Holelike analytic mean-field solutions in the fractional quantum Hall regime

Alejandro Cabo,[1,2] Francisco Claro,[2] Alejandro Pérez,[2] and Jerónimo Maze[1,2,3]
[1]*Grupo de Física Teórica, Instituto de Cibernética, Matemática y Física, Calle E No. 309, Vedado, La Habana, Cuba*
[2]*Facultad de Física, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago 6904411, Chile*
[3]*Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA*
(Received 25 August 2006; revised manuscript received 20 April 2007; published 6 August 2007)

We show that besides the Wigner crystal, the lowest Landau level supports a state with the same unit cell but a qualitatively different charge density distribution. Instead of periodic peaks, the new state forms percolating ridges that may favor an energy decrease through correlated ring exchange contributions. It is found after developing a general framework for the determination of the mean-field states at arbitrary filling factor p/q, q odd. For fillings $1/q$, the eigenstate is given in complete analytical form. In the case for which the unit cell contains one half electron, a crossover is found close to filling $1/7$ between this state and the periodic peak solid, suggesting that it may compete with the Wigner solid after correlations are included.

DOI: 10.1103/PhysRevB.76.075308 PACS number(s): 73.43.Nq, 73.43.Cd, 73.22.Gk

I. INTRODUCTION

The quantum Hall effect has been the source of continued interest for over a quarter century.[1] As the samples have become cleaner and the temperature made lower, unexpected structures have been seen, suggesting a richer physics than originally thought. Although the system was first believed to simply transit from a liquid state to a Wigner crystal (WC) as the electron density is decreased below about filling fraction $1/6.5$,[2–6] magneto-optical measurements soon displayed quantum Hall features down to filling $1/9$.[7] More recently, observation has shown that new structures that may signal transitions between states of different symmetries develop near major fractions.[8–11] Numerical calculations done in small samples also find a variety of states of nonuniform charge density that are found to be the lowest energy states in certain filling fraction ranges, including striped phases, bubble crystals, and others.[12–15]

One of the most studied is the two-dimensional WC. In it, electrons form a hexagonal lattice with a Gaussian-like periodic density amounting to one whole electron per unit cell. As the magnetic field is increased, such peaks become sharper and the electrons approach a classical ensemble of well separated charged particles forming a crystal.[16] Although the exchange energy makes this state a stable mean-field solution in a high magnetic field, early results showed it to be incapable of explaining the odd-denominator rule observed in the lowest Landau level in transport experiments, e.g., that only filling fractions of odd denominator exhibit the quantum Hall effect.[17]

It was later discovered that another mean-field solution, a crystalline charge density wave with only half an electron per unit cell, predicted the Fermi energy to be in a gap at all odd-denominator filling fractions and at the center of a band if the denominator is even, thus providing an odd-denominator rule consistent with experiment.[18,19] In spite of such remarkable property, this state was found to have higher energy than the Wigner crystal, a fact that placed a question mark in its actual relevance to the ground state problem. As the electron number per unit cell γ is varied at filling $1/3$, for example, the energy goes through pronounced cusplike minima at $\gamma=1$ and $1/2$, but the minimum in the former is deeper than in the latter.[20] However, the $\gamma=1/2$ state has a charge distribution qualitatively different from that of the WC, being made of ridges that percolate the entire sample even at low fillings. This fact has prompted the proposal that correlations could affect this state more than it does the WC possibly changing the energy ordering,[19] owing, in particular, to the high degree of wave function overlap exhibited by such mean-field solution.[21,22] Because a class of crystalline states compatible with the quantum Hall effect has been called Hall crystals in the past, we adopt here this same nomenclature for the particular state with half electron per unit cell described above.[23]

In this work, we develop a formalism that allows for a closer study of the Hall crystal state. We derive analytic expressions for the Hartree-Fock problem in the lowest Landau level (LLL) at arbitrary filling fraction $\nu=p/q$, q odd, generalizing work reported in the past for filling $1/3$.[24–26] Regardless of how many electrons are trapped in each cell, we find that two crystalline solutions are always possible, one whose charge density exhibits a single peak at a point of high symmetry in the unit cell, and another whose charge density forms hexagonal ridges that become sharper as the electron density is lowered, increasingly resembling a honeycomb. The first we call electronlike, while the second shall be referred to as holelike in what follows.[20] The Hall crystal belongs to this latter class in the case $\gamma=1/2$. The percolating charge density that characterizes it makes strong correlated ring exchange effects more likely.[27,28] We find this mean-field solution to have lower energy than the electronlike state at this value of γ, except below filling $1/7$ where the latter becomes lowest in energy.

In Sec. II, a formula for the single-particle Fock operator is presented for the case of a hexagonal lattice and arbitrary electron number γ. In Sec. III, a basis is constructed in terms of which the Fock operator matrix representation reduces to q-dimensional blocks. Section IV discusses the holelike state, characterized by a region of very low charge density around a high symmetry point in the unit cell. The depression becomes a zero of order $2\gamma(q-1)$ if the filling is of the form $1/q$, a property that is used to completely determine the

analytic form of the Hartree-Fock state. Finally, Sec. V presents our conclusions.

