

Impurity spin textures across conventional and deconfined quantum critical points of two-dimensional antiferromagnets

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We describe the spin distribution in the vicinity of a nonmagnetic impurity in a two-dimensional antiferromagnet undergoing a transition from a magnetically ordered Néel state to a paramagnet with a spin gap. The quantum critical ground state in a finite system has total spin $S=1/2$ (if the system without the impurity had an even number of $S=1/2$ spins), and recent numerical studies in a double layer antiferromagnet [K. H. Höglund *et al.*, Phys. Rev. Lett. **98**, 087203 (2007)] have shown that the spin has a universal spatial form delocalized across the entire sample. We present the field theory describing the uniform and staggered magnetizations in this spin texture for two classes of antiferromagnets: (i) the transition from a Néel state to a paramagnet with local spin singlets, in models with an even number of $S=1/2$ spins per unit cell, which are described by a O(3) Landau-Ginzburg-Wilson field theory; and (ii) the transition from a Néel state to a valence bond solid, in antiferromagnets with a single $S=1/2$ spin per unit cell, which are described by a “deconfined” field theory of spinons.

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I. INTRODUCTION

There have been many experimental studies of nonmagnetic Zn impurities substituting for the spin $S=1/2$ Cu ions in spin-gap and superconducting compounds.^{1–6} These have stimulated many theoretical studies of the spin dynamics in the vicinity of a vacancy (i.e., a site with no spin) in $S=1/2$ square lattice antiferromagnets.^{7–21}

An important feature of the impurity response escaped²² theoretical attention until recently.²⁰ Consider the regime where the bulk antiferromagnet preserves global rotational symmetry and has a $S=0$ ground state. Such states can be reached by deforming the nearest-neighbor antiferromagnet into a coupled-ladder or coupled-dimer antiferromagnet,^{23,24} in a double-layer antiferromagnet,²⁵ or by adding additional ring-exchange interactions while preserving full square lattice symmetry.²⁶ Now remove a single $S=1/2$ spin in a system with an even number of spins, leaving an antiferromagnet with a vacancy and an odd number of $S=1/2$ spins. We expect this antiferromagnet to have a doubly degenerate ground state with total spin $S=1/2$. Without loss of generality, we can examine the ground state with spin projection $S_z=1/2$. In such a state, even though there is no broken symmetry and no applied magnetic field [the Hamiltonian has full SU(2) spin symmetry], the expectation values of the spin projection on the site i , $\langle S_{zi} \rangle$, is nonzero on all i for any finite system of size L . The question of interest in this paper is the following: What is the spatial form of $\langle S_{zi} \rangle$? It is possible that the $S=1/2$ magnetization is pushed out to the boundaries of the system, far from the impurity: in this case, it will not be relevant to the impurity properties in the limit $L \rightarrow \infty$. However, we will find that this is not the case for the antiferromagnets examined in this paper. For the spin-gap antiferromagnets we consider, the $S=1/2$ magnetization is bound to the impurity over a length scale inversely proportional to the spin gap. At the quantum critical points separating the spin-gap states from the Néel state, which define “algebraic spin liquids,” we will find, as in Ref. 20, that the

impurity magnetization is delocalized over the entire system, forming a spin texture with a universal spatial form determined only by the system size L .

We will divide our introductory discussion here into two sections. The first section will consider the models which have been numerically studied in Ref. 20. These are antiferromagnets which have an even number of $S=1/2$ spins per unit cell (such as the coupled-dimer^{23,24} or double-layer²⁵ models), which exhibit a transition between a Néel state and a simple spin-gap state; the latter state is adiabatically connected to a state in which the spins in each unit cell are separately locked into singlets, with negligible resonance between unit cells. This is a “conventional” transition, described by a Landau-Ginzburg-Wilson (LGW) theory.

In the second section, we consider the more interesting and much more subtle case of a “deconfined” critical point.^{27,28} Here, we are considering antiferromagnets with a single $S=1/2$ spin per unit cell, and so there is no simple spin-gap state with local singlets. For the models studied in Refs. 26–28, the spin-gap state has singlet valence bonds which crystallize into a regular arrangement, breaking the space group symmetry of the square lattice while preserving spin rotation invariance. Such a state is a valence bond solid (VBS), and we will be interested in the impurity response across the Néel-VBS transition.

A. Landau-Ginzburg-Wilson transition

As noted above, we consider a transition in a dimerized antiferromagnet (with an even number of $S=1/2$ spins per unit cell of the Hamiltonian) from a spin-gap state to a Néel state. A convenient description of both phases and the quantum phase transition is provided by the O(3) nonlinear sigma model, expressed in terms of a unit vector field $\mathbf{n}(\vec{x}, \tau)$ representing the local orientation of the Néel order parameter. Here, \vec{x} is the two-dimensional spatial position, τ is imaginary time, and $\mathbf{n}^2=1$ everywhere in space-time. The bulk

action in the absence of the impurity is the O(3) nonlinear sigma model

$$\mathcal{S}_b^n = \frac{1}{2g} \int d\tau \int d^2x (\partial_\mu \mathbf{n})^2, \quad (1.1)$$

where g is the coupling constant which tunes the antiferromagnet from the Néel state ($g < g_c$) to the spin-gap state ($g > g_c$), μ is a three-dimensional space-time index, and a spin-wave velocity has been set to unity. In this formulation, the influence of the impurity is represented universally by the following Berry phase term alone¹⁵ (provided the antiferromagnet is not too far from the critical point):

$$\mathcal{S}_{\text{imp}}^n = iS \int d\tau \mathbf{A}[\mathbf{n}(0, \tau)] \cdot \frac{d\mathbf{n}(0, \tau)}{d\tau}, \quad (1.2)$$

for a spin $S=1/2$ antiferromagnet, where \mathbf{A} is the Dirac monopole function in spin space with $\nabla_{\mathbf{n}} \times \mathbf{A} = \mathbf{n}$. Note that $\mathcal{S}_{\text{imp}}^n$ does not include any coupling constants, and it depends on the value of \mathbf{n} only at $\vec{x}=0$, which is the position of the impurity.

Now we need to describe the $S=1/2$ ground state of $\mathcal{S}_b^n + \mathcal{S}_{\text{imp}}^n$ for $g \geq g_c$. First, we need a proper discussion of the rotationally invariant $S=0$ ground state without the impurity. While it may be possible to do this within the context of a small g expansion of the O(3) nonlinear sigma model, the procedure is quite cumbersome and delicate, requiring a global average over all possible locally ordered states. We shall instead follow a simpler procedure which is described in more detail in Sec. II: we use an alternative soft-spin LGW formulation of \mathcal{S}_b^n in terms of a vector order parameter ϕ , whose length is unconstrained. The $\phi=0$ saddle point then is an appropriate starting point for describing the physics of the $S=0$ ground state of the bulk theory and its excitations. Next, we include the impurity term described by $\mathcal{S}_{\text{imp}}^n$, and also apply an infinitesimal magnetic field in the z direction. As we will show in Sec. II, the Berry phase effectively localizes the order parameter at the impurity site, $\mathbf{n}(\vec{x}=0, \tau)$, to a specific orientation on the unit sphere; in particular, for the $S_z=1/2$ state chosen by the applied field, we may perform an expansion about a saddle point with $\mathbf{n}(\vec{x}=0, \tau) = (1, 0, 0)$. This expansion quantizes, at each order, the total spin at $S_z=1/2$: this was established in Sec. II.C.2 of Ref. 10 for $g < g_c$, and the same result also applies here for $g \geq g_c$. The infinitesimal magnetic field is set to zero at the end, but the spin density of the $S_z=1/2$ state remains nonzero in this limit.

The results in Sec. II provide an explicit analytic realization for the scaling forms presented in Ref. 20 for the spin texture near the impurity. For the magnetization density \mathbf{Q} , which is the conserved Noether “charge” density associated with the O(3) symmetry of the antiferromagnet, we have at $g=g_c$ and zero temperature (T) and in the $S_z=1/2$ state:

$$\langle Q_z(\vec{x}) \rangle = \frac{1}{L^2} \Phi_Q \left(\frac{\vec{x}}{L} \right), \quad (1.3)$$

where $\Phi_Q(\vec{r})$ is a universal function obeying the quantized total spin condition

$$\int d^2r \Phi_Q(\vec{r}) = S. \quad (1.4)$$

Similarly, the staggered magnetization associated with the Néel order parameter obeys the scaling form

$$\langle n_z(\vec{x}) \rangle = \frac{1}{L^{(1+\eta)/2}} \Phi_n \left(\frac{\vec{x}}{L} \right) \quad (1.5)$$

at $g=g_c$, where $\Phi_n(\vec{r})$ is another universal function, but its overall scale is nonuniversal. The exponent η is the anomalous dimension of \mathbf{n} at $g=g_c$ in the absence of the impurity.

B. Deconfined transition

Now let us turn to the more interesting case of a transition in an antiferromagnet with an odd number of $S=1/2$ spins per unit cell, such as the square lattice antiferromagnet. In this case, there is no *a priori* obvious choice for the spin-gap state, and the paramagnetic state exhibits spin liquid behavior over all but the largest length scales.^{27,28} The spin liquid state has a bosonic spinon excitation represented by a complex spinor field $z_\alpha(\vec{x}, \tau)$, where $\alpha = \uparrow, \downarrow$, and the constraint $\sum_\alpha |z_\alpha|^2 = 1$ is obeyed everywhere in space-time. There is also a noncompact U(1) gauge field A_μ , which encodes collective singlet excitations. As argued in Refs. 27 and 28, the vicinity of the quantum critical point to the Néel phase is described by the CP¹ field theory of these degrees of freedom. The Néel order parameter \mathbf{n} is related to z_α by

$$\mathbf{n} = z_\alpha^\dagger \vec{\sigma}_{\alpha\beta} z_\beta, \quad (1.6)$$

where $\vec{\sigma}$ are the Pauli matrices. Also, in our analysis, we find it useful to generalize to the CP ^{$N-1$} model with SU(N) symmetry, where $\alpha=1, \dots, N$, and then the Pauli matrices are replaced by the generators of SU(N). The action of the CP ^{$N-1$} model also involves a noncompact U(1) gauge field A_μ , and is given by

$$\mathcal{S}_b^z = \int d\tau \int d^2x \left[\frac{1}{g} |(\partial_\mu - iA_\mu) z_\alpha|^2 + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]. \quad (1.7)$$

This theory describes a Néel-ordered phase for $g < g_c$, and a spin-gap state with VBS order for $g \geq g_c$ (additional Berry phase terms are needed to obtain the fourfold square-lattice symmetry of the VBS order²⁹). It is crucial to note that, unlike the situation in 1+1 dimensions,^{30,31} the models \mathcal{S}_b^n [in Eq. (1.1)] and \mathcal{S}_b^z are *not* equivalent to each other in 2+1 dimensions. This was established in Ref. 32, and is a consequence of the proliferation of “hedgehog” or “monopole” defects at the critical point of \mathcal{S}_b^n ; such defects are absent in the \mathcal{S}_b^z theory.

Now let us add an impurity to the field theory in Eq. (1.7). It was argued in Ref. 33 that the impurity is now represented by a source term for a static charge $Q=2S$ at $\vec{x}=0$. Thus

$$\mathcal{S}_{\text{imp}}^z = iQ \int d\tau A_\tau(\vec{x}=0, \tau). \quad (1.8)$$

As before, we are now interested in describing the ground state of $\mathcal{S}_b^z + \mathcal{S}_{\text{imp}}^z$, which we expect carries total spin $S=1/2$.

