K\phi_2, \\ \frac{\chi^F}{\chi_0^F} - \frac{\gamma}{\gamma_0} &= \sum_{\alpha\alpha'\nu\nu'} \tilde{\nu}\tilde{\nu}'\phi_{\alpha\nu,\alpha'\nu'} = -\phi_1 + N\phi_2, \\ \frac{\chi^C}{\chi_0^C} - \frac{\gamma}{\gamma_0} &= \sum_{\alpha\alpha'\nu\nu'} \phi_{\alpha\nu,\alpha'\nu'} = (NK-1)\phi_1 + (N-K)\phi_2, \end{aligned} \quad (86)$$

where the Pauli susceptibilities are $\chi_0^A = NK\rho_0/\pi$. By comparing the latter equations to the set of Eqs. (81), it is easy to read off the relations between the Landau parameters and the on-shell interaction vertex,

$$\phi_{\alpha\nu,\alpha'\nu'} = -\frac{\mathrm{Im}\,\mathcal{G}_c}{\pi NK}\Gamma_{c\alpha\nu,c\alpha'\nu'}(0,0). \quad (87)$$

From Eqs. (86), it is possible to express the Fermi liquid parameters in terms of the reduced susceptibilities

$$\begin{aligned}\phi_1 &= \frac{\chi^C/\chi_0^C}{NK} - \frac{1}{N+K}\left(\frac{\chi^F/\chi_0^F}{K} + \frac{\chi^S/\chi_0^S}{N}\right), \\ \phi_2 &= \frac{\chi^F/\chi_0^F - \chi^S/\chi_0^S}{N+K}, \end{aligned} \quad (88)$$

which produce the following relation between the susceptibilities and specific heat coefficient:

$$NK\frac{\gamma}{\gamma^0} = K\frac{N^2-1}{N+K}\frac{\chi_s}{\chi_s^0} + N\frac{K^2-1}{K+N}\frac{\chi_f}{\chi_f^0} + \frac{\chi_c}{\chi_c^0}, \quad (89)$$

generalizing the Yamada-Yosida relation[12,13,31] to our infinite-U model with K channels. Note that also the on-shell interaction vertex can be parametrized in a similar way,

$$\Gamma_{c\alpha\nu,\alpha'\nu'}(0,0) = \Gamma_1(1-\delta_{\alpha\alpha'}\delta_{\nu\nu'}) + \Gamma_2(\delta_{\nu\nu'} - \delta_{\alpha\alpha'}), \quad (90)$$

with $\Gamma_{1,2} = -\pi NK\phi_{1,2}/\mathrm{Im}\,\mathcal{G}_c$. These two Fermi liquid parameters control the low-temperature thermodynamics and are renormalized due to intermediate spinon and holon states.

An interesting observation is that the prefactors of the susceptibility terms in Eq. (89) correspond to the three central charges of the $\widehat{SU(N)_K}$, $\widehat{SU(K)_N}$, and $\widehat{U(1)}$ Kac-Moody algebras, respectively.[32] In terms of the relevant Kac-Moody currents, the free fermion problem is decomposed to the free current Hamiltonians $\mathcal{H}_S^0$, $\mathcal{H}_F^0$, and $\mathcal{H}_C^0$. The fraction of the degrees freedom in section A, γ_A^0/γ^0, is given by the ratio of the central charge of section A to the total central charge NK.[33] Equation (89) emerges under the assumption that for the fully screened impurity problem, the fixed point Hamiltonians have a similar form to the bare Hamiltonians with renormalized Fermi velocities and no residual degrees of freedom,[34]

$$\mathcal{H}_A^{FP} = \left(1 + \frac{3\pi\lambda_A}{\ell}\right)\mathcal{H}_A^0, \tag{91}$$

where λ_A represents the impurity effective couplings and ℓ is the size of the system. Under this assumption, $\chi_A/\chi_A^0 = \gamma_A/\gamma_A^0 = 3\pi\lambda_A/\ell$, where γ_A is the impurity contribution to the specific heat coefficient in section A. Summing up the impurity contribution to the specific heat coefficient $\gamma = \Sigma_A \gamma_A$, we recover Eq. (89),