II. FOCK OPERATOR

We consider N_e electrons on a plane, in a strong perpendicular magnetic field. For such a system, it was proven long ago that at filling less than 1, the only consistent mean-field solutions have space fluctuations.[19] We shall here assume the charge density to form a periodic lattice with a unit cell containing γ electrons, this number being fractional or integer. Denoting by ϕ the flux through such cell in units of the flux quantum $\phi_0=hc/e$, one can readily verify the simple relation

$$\phi\nu = \gamma. \tag{1}$$

The Fock operator in the LLL may then be written in the form[26]

$$H_{HF} = \sum_{Q} v(\mathbf{Q})\exp\left(-\frac{r_o^2\mathbf{Q}^2}{4}\right)T_{r_o^2\mathbf{n}\times\mathbf{Q}}, \tag{2}$$

where

$$v(\mathbf{Q}) = 2\pi r_o^2\rho(\mathbf{Q})\exp\left(\frac{r_o^2\mathbf{Q}^2}{4}\right)\left[\frac{1-\delta_{\mathbf{Q},\mathbf{0}}}{r_o|\mathbf{Q}|}\exp\left(-\frac{r_o^2\mathbf{Q}^2}{4}\right) - \sqrt{\frac{\pi}{2}}I_o\left(\frac{r_o^2\mathbf{Q}^2}{4}\right)\right]\frac{e^2}{\varepsilon_o r_o}. \tag{3}$$

Here, $r_0=(\hbar c/eB)^{1/2}$ is the magnetic length, $I_0(u)$ a modified Bessel function, and ε_0 the background dielectric constant. The operator $T_{\mathbf{a}}$ displaces the function it acts upon in $-\mathbf{a}$ adding a magnetic phase factor, as defined in the Appendix. The Fourier components of the periodic charge density $\rho(\mathbf{x})$ are defined as usual as

$$\rho(\mathbf{Q}) = \frac{1}{A_{cell}}\int \mathbf{dx}\rho(\mathbf{x})\exp(i\mathbf{Q}\cdot\mathbf{x}), \tag{4}$$

where A_{cell} is the unit cell area

$$A_{cell} = \mathbf{n}\cdot\mathbf{a}_1\times\mathbf{a}_2 = 2\pi r_o^2\phi, \tag{5}$$

$\mathbf{n}$ being a unit vector normal to the plane containing the electrons. Assuming triangular symmetry, the lattice formed by the electrons is then invariant under translations in the set of vectors

$$\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2, \quad n_1,n_2 = 0,\pm 1,\pm 2,\ldots, \tag{6}$$

$$\mathbf{a}_1 = a(1,0), \tag{7}$$

$$\mathbf{a}_2 = a\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right), \quad a = \sqrt{\frac{4\pi\phi}{\sqrt{3}}}r_o. \tag{8}$$

The vectors $\mathbf{Q}$ in Eq. (4) span all points in the reciprocal lattice and are given by

$$\mathbf{Q} = Q_1\mathbf{s}_1 + Q_2\mathbf{s}_2, \tag{9}$$

$$Q_1,Q_2 = 0,\pm 1,\pm 2,\ldots,$$

$$\mathbf{s}_1 = -\frac{1}{\phi r_o^2}\mathbf{n}\times\mathbf{a}_2,$$

$$\mathbf{s}_2 = \frac{1}{\phi r_o^2}\mathbf{n}\times\mathbf{a}_1,$$

$$\mathbf{s}_i\cdot\mathbf{a}_j = 2\pi\delta_{ij}.$$

The Fourier components of the density obey the sum rule[19]

$$\sum_{Q}{}' |2\pi r_o^2\rho(\mathbf{Q})|^2\exp\left(\frac{r_o^2\mathbf{Q}^2}{2}\right) = \nu(1-\nu), \tag{10}$$

where the term $\mathbf{Q}=0$ is omitted from the sum. This relation states that within the mean-field approximation, the liquid state of uniform density, $\rho(\mathbf{Q})=0$ all finite $\mathbf{Q}$, is only possible at filling 1 or 0 in the LLL. At fractional fillings, the right hand side is finite and so must be at least one wave-vector Fourier component of the charge density associated with some $\mathbf{Q}\neq 0$.

III. BLOCK DIAGONALIZATION OF THE FOCK OPERATOR

The mean-field Hamiltonian (2) describes an electron in a periodic potential and a perpendicular magnetic field, a case for which many results are known.[29] An important property is that the single-particle spectrum in the LLL is arranged in nonoverlapping bands, each with the same number of states.[30] The number of bands equals the numerator of the flux per cell, the latter assumed a rational. For filling $\nu=p/q$, with p,q prime to each other, Eq. (1) yields a flux $\phi=\gamma q/p$ per plaquette. Assuming γ to be a rational, this flux is then a rational number as well.

We consider in what follows two simple cases that illustrate how different values of the flux ϕ are to be treated. One is when this number is an integer, and another when it is half an integer. For simplicity, we set $\gamma=1,1/2$ and $\nu=1/q$, so that $\phi=q,q/2$, respectively. Our results cover the more general case $\gamma=p,p/2$ and $\nu=p/q$, giving rise to the same values of the flux we include in the following discussion. Other cases may be treated using similar methods to the ones described below.

A. Integer flux quanta per unit cell

Now, $\phi=q$, $\nu=1/q$, and $\gamma=1$. This case is important since it corresponds to the standard WC, in which each unit cell captures one whole electron charge. The flux traversing a plaquette is q so that the single electron spectrum will have q bands, one of which is completely filled and the others empty. Since these bands do not overlap,[30] the WC state has a gap for all values of q, whether even or odd. Because an essential feature of experiment at not too low filling fractions is the different behavior at even and odd values of the denominator of the filling fraction—q in our case—the WC state is not a good candidate for being the mean-field precursor to the true ground state.