However, now the projection onto the state with $S=1/2$ cannot be done by the method used for the LGW theory. For $g \geq g_c$, we begin with a $S=0$ ground state of \mathcal{S}_b^z , but now do not find that the impurity term in Eq. (1.8) introduces any net spin: the total spin remains at $S=0$ to all orders in perturbation theory. Clearly, we need the impurity charge Q to non-perturbatively bind a $S=1/2$ z_α spinon. For $g > g_c$, such binding can be addressed via a nonrelativistic Schrödinger equation;³⁴ the analysis does not appear appropriate at the main point of interest, $g=g_c$, where we have a conformal field theory with no sharp quasiparticle excitations. Here, we

expect the spinon to be smeared over the whole system of size L . We shall describe this spinon state by explicitly beginning with a $S=1/2$ state of \mathcal{S}_b^z and then perturbatively examining the influence of $\mathcal{S}_{\text{imp}}^z$: this is expected to yield correlations in the true $S=1/2$ ground state of $\mathcal{S}_b^z + \mathcal{S}_{\text{imp}}^z$.

Using the language of general $SU(N)$, let the ground states of $\mathcal{S}_b^z + \mathcal{S}_{\text{imp}}^z$ be $|\alpha\rangle$; these transform under the fundamental representation of $SU(N)$. To find the matrix element of some operator $O(\vec{x})$ between states $|\alpha\rangle$ and $|\beta\rangle$ of the $SU(N)$ multiplet, we compute

$$\langle \alpha | O(\vec{x}) | \beta \rangle = \lim_{T \rightarrow \infty} \frac{\left\langle z_\alpha(0, T/2) \exp \left[-i \int_{-T/2}^{T/2} A_\tau(0, \tau) d\tau \right] O(\vec{x}, 0) z_\beta^\dagger(0, -T/2) \right\rangle_{\mathcal{S}_b^z}}{\left\langle z_\alpha(0, T/2) \exp \left[-i \int_{-T/2}^{T/2} A_\tau(0, \tau) d\tau \right] z_\alpha^\dagger(0, -T/2) \right\rangle_{\mathcal{S}_b^z}}. \quad (1.9)$$

Effectively, we start with external charge free vacuum, and then at time $\tau = -T/2$ create a spinon together with the Wilson line, the latter representing the effect of the external charge $Q=1$. We wait for a long time $T/2$ to single out the lowest energy state with the quantum numbers of the operator z_α^\dagger . We then measure the operator $O(\vec{x})$, again wait time $T/2$, and annihilate our spinon together with the external charge. The denominator in Eq. (1.9) serves to cancel out the matrix element for creating the spinon-external charge bound state out of the vacuum (no sum over α is implied in the denominator). Expressions of type (1.9) are common when studying the properties of heavy-light mesons in quantum chromodynamics.

The time T must be much larger than the gap between states with the quantum numbers that we are studying. In the spin-gap phase, $g > g_c$, this gap is finite in the infinite volume limit. However, at the critical point, the gap will be of order $1/L$. So one has to choose $T \gg L$. Although unusual, this condition can always be satisfied as we work at zero temperature.

To discuss higher charge impurity ($Q > 1$), one needs to act on the vacuum with higher $U(1)$ charge composite operators of the z field. The resulting states can form higher representations of $SU(N)$ symmetry. For simplicity, we limit ourselves to $Q=1$ below.

Details of our evaluation of Eq. (1.9) in the $1/N$ expansion appear in Sec. III A. We will obtain results for the scaling functions appearing in Eqs. (1.3) and (1.5) describing the spin distribution at the deconfined quantum critical point.

In addition, in Sec. III B, we compute the uniform and staggered spin distributions in the Néel phase of the $\mathbb{C}P^{N-1}$ model. We find that the short distance behavior of spin distributions both at the critical point and in the Neel phase is in agreement with the impurity scaling theory postulated in Ref. 33. In particular, we obtain substantial additional evidence

that the uniform and staggered spin operators flow to the same impurity spin operator upon approaching the impurity site. Results of the $1/N$ expansion for the impurity critical exponents of uniform and staggered magnetizations are obtained.

II. LANDAU-GINZBURG-WILSON CRITICALITY

This section will study the field theory $\mathcal{S}_b^n + \mathcal{S}_{\text{imp}}^n$ describing an impurity in an antiferromagnet with an even number of $S=1/2$ spins per unit cell. As discussed in Sec. I, the $O(3)$ nonlinear sigma model formulation in Eqs. (1.1) and (1.2) is not appropriate for our purposes. Instead, we shall use a “soft-spin” approach, which yields a convenient description of the rotationally invariant state of the bulk antiferromagnet for $g \geq g_c$ and of its impurity-induced deformations. The universal results appear in an expansion in

$$\epsilon = (3 - d), \quad (2.1)$$

where d is the spatial dimensionality.

This dimensionality expansions allow us to compute, in principle, the universal scaling functions appearing in Eqs. (1.3) and (1.5), which were numerically computed recently in Ref. 20. The scaling functions clearly depend on the geometry of the sample and on the nature of the finite-size boundary conditions. Such features are not easily captured in a dimensionality expansion. Consequently, the results in this section are more a “proof of principle” that the scaling results apply. Direct comparison of the results below for scaling functions to the numerical results are not very useful.

As discussed in Ref. 10, the ϵ expansion is obtained by replacing the fixed length field \mathbf{n} by a field ϕ whose amplitude is allowed to vary freely. However, we do not have the freedom to relax the length constraint on the impurity site

because the Berry phase term is only defined for a unit length field. Consequently, we retain an independent field $\mathbf{n}(\tau)$ representing the impurity spin, which is now linearly coupled to ϕ . So we consider the theory

$$\begin{aligned} \mathcal{Z}_\phi &= \int \mathcal{D}\phi(\vec{x}, \tau) \mathcal{D}\mathbf{n}(\tau) \delta(\mathbf{n}^2 - 1) \exp(-S_b^\phi - S_{\text{imp}}^\phi), \\ S_b^\phi &= \int d^d x d\tau \left\{ \frac{1}{2} [(\partial_\mu \phi)^2 + s\phi^2] + \frac{g_0}{4!} (\phi^2)^2 \right\}, \\ S_{\text{imp}}^\phi &= iS \int d\tau \left[\mathbf{A}[\mathbf{n}(\tau)] \cdot \frac{d\mathbf{n}(\tau)}{d\tau} - \gamma_0 S \mathbf{n}(\tau) \cdot \phi(0, \tau) \right]. \end{aligned} \quad (2.2)$$

Here, $s \sim g$ is the coupling that tunes the system across the bulk quantum phase transition, and g_0 and γ_0 are the couplings which were shown in Ref. 10 to approach fixed point values in the vicinity of the quantum critical point. In the $(3-d)$ expansion, these fixed point values are small with $g_0 \sim \gamma_0^2 \sim \epsilon$. It was argued in Ref. 15 that this fixed point is identical to that obtained from the $O(3)$ nonlinear sigma model theory appearing in Eqs. (1.1) and (1.2).

We will be interested here in the $s \gg s_c$ regime of \mathcal{Z}_ϕ , where $\langle \phi \rangle = 0$ and full rotational symmetry is preserved in the absence of the impurity. As discussed in Sec. I, we need to project onto the state with total $S_z = 1/2$ in the presence of the impurity. This is easily done here by choosing the following parametrization for the impurity degree of freedom $\mathbf{n}(\tau)$ in terms of a complex scalar $\psi(\tau)$:

$$\mathbf{n} = \left(\frac{\psi + \psi^*}{2} \sqrt{2 - |\psi|^2}, \frac{\psi - \psi^*}{2i} \sqrt{2 - |\psi|^2}, 1 - |\psi|^2 \right). \quad (2.3)$$

The advantage of the representation (2.3) is that with the gauge choice

$$\mathbf{A}(\mathbf{n}) = \frac{1}{1 + n_z} (-n_y, n_x, 0), \quad (2.4)$$

the Berry phase takes the following form:

$$i\mathbf{A}(\mathbf{n}) \cdot \frac{d\mathbf{n}}{d\tau} = \frac{1}{2} \left(\psi^* \frac{\partial \psi}{\partial \tau} - \psi \frac{\partial \psi^*}{\partial \tau} \right). \quad (2.5)$$

Furthermore, the measure term in the functional integral also has the simple form

$$\int \mathcal{D}\mathbf{n} \delta(\mathbf{n}^2 - 1) = \int \mathcal{D}\psi \mathcal{D}\psi^*. \quad (2.6)$$

Now, an expansion of the correlators of \mathcal{Z}_ϕ , in a functional integral over ϕ and ψ about the saddle point with $\phi=0$ and $\psi=0$, in powers of the couplings γ_0 and g_0 , automatically projects onto the state with total spin projection $S_z = 1/2$. This is easily established by applying a uniform magnetic field and verifying by the methods of Refs. 10 and 15 that the total magnetization is quantized by a Ward identity associated with the conservation of spin.

We can now use the above perturbative expansion, using methods explained at length elsewhere,^{10,15} to compute the

expectation values of the magnetization density $\langle Q_z(\vec{x}) \rangle$ and the Néel order parameter $\langle \phi_z(\vec{x}) \rangle$. We perform this computation on a sample with periodic boundary conditions and length L in each spatial dimension, i.e., a torus T^d . The main effect of the finite boundary conditions is that the momenta \vec{p} are discrete, and each momentum component is quantized in integer multiples of $2\pi/L$. The results below are easily generalized to other finite-size geometries and boundary conditions. To leading order in ϵ , the results are

$$\begin{aligned} \langle Q_z(\vec{x}) \rangle &= S \delta^d(\vec{x}) - \gamma_0^2 S \delta^d(\vec{x}) \int \frac{d\omega}{2\pi} \frac{1}{(i\omega + \epsilon)^2} G(\omega, 0) \\ &\quad + 2\gamma_0^2 S \int \frac{d\omega}{2\pi} G(\omega, \vec{x}) G(\omega, -\vec{x}), \end{aligned}$$

$$\langle \phi_z(\vec{x}) \rangle = \gamma_0 S G(0, \vec{x}) \left[1 - \gamma_0^2 \int \frac{d\omega}{2\pi} \frac{1}{(i\omega + \epsilon)^2} G(\omega, 0) \right], \quad (2.7)$$

where ϵ is a positive infinitesimal proportional to an applied magnetic field which selects the $S_z = 1/2$ state. We may set $\epsilon = 0$ after the frequency integrals have been performed. The Green's function of the ϕ field is

$$G(\omega, \vec{x}) = \frac{1}{L^d} \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{x}}}{\omega^2 + \vec{p}^2 + \Delta^2}, \quad (2.8)$$

where Δ is the spin gap of the bulk antiferromagnet in the absence of the impurity. Other boundary conditions will only change the form of G , requiring expressions involving different normal mode wave functions, but the form in Eq. (2.7) will remain unchanged. It is easy to check that the spatial integral of $\langle Q_z \rangle$ is quantized at S .