$$\frac{\gamma}{\gamma^0} = \sum_A \frac{\gamma_A^0}{\gamma^0}\frac{\chi_A}{\chi_A^0} = \frac{1}{NK}\left[K\frac{N^2-1}{N+K}\frac{\chi_S}{\chi_S^0} + N\frac{K^2-1}{N+K}\frac{\chi_F}{\chi_F^0} + \frac{\chi_C}{\chi_C^0}\right].$$

IX. SHIBA-KORRINGA RELATIONS

The Korringa-Shiba relations relate the impurity's nuclear-spin–lattice-relaxation time T_1 with the Knight shift K at small magnetic fields. Alternatively, it also relates the impurity static susceptibility with the low frequency impurity spin power spectrum. In the context of our paper, this relation is important as a manifestation of the soft mode formation in the susceptibility power spectrum even though the spinon spectrum is gapped. This relation was first deduced from experimental data by Korringa, and only later it was proven[14] by Shiba for the finite-U Anderson model summing up the perturbative expansion in U to arbitrary order. Below, we show how Shiba's arguments can be extended to the infinite-U Anderson model and to the Kondo model by summing up the perturbative expansion in $1/N$.

The expression deduced by Korringa reads

$$K^2 T_1 T = C, \tag{92}$$

where C is a constant. The Knight shift K describes the mean-field response observed by the nuclear moment, which, for low Larmor frequency, is proportional to the static impurity susceptibility χ_S. On the other hand, the relaxation rate T_1^{-1} is proportional to the low frequency power spectrum of the impurity susceptibility. Explicitly, in terms of impurity susceptibilities, K and T_1^{-1} are given by

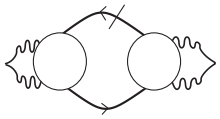
$$K = A_{\mathrm{HF}}\chi_S^{zz\prime}(\Omega),$$

$$T_1^{-1} = -T(g_N\mu_N)^2 A_{\mathrm{HF}}^2 \frac{\chi_S^{+-\prime\prime}(\Omega - i0^+)}{\Omega},$$

where A_{HF} is the hyperfine coupling and g_N and μ_N are the nuclear spin g factor and Bohr magneton, respectively, and Ω is the Larmor frequency. The latter expression with Korringa's empirical rule immediately gives the Shiba relation between the static susceptibility and the spin power spectrum at low frequencies where the susceptibilities become isotropic,

$$\left.\frac{\chi_S''(\Omega - i0^+)}{\Omega}\right|_{\Omega\to 0} = -\frac{\chi_S^2}{C(g_N\mu_N)^2}. \tag{93}$$

Extending Shiba's arguments to the infinite-U Anderson model, we see that by taking the derivative of the dynamic susceptibility $d\chi''/d\Omega$ and substituting $\Omega \to 0$, the only contribution to the low frequency power spectrum comes from the following diagram:



The line that intersects the upper conduction electron line denotes the nonanalytic part of the derivative of the Green's function, which arises from the jump across the real axis $d\mathcal{G}_c/d\omega|_{\mathrm{na}} = -2i\mathcal{G}_c''(-i0^+)\delta(i\omega)$. Since the boson and holons are gapped, there are no similar contributions from derivatives of the boson and holon Green's functions. The analytic part of the Green's function derivatives produces no imaginary contribution and should not be considered here. The other diagrams arise from derivatives of bare interaction vertex Λ. These diagrams have more than one electron-hole pair; hence, their contribution to the imaginary part is superlinear in Ω and the derivatives vanish as $\Omega \to 0$.[14]