Owing to definition (9), the magnetic translations entering the Fock operator have the form

$$T_{r_o^2\mathbf{n}\times\mathbf{Q}} = T_{-(Q_2/q)\mathbf{a}_1+(Q_1/q)\mathbf{a}_2}. \tag{11}$$

Since the flux piercing the unit cell is an integral number q of flux quanta, the set of translation operators $T_{\mathbf{R}}$ for all the $\mathbf{R}$ defined in Eq. (6) commute among themselves, allowing to find common eigenfunctions to all of them. This is not the case for the translations (11) since the original unit cell is partitioned in smaller sectors if q is greater than 1. The basis we shall construct defines a set of q-dimensional subspaces, which are closed under the action of translations (11) for all values of $\mathbf{Q}$.

A first step in finding the basis is to define a set of eigenfunctions $\chi_{\mathbf{k}}(\mathbf{x})$ of a translation in the vector $-\mathbf{a}_1/q$ for each value of the momentum $\mathbf{p}=\hbar\mathbf{k}$. Expressed as linear combinations of the functions $\varphi_{\mathbf{k}}(\mathbf{x})$ defined in the Appendix, we write them in the form

$$\chi_{\mathbf{k}}(\mathbf{x}) = \sum_{s=-(q-1)/2}^{(q-1)/2} c_s(\mathbf{k}) T_{-(s/q)\mathbf{a}_1}\varphi_{\mathbf{k}}(\mathbf{x}), \tag{12}$$

where, for definiteness, we have assumed q to be odd. These functions must obey the condition

$$T_{-(1/q)\mathbf{a}_1}\chi_{\mathbf{k}}(x) = \lambda\chi_{\mathbf{k}}(\mathbf{x}).$$

One finds for the eigenvalues λ and coefficients c_s the set of q solutions

$$\lambda^{(r)}(\mathbf{k}) = \exp\left(i\frac{\mathbf{k}\cdot\mathbf{a}_1}{q} + i\frac{2\pi r}{q}\right),$$

$$c_s^r(\mathbf{k}) = \frac{1}{\sqrt{q}}\exp\left(-i\frac{s\mathbf{k}\cdot\mathbf{a}_1}{q} - i\frac{2\pi rs}{q}\right),$$

$$r = -\frac{q-1}{2}, \ldots, \frac{q-1}{2}. \tag{13}$$

Substituting in Eq. (12) yields the q eigenfunctions

$$\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \frac{1}{\sqrt{q}}\sum_{s=-(q-1)/2}^{(q-1)/2}\exp\left(-i\frac{s\mathbf{k}\cdot\mathbf{a}_1}{q} - i\frac{2\pi rs}{q}\right)T_{-(s/q)\mathbf{a}_1}\varphi_{\mathbf{k}}(\mathbf{x}), \tag{14}$$

$$\mathbf{k} \equiv \mathbf{k} + n\mathbf{s}_1 + m\mathbf{s}_2, \quad n,m = 0, \pm 1, \pm 2, \ldots. \tag{15}$$

The last relation expresses the fact that the states in the new basis are equivalent upon a shift of $\mathbf{k}$ in any linear combination with integer coefficients, of the unit cell vectors of the reciprocal lattice corresponding to the periodicity of the density. The equivalence follows from the following properties: (a) the functions $\varphi_{\mathbf{k}}(\mathbf{x})$ are eigenfunctions of any translation $T_{\mathbf{R}}$ for lattice vectors $\mathbf{R}$ given by Eq. (6), (b) the operator $T_{\mathbf{R}}$ commutes with all translations entering in the definition of $\chi_{\mathbf{k}}^{(r)}(\mathbf{x})$, and (c) relation (A8) in the Appendix, stating the equivalence between magnetic translations acting on $\varphi_{\mathbf{k}}(\mathbf{x})$ and a shift in the momentum labeling these functions.

Let us now inspect the effect of a magnetic translation in $\mathbf{a}_2/q$ on the new functions. If such a transformation leaves the q-plets invariant, then the matrix reduction of the Hartree-Fock Hamiltonian will follow. One has

$$T_{(1/q)\mathbf{a}_2}\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \frac{1}{\sqrt{q}}\sum_{s=-(q-1)/2}^{(q-1)/2}\exp\left(-i\frac{s\mathbf{k}\cdot\mathbf{a}_1}{q} - i\frac{2\pi rs}{q}\right) \times T_{(1/q)\mathbf{a}_2}T_{-(s/q)\mathbf{a}_1}\varphi_{\mathbf{k}}(\mathbf{x}). \tag{16}$$

After using Eq. (A4) for changing the order of the two operators within the sum, it follows that

$$T_{(1/q)\mathbf{a}_2}\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \exp\left(-i\frac{\mathbf{k}\cdot\mathbf{a}_2}{q}\right)\chi_{\mathbf{k}}^{([r-1])}(\mathbf{x}), \tag{17}$$

where the square bracket defines the number in the set $\left\{-\frac{(q-1)}{2}, \ldots, \frac{(q-1)}{2}\right\}$ which is equivalent, modulo q, to the integer in the argument. Thus, a magnetic translation in $\mathbf{a}_2/q$ just turns one function in the q-plet into another. Besides the properties already discussed, the basis can be checked to obey

$$T_{\mathbf{a}_1}\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \exp(-i\mathbf{k}\cdot\mathbf{a}_1)\chi_{\mathbf{k}}^{(r)}(\mathbf{x}), \tag{18}$$

$$T_{\mathbf{a}_2}\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \exp(-i\mathbf{k}\cdot\mathbf{a}_2)\chi_{\mathbf{k}}^{(r)}(\mathbf{x}), \tag{19}$$

$$P\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \chi_{-\mathbf{k}}^{(-r)}(\mathbf{x}), \tag{20}$$

where the parity transformation P is defined as usual, $P\chi_{\mathbf{k}}^{(r)}(\mathbf{x}) = \chi_{\mathbf{k}}^{(r)}(-\mathbf{x})$.