To leading order in ϵ , it would appear that we can set Δ equal to the spin gap in the infinite bulk antiferromagnet, and in particular, set $\Delta = 0$ at the critical point $s = s_c$. However, we will see below that for the particular boundary conditions we are using here, there are infrared divergencies at $\Delta = 0$ in the expressions for the impurity-induced spin textures. In such a situation, we have to examine the finite L corrections to the value of Δ at $s = s_c$, which yield a nonzero Δ even at the bulk quantum critical point. The value of Δ can be computed as described elsewhere,³⁵ and to leading order in ϵ , the equation determining Δ at the quantum critical point $s = s_c$ is

$$\Delta^2 = \frac{5g_0}{6} \frac{1}{L^d} \sum_{\vec{p}} \int \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \vec{p}^2 + \Delta^2}. \quad (2.9)$$

To leading order in ϵ , only the $\vec{p} = 0$ term on the right-hand side has to be included; setting g_0 equal to its fixed point value,³⁵ we find for small ϵ

$$\Delta = \left(\frac{20\pi^2 \epsilon}{11} \right)^{1/3} \frac{1}{L}. \quad (2.10)$$

Note that $L\Delta$ is a universal number at $s = s_c$, which is the main result we will need below to establish the universality of the spin texture.

Returning to the expressions in Eq. (2.7), we now want to manipulate them into the forms of Eqs. (1.3) and (1.5). However, the presence of the $\delta^l(\vec{x})$ in Eq. (2.7) makes the \vec{x} dependence singular. These singularities are, in fact, an artifact of the present perturbative expansion in real space and are not expected to be present once the expansion is resummed. This is evident by examining the results in momentum space, where the results are a smooth function of momentum. In this manner, we obtain after applying Eq. (2.8) to Eq. (2.7)

$$\begin{aligned} \langle Q_z(\vec{p}) \rangle &= S \left\{ 1 - \frac{\gamma_0^2}{L^d} \sum_{\vec{q}} \frac{1}{2E_{\vec{q}}} \left[\frac{1}{E_{\vec{q}}^2} - \frac{2}{E_{\vec{p}+\vec{q}}(E_{\vec{p}+\vec{q}} + E_{\vec{q}})} \right] \right\}, \\ \langle \phi_z(\vec{p}) \rangle &= \frac{\gamma_0 S}{\vec{p}^2 + \Delta^2} \left[1 - \frac{\gamma_0^2}{L^d} \sum_{\vec{q}} \frac{1}{2E_{\vec{q}}^3} \right], \end{aligned} \quad (2.11)$$

where $E_{\vec{p}} = \sqrt{\vec{p}^2 + \Delta^2}$. Now Eqs. (2.11) can be evaluated at the fixed point value of γ_0 , and to leading order in ϵ , they are seen to yield results consistent with the following scaling forms which can be deduced from Eqs. (1.3) and (1.5):

$$\begin{aligned} \langle Q_z(\vec{p}) \rangle &= \tilde{\Phi}_Q(\vec{p}L), \\ \langle \phi_z(\vec{p}) \rangle &= L^{(d+1-\eta)/2} \tilde{\Phi}_n(\vec{p}L). \end{aligned} \quad (2.12)$$

The explicit results for the scaling functions to leading order in ϵ are

$$\begin{aligned} \tilde{\Phi}_Q(\vec{y}) &= S \left\{ 1 - 2\pi^2 \epsilon \sum_{\vec{x}} \frac{1}{2\mathcal{E}_{\vec{x}}} \left[\frac{1}{\mathcal{E}_{\vec{x}}^2} - \frac{2}{\mathcal{E}_{\vec{y}+\vec{x}}(\mathcal{E}_{\vec{y}+\vec{x}} + \mathcal{E}_{\vec{x}})} \right] \right\}, \\ \tilde{\Phi}_n(\vec{y}) &= \frac{\pi S \sqrt{2\epsilon}}{\vec{y}^2 + L^2 \Delta^2} [1 - 2\pi^2 \epsilon (\text{a finite number})], \end{aligned} \quad (2.13)$$

where, now, \vec{x} and \vec{y} are *three-dimensional* momenta whose components are quantized in integer multiples of 2π and $\mathcal{E}_{\vec{x}} = \sqrt{\vec{x}^2 + L^2 \Delta^2}$. It is easily checked that these expressions are free of infrared (IR) and ultraviolet (UV) divergencies, and so yield universal results because $L\Delta$ is a universal number.

From the above expression, we observe that $\tilde{\Phi}_Q(|\vec{y}| \rightarrow \infty) = S[1 - (\epsilon/2)\ln|\vec{y}|]$, which we assume exponentiates to $\tilde{\Phi}_Q(|\vec{y}| \rightarrow \infty) \sim |\vec{y}|^{-\epsilon/2}$. From the short distance behavior of the spin texture discussed in Ref. 20, we expect that $\tilde{\Phi}_Q(|\vec{y}| \rightarrow \infty) \sim |\vec{y}|^{-\eta'/2}$, where η' is the scaling dimension of the boundary spin.¹⁰ So we obtain the value $\eta' = \epsilon$, which is consistent with earlier results.¹⁰ Similarly, from the short distance behavior discussed in Ref. 20, we also have $\tilde{\Phi}_n(|\vec{y}| \rightarrow \infty) \sim |\vec{y}|^{-2+(\epsilon+\eta-\eta')/2}$. So with $\eta \sim \mathcal{O}(\epsilon^2)$ and $\eta' = \epsilon$, we have $\tilde{\Phi}_n(|\vec{y}| \rightarrow \infty) \sim |\vec{y}|^{-2}$, which is consistent with Eq. (2.13).

III. DECONFINED CRITICALITY

This section describes the Néel-VBS transition in square-lattice quantum antiferromagnets with a single $S=1/2$ per unit cell. As discussed in Sec. I, the response of a nonmag-

netic impurity is described by the action $\mathcal{S}_b^z + \mathcal{S}_{\text{imp}}^z$ in Eqs. (1.7) and (1.8) for a complex $SU(N)$ spinon field z_α and a noncompact $U(1)$ gauge field A_μ . Here, we will describe the $1/N$ expansion of its universal critical properties. Note that in what follows we have rescaled the spinon field z , to remove the coupling constant g from the action (1.7), in favor of a rescaled constraint $z_\alpha^\dagger z_\alpha = 1/g$. This constraint is enforced with a local Lagrange multiplier λ , so that the bulk action becomes

$$\begin{aligned} \mathcal{S}_b^z &= \int d\tau \int d^2x \left[|(\partial_\mu - iA_\mu)z_\alpha|^2 + i\lambda \left(|z_\alpha|^2 - \frac{1}{g} \right) \right. \\ &\quad \left. + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda)^2 \right]. \end{aligned} \quad (3.1)$$

It is useful to define $SU(N)$ generalizations of the $SU(2)$ observables introduced in Sec. I. The uniform magnetization density \mathbf{Q} generalizes to Q^a , which is the temporal component of a current associated with the $SU(N)$ rotation symmetry,

$$Q^a = z^\dagger T^a D_\tau z - (D_\tau z)^\dagger T^a z \quad (3.2)$$

(where $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative), while the Néel order \mathbf{n} in Eq. (1.6) becomes the staggered magnetization operator

$$n^a = z^\dagger T^a z, \quad (3.3)$$

where T^a are generators of the $SU(N)$ algebra. We will describe the spatial dependence of the expectation values of these operators for two cases: a finite system of size L at the critical point $g=g_c$ in Sec. III A, and the infinite system in the Néel phase with broken $SU(N)$ symmetry in Sec. III B.

A. Critical point in a finite system

We tune the system to the critical point $g=g_c$ of the infinite volume zero temperature model, and then consider the system on a spatial torus of length L . We use periodic boundary conditions for all fields.³⁶ As we discussed in Sec. I, the ground state in the absence of an impurity is a spin singlet, while adding an impurity yields a ground state which transforms under the fundamental representation of $SU(N)$. This ground state has a single spinon in it, and we argued that the projection onto this state can be performed by Eq. (1.9). For an additional test of our projection formalism, see the Appendix, where we compute the $U(1)$ (electric) charge density in the presence of the impurity.

Before we address the explicit computation of Eq. (1.9), we discuss scaling forms that our results should obey.

1. Scaling forms

We are interested in computing the uniform and staggered magnetization densities. Recall that since the uniform magnetization is a zeroth component of a conserved current, it receives no renormalizations. Therefore, utilizing the $SU(N)$ symmetry, we have the general scaling form,

$$\langle \alpha | Q^a(\vec{x}) | \beta \rangle = \frac{1}{L^2} \Phi_Q \left(\frac{\vec{x}}{L} \right) T_{\alpha\beta}^a. \quad (3.4)$$

The leading $1/L^2$ prefactor corresponds to the scaling dimension $\Delta_Q = d = 2$ of the magnetization density. Moreover, by conservation of total $SU(N)$ charge,

$$\int d^2r \Phi_Q(\vec{r}) = -1, \quad (3.5)$$

where the integral is over $0 < r_1, r_2 < 1$. Similarly, for the case of the staggered magnetization,

$$\langle \alpha | n^a(\vec{x}) | \beta \rangle = \Lambda \eta_n \left(\frac{1}{L} \right)^{1-\eta_n} \Phi_n \left(\frac{\vec{x}}{L} \right) T_{\alpha\beta}^a. \quad (3.6)$$

Here, η_n is the anomalous dimension of the staggered magnetization operator $n^a(x)$, $\Delta_n = \dim[n^a] = 1 - \eta_n$. This exponent is related to the exponent η in Eq. (1.5), and their values were computed previously³⁷ in the $1/N$ expansion for arbitrary space-time dimension $2 < D < 4$:

$$\begin{aligned} \eta_n &= \frac{1}{2}(D - 2 - \eta) = \frac{1}{N} \frac{16\Gamma(D-2)}{\Gamma(2-D/2)\Gamma(D/2-1)^3} + O(1/N^2) \\ &\stackrel{D=3}{=} \frac{16}{\pi^2 N} + O(1/N^2). \end{aligned} \quad (3.7)$$

The function Φ_Q is completely universal, whereas Φ_n is universal only up to an overall scale. In particular, Φ_n does not have any property analogous to Eq. (3.5).

Of particular interest is the behavior of the functions $\Phi_Q(\vec{r})$, $\Phi_n(\vec{r})$ for $\vec{r} \rightarrow 0$. We make a hypothesis that $n^a(\vec{x}, \tau)$ and $Q^a(\vec{x}, \tau)$ flow to the same operator $S^a(\tau)$ as \vec{x} approaches the Wilson line,

$$\begin{aligned} \lim_{|\vec{x}| \rightarrow 0} Q^a(\vec{x}, \tau) &= \frac{c_Q}{|\vec{x}|^{-\Delta_{\text{imp}}^Q}} S^a(\tau), \\ \lim_{|\vec{x}| \rightarrow 0} n^a(\vec{x}, \tau) &= \frac{c_n}{|\vec{x}|^{-\Delta_{\text{imp}}^n}} S^a(\tau). \end{aligned} \quad (3.8)$$

Calculations in the ϵ expansion supporting this hypothesis have been given in Ref. 33. We have performed analogous calculations in the $1/N$ expansion, again confirming the OPE (3.8). Technically, this impurity operator product expansion (OPE) program consists of the following steps. First, one considers the (multiplicative) renormalization of the operator $n^a(\vec{x}=0)$ by studying its insertion into the two point function of the z field (this consists of the usual bulk renormalization, plus an additional renormalization of the logarithmic divergences that appear as $\vec{x} \rightarrow 0$). Once $n^a(\vec{x}=0)$ operator is renormalized, one considers the insertion of $Q^a(\vec{x} \rightarrow 0)$ into the two point function of the z field. The highest divergence as $|\vec{x}| \rightarrow 0$ is powerlike, $1/|\vec{x}|$, modified by logarithms at higher orders in $1/N$. This leading divergence can be cancelled by an $n^a(\vec{x}=0)$ counterterm (with a coefficient that diverges as $\vec{x} \rightarrow 0$). This procedure gives one a way to construct, order by order in $1/N$, the impurity operator $S^a(\tau)$ [which is essentially a regularized $n^a(\vec{x}=0, \tau)$] and compute the anomalous

dimensions Δ_{imp}^Q and Δ_{imp}^n as well as coefficients c_Q and c_n (the latter are renormalization scheme dependent). As the computation of the OPE in the $1/N$ expansion essentially follows that in the ϵ expansion presented in Ref. 33, we shall not include it here. We only note that, in this way, we have been able to explicitly check the OPE (3.8) to order $1/N^2$, obtaining Δ_{imp}^n to order $1/N^2$ and Δ_{imp}^Q to order $1/N$ (this is lower order than the corresponding result for Δ_{imp}^n as c_Q/c_n is of order $1/N$). Explicit results in this expansion will appear in Sec. III B.