A straightforward calculation of the diagram gives

$$\left.\frac{\chi_S''(\Omega - i0^+)}{\Omega}\right|_0 = \frac{K(\mathcal{G}_c''(-i0^+))^2}{\pi}\sum_{\alpha\alpha'\alpha''}\tilde{\alpha}\tilde{\alpha}' \times \int_{-i\infty}^{i\infty}\frac{d\omega d\omega'}{(2\pi i)^2}\mathcal{G}_b^2(\omega)\Gamma_{b\alpha c\alpha''}(\omega,0) \times \Gamma_{c\alpha'' b\alpha'}(0,\omega')\mathcal{G}_b^2(\omega'). \tag{94}$$

In order to obtain the relation between the power spectrum of Eq. (94) and the static susceptibility of Eq. (57), we need to resort to the symmetry properties of the vertex function Γ. In addition to the symmetry to the exchange of variables which is inherited from the bare interaction vertex Λ, there is an additional symmetry of the vertex function as a two-particle vertex. In general, a two-particle vertex $\Gamma_{\alpha\bar{\alpha},\alpha'\bar{\alpha}}$ is a function of the incoming spins α and α' and of the outgoing spins $\bar{\alpha}$ and $\bar{\alpha}'$. In our model, there is no annihilation or creation of spin; therefore, each incoming spin must identify with an outgoing spin,

$$\Gamma_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} = \Gamma^{(1)}\delta_{\alpha\bar{\alpha}}\delta_{\alpha'\bar{\alpha}'} + \Gamma^{(2)}\delta_{\alpha\bar{\alpha}'}\delta_{\alpha'\bar{\alpha}'}. \tag{95}$$

In the full interaction vertex function, the incoming and outgoing spins of each particle are already identified as $\alpha = \bar{\alpha}$ and $\alpha' = \bar{\alpha}$; hence, we can write

$$\Gamma_{c\alpha b\alpha'}(\omega,\omega') = \Gamma_{cb}^{(1)}(\omega,\omega') + \Gamma_{cb}^{(2)}(\omega,\omega')\delta_{\alpha\alpha'}. \tag{96}$$

The static susceptibility of Eq. (57) gets a contribution only from the exchange part of the vertex $\Gamma^{(2)}$ as the direct part $\Gamma^{(1)}$ drops out of the summation $\Sigma\tilde{\alpha}\tilde{\alpha}'\Gamma$. The resulting simplified expression is

$$\chi_S = NK\frac{\mathcal{G}_c''(-i0^+)}{\pi}\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\mathcal{G}_b^2(\omega)\Gamma_{cb}^{(2)}(0,\omega). \tag{97}$$

A similar simplification can be presented for the power spectrum of Eq. (94) by noticing that

$$\sum_{\alpha} \tilde{\alpha}\Gamma_{b\alpha c\alpha''} = \tilde{\alpha}''\Gamma^{(2)}_{bc}, \tag{98}$$

$$\sum_{\alpha'} \tilde{\alpha}'\Gamma_{c\alpha'' b\alpha'} = \tilde{\alpha}''\Gamma^{(2)}_{cb}. \tag{99}$$

Taking further advantage of the symmetry to variables exchange, we can rewrite the power spectrum as

$$\left.\frac{\chi''_S(\Omega - i0^+)}{\Omega}\right|_0 = \frac{NK(\mathcal{G}''_c)^2}{\pi}\left(\int \frac{d\omega}{2\pi i}\mathcal{G}^2_b(\omega)\Gamma^{(2)}_{cb}(0,\omega)\right)^2.$$

The right-hand side of the latter equation can be expressed in terms of the static susceptibility of Eq. (97), which gives the Shiba relation

$$\left.\frac{\chi''_S(\Omega - i0^+)}{\Omega}\right|_0 = \frac{\pi}{NK}\chi^2_S. \tag{100}$$

According to this derivation, the Korringa constant C of Eq. (92) is $NK/[\pi(g_N\mu_N)^2]$.