From the above considerations, it follows that the q-dimensional subspace spanned by the functions $\chi_{\mathbf{k}}^{(r)}(\mathbf{x})$ at fixed values of $\mathbf{k}$ is left invariant by the action of the operators (11) for arbitrary values of the integers Q_1 and Q_2. Since the Fock Hamiltonian involves just a sum of such translations, it leaves invariant these q-dimensional subspaces as well. It is of interest to note that this is a purely kinematic result which does not depend on the form of the interaction potential.

In the new basis, the q^2 matrix elements of the Hamiltonian (2) can be readily found to have the convenient form

$$h_{\mathbf{k}}^{(r',r)} = \langle\chi_{\mathbf{k}}^{(r')}|H_{HF}|\chi_{\mathbf{k}}^{(r)}\rangle = \sum_{Q} v(\mathbf{Q})\exp\left(-\frac{r_0^2\mathbf{Q}^2}{4}\right) \times \exp\left[-i\mathbf{k}\cdot\mathbf{n}\times\mathbf{Q}r_0^2 + i\frac{\pi}{q}Q_2(2r+Q_1)\right]\delta_{r',[r-Q_1]}. \tag{21}$$

The problem has thus been reduced to the self-consistent diagonalization of a q-dimensional matrix for each value of the wave vector $\mathbf{k}$. For a sample of surface S, the degeneracy $D=BS/\phi_0$ of the Landau level of the noninteracting problem is then split into q bands that span their range as $\mathbf{k}$ covers the Brillouin zone, each holding exactly D/q single-particle states.

B. Half integer flux quanta per unit cell

We next turn our attention to the case $\nu=1/q$, $\gamma=1/2$ for which, following Eq. (1), $\phi=q/2$. This case is particularly interesting because if q is even, say, $q=2r$, then $\phi=r$ and the single-particle spectrum has just r bands. Since $\nu=1/2r$, only the lowest of these energy bands has any occupied states, being half filled. The Fermi energy is at the center of the band and the state is metallic. By contrast, if q is odd, the fraction $\phi=q/2$ is not reduced and there are q bands in the spectrum, one of which is completely filled and the others empty, leaving the Fermi level in a gap. Even and odd filling fraction denominators thus show qualitatively different behavior, a metal or an insulator, as experiment requires. Such remarkable property makes this state a reasonable candidate to be the mean-field precursor to the true ground state of the system.

The magnetic translations defining the Fock operator have now the form

$$T_{r_0^2\mathbf{n}\times\mathbf{Q}} = T_{-(2/q)Q_2\mathbf{a}_1+(2/q)Q_1\mathbf{a}_2}. \tag{22}$$

We define a doublet invariant under magnetic translations in the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$,

$$\varphi_{\mathbf{k}}^{\sigma}(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\varphi_{\mathbf{k}}(\mathbf{x}) + \frac{\sigma}{\exp(-i\mathbf{a}_2\cdot\mathbf{k})}T_{\mathbf{a}_2}\varphi_{\mathbf{k}}(\mathbf{x})\right], \quad \sigma = \pm 1. \tag{23}$$

These functions obey

$$T_{\mathbf{a}_1}\varphi_{\mathbf{k}}^{\sigma}(\mathbf{x}) = \exp(-i\mathbf{a}_1\cdot\mathbf{k})\varphi_{\mathbf{k}}^{-\sigma}(\mathbf{x}),$$

$$T_{\mathbf{a}_2}\varphi_{\mathbf{k}}^{\sigma}(\mathbf{x}) = \sigma\exp(-i\mathbf{a}_2\cdot\mathbf{k})\varphi_{\mathbf{k}}^{\sigma}(\mathbf{x}), \tag{24}$$

being turned into each other by translations along the axes, save for a phase factor. In analogy with the previous section, we define next a set of eigenfunctions of the translations $T_{(2/q)\mathbf{a}_1}$ with the form

$$\chi_{\mathbf{k}}^{(r,\sigma)}(\mathbf{x}) = \frac{1}{\sqrt{q}}\sum_{s=-(q-1)/2}^{(q-1)/2}\exp\left(i\frac{2\mathbf{k}\cdot\mathbf{a}_1 s}{q} - i\frac{2\pi rs}{q}\right)T_{(2s/q)\mathbf{a}_1}\varphi_{\mathbf{k}}^{\sigma}(\mathbf{x}),$$

$$\mathbf{k} \equiv \mathbf{k} + n\mathbf{s}_1/2 + m\mathbf{s}_2/2, \quad n,m = 0, \pm 1, \pm 2, \ldots. \tag{25}$$

Note that the last line indicates for this case that the equivalence of states is now under shifts in half the reciprocal lattice unit cell vectors. This is related to the fact that magnetic translations in vectors (6) are noncommuting, so that they have no common eigenfunctions. However, translations in twice the spatial unit cell vectors are commuting operations. Similarly, as in the previous section, the equivalence follows after considering that (a) the functions $\varphi_{\mathbf{k}}(\mathbf{x})$ are eigenfunctions of any operator $T_{2\mathbf{R}}$ for lattice vectors $\mathbf{R}$ given by Eq. (6), (b) $T_{2\mathbf{R}}$ commutes with all the translations entering in the definition of $\chi_{\mathbf{k}}^{(r,\sigma)}(\mathbf{x})$ through the original functions $\varphi_{\mathbf{k}}(\mathbf{x})$, and, as before, (c) relation (A8) in the Appendix, expressing the equivalence between magnetic translations acting on $\varphi_{\mathbf{k}}(\mathbf{x})$ and a shift in the momentum labeling these functions.