Calculations of Φ_Q and Φ_n given below provide additional support for the OPE (3.8). Note that the exponents Δ_{imp}^Q and Δ_{imp}^n are not independent. Indeed, let the correlator

$$\langle S^a(\tau) S^b(0) \rangle \sim \frac{1}{\tau^{2\Delta_S}} \delta_{ab}. \quad (3.9)$$

The exponent Δ_S is related to the boundary spin exponent η' used in Refs. 10 and 20 by $\eta' = 2\Delta_S$. Then,

$$\Delta_S = \Delta_Q + \Delta_{\text{imp}}^Q = \Delta_n + \Delta_{\text{imp}}^n. \quad (3.10)$$

Recalling $\Delta_Q = 2$ and $\Delta_n = 1 - \eta_n$,

$$\Delta_{\text{imp}}^Q = \Delta_{\text{imp}}^n - 1 - \eta_n. \quad (3.11)$$

Our explicit results for the profiles Φ_Q and Φ_n confirm the relation (3.11) to leading (zeroth) order in $1/N$ (see below). We have also been able to check this relation to order $1/N$ using the impurity OPE program summarized above: to this order, $\Delta_{\text{imp}}^Q = -1 - \eta_n$, as $\Delta_{\text{imp}}^n \sim O(1/N^2)$. The result of our evaluation of Δ_{imp}^n to $O(1/N^2)$ will appear later in Eqs. (3.98) and (3.99).

Note that the OPE (3.8) is sensitive only to short distance physics, and, thus, coefficients c_Q and c_n should be independent of the system size L as well as the deviation from the critical point (all this IR information is, however, contained in the impurity operator S^a). Thus, the ratio

$$\frac{c_Q}{c_n} = \lim_{|\vec{x}| \rightarrow 0} |\vec{x}|^{\Delta_{\text{imp}}^n - \Delta_{\text{imp}}^Q} \frac{\langle Q^a(\vec{x}) \rangle}{\langle n^a(\vec{x}) \rangle} = \lim_{|\vec{x}| \rightarrow 0} |\vec{x}|^{1+\eta_n} \frac{\langle Q^a(\vec{x}) \rangle}{\langle n^a(\vec{x}) \rangle}, \quad (3.12)$$

although nonuniversal, should be constant throughout the scaling regime (once the regularization scheme is chosen). We shall check this fact below to leading order in $1/N$ by comparing the short distance behavior (controlled by the OPE) of uniform and staggered magnetization densities at the critical point and in the Néel phase.

2. Projection onto the single spinon state

Now we return to the evaluation of the matrix elements (1.9). Although it is possible to obtain all the results presented below directly from Eq. (1.9), it is technically somewhat simpler to use instead

$$\langle \alpha | O(\vec{x}) | \beta \rangle = \lim_{T \rightarrow \infty} \frac{\langle z_\alpha(\vec{k}, T/2) O(\vec{x}, 0) z_\beta^\dagger(\vec{k}', -T/2) \rangle_{\text{imp}}}{\langle z_\alpha(\vec{k}, T/2) z_\alpha^\dagger(\vec{k}', -T/2) \rangle_{\text{imp}}}. \quad (3.13)$$

Here, $z_\alpha(\vec{k}, \tau) = \int d^2x z_\alpha(\vec{x}, \tau) e^{-i\vec{k}\vec{x}}$ and the subscript ‘‘imp’’ indicates that the correlator should be computed in a theory with the action $\mathcal{S}_b^c + \mathcal{S}_{\text{imp}}^c$, which includes the impurity term. Effectively, we have extended the Wilson line, which in Eq. (1.9) stretched from the point where a spinon was created to the point where it was destroyed, to run from $\tau = -\infty$ to $\tau = \infty$. In addition, we have taken our ‘‘incoming’’ and ‘‘outgoing’’ spinons to be in momenta \vec{k} and \vec{k}' states. This makes the numerator and denominator of Eq. (3.13) nongauge invariant. Nevertheless, we expect that this nongauge invariance comes solely from the matrix element for creating the ground state of the system by acting on the vacuum with z^\dagger and cancels out between the numerator and denominator of Eq. (3.13).

Since the impurity term Eq. (1.8) breaks spatial (but not temporal) translational invariance, for $T \rightarrow \infty$ we expect to

obtain the ground state irrespective of which \vec{k}, \vec{k}' we started with. Nevertheless, it will be most convenient in our perturbative treatment to work with $\vec{k} = \vec{k}' = 0$.

Since the external charge does not break $SU(N)$ symmetry and time translation symmetry, we have

$$\langle z_\alpha(x) z_\beta^\dagger(x') \rangle_{\text{imp}} = \delta_{\alpha\beta} D(\vec{x}, \vec{x}', \tau - \tau'). \quad (3.14)$$

We let

$$D(\vec{x}, \vec{x}', \tau) = \frac{1}{L^2} \sum_{\vec{p}, \vec{p}'} \int \frac{d\omega}{2\pi} D(\vec{p}, \vec{p}', \omega) e^{i\vec{p}\vec{x}} e^{-i\vec{p}'\vec{x}'} e^{i\omega\tau}. \quad (3.15)$$

We write

$$\langle z_\alpha(y) O(x) z_\beta^\dagger(y') \rangle_{\text{imp}} = \int dv dv' D(y, v) O_{\alpha\beta}(v, x, v') D(v', y'). \quad (3.16)$$

Fourier transforming,

$$O_{\alpha\beta}(v, x, v') = \frac{1}{L^2} \sum_{\vec{p}} \frac{1}{L^2} \sum_{\vec{p}'} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} O_{\alpha\beta}(\vec{p}, \vec{q}, \vec{p}', \omega, \omega') e^{i\vec{p}\vec{v}} e^{-i\vec{p}'\vec{v}'} e^{i\vec{q}\vec{x}} e^{i\omega v} e^{-i\omega' v'} e^{i(\omega' - \omega)x_\tau}, \quad (3.17)$$

where we use the notation that the three-vector x has spatial components \vec{x} and temporal component x_τ . So,

$$\left\langle z_\alpha\left(\vec{k}, \frac{T}{2}\right) O(\vec{x}, 0) z_\beta^\dagger\left(\vec{k}', -\frac{T}{2}\right) \right\rangle_{\text{imp}} = \sum_{\vec{p}, \vec{p}', \vec{q}} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} D(\vec{k}, \vec{p}, \omega) O_{\alpha\beta}(\vec{p}, \vec{q}, \vec{p}', \omega, \omega') D(\vec{p}', \vec{k}', \omega') e^{i\omega T/2} e^{i\omega' T/2} e^{i\vec{q}\vec{x}}. \quad (3.18)$$

As we perform the integral over ω, ω' , we pick up poles of the propagators D in the $\Im(\omega) > 0, \Im(\omega') > 0$ planes (we expect that $O_{\alpha\beta}$ is analytic in ω). In the limit $T \rightarrow \infty$, only the contribution from the pole with smallest imaginary part survives. Let this pole be at $\omega = im$ and denote by $\text{Res}(\vec{k}, \vec{p})$ the residue of $D(\vec{k}, \vec{p}, \omega)$ at this pole. Then,

$$\langle z_\alpha(\vec{k}, T/2) O(\vec{x}, 0) z_\beta^\dagger(\vec{k}', -T/2) \rangle_{\text{imp}} \rightarrow \sum_{\vec{p}, \vec{p}', \vec{q}} [i \text{Res}(\vec{k}, \vec{p})] \times [i \text{Res}(\vec{p}', \vec{k}')] O_{\alpha\beta}(\vec{p}, \vec{q}, \vec{p}', im, im) e^{i\vec{q}\vec{x}} e^{-mT}. \quad (3.19)$$

Similarly, the denominator of Eq. (3.13) is

$$\langle z_\alpha(\vec{k}, T/2) z_\alpha^\dagger(\vec{k}', -T/2) \rangle_{\text{imp}} \rightarrow L^2 i \text{Res}(k, k') e^{-mT}. \quad (3.20)$$

Finally,

$$\langle \alpha | O(\vec{x}) | \beta \rangle = \frac{1}{L^2} \sum_q \langle \alpha | O(\vec{q}) | \beta \rangle e^{i\vec{q}\vec{x}} \quad (3.21)$$

with

$$\langle \alpha | O(\vec{q}) | \beta \rangle = \sum_{\vec{p}, \vec{p}'} \frac{[i \text{Res}(\vec{k}, \vec{p})][i \text{Res}(\vec{p}', \vec{k}')] }{i \text{Res}(k, k')} O_{\alpha\beta}(\vec{p}, \vec{q}, \vec{p}', im, im). \quad (3.22)$$

3. Large N expansion of CP^{N-1} theory in finite volume

We now compute the expression (3.22) using the large N expansion in finite volume. First, consider the $N = \infty$ limit. The gap equation reads

$$\frac{1}{L^2} \sum_{\vec{p}} \int \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \vec{p}^2 + m_0^2} = \frac{1}{gN}, \quad (3.23)$$

and to this order in N , $m_0^2 = i\langle \lambda \rangle$. In the infinite volume, the critical coupling $g = g_c$ is obtained when the gap m_0 vanishes,

$$\frac{1}{g_c N} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}. \quad (3.24)$$

However, once we make the spatial volume finite, a nonzero m_0 is generated even at the critical point. Thus, setting g

$=g_c$, using Eq. (3.24), and Poisson resumming, we obtain

$$\begin{aligned} & \sum_{\vec{n} \in \mathbb{Z}^2} \int \frac{d\omega}{2\pi} \int \frac{d^2 p}{(2\pi)^2} e^{i\vec{p}\vec{n}L} \frac{1}{\omega^2 + \vec{p}^2 + m_0^2} \\ &= \int \frac{d\omega}{2\pi} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\omega^2 + \vec{p}^2}. \end{aligned} \quad (3.25)$$

On the left-hand side, only the $\vec{n}=0$ term diverges in the UV. However, this divergence cancels with the divergence of the right-hand side. Thus, performing all integrals,

$$\sum_{\vec{n} \neq 0} \frac{1}{4\pi|\vec{n}|} e^{-m_0|\vec{n}|L} = \frac{m_0 L}{4\pi}. \quad (3.26)$$

The solution of Eq. (3.26) is

$$m_0 = \theta \frac{1}{L}, \quad (3.27)$$

where θ is a constant that can be obtained by solving Eq. (3.26) numerically to be $\theta \approx 1.51196$.