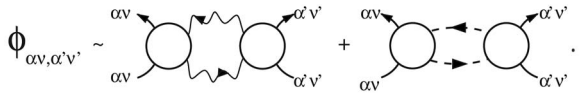
This result implies that the dynamical susceptibility is dominated at low frequencies by intermediate electron-hole pair states. These intermediate states of the Fermi liquid dress the dynamical susceptibility to produce a nonvanishing power spectrum at low frequencies. While the spin-charge decoupled excitations remain gapped, there are soft magnetic modes that are carried by intermediate quasiparticles.

X. DISCUSSION

This paper was motivated by an interest in the way a Landau Fermi liquid can emerge within the constrained Hilbert space of highly correlated electron systems. In such highly constrained systems, we can no longer appeal to adiabaticity or the use of infinite order expansions in the strength of the interaction. By combining the Luttinger-Ward approach with a new class of large N expansion for the infinite-U impurity Anderson model, we have obtained a perspective on the Fermi liquid that may have implications for our understanding of the corresponding Anderson and Kondo lattice models.

In the single impurity model, we find that the Fermi liquid contains low lying electronic quasiparticles that break up into fractionalized excitations above a certain gap energy Δ_g. The assumption that this gap holds to all orders in our analysis leads to a full set of conserving relationships between dynamic and thermodynamic variables, but it also has qualitative implications for our understanding of the Landau Fermi liquid.

Indeed, our results indicate a rather intimate relation between the opening of a gap in the spinon and holon spectra and the formation of a heavy Fermi liquid. In particular, the Landau scattering amplitudes that appear in our treatment contain, as intermediate states, the virtual excitation of spinon or holon pairs, for example,



These amplitudes are proportional to the inverse of the gap and are finite as long as the gap is present. Through this description, we showed how these virtual excitations give rise to the renormalization of the electron-electron interaction and a whole host of conserving relationships known to be valid for the Fermi liquid.

Is this really a Fermi liquid? Landau's notion of a Fermi liquid certainly has no place for fractional quasiparticle excitations, even if those particles are gapped. On the other hand, the fractional particles that appear in this theory involve fields with matrix elements that link states of *different* Q: states that correspond to models with different impurity spins $S=Q/2$. The excitation "gap" in the spinon spectrum involves excitations from the ground state of the fully screened model with spin $S_0=K/2$ to excited states of the underscreened model, with $S=S_0+\frac{1}{2}$,

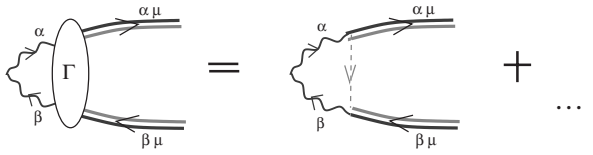
$$G_b(t) \sim \sum_{\alpha}\left\langle S_0 + \frac{1}{2}, \alpha | b^{\dagger}_{\alpha} | S_0 \right\rangle e^{-i(E_{\alpha}[S_0+1/2]-E_g[S_0])t}$$
$$\sim e^{-i(E_g[S_0+1/2]-E_g[S_0])t}, \tag{101}$$

where $E_g[S_0]$ and $E_g\left[S_0+\frac{1}{2}\right]$ are the ground-state energies of the fully screened and underscreened models, respectively. Thus, the appearance of the gap in the spinon-holon spectrum is purely a consequence of the stability of the fully screened ground state.

Physical quantities, such as the spin or charge correlation functions, or the f-spectral function involve combinations of spinons or holons which do not change Q and do not display this gap. For example, the spin raising operator creates a spinon-antispinon pair. Such excitations are "gauge neutral" and are contained in the physical Hilbert space of definite Q. These pair excitations couple to the conduction sea, converting the sharp gap of the infinite N limit into a pseudogap at finite N. The associated time scale in this gap, $t_c \sim \hbar/\Delta_g$ gap, can be interpreted as a confinement time scale, for after this period the spinon-antispinon pair combines with holons from the sea to produce an electron-hole pair,

$$s_{\alpha} + \bar{s}_{\beta} \xrightarrow{t \gtrsim \hbar/\Delta_g} e_{\alpha\mu} + \bar{e}_{\beta\mu}. \tag{102}$$