Aside from relations (24), our functions satisfy the following transformations:

$$T_{(2/q)\mathbf{a}_1}\chi_{\mathbf{k}}^{(r,\sigma)}(x) = \exp\left(i\frac{2\mathbf{k}\cdot\mathbf{a}_1}{q} + i\frac{2\pi r}{q}\right)\chi_{\mathbf{k}}^{(r,\sigma)}(x), \tag{26}$$

$$T_{(2/q)\mathbf{a}_2}\chi_{\mathbf{k}}^{(r,\sigma)}(x) = \exp\left(-i\frac{2}{q}\mathbf{k}\cdot\mathbf{a}_2\right)\chi_{\mathbf{k}}^{([r-2],\sigma)}(x). \tag{27}$$

Consider now fixed values of the quantum numbers $(\mathbf{k},\sigma)$. Relations (26) and (27) directly show that all translations (22) included in the Fock operator (2) leave invariant the q-dimensional subspace spanned by the set $\chi_{\mathbf{k}}^{(r,\sigma)}(x)$, all r.

Using the commutation properties (A4) and relations (26) and (27), the matrix elements of the Hamiltonian (2) can now be written as

$$h_{\mathbf{k}}^{(r',r)} = \langle\chi_{\mathbf{k}}^{(r',\sigma)}|H_{HF}|\chi_{\mathbf{k}}^{(r,\sigma)}\rangle = \sum_{Q} v(\mathbf{Q})\exp\left(-\frac{r_0^2\mathbf{Q}^2}{4}\right) \times\exp\left[-i\mathbf{k}\cdot\mathbf{n}\times\mathbf{Q}r_0^2 + i\frac{2\pi Q_2}{q}(r+Q_1)\right]\delta_{r',[r+2Q_1]}. \tag{28}$$

Again, the mean-field problem has been reduced to the diagonalization of a q-dimensional matrix, but now, this must be done for each value of the wave vector $\mathbf{k}$ and the index σ. An interesting outcome is that the matrix representing the Hamiltonian is identical for the two values of σ, so that its q eigenvalues are twice degenerate .

IV. HOLELIKE STATE AT $\nu=1/q$

Further progress will normally require a numerical routine that diagonalizes self-consistently either Eq. (21) or (28), where the solutions obtained at the end generate the same Fourier coefficients of the potential $v(\mathbf{Q})$ that gave rise to them. There is a special case, however, for which a completely analytic result is possible. Because of the form the charge density acquires in the unit cell, we have called it a holelike state.

Its characterization rests on the following observation. Earlier numerical work showed that there are two self-consistent solutions within the Hartree-Fock approximation.[19] In one solution, the charge density has Gaussian-like peaks centered at equivalent symmetry points in the lattice. In the other, the charge density has a deep depression at the point of highest symmetry in the unit cell and actually vanishes there if the filling fraction is of the form $\nu=1/q$. Further inspection showed this zero to be of order $2\gamma(q-1)$. Since the total particle density is a sum of the positive definite particle densities of the single-particle occupied states, all filled orbitals must vanish at the special points at least as the power $\gamma(q-1)$. We use this condition to determine completely the state of interest.

In terms of the basis states defined previously, the eigenfunctions of the Fock operator may be written in the form

$$\Psi_{\mathbf{k}}^{(b,\sigma)}(\mathbf{x}) = \sum_{r=-(q-1)/2}^{(q-1)/2} g_r^b(\mathbf{k})\chi_{\mathbf{k}}^{(r,\sigma)}(\mathbf{x}), \tag{29}$$