Thus, at leading order, the propagator

$$D_0(\vec{k}, \vec{k}', \omega) = \delta_{\vec{k}, \vec{k}'} \frac{1}{\omega^2 + \vec{k}^2 + m_0^2} \quad (3.28)$$

and the lowest pole is at $\vec{k}=0$, $\omega = im_0$, and $i \text{Res}(\vec{k}, \vec{p}) = \delta_{\vec{k}, 0} \delta_{\vec{p}, 0} \frac{1}{2m_0}$.

To develop the $1/N$ expansion, we will need to find the A_μ and λ propagators. The dynamically generated self-energy for A_μ is to leading order

$$\begin{aligned} K_{\mu\nu}(p) = & -N \frac{1}{L^2} \sum_{\vec{q}} \int \frac{dq_\tau}{2\pi} \left[\frac{(2q-p)_\mu (2q-p)_\nu}{[(q-p)^2 + m_0^2](q^2 + m_0^2)} \right. \\ & \left. - \frac{2\delta_{\mu\nu}}{(q^2 + m_0^2)} \right]. \end{aligned} \quad (3.29)$$

This self-energy is always more singular near the critical point than the bare Maxwell term in \mathcal{S}_b^z , and so we will work with $e^2 = \infty$ for the rest of this paper. To find the photon propagator, $\mathcal{D}_{\mu\nu}(p)$, we also need to fix a gauge. Practically, for the calculations to follow, we will only need the static electromagnetic propagator $\mathcal{D}_{\tau\tau}(\vec{p}, p_\tau=0) = K_{\tau\tau}(\vec{p}, p_\tau=0)^{-1}$, which is a gauge invariant quantity. We also note that in the infinite volume limit,

$$K_{\mu\nu}(q) = K(q)(q^2 \delta_{\mu\nu} - q_\mu q_\nu), \quad (3.30)$$

$$K(q) = NAq^{D-4}, \quad (3.31)$$

where the constant A is given by

$$A = \frac{1}{(4\pi)^{D/2}} \frac{(D-2)\Gamma(2-D/2)\Gamma(D/2-1)^2}{\Gamma(D)}. \quad (3.32)$$

Here, D is the space-time dimension. In our case, $D=3$ and $A = \frac{1}{16}$.

Likewise, the self-energy for λ is to leading order



FIG. 1. The insertion of Q^a into the z propagator.

$$\Pi(p) = N \frac{1}{L^2} \sum_{\vec{q}} \int \frac{dq_\tau}{2\pi} \frac{1}{(q^2 + m_0^2)[(q-p)^2 + m_0^2]}. \quad (3.33)$$

In the infinite volume limit,

$$\Pi(p) = NBp^{D-4}, \quad (3.34)$$

where the constant B is given by

$$B = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-D/2)\Gamma(D/2-1)^2}{\Gamma(D-2)}. \quad (3.35)$$

For $D=3$, $B = \frac{1}{8}$.

4. Matrix elements

Now, let us compute the matrix elements of operator $Q^a(x)$. The insertion of Q^a into the z propagator, to leading order in $1/N$, is given by the diagram in Fig. 1, so

$$Q_{\alpha\beta}^a(\vec{p}, \vec{q}, \vec{p}', \omega, \omega') = i(\omega + \omega') T_{\alpha\beta}^a \delta_{\vec{q}, \vec{p}' - \vec{p}}. \quad (3.36)$$

So utilizing formula (3.22), with $\vec{k} = \vec{k}' = 0$, we obtain

$$\langle \alpha | Q^a(\vec{q}) | \beta \rangle = -T_{\alpha\beta}^a \delta_{\vec{q}, 0}, \quad (3.37)$$

i.e.,

$$\langle \alpha | Q^a(\vec{x}) | \beta \rangle = -\frac{1}{L^2} T_{\alpha\beta}^a, \quad (3.38)$$

and the function $\Phi_Q(\vec{r}) = -1$ satisfies the normalization condition (3.5). So, at leading order in the $1/N$ expansion, the magnetization in the presence of an impurity is spatially uniform. The system with the impurity simply consists of a free spinon in the zero momentum state. The effects of the interaction with the impurity appear only at next order in $1/N$.

Similarly, for the staggered magnetization, the insertion of $n^a(x)$ into the z propagator, to leading order, is given by the same diagram in Fig. 1, except that the cross now stands for n^a ,

$$n_{\alpha\beta}^a(\vec{p}, \vec{q}, \vec{p}', \omega, \omega') = \delta_{\vec{q}, \vec{p}' - \vec{p}} T_{\alpha\beta}^a, \quad (3.39)$$

so that

$$\langle \alpha | n^a(\vec{q}) | \beta \rangle = \frac{1}{2m_0} \delta_{\vec{q}, 0} T_{\alpha\beta}^a \quad (3.40)$$

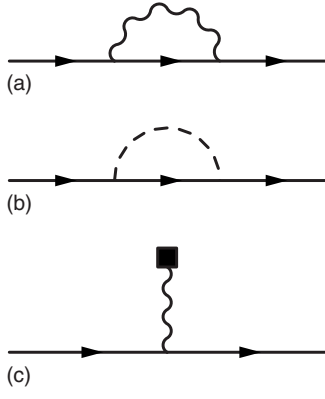
and

$$\langle \alpha | n^a(\vec{x}) | \beta \rangle = \frac{1}{2\theta L} T_{\alpha\beta}^a. \quad (3.41)$$

So the staggered magnetization at leading order in $1/N$ is also uniform, $\Phi_n(\vec{r}) = \frac{1}{2\theta}$.

Now, let us include the $1/N$ corrections.

We will concentrate on corrections to $\langle \alpha | O(\vec{q}) | \beta \rangle$ for $O = Q^a, n^a$, with $\vec{q} \neq 0$ [where the leading $O(1)$ term vanishes].


 FIG. 2. $1/N$ corrections to z self-energy.

These turn out to be much simpler to compute than corrections for $\vec{q}=0$. Moreover, for Q^a , we know by $SU(N)$ charge conservation that the $N=\infty$ result (3.37) at $\vec{q}=0$ receives no further corrections. Thus, to order $1/N$,

$$\begin{aligned} \langle \alpha | Q(\vec{q}) | \beta \rangle &= i \operatorname{Res}(0, -\vec{q})_1 O_{\alpha\beta}(-\vec{q}, \vec{q}, 0, im_0, im_0)_0 \\ &+ i \operatorname{Res}(\vec{q}, 0)_1 O_{\alpha\beta}(0, \vec{q}, \vec{q}, im_0, im_0)_0 \\ &+ i \operatorname{Res}(0, 0)_0 O_{\alpha\beta}(0, \vec{q}, 0, im_0, im_0)_1, \end{aligned} \quad (3.42)$$

where the subscripts 0 and 1 indicate the order in $1/N$ to which the quantity has to be computed.

The $1/N$ corrections to the z self-energy are shown in Fig. 2 (we drop λ tadpole diagrams). Of these, only the last one couples to the impurity and, therefore, breaks translational invariance. So, letting

$$\begin{aligned} D(\vec{k}, \vec{k}', \omega) &= D_0(\vec{k}, \vec{k}', \omega) \\ &- \sum_{\vec{p}, \vec{p}'} D_0(\vec{k}, \vec{p}, \omega) \Sigma(\vec{p}, \vec{p}', \omega) D(\vec{p}', \vec{k}', \omega), \end{aligned} \quad (3.43)$$

$$\Sigma(\vec{k}, \vec{k}', \omega) = \frac{\vec{k} \neq \vec{k}'}{L^2} 2i\omega \mathcal{D}_{\tau\tau}(\vec{k} - \vec{k}', 0) + O\left(\frac{1}{N^2}\right), \quad (3.44)$$

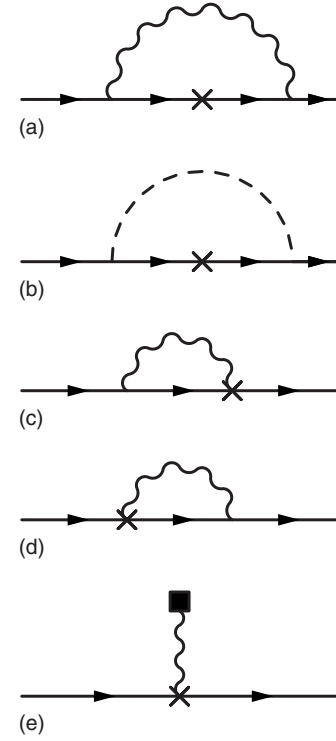
and

$$\begin{aligned} D(\vec{k}, \vec{k}', \omega) &= -\frac{\vec{k} \neq \vec{k}'}{L^2} 2i\omega \mathcal{D}_{\tau\tau}(\vec{k} - \vec{k}', 0) \\ &\times \frac{1}{\omega^2 + \vec{k}^2 + m_0^2} \frac{1}{\omega^2 + \vec{k}'^2 + m_0^2} + O\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.45)$$

So the residue

$$i \operatorname{Res}(0, -\vec{q}) = i \operatorname{Res}(\vec{q}, 0) = \frac{\vec{q} \neq 0}{L^2} \frac{1}{\vec{q}^2} \mathcal{D}_{\tau\tau}(\vec{q}, 0) + O\left(\frac{1}{N^2}\right). \quad (3.46)$$

Note that at this order, renormalization of the location of the pole $\omega = im = im_0$ can be neglected.


 FIG. 3. $1/N$ corrections to the insertion of Q^a into the z propagator.

The $1/N$ corrections to the insertion of Q^a into the z propagator are shown in Fig. 3.

Again, only the last one of these couples to the impurity and breaks translational invariance, so

$$\begin{aligned} Q_{\alpha\beta}^a(\vec{p}, \vec{q}, \vec{p}', \omega, \omega') &= -2 \frac{\vec{q} \neq \vec{p}' - \vec{p}}{L^2} \mathcal{D}_{\tau\tau}(\vec{q} + \vec{p} - \vec{p}', 0) T_{\alpha\beta}^a \\ &+ O\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.47)$$

Combining Eqs. (3.42), (3.46), and (3.47),

$$\begin{aligned} \langle \alpha | Q^a(\vec{q}) | \beta \rangle &= - \left[\delta_{\vec{q}, 0} + (1 - \delta_{\vec{q}, 0}) \frac{1}{\theta L} \left(1 \right. \right. \\ &\left. \left. + \frac{4m_0^2}{\vec{q}^2} \right) \mathcal{D}_{\tau\tau}(\vec{q}, 0) \right] T_{\alpha\beta}^a + O\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.48)$$

The calculation of $1/N$ corrections to result (3.40) for impurity-induced staggered magnetization $n^a(x)$ proceeds in the same fashion. The corrections to insertion of $n^a(x)$ into the z propagator are given by the first two diagrams in Fig. 3 (except now the cross stands for n^a insertion). None of these break translational invariance (as the last diagram in Fig. 3 is present only for Q^a but not for n^a). Therefore,

$$\langle \alpha | n^a(\vec{q}) | \beta \rangle = \frac{1}{2m_0} \left\{ \delta_{\vec{q},0} \left[1 + O\left(\frac{1}{N}\right) \right] + \left(-\delta_{\vec{q},0} \frac{1}{L^2} \frac{4m_0}{\vec{q}^2} \mathcal{D}_{\tau\tau}(\vec{q},0) \right) T_{\alpha\beta}^a + O\left(\frac{1}{N^2}\right) \right\} \quad (3.49)$$

Note again that in the case of $\langle \alpha | n^a(\vec{q}) | \beta \rangle$, we have computed the $1/N$ corrections only to $\vec{q} \neq 0$. Unlike the case of uniform magnetization, here the $N=\infty$ result for $\langle \alpha | n^a(\vec{q}=0) | \beta \rangle$ is expected to receive corrections.