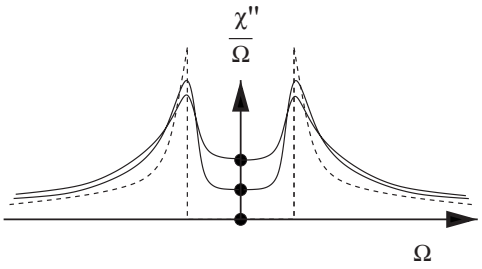
Diagrammatically,



. (103)

(This process can be loosely compared to "quark jets" in a collider, where quark-antiquark pairs rapidly combine with virtual quarks from the vacuum to form mesons or baryons.)

The processes that confine the spinons are precisely those that gave rise to the gapless spin spectral function. From our derivation of the Shiba relationship, we know that the decay of spinon pairs into the electron-hole continuum is constrained to give rise to a spectral function with a pseudogap whose zero-frequency intercept is determined by the relation $\chi''(\omega)/\omega|_{\omega=0} = \chi^2\pi/(NK)$:



where the dots will represent the dc limit $\chi''(\omega)/\omega \rightarrow \chi^2\pi/(NK)$ predicted by the Shiba and the dashed line represents the gapped, $N\rightarrow\infty$ limit. A detailed calculation of this conserving spectrum within the leading Kadanoff-Baym approximation[11] would be an interesting exercise for future work. The Fourier transform of this power spectrum leads to the correct $1/t^2$ decay of the spin correlation function $\langle S(t)S(0)\rangle$. Thus, the confinement of the spinons through their coupling to the electron continuum is an integral part of the emergence of Fermi liquid behavior in the single impurity model. The pseudogap in this function is quite unremarkable from the Fermi liquid perspective, where the offset peaks in the dynamical susceptibility are a well-known signature of the asymmetric Kondo resonance.

In this way, the confinement of spinons and holons to form a Landau Fermi liquid appears very natural in the Schwinger boson approach to the single impurity, and it can be seen as a direct consequence of the way the gapped spin fluid couples to the conduction fluid. This is a very different kind of confinement to that considered in the context of insulating spin liquids, where there is a broken gauge symmetry and topological tunneling effects of the gauge field are required to understand the confinement mechanism.[5,6,35]

Lattice: Heavy fermions

There are some interesting implications of this kind of picture for the more general lattice models. Consider, for example, the (K-channel) infinite-U Anderson-Heisenberg lattice model

$$\mathcal{H}_{lattice} = \sum_{\vec{k}\nu\alpha} \epsilon_{\vec{k}} c^\dagger_{\vec{k}\nu\alpha} c_{\vec{k}\nu\alpha} + \sum H_I(j) + \sum_{\langle i,j\rangle} \mathcal{H}_m(i,j).$$

Here,

$$H_I(j) = \frac{V}{\sqrt{N}} \sum_{\nu\alpha} [\psi^\dagger_{j\nu\alpha} \chi^\dagger_{j\nu} b_{j\alpha} + \text{H.c.}] + \epsilon_0 \sum_\alpha b^\dagger_{j\alpha} b_{j\alpha}$$

describes the atomic orbitals at the different sites and their hybridization with the conduction band, where $\psi^\dagger_{j\nu\alpha} = \Sigma_{\vec{k}} c^\dagger_{\vec{k}\nu\alpha} e^{-i\vec{k}\cdot\vec{R}_j}$ creates a conduction electron at site j. The final terms $\mathcal{H}_m(i,j)$ describe the antiferromagnetic spin interaction between nearest neighbors, where

$$\mathcal{H}_m(i,j) = -\frac{J_H}{N} B^\dagger_{ij} B_{ij}$$

and $B^\dagger_{ij} = \Sigma_\alpha \text{sgn}(\alpha) b^\dagger_{i\alpha} b^\dagger_{j-\alpha}$ creates a singlet pair of bosons along the bond (i,j). In the lattice model, the conserved charges $Q_j = n_b(j) + n_\chi(j)$ replace the no-double occupancy constraint at each site, and as before, the fully screened case corresponds to $Q_j = K$.