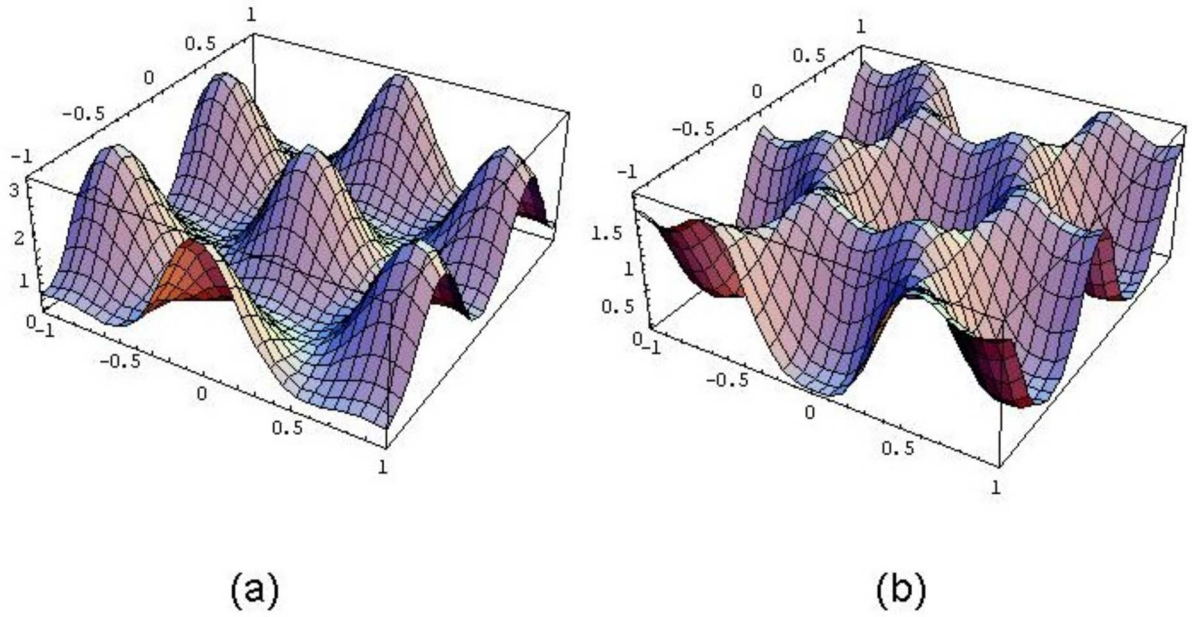

FIG. 1. (Color online) Particle density for the (a) WC and (b) holelike states at $\nu=1/3$ and $\gamma=1$ (one electron per cell). Notice the percolating hexagonal ridges and the absence of any Gaussian-like peaks in the latter.

$$\sum_{r=-(q-1)/2}^{(q-1)/2} g_r^{*b}(\mathbf{k}) g_r^b(\mathbf{k}) = 1, \tag{30}$$

where $b=0,1,\ldots,q-1$ is the band index, and the label $\sigma=\pm 1$ is to be omitted if $\gamma=1$. Because these functions are in the LLL, in our sign convention ($e=-|e|$), they must be of the form[31]

$$\Psi(z,z^*) = F(z^*)\exp\left(-\frac{zz^*}{4r_0^2}\right), \tag{31}$$

where $z=x+iy$, $z^*=x-iy$, and $F(z^*)$ is an analytic function of its argument. With no loss of generality, we choose one of the special zeros to be at the origin. As one approaches this point, one expects that asymptotically $F(z^*)\sim(z^*)^s$, with $s=\gamma(q-1)$. Thus, the function Ψ itself and its first $s-1$ derivatives must vanish at the origin, giving in all s independent equations to be satisfied. These, together with the normalization condition (30) total $s+1$ equations, being sufficient to determine the q coefficients $\{g_r^0(\mathbf{k}), r=-\frac{q-1}{2},\ldots,\frac{q-1}{2}\}$ of the occupied orbitals. Specifically, for $\gamma=1$, one has $s+1=q$, while for $\gamma=1/2$, one has $2s+1=q$, the factor of 2 arising from the double valued index σ.

As an example, we discuss the simple case $q=3$, $\gamma=1$. The electronlike state—the WC—has been reported in the literature for these parameter values,[17,20] whereas the holelike state has not. Figure 1(a) shows the charge density for the WC, obtained after achieving numerical self-consistency in the Hartree-Fock equations. The holelike state density may be obtained analytically using the above procedure by way of the coefficients $g_{-1}^0(\mathbf{k}), g_0^0(\mathbf{k}), g_1^0(\mathbf{k})$ defining the filled orbitals, and after substitution in the expression

$$\rho(\mathbf{x}) = \sum_{\mathbf{k}} \left| \sum_{r=-1}^{1} g_r^0(\mathbf{k}) \chi_{\mathbf{k}}^{(r)}(\mathbf{x}) \right|^2. \tag{32}$$

Here, the first sum runs over all momenta $\mathbf{k}$ in the Brillouin zone. The result is shown in Fig. 1(b). Notice the presence of sharp hexagonal ridges surrounding low density regions leading to the origin, where the density vanishes as the fourth power of the distance. The fact that the density percolates the structure much like the wax in a honeycomb marks the essential difference with the WC solution, whose charge density is made up of essentially of Gaussian functions centered at lattice points, as seen in Fig. 1(a).

Further, insertion of the calculated density in Eq. (4) and use of this result in Eq. (3) allow finding the associated eigenvalues ϵ^r, $r=0,1,2$ by diagonalization of the 3×3 matrix (21) for each value of $\mathbf{k}$. Three bands are obtained that span their range as $\mathbf{k}$ covers the Brillouin zone. The band dispersion relations are illustrated in Fig. 2. They are quite narrow, with the lowest—the filled one—well separated from the rest by a sizable gap. The same pattern was found for larger values of q. The LLL having been split into q separate bands appears to yield the spectrum associated with a value of the magnetic field reduced by a factor of $1/q$. This feature is reminiscent of the composite fermion theory, which interprets the fractional quantum Hall effect as the integer quantum Hall effect of composite fermions in a field reduced in the same factor.[32,33]

The energy per particle of the holelike solution we have just discussed is found through

$$\epsilon = \frac{1}{N}\sum_{\mathbf{k}} \frac{\epsilon^0(\mathbf{k})}{2} = -0.362\frac{e^2}{\varepsilon_0 r_0}. \tag{33}$$

This value is higher than that of the WC state, for which case $\epsilon=-0.388e^2/\varepsilon_0 r_0$. We find the WC to have lower energy at

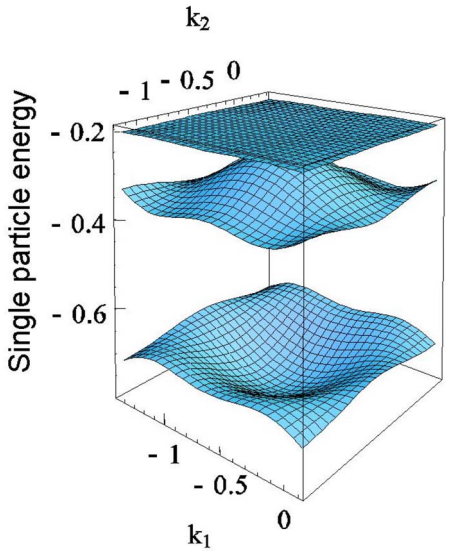

FIG. 2. (Color online) Dispersion relation for the three bands into which the lowest Landau level is split by the action of the Coulomb interaction, for the same case as in Fig. 1. Energy units are $e^2/\varepsilon_0 r_0$.

all fillings, save for $\nu=1/2$ for which value both states have the same energy.