Thus, the scaling functions

$$\Phi_Q\left(\frac{\vec{x}}{L}\right) = -1 - \frac{1}{\theta L} \sum_{\vec{q} \neq 0} \left(1 + \frac{4m_0^2}{\vec{q}^2} \right) \mathcal{D}_{\tau\tau}(\vec{q},0) e^{i\vec{q}\vec{x}} + O\left(\frac{1}{N^2}\right), \quad (3.50)$$

$$\Phi_n\left(\frac{\vec{x}}{L}\right) = \frac{1}{2\theta} + c_1 + \frac{1}{L^3} \sum_{\vec{q} \neq 0} \frac{2}{\vec{q}^2} \mathcal{D}_{\tau\tau}(\vec{q},0) e^{i\vec{q}\vec{x}} + O\left(\frac{1}{N^2}\right), \quad (3.51)$$

where c_1 is an \vec{x} -independent constant of order $1/N$ (c_1 should also be independent of Λ ; we have not verified this fact as we did not compute the $1/N$ corrections to $\langle \alpha | n^a(\vec{q}=0) | \beta \rangle$). We may write

$$\Phi_Q(\vec{r}) = - \left[1 + \frac{1}{N} f_Q(\vec{r}) \right] + O\left(\frac{1}{N^2}\right), \quad (3.52)$$

$$\Phi_n(\vec{r}) = \frac{1}{2\theta} \left[1 + 2c_1\theta + \frac{1}{N} f_n(\vec{r}) \right] + O\left(\frac{1}{N^2}\right). \quad (3.53)$$

We have evaluated the functions f_Q, f_n numerically and plotted them along the diagonal of our spatial torus in Fig. 4.

Now, we would like to find the $\vec{q} \rightarrow \infty, \vec{x} \rightarrow 0$ asymptotes of Eqs. (3.50) and (3.51). For this purpose, we may replace the finite box propagator $\mathcal{D}_{\tau\tau}(q)$ by the infinite box propagator,

$$\mathcal{D}_{\tau\tau}(\vec{q},0) \xrightarrow{\vec{q} \rightarrow \infty} \frac{1}{NA} \frac{1}{|\vec{q}|}. \quad (3.54)$$

Writing $\Phi_{Q,n}(\vec{x}/L) = \frac{1}{L^2} \sum_{\vec{q}} \Phi_{Q,n}(\vec{q}) e^{i\vec{q}\vec{x}}$,

$$\Phi_Q(\vec{q}) \xrightarrow{\vec{q} \rightarrow \infty} -\frac{1}{NA\theta} \frac{L}{|\vec{q}|} + O\left(\frac{1}{N^2}\right), \quad (3.55)$$

$$\Phi_n(\vec{q}) \xrightarrow{\vec{q} \rightarrow \infty} \frac{2}{NA} \frac{1}{L|\vec{q}|^3} + O\left(\frac{1}{N^2}\right). \quad (3.56)$$

Fourier transforming,

$$\Phi_Q(\vec{r}) \xrightarrow{|\vec{r}| \rightarrow 0} -\frac{1}{2\pi\theta NA} \frac{1}{|\vec{r}|} + O\left(\frac{1}{N^2}\right), \quad (3.57)$$

$$\Phi_n(\vec{r}) \xrightarrow{|\vec{r}| \rightarrow 0} \frac{1}{2\theta} + c_2 + O\left(\frac{1}{N^2}\right), \quad (3.58)$$

where c_2 is a constant of order $1/N$.

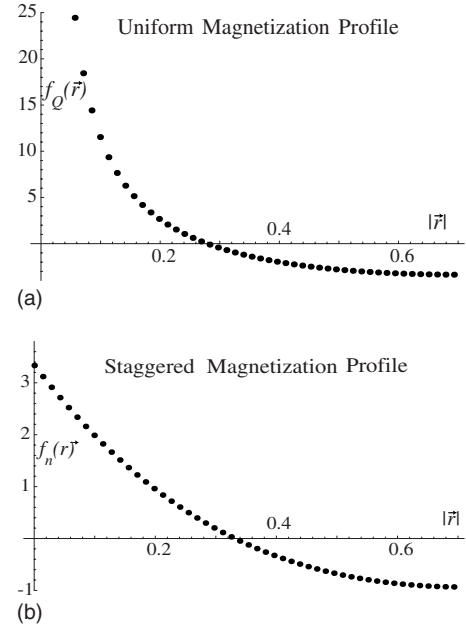


FIG. 4. (a) Uniform and (b) staggered magnetization distribution functions $f_Q(\vec{r})$ and $f_n(\vec{r})$ plotted along the torus diagonal.

Thus, we conclude that

$$\Delta_{\text{imp}}^Q = -1 + O(1/N), \quad \Delta_{\text{imp}}^n = O(1/N^2), \quad (3.59)$$

which is consistent with the relation between impurity exponents (3.11). Note that the present calculation shows that Δ_{imp}^n is zero to order $1/N$. We shall verify this fact in a different way in Sec. III B and compute Δ_{imp}^n to order $1/N^2$. Moreover, the ratio

$$\frac{c_Q}{c_n} = -\frac{1}{\pi NA} + O\left(\frac{1}{N^2}\right) \quad (3.60)$$

is independent of regularization at this order in N .

B. Néel phase

In this section, we compute the uniform and staggered magnetizations in the presence of an impurity of charge Q in the symmetry broken phase, $g < g_c$. We work in infinite volume. We develop the $1/N$ expansion around the symmetry broken vacuum,

$$\langle z_1 \rangle = \frac{1}{\sqrt{2}} v. \quad (3.61)$$

Note that, in general, v is not a gauge invariant quantity. However, this fact does not manifest itself at the order at which we are working. To leading order in N ,

$$\frac{1}{2} v^2 = \frac{1}{g} - \frac{1}{g_c}. \quad (3.62)$$

Note that $v^2 \sim O(N)$. Moreover, we take $Q \sim O(1)$ in N .

We now must quantize our theory around the symmetry broken state. We write

$$z_1 = \frac{1}{\sqrt{2}}(h + v + i\phi), \quad z_\alpha = \pi_\alpha, \alpha = 2, \dots, N. \quad (3.63)$$

We work in the so-called R_ξ gauge, in which the mixing between the Goldstone ϕ and the photon A_μ is absent, at the expense of introducing a ghost field c . In what follows, we have eliminated the mixing only to leading order in $1/N$. This is achieved by using the gauge-fixing condition,

$$\partial_\mu A_\mu = \xi v K^{-1} \phi + w, \quad (3.64)$$

where the action for the auxiliary field w , which appears in the Fadeev-Popov formalism, is

$$\mathcal{S}_w = \frac{1}{2\xi} \int dx dy w(x) K(x-y) w(y). \quad (3.65)$$

Here, $K(x-y)$ is the photon polarization function given by Eq. (3.31). Similarly, in what follows, $\Pi(x-y)$ is the λ self-energy given by Eq. (3.33).

At the end of the day, the action one obtains is

$$\begin{aligned} \mathcal{S}_\xi = & \frac{1}{2} \int dx dy A_\mu(x) \left[K_{\mu\nu}(x-y) - \frac{1}{\xi} \partial_\mu \partial_\nu K(x-y) \right. \\ & \left. + \delta_{\mu\nu} v^2 \right] A_\nu + \frac{1}{2} \int dx dy \phi(x) [-\partial^2 \delta(x-y) + \xi v^2 K^{-1}(x \\ & -y)] \phi(y) + \frac{1}{2} \int dx dy \lambda(x) \Pi(x-y) \lambda(y) + \int dx dy \{ \bar{c} [\\ & -\partial^2 \delta(x-y) + \xi v^2 K^{-1}(x-y)] c(y) + \xi v \bar{c}(x) K^{-1}(x \\ & -y) h(y) c(y) \} + \int dx \left[|D_\mu \pi|^2 + \frac{1}{2} (\partial_\mu h)^2 + i v \lambda h + (\phi \partial_\mu h \right. \\ & \left. - \partial_\mu \phi h) A_\mu + \left(v h + \frac{1}{2} h^2 + \frac{1}{2} \phi^2 \right) A_\mu^2 \right] + \int dx \left[i \lambda |\pi|^2 \right. \\ & \left. + \frac{1}{2} i \lambda (\phi^2 + h^2) \right]. \end{aligned} \quad (3.66)$$

As usual, we avoid double counting by dropping any diagrams, which are already included in the dynamically generated $N=\infty$ self-energies for A_μ , λ , etc. The propagators for our fields are shown in Fig. 5. Note that in the Néel phase, we get mixing between the λ and h fields.

Now, having set up the perturbation theory, we wish to compute $\langle Q^a(\vec{x}) \rangle$ and $\langle n^a(\vec{x}) \rangle$. Utilizing the pattern of spontaneous symmetry breaking, $U(N) \rightarrow U(N-1)$ (here, we look only at global symmetry), one can show that

$$\langle n^a \rangle = T_{11}^a \langle n^0 \rangle, \quad (3.67)$$

where $n^0 = z^\dagger T^0 z$ and T^0 is any generator of $SU(N)$ with $T_{11}^0 = 1$. Similarly for Q^a . For definiteness, we may choose $T_{11}^0 = 1$, $T_{1\alpha}^0 = T_{\alpha 1}^0 = 0$, $T_{\alpha\beta}^0 = -\frac{1}{N-1} \delta_{\alpha\beta}$, $\alpha, \beta = 2, \dots, N$.

Let us start with computing the uniform magnetization,

$$Q^0 = \frac{N}{N-1} j_\tau^1 - \frac{1}{N-1} j_\tau, \quad (3.68)$$

where

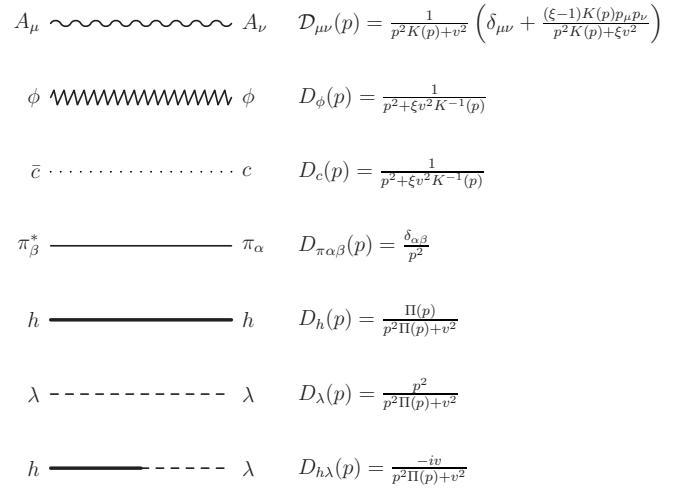


FIG. 5. Propagators in the Néel phase.

$$j_\tau^1 = z_1^\dagger D_\tau z_1 - (D_\tau z_1)^\dagger z_1 \quad (3.69)$$

and j_τ is the $U(1)$ charge density discussed in the Appendix [see Eq. (A1)]. By equation of motion (A4),

$$\langle j_\tau(\vec{x}) \rangle = -J_\tau^{\text{ext}}(\vec{x}) = -Q \delta^2(\vec{x}). \quad (3.70)$$