The manipulations that we have carried out in this paper can be extended to the model, but certain caveats apply. One of the key differences in the lattice model is that in the large N limit, the antiferromagnetic interactions induce pair condensation $[-(J_H/N)\langle B_{ij}\rangle = \Delta_{ij}]$ of the Schwinger bosons, which mark the onset of short-range magnetic correlations.[36] The important effect of the antiferromagnetism at a mean-field level is described by

$$\mathcal{H}_m(i,j) \rightarrow \left\{ \Delta_{ij} B^\dagger_{ij} + \bar{\Delta}_{ij} B_{ij} + N\frac{\bar{\Delta}_{ij}\Delta_{ij}}{J} \right\}. \qquad (104)$$

The appearance of a spinon-pair condensate is restricted to regions of large $J_H \gtrsim T_K$. Once pair condensation occurs, the local gauge symmetry associated with conservation of the $Q_i = K$ is broken, replaced by a global U(1) symmetry (where the difference of the Q_i on even and odd sublattices is conserved). Now, individual spinon and holon excitations start to propagate between sites. The phase of the bond variables becomes elevated into a compact U(1) gauge theory.[35]

We now discuss the qualitative phase diagram of this model using some of the insights gained from our impurity treatment. A rigorous application of our methods to the lattice requires that we include the U(1) gauge fields into the Luttinger-Ward functional, a task that is beyond the scope of the current discussion. However, we expect that many qualitative aspects of the current analysis will survive.

There are two variables that tune quantum fluctuations in the lattice model (Fig. 3): "spin fluctuation parameter" $1/\tilde{S} = \frac{N}{2S}$ and "Doniach parameter" $t = T_K/J_H$, where T_K is the impurity Kondo temperature.

The first parameter tunes the strength of the quantum spin fluctuations and is loosely equivalent to the effect of lattice frustration. By tuning the value of $1/\tilde{S}$, we can tune from classical antiferromagnetism at small $1/\tilde{S}$, where the Schwinger boson individually condenses to form an antiferromagnet, to a quantum antiferromagnetism at large $1/\tilde{S}$, where a spin liquid with a gap develops.

The second parameter tunes the strength of the Kondo effect. When t is small, boson pair condensation takes place and the spin physics is that of an insulating antiferromagnet or spin liquid with a decoupled background of conduction electrons. At large $t > t_2 \sim 1$, no pair condensation occurs, preserving the locality of the spinon and holon propagators. The direct extension of our methods to this case predicts a local heavy electron state which develops a large Fermi-surface volume and enhanced momentum independent scattering amplitudes. In the single impurity model, the charge of the localized state is given by the difference of the electron and the holon phase shifts [Eq. (34)],

$$K - n_\chi = NK\frac{\delta_c}{\pi} - K\frac{\delta_\chi}{\pi}.$$

The corresponding Luttinger theorem for the lattice[38] tells us that the charge density is given by the difference of the conduction electron and holon Fermi-surface volumes,