A more interesting case corresponds to the value $\gamma=1/2$ which, as discussed earlier, gives rise to the odd-denominator rule observed in experiment in the lowest Landau level. Following a similar procedure as above, the energy for both the electronlike and the holelike states may be obtained. Results are shown in Fig. 3 for a few values of q. Above filling 1/7, the state of lowest energy is the holelike state, and below, the electronlike crystal. It is interesting that the crossover occurs at about the same value as obtained for the transition between the Laughlin state and the Wigner crystal.[2] This is likely to be just a coincidence, however, since correlations are expected to affect differently both mean-field solutions.

We end by noting that in the more general case, when $\nu=p/q$, the electrons fill p of the q bands so that the charge density is obtained from an expression such as Eq. (32), but with an additional sum over the p occupied bands. The odd-denominator rule still holds for $\gamma=1/2$ but the zero found in the holelike state for the case $p=1$ is no longer present.

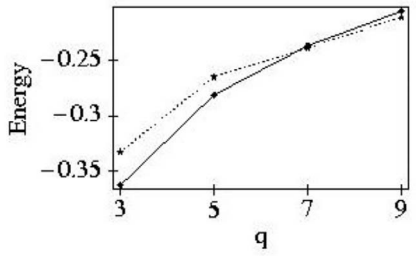

FIG. 3. Energy per particle for different values of $q=1/\nu$ for the electronlike (points line) and holelike (full line) solutions, both for $\gamma=1/2$. Units are as in Fig. 2. Lines are drawn to guide the eyes only.

V. SUMMARY AND CONCLUSIONS

Within mean-field theory, it is possible to find a variety of crystalline solutions to the problem of 2D interacting electrons moving in a high magnetic field. These may differ in the symmetry of the unit cell, square or hexagonal, and in the number of electrons that it contains. We find that once these parameters have been fixed, there are still two possible solutions to the mean-field problem, one where the electron distribution shows a single peak in the unit cell and another that appears to have a hole in the point of highest symmetry within the unit cell.

As we have shown, the holelike states may be completely determined analytically if the filling is of the form $\nu=1/q$, a truly unusual situation for a many-body Coulomb system in the thermodynamic limit. This result follows from symmetry considerations and the construction of a special set of common eigenfunctions to all magnetic translations in a lattice vector. The spectrum of single-particle states is organized in multiplets of dimension equal to the denominator q of the filling fraction in such a basis, which are left invariant by the action of the Fock operator. The solution is finally obtained under the condition that the wave function and a certain number of its derivatives must vanish at a point of high symmetry in the unit cell and all its periodic replicas in the crystal. The original degenerate Landau level is split into q narrow subbands that may be interpreted as resulting from weakly interacting particles either holding a reduced charge e/q or placed in the reduced magnetic field B/q. These alternative pictures conform with features contained in well established theories based on ansatz wave functions.[34,35] Also, if the electron number per cell equals 1/2, the mean-field solution provides naturally the experimental odd-denominator rule.

In spite of its virtues, the holelike solution has higher energy than the Wigner crystal. One is then in the presence of a state that exhibits many features required by experiment and found in other theories but not conforming with the usual criterion of it being lowest in energy. As we have illustrated, however, the honeycomblike charge density it gives rise to is indicative of strong overlap favoring correlations through ring exchange. The state is a well understood approximation derived from the original many-body Hamiltonian, and it may very well be that correlations will affect it differently than they do the Wigner crystal, which has qualitatively different features. The correlation energy of these states has been estimated systematically through second order perturbation theory only, yielding too small a correction to make a final judgment possible.

States in the lowest Landau level obeying periodic boundary conditions must vanish at a number of points equaling the degeneracy of the noninteracting system.[31] For $\nu=1/q$, this number is qN_e, where N_e is the total number of electrons in the sample. The holelike states attach $(q-1)N_e$ of them to fixed high symmetry periodic points in the electron lattice, leaving N_e zeros whose spatial location depends on the quantum number $\mathbf{k}$ labeling each occupied state. This degree of freedom is necessary to construct a Slater determinant that makes the many particle wave function antisymmetric. The determinant will vanish as the first power in the relative co-

ordinate as two particles approach each other, while the Laughlin state does so as a power q. This feature decreases the direct Coulomb energy in the latter state by keeping particles as far as possible from each other, exhibiting built-in correlations which mean field solutions lack. We hope this work will stimulate further research on corrections to the mean-field results discussed above, which may account for correlated ring exchange and other effects ignored by the Hartree-Fock approximation.

ACKNOWLEDGMENTS

We acknowledge partial support from Fondecyt, Grants No. 1060650 and No. 7020829, the Catholic University of Chile, and the Caribbean Network on Quantum Mechanics, Particles and Fields of the ICTP Office of External Activities (OEA). One of us (F.C.) would like to thank the ICTP for hospitality while part of this work was being done.