So, it remains to compute $\langle j_\tau^1(\vec{x}) \rangle$. Expanding j_τ^1 in terms of ϕ , h , and A_μ ,

$$j_\tau^1 = -i v^2 A_\tau + i v (\partial_\tau \phi - 2A_\tau h) + i [h \partial_\tau \phi - \phi \partial_\tau h - A_\tau (h^2 + \phi^2)]. \quad (3.71)$$

In the $1/N$ expansion, the leading contribution to $\langle j_\tau^1 \rangle$ is of $O(1)$ and comes from the first term on the right-hand side of Eq. (3.71) (see Fig. 6),

$$\langle j_\tau^1(\vec{p}) \rangle = -Q v^2 \mathcal{D}_{\tau\tau}(\vec{p}, 0) = -Q v^2 \frac{1}{\vec{p}^2 K(\vec{p}) + v^2}. \quad (3.72)$$

Thus, $\langle Q^0(\vec{p}) \rangle = \langle j_\tau^1(\vec{p}) \rangle + O(1/N)$. Fourier transforming,

$$\begin{aligned} \langle Q^0(\vec{x}) \rangle = & -\frac{Q v^2}{2\pi N A} \frac{1}{|\vec{x}|} + \frac{Q v^4}{4N^2 A^2} \left[\mathbf{H}_0 \left(\frac{v^2 |\vec{x}|}{N A} \right) - Y_0 \left(\frac{v^2 |\vec{x}|}{N A} \right) \right] \\ & + O\left(\frac{1}{N}\right), \end{aligned} \quad (3.73)$$

where \mathbf{H}_0 is the Struve function and Y_0 is the Bessel function. Taking the short and long distance asymptotes,

$$\langle Q^0(\vec{x}) \rangle \xrightarrow{|\vec{x}| \rightarrow 0} -\frac{Q v^2}{2\pi N A} \frac{1}{|\vec{x}|}, \quad (3.74)$$



FIG. 6. Leading contribution to uniform magnetization in the symmetry broken phase.

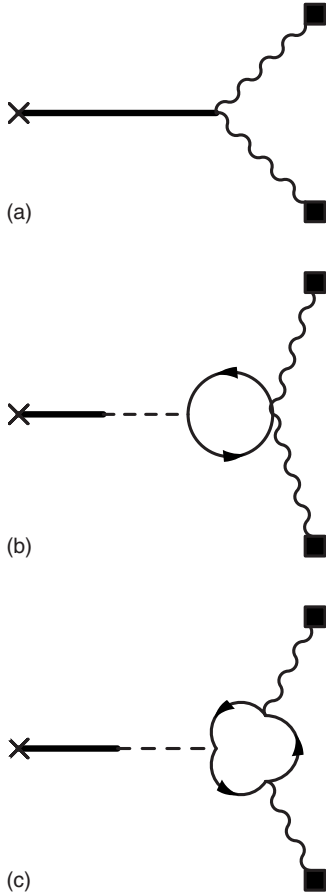


FIG. 7. Leading \vec{x} -dependent contribution to staggered magnetization in the symmetry broken phase.

$$\langle Q^0(\vec{x}) \rangle \xrightarrow{\vec{x} \rightarrow \infty} -\frac{QNA}{2\pi v^2} \frac{1}{|\vec{x}|^3}. \quad (3.75)$$

The long distance decay is a consequence of the Goldstone physics of the spin waves, and the $1/|\vec{x}|^3$ decay is expected to be exact. At short distances, we have the physics of the critical point, and the exponent will have corrections at higher order. From the present result, we can conclude that the impurity exponent

$$\Delta_{\text{imp}}^Q = -1 + O(1/N), \quad (3.76)$$

which is consistent with the result obtained at the critical point (3.59).

Now, let us discuss the staggered magnetization,

$$n^0 = \frac{N}{N-1} z_1^\dagger z_1 - \frac{1}{N-1} z^\dagger z. \quad (3.77)$$

By equations of motion,

$$z^\dagger z = \frac{1}{g}, \quad (3.78)$$

thus,

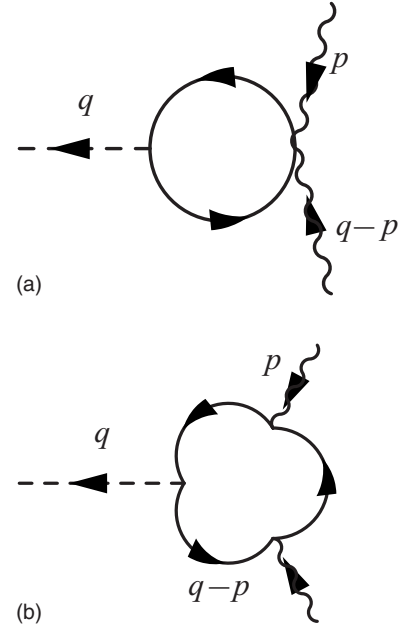


FIG. 8. Leading contribution to the three point vertex of A_μ , A_ν , and λ fields, $\Gamma^{\mu\nu}(q, p, q-p)$.

$$n^0 = \frac{N}{N-1} z_1^\dagger z_1 - \frac{1}{(N-1)g} \quad (3.79)$$

and

$$z_1^\dagger z_1 = \frac{1}{2}v^2 + vh + \frac{1}{2}(h^2 + \phi^2). \quad (3.80)$$

Thus, at leading order, $\langle z_1^\dagger z_1(\vec{x}) \rangle = \frac{1}{2}v^2$, and

$$\langle n^0(\vec{x}) \rangle = \frac{1}{2}v^2 + O(1). \quad (3.81)$$

Moreover, the \vec{x} -dependent corrections to $\langle n^0(\vec{x}) \rangle$ come only at $O(1/N)$, with diagrams of Fig. 7 (the part of n^0 which contributes at this order, denoted by \times , is vh).

We will discuss the diagrams in Fig. 7 shortly. For now, we can conclude that

$$\Delta_{\text{imp}}^n = O(1/N^2), \quad (3.82)$$

in agreement with the result (3.59) obtained at the critical point. Moreover, we can now compute the ratio

$$\frac{c_Q}{c_n} = -\frac{Q}{\pi NA} + O\left(\frac{1}{N^2}\right), \quad (3.83)$$

which exactly agrees with the result obtained at the critical point (3.60) for $Q=1$. Notice that this is a highly nontrivial check of the OPE (3.8) as $\langle Q^a \rangle$ and $\langle n^a \rangle$ depend on v in the Néel phase and on L at the critical point. Nevertheless, all the dependence on the IR scale cancels out in the ratio c_Q/c_n , which is constant throughout the scaling regime.

Going back to the diagrams in Fig. 7,

$$\begin{aligned} \langle n^0(\vec{q}) \rangle^{\vec{q} \neq 0} &= Q^2 v^2 D_h(\vec{q}, 0) \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left[1 \right. \\ &\quad \left. + \frac{i}{2} \Pi^{-1}(\vec{q}, 0) \Gamma^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} - \vec{p}, 0) \right] \mathcal{D}_{\tau\tau}(\vec{p}, 0) \mathcal{D}_{\tau\tau}(\vec{q} \\ &\quad - \vec{p}, 0). \end{aligned} \quad (3.84)$$

We keep the space-time dimension D arbitrary in what follows, as we wish to compare our result for Δ_{imp}^n obtained in the $1/N$ expansion with the result obtained using ϵ expansion.³³ Here, $\Gamma^{\mu\nu}(q, p, q-p)$ is the lowest order contribution to the A_μ, A_ν, λ vertex, given by the sum of the loops

$$\begin{aligned} \Gamma_2^{\mu\nu}(q, p, q-p) &= -2iN \int \frac{d^D l}{(2\pi)^D} \frac{(2l-p)_\mu (2l-p-q)_\nu}{l^2(l-p)^2(l-q)^2} = -\frac{4iN\Gamma(2-D/2)}{(4\pi)^{D/2}} \int dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3) (\Delta^2)^{D/2-2} \left[\delta_{\mu\nu} \right. \\ &\quad \left. + \frac{(4-D)(2x_1 q + (2x_2-1)p)_\mu ((2x_1-1)q + (2x_2-1)p)_\nu}{4\Delta^2} \right], \end{aligned} \quad (3.87)$$

where

$$\Delta^2 = x_1(1-x_1)q^2 + x_2(1-x_2)p^2 - 2x_1 x_2 p q. \quad (3.88)$$

We are interested only in $\Gamma_2^{\tau\tau}$, with $p^0 = q^0 = 0$. Thus,

$$\begin{aligned} \Gamma_2^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} - \vec{p}, 0) &= -\frac{4iN\Gamma(2-D/2)}{(4\pi)^{D/2}} \int dx_1 dx_2 dx_3 \delta(1 \\ &\quad - x_1 - x_2 - x_3) (\Delta^2)^{D/2-2}. \end{aligned} \quad (3.89)$$

For $|\vec{p}| \gg |\vec{q}|$, $\Gamma_2^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} - \vec{p}, 0) \sim |\vec{p}|^{D-4}$; so for $\vec{p} \rightarrow \infty$, the integrand in Eq. (3.86) behaves as $|\vec{p}|^{-D}$ and the integral is UV convergent.

We now attempt to understand the behavior of Eq. (3.86) for $\vec{q} \rightarrow \infty$, from which we should be able to extract the impurity anomalous dimension Δ_{imp}^n . For this purpose, we may set $v=0$ in the propagators $D_h(\vec{q}, 0)$, $\mathcal{D}_{\tau\tau}(\vec{p}, 0)$, and $\mathcal{D}_{\tau\tau}(\vec{q} - \vec{p}, 0)$ (this does not introduce any IR divergences),

$$\begin{aligned} \langle n^0(\vec{q}) \rangle^{\vec{q} \rightarrow \infty} &= \frac{Q^2 v^2}{N^3 A^2 B} |\vec{q}|^{2-D} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{i}{2} \Gamma_2^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} \\ &\quad - \vec{p}, 0) \frac{1}{|\vec{p}|^{D-2}} \frac{1}{|\vec{p} - \vec{q}|^{D-2}}. \end{aligned} \quad (3.90)$$

Let us first discuss the limit $D=4-\epsilon$, $\epsilon \rightarrow 0$. In this regime, to leading order in ϵ ,

$$\Gamma_2^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} - \vec{p}, 0) = -2iN \frac{1}{(4\pi)^2} \Gamma\left(2 - \frac{D}{2}\right) \quad (3.91)$$

and

$$\langle n^0(\vec{q}) \rangle^{\vec{q} \rightarrow \infty} = \frac{72\pi^4 \epsilon^2 Q^2 v^2}{N^2} \frac{1}{|\vec{q}|^3}. \quad (3.92)$$

Fourier transforming,

in Fig. 8. The diagram in Fig. 8(a) is given by

$$\Gamma_1^{\mu\nu}(q, p, q-p) = 2i\delta_{\mu\nu}\Pi(q). \quad (3.85)$$

Thus, diagrams in Figs. 7(a) and 7(b) cancel (by the way, these diagrams are individually UV divergent for $D \leq 3$). So, following the diagram in Fig. 8(b), $\Gamma_2^{\mu\nu}(q, p, q-p)$,

$$\begin{aligned} \langle n^0(\vec{q}) \rangle^{\vec{q} \neq 0} &= Q^2 v^2 D_h(\vec{q}, 0) \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{i}{2} \Pi^{-1}(\vec{q}, 0) \Gamma_2^{\tau\tau}(\vec{q}, 0, \vec{p}, 0, \vec{q} \\ &\quad - \vec{p}, 0) \mathcal{D}_{\tau\tau}(\vec{p}, 0) \mathcal{D}_{\tau\tau}(\vec{q} - \vec{p}, 0). \end{aligned} \quad (3.86)$$