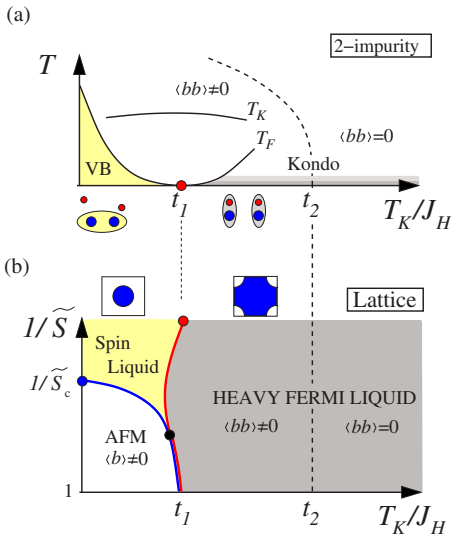


FIG. 3. (Color online) The large N phase diagram of two impurity and lattice models as a function of the ratio $t=T_K/J_H$. Boson pair condensation $\langle bb\rangle \neq 0$ develops for $t<t_2$ inducing short-range magnetic correlations that compete with the "Kondo" screening. (a) Computed large N phase diagram of the two impurity Kondo problem (Ref. 10) displaying the relative temperature scales. $T=T_K$ and $T=T_F$ denote the lower and upper scales associated with the two stage screening process (Ref. 10). The Fermi temperature and spinon gap close at the Varma-Jones quantum critical point at $t=t_1$, which separates the valence bond state from the two impurity Kondo state. (b) Speculative sketch of the zero temperature phase diagram for the Kondo lattice, based on the discussion in the text, showing the dependence on the spin fluctuation parameter $1/\tilde{S}=N/(2S)$. At large $1/\tilde{S}$ and small t, a spin liquid forms. At small $1/\tilde{S}$ and small $t=T_K/J_H$, the Schwinger boson pair condenses to produce an antiferromagnet. The spin-liquid/AFM and spin-liquid/HF (Refs. 6 and 37) phase transition are expected to merge into a single quantum critical point that separates the heavy electron state from the antiferromagnet. The spinon and holon gaps close at each of these quantum phase transitions.

$$n_e = NK\frac{v_{FS}}{(2\pi)^D} - K\frac{v_\chi}{(2\pi)^D}.$$

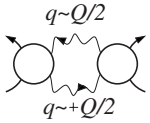
This expression holds some interest for us, because in the spin liquid phases of the model, where the Kondo effect is inactive, we expect the Fermi-surface volume of the holons to be zero, giving a small Fermi surface for the electrons. By contrast, at large $t=T_K/J_H$, where the Kondo effect occurs, the Fermi-surface volume of the gapped holons is now at a maximum $v_\chi/(2\pi)^D=1$, causing the electron Fermi surface to expand by an amount $\Delta v_{FS}=1/N$, forming a large Fermi surface. As long as the holons preserve their gap, the volume of these Fermi surfaces will remain fixed.

The effect of tuning the ratio $t=T_K/J_H$ has already been considered in the context of a two impurity Kondo model[10] [Fig. 3(a)]. Figure 3(b) shows the schematic behavior expected in the large N lattice model. Here, depending on the size of $1/\tilde{S}$, either a spin liquid (large $1/\tilde{S}$) or an antiferromagnet (small $1/\tilde{S}$) develops at small Kondo coupling $t=T_K/J_H$. At large t, however, the heavy Fermi liquid with a gap to spinons and holons will remain stable, with Fermi liquid scattering amplitudes that are determined by the exchange of virtual spinons and holons. Provided the phase transition between these two limits is of second order, then the spinon-holon gap must close at the transition line, with the concomitant divergence of the Landau scattering amplitudes.

The closing of the spinon gap may have important implications for the evolution of the Fermi liquid parameters near the antiferromagnetic instability. In the Schwinger boson scheme, at the antiferromagnetic instability, the spinons condense at one-half the magnetic wave vector, becoming gapless at two points in momentum space $\vec{q}=\pm\vec{Q}_0/2$. This will give rise to strong electron-electron scattering at

$$\vec{q} = \vec{Q}_0/2 \pm \vec{Q}_0/2 = \begin{cases} 0 \\ \vec{Q}_0, \end{cases} \tag{105}$$

as illustrated in



The appearance of strong scattering at a staggered wave vector is well known in a spin-density wave scenario. However, the additional appearance of strong scattering at $\vec{q}=0$ is a new feature. In other words, a subleading effect of the antiferromagnetic instability driven by spinon condensation is the development of strong forward scattering reminiscent of an almost *ferromagnetic* metal. Features of this sort have been observed near the field-induced quantum critical point in $YbRb_2Si_2$.[39]

These qualitative features of the large N limit of our model are a topic of current active examination, and we hope to report on them in greater detail in forthcoming work.

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