APPENDIX: EIGENFUNCTIONS OF MAGNETIC TRANSLATIONS

It is well known that Bloch-like states may be constructed by placing a seed function at each lattice point and attaching to it an appropriate phase factor. In the presence of a magnetic field, this procedure may be implemented using the zero angular momentum eigenfunction

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{2\pi}r_0}\exp\left(-\frac{\mathbf{x}^2}{4r_0^2}\right) \tag{A1}$$

and forming the sum[24,25,36]

$$\varphi_{\mathbf{k}}(\mathbf{x}) = \frac{1}{N_{\mathbf{k}}}\sum_{\ell}(-1)^{\ell_1\ell_2}\exp(i\mathbf{k}\cdot\ell)T_{\ell}\phi(\mathbf{x}),$$

$$N_{\mathbf{k}} = \sqrt{N_{\phi_0}}\sqrt{\sum_{\ell}(-1)^{\ell_1\ell_2}\exp\left(i\mathbf{k}\cdot\ell-\frac{\ell^2}{4r_0^2}\right)}. \tag{A2}$$

The summation indices ℓ_1,ℓ_2 run over all integers, defining a planar lattice L through $\ell=\ell_1\mathbf{b}_1+\ell_2\mathbf{b}_2$ with the unit cell intercepting one flux quantum, so that $\mathbf{n}\cdot\mathbf{b}_1\times\mathbf{b}_2=2\pi r_0^2$. Displacements of the seed function are effected by the magnetic translation operator $T_{\mathbf{a}}$ in vector $\mathbf{a}$, whose action on any function f is defined by[37]

$$T_{\mathbf{a}}f(\mathbf{x}) = \exp\left[\frac{ie}{\hbar c}\mathbf{A}(\mathbf{a})\cdot\mathbf{x}\right]f(\mathbf{x}-\mathbf{a}). \tag{A3}$$

Here, the vector potential is assumed in the axial gauge $\mathbf{A}(\mathbf{x})=B(-x_2,x_1,0)/2$ and the electron charge e is taken with its negative sign ($e=-|e|$). These translations in general do not commute,

$$T_{\mathbf{a}_1}T_{\mathbf{a}_2} = \exp\left[\frac{2ie}{\hbar c}\mathbf{A}(\mathbf{a}_1)\cdot\mathbf{a}_2\right]T_{\mathbf{a}_2}T_{\mathbf{a}_1}$$
$$= \exp\left[\frac{ie}{\hbar c}\mathbf{A}(\mathbf{a}_1)\cdot\mathbf{a}_2\right]T_{\mathbf{a}_1+\mathbf{a}_2}. \tag{A4}$$

In the special case of displacements in any vector belonging to L, however, since the flux trapped by any parallelogram bounded by lattice vectors is an integral number of flux quanta, all translations commute.

One can easily check that the functions $\varphi_{\mathbf{k}}$ are eigenstates of translations in any lattice vector, satisfying the eigenvalue equation

$$T_{\ell}\varphi_{\mathbf{k}}(\mathbf{x}) = \lambda_{\mathbf{k}}(\ell)\varphi_{\mathbf{k}}(\mathbf{x}), \tag{A5}$$

$$\lambda_{\mathbf{k}}(\ell) = (-1)^{\ell_1\ell_2}\exp(-i\mathbf{k}\cdot\ell). \tag{A6}$$

Arranged in a Slater determinant, these functions are exact solutions of the Hartree-Fock problem.[21,22,25,38] This strong property arises from the fact that the HF single-particle Hamiltonian commutes with all translations leaving L invariant.[25] The functions (A2) are common eigenfunctions of the commuting magnetic translations. Moreover, the set of eigenvalues (A6) uniquely determines them. Therefore, the HF Hamiltonian associated with the Slater determinant cannot change those eigenvalues and the $\varphi_{\mathbf{k}}$ should be eigenfunctions.

An important property of the basis functions (A2) is that an arbitrary translation is equivalent to a shift in the momentum label, modulo a phase factor.[24] Operating twice with the translation operator involving an arbitrary vector $\mathbf{a}$ and a vector in the lattice ℓ, and using Eqs. (A4) and (A5), one readily gets

$$T_{\mathbf{a}}T_{\ell}\varphi_{\mathbf{k}}(\mathbf{x}) = \lambda_{\mathbf{k}}(\ell)T_{\mathbf{a}}\varphi_{\mathbf{k}}(\mathbf{x})$$
$$= \exp\left[\frac{2ie}{\hbar c}\mathbf{A}(\mathbf{a})\cdot\ell\right]T_{\ell}T_{\mathbf{a}}\varphi_{\mathbf{k}}(\mathbf{x}), \tag{A7}$$

which can also be written as

$$T_{\ell}T_{\mathbf{a}}\varphi_{\mathbf{k}}(\mathbf{x}) = \lambda_{\mathbf{k}+(2e/\hbar c)\mathbf{A}(\mathbf{a})}(\ell)T_{\mathbf{a}}\varphi_{\mathbf{k}}(\mathbf{x}).$$

Then, taking into account that the set of eigenvalues defines uniquely the wave functions modulo a phase, it follows that

$$T_{\mathbf{a}}\varphi_{\mathbf{k}}(\mathbf{x}) = \mathcal{F}_{\mathbf{k}}(\mathbf{a})\varphi_{\mathbf{k}+(2e/\hbar c)\mathbf{A}(\mathbf{a})}(\mathbf{x}), \tag{A8}$$

where

$$\mathcal{F}_{\mathbf{k}}(\mathbf{a}) = \frac{\varphi_{\mathbf{k}}(\mathbf{0})}{\varphi_{\mathbf{k}+(2e/\hbar c)\mathbf{A}(\mathbf{a})}(\mathbf{a})}. \tag{A9}$$

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