Evaluating $\Gamma_2^{\mu\nu}$,

$$\langle n^0(\vec{x}) \rangle^{\vec{x} \rightarrow 0} = \frac{1}{2} v^2 + c_3 - \frac{36\pi^2 \epsilon^2 Q^2}{N^2} v^2 \log(v|\vec{x}|) + c_4 + \mathcal{O}\left(\frac{1}{N^2}\right), \quad (3.93)$$

where c_3 and c_4 do not depend on \vec{x} and are of order 1 and $1/N$, respectively. Thus, to leading order in $1/N$ and ϵ ,

$$\Delta_{\text{imp}}^n = -\frac{72\pi^2 Q^2 \epsilon^2}{N^2} \quad (3.94)$$

in agreement with the calculations of Ref. 33, where the impurity exponents were obtained by performing the impurity operator renormalizations as summarized in Sec. III A 1.³⁸

For arbitrary D , Δ_{imp}^n is difficult to calculate analytically, as $\Gamma^{\tau\tau}$ is no longer a constant. However, combining Eqs. (3.86) and (3.89) and introducing a new set of Feynman parameters,

$$\langle n^0(\vec{q}) \rangle^{\vec{q} \rightarrow \infty} = \frac{Q^2 v^2}{N^2} \frac{1}{|\vec{q}|^{D-1}} f(D), \quad (3.95)$$

where the numerical constant $f(D)$ is given by

$$f(D) = \frac{1}{A^2 B (4\pi)^{D-1} \Gamma(D/2 - 1)^2} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^1 dy_1 \int_0^{1-y_1} dy_2 x_2^{(D-3)/2} (1-x_2)^{D/2-1} y_1^{1-D/2} y_2^{D/2-2} (1-y_1-y_2)^{D/2-2} \times \{x_2(1-x_2)^2 y_2(1-y_2) + x_1 y_1 [(1-x_1)(1-x_2) - 2y_2 x_2(1-x_2) - y_1 x_1 x_2]\}^{-1/2}. \quad (3.96)$$

Consequently,

$$\langle n^0(\vec{x}) \rangle = \frac{1}{2} v^2 + c_3 - \frac{2}{(4\pi)^{(D-1)/2} \Gamma[(D-1)/2]} \times f(D) \frac{Q^2}{N^2} v^2 \log(v^{2(D-2)} |\vec{x}|) + c_4 + O\left(\frac{1}{N^2}\right) \quad (3.97)$$

and

$$\Delta_{\text{imp}}^n = - \frac{4}{(4\pi)^{(D-1)/2} \Gamma[(D-1)/2]} f(D) \frac{Q^2}{N^2} + O\left(\frac{1}{N^3}\right). \quad (3.98)$$

Evaluating $f(D)$ numerically for $D=3$,

$$\Delta_{\text{imp}}^n \approx -25.9 \frac{Q^2}{N^2} + O\left(\frac{1}{N^3}\right). \quad (3.99)$$

We note that we have separately verified the result (3.98) by performing the impurity OPE program as summarized in Sec. III A 1.

IV. CONCLUSIONS

A recent numerical study²⁰ examined the spin distribution in the vicinity of a nonmagnetic impurity in a double-layer, $S=1/2$ square-lattice antiferromagnet at its quantum critical point. The ground state of the system has total spin $S=1/2$, and the spin distribution of this $S=1/2$ was found to be extended across the entire system. Universal scaling forms [Eqs. (1.3) and (1.5)] for the uniform and staggered spin distributions were postulated²⁰ and found to be in excellent agreement with the numerical results.

This paper has presented the field-theoretic foundation of the above results. Using the soft-spin $O(3)$ LGW field theory in Eq. (2.2), we found that the universal scaling forms in Eqs. (1.3) and (1.5) were indeed obeyed in an expansion in $(3-d)$ (where d is the spatial dimensionality), and explicit results for the universal scaling functions appear in Eq. (2.13).

Next, we examined a similar nonmagnetic impurity in $S=1/2$ antiferromagnets, which have a single $S=1/2$ spin per unit cell. Such antiferromagnets can display a deconfined quantum phase transition^{27,28} between Néel and valence bond solid (VBS) states. An explicit example of such a transition was found recently in Ref. 26. We expect that such studies will be extended to include nonmagnetic impurities in the future, and so we have provided our theoretical predictions here. The field theory for this situation is $S_b^z + S_{\text{imp}}^z$ in Eqs. (1.7) and (1.8). It describes the dynamics of a $SU(N)$ spinor

field, z_α (the spinon), and we obtained its critical properties in a $1/N$ expansion. Projecting onto the total spin $S=1/2$ sector of this theory (which contains the ground state in the presence of the impurity) was not straightforward here, and we achieved this by the relation Eq. (1.9). Our results obey scaling forms which appear in Sec. III A 1. The scaling functions are in Eqs. (3.50) and (3.51), and are plotted in Fig. 4. The boundary spin exponent for the deconfined critical point appears in Eqs. (3.98) and (3.99). We also obtained substantial evidence for the structure of the operator product expansion near the impurity and the fact that the staggered and uniform magnetizations flow to the same impurity spin operator.

In the context of the present study, the basic scaling structures of the uniform and staggered magnetizations in the two cases above are quite similar. In both cases, there is a single impurity spin operator which determines the exponents characterizing the spatial form of the spin texture. The main observable difference is in the very different values of the exponents. However, a more significant difference arises when we consider the form of the VBS order near the impurity, as this is an issue only for the deconfined critical point. Results on the structure of the VBS order will be presented in a forthcoming paper.

ACKNOWLEDGMENTS

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APPENDIX: U(1) CHARGE DENSITY

Throughout the paper we have concentrated on computing matrix elements of uniform and staggered magnetizations $Q^a(x)$ and $n^a(x)$. However, for the deconfined critical point, it is also interesting to compute the charge density associated with the $U(1)$ local symmetry of the $\mathbb{C}P^{N-1}$ model. This charge density is the zeroth component of the current,

$$j_\mu(x) = z^\dagger D_\mu z - (D_\mu z)^\dagger z. \quad (A1)$$

As we shall see, this computation serves as an additional test of our procedure for projecting onto the single spinon state.

Consider the $\mathbb{C}P^{N-1}$ model coupled to an external current,

$$S = S_b^z + i \int d^3x A_\mu J_\mu^{\text{ext}}. \quad (A2)$$

As in the rest of the paper, we set $e^2 = \infty$, so that the gauge field has no bare kinetic term. Then, by equations of motion,

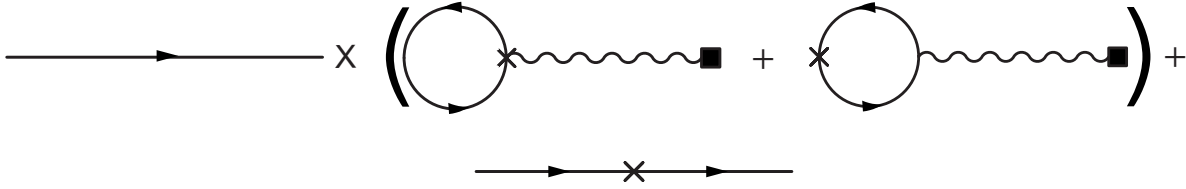


FIG. 9. Diagrams contributing to U(1) charge induced.

$$0 = \frac{\delta \mathcal{S}}{\delta A_\mu} = i(j^\mu + J_\mu), \quad (\text{A3})$$

$$j_\mu = -J_\mu^{\text{ext}}. \quad (\text{A4})$$

Thus, the dynamical current completely screens (locally!) the external current. Equation (A4) is an operator identity and should, in particular, hold in the ground state of the system with a single impurity. Let us check this statement in the $1/N$ expansion.

We start from Eq. (1.9), with $O(x) = j_0(x)$. We write

$$\langle \alpha | j_0(\vec{x}) | \beta \rangle = \rho(\vec{x}) \delta_{\alpha\beta}, \quad (\text{A5})$$

with $\rho(\vec{x}) = \frac{1}{L^2} \sum_{\vec{q}} \rho(\vec{q}) e^{i\vec{q}\vec{x}}$. The Wilson line term in Eq. (1.9) can be incorporated into the action as coupling to an external current, $J_\mu^{\text{ext}}(\vec{x}, \tau) = \delta_{\mu 0} \delta^2(\vec{x}) \theta(T/2 - \tau) \theta(\tau + T/2)$. At leading order in $1/N$, the numerator of Eq. (1.9) is given by diagrams shown in Fig. 9, while the denominator is given by the bare propagator $D(\vec{x}=0, \mathcal{T})$.

Thus, we can distinguish two contributions: the disconnected one, $\rho^1(\vec{q})$, coming from the first line in Fig. 9, and the connected one, $\rho^2(\vec{q})$, coming from the second line. We note that the scalar loops contributing to $\rho^1(\vec{q})$ are precisely the same as those contributing to the self-energy of A_μ field, thus,

$$\rho^1(\vec{q}) = - \int \frac{dq_\tau}{2\pi} K_{0\nu}(\vec{q}, q_\tau) \mathcal{D}_{\nu\lambda}(\vec{q}, q_\tau) J_\lambda^{\text{ext}}(\vec{q}, q_\tau). \quad (\text{A6})$$

Now,

$$K_{\mu\nu}(q) \mathcal{D}_{\nu\lambda}(q) = \delta_{\mu\lambda} - \frac{q_\mu q_\lambda}{q^2}. \quad (\text{A7})$$

Thus,

$$\rho^1(\vec{q}) = - \int \frac{dq_\tau}{2\pi} \frac{\vec{q}^2}{q_\tau^2 + \vec{q}^2} J^{\text{ext}}(\vec{q}, q_\tau). \quad (\text{A8})$$

$$\text{Noting } J^{\text{ext}}(\vec{q}, q_\tau) \xrightarrow{T \rightarrow \infty} 2\pi \delta(q_\tau),$$

$$\rho^1(\vec{q}) = - (1 - \delta_{\vec{q},0}). \quad (\text{A9})$$

For the $\vec{q}=0$ part, the order of the limits $\vec{q} \rightarrow 0$, $q_\tau \rightarrow 0$ is very important. In our finite system, the $\vec{q}=0$ mode is isolated. Moreover, in our present treatment, the Wilson line is of finite length, so we must take the $\vec{q}=0$ limit first and then $q_\tau \rightarrow 0$. Hence, $\rho^1(\vec{q}=0) = 0$. This is not surprising. In perturbation theory, we start with the vacuum which has charge 0. Unless we manually project the system into a finite charge subspace (as we do in our treatment by acting on the vacuum with z and z^\dagger operators), we will never be able to see global screening of charge. Since the diagrams contributing to $\rho^1(\vec{q})$ are disconnected from the external z line, $\rho^1(\vec{q}=0) = 0$.

Now, the connected contribution simply gives the charge density of one spinon in the $\vec{k}=0$ state,

$$\rho^2(\vec{q}) = - \delta_{\vec{q},0}. \quad (\text{A10})$$

Putting the two contributions together,

$$\rho(\vec{q}) = -1, \quad (\text{A11})$$

$$\rho(\vec{x}) = - \delta^2(\vec{x}), \quad (\text{A12})$$

as expected by equations of motion (A4).

Thus, we have been able to check the exact screening of external charge, which follows from equation of motion (A4), to leading order in $1/N$. We see that local and global parts of the screening charge come from very different Feynman diagrams.

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