Survival probability of multibarrier resonance systems: Exact analytical approach

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We study the properties of the survival probability in multibarrier resonance systems using an exact analytical approach that involves a representation in terms of the resonance poles and resonant states of the system. We find a mechanism that modifies the usual exponential decaying regime into a nonexponential decay behavior of the survival probability. This mechanism corresponds to the decay of Rabi oscillations originating from transitions among the closely lying resonances characterizing the energy spectra of multibarrier resonance systems.

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I. INTRODUCTION

The theoretical description of the time evolution of quantum decay originated in the old days of quantum mechanics with the work by Gamow[1] and by Condon and Gurney[2] on α decay in radioactive nuclei. As it is well known, it was found that the disintegration rate of α particles follows the exponential decay law. Subsequent work by Khalfin[3] in the 1950s and since then by a number of authors, however, pointed out the approximate validity of the exponential decay law. It was argued that deviations from this law should be expected at very short and very long times compared with the lifetime of the system. These theoretical predictions have been confirmed experimentally in recent years: The short-time deviations were observed experimentally a few years ago with ultracold atoms,[4] and, very recently, the long-time deviations have been observed in dissolved organic materials.[5] The advent of artificial quantum structures has indeed widened the realm of possibilities of studying the time evolution of decay beyond the systems provided by nature. The exponential decay law has been verified for electronic decay in thin double-barrier semiconductor heterostructures,[6] and in recent work, it is argued that these structures with appropriate parameters could exhibit nonexponential decay in the full time interval.[7,8]

The simple theoretical treatments of decay usually refer to the escape or survival of a particle from a trapping region by tunneling through a single barrier.[7–12] However, the possibility of designing the potential parameters of artificial quantum systems raises the possibility of exploring the time evolution of decay in more complex potential profiles. In this work, we address this issue for multibarrier quantum systems in one dimension, that is, systems formed by a number $(N+1)$ of alternating barriers with N wells. Our approach rests on a resonant formalism that yields an analytical expression for the survival probability and hence allows us to study and assess the different contributions to decay in these systems. The paper is organized as follows. Section II provides a résumé of the formalism. Section III discusses the multiresonance survival probability. Section IV presents and discusses the results for some model calculations. Finally, Sec. V provides the concluding remarks.

II. FORMALISM

For the sake of completeness and to establish the notation, in this section we present a résumé of the resonant formalism employed to describe the time evolution of decay. We shall derive the solution $\psi(x,t)$, at a time $t>0$, for the decay of an initial arbitrary state at $t=0$, $\psi(x,t=0)$, confined along the internal region, $0\leq x\leq L$, of a one-dimensional finite range potential $V(x)$. The solution $\psi(x,t)$ can be written in terms of the retarded Green's function $g(x,x';t)$ as

$$\psi(x,t)=\int_0^L \psi(x',0)g(x,x';t)dx'. \tag{1}$$

As discussed elsewhere,[11,12] a convenient approach to solve the above problem is by Laplace transforming $g(x,x';t)$ into the k plane to express it in terms of the corresponding outgoing Green's function $G^+(x,x';k)$ of the problem, namely,

$$g(x,x';t)=\frac{i}{2\pi}\int_c G^+(x,x';k)e^{-i\hbar k^2 t/2m}2kdk, \tag{2}$$

where c stands for the corresponding Bromwich contour along the first quadrant on the complex k plane. Equation (2) may be evaluated by using an expansion of $G^+(x,x';k)$ in terms of its complex poles k_n and the resonant states $u_n(x)$, which follow from the residues at these poles,[11–14]

$$G^+(x,x';k)=\sum_{n=-\infty}^{\infty}\frac{u_n(x)u_n(x')}{2k_n(k-k_n)},\quad 0\leq[x,x']\leq L, \tag{3}$$

where the notation $0\leq[x,x']\leq L$ means that all the values of x and x' are allowed in the above interval except $x=x'=0$ $=L$. The above representation holds provided that the resonant states are normalized according to the condition[15]

$$\int_0^L u_n^2(x)dx+i\frac{u_n^2(0)+u_n^2(L)}{2k_n}=1. \tag{4}$$

The representation given by Eq. (3) is appealing from a physical point of view because it takes into account, through the boundary conditions imposed on the resonant solution

$u_n(x)$, that the system is an open decaying system. Indeed, the resonant states $\{u_n\}$ satisfy the Schrödinger equation of the problem with outgoing boundary conditions at $x=0$ and $x=L$.[15] The substitution of Eq. (3) into Eq. (2) allows us to write $g(x,x';t)$ as an expansion over all the resonant states of the problem,

$$g(x,x';t) = \sum_{n=-\infty}^{\infty} u_n(x)u_n(x')M(k_n,t), \quad 0 \leqslant [x,x'] \leqslant L, \tag{5}$$

where $M(k_n,t) \equiv M(0,k_n,t)$ is the M function defined, in general, as

$$M(x,q,t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\hbar k^2 t/2m}e^{ikx}}{k-q}dk = \frac{1}{2}e^{imx^2/(2\hbar t)}w(iy_q), \tag{6}$$

where $w(iy_q)$ is defined as $w(iy_q)=\exp(y_q^2)\mathrm{erfc}(y_q)$,[17,18] with

$$y_q = e^{-i\pi/4}\left(\frac{m}{2\hbar t}\right)^{1/2}\left[x - \frac{\hbar q}{m}t\right], \tag{7}$$

and q is either k_n or k_{-n}.

By substituting the expansion for $g(x,x';t)$ given by Eq. (5) into Eq. (1), we obtain a representation for the time-dependent solution $\Psi(x,t)$ along the internal region given by

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} C_n u_n(x)M(k_n,t), \quad 0 \leqslant x \leqslant L, \tag{8}$$

where the coefficient C_n is defined as follows:

$$C_n = \int_0^L \psi(x,0)u_n(x)dx. \tag{9}$$

Similarly, we can define the coefficients $\bar{C}_n$ as

$$\bar{C}_n = \int_0^L \psi^*(x,0)u_n(x)dx. \tag{10}$$

Both coefficients, C_n and $\bar{C}_n$, obey the following useful relations:[11,12]

$$\sum_{n=-\infty}^{\infty} \frac{C_n\bar{C}_n}{k_n} = 0 \tag{11}$$

and

$$\sum_{n=-\infty}^{\infty} C_n\bar{C}_n k_n = 0, \tag{12}$$

and the sum rule

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} C_n\bar{C}_n = 1. \tag{13}$$

The survival amplitude $A(t)$ is defined as

$$A(t) = \int_0^a \psi^*(x,0)\psi(x,t)dx, \tag{14}$$

and the survival probability as $S(t)=|A(t)|^2$. It provides the probability of finding the state $\Psi(x,t)$ in its initial value $\Psi(x,0)$. By substituting Eq. (8) into Eq. (14), one may obtain an expression for the survival amplitude,

$$A(t) = \sum_{n=-\infty}^{\infty} C_n\bar{C}_n M(k_n,t), \tag{15}$$

and since $S(t)=|A(t)|^2$, we obtain

$$S(t) = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} C_n\bar{C}_n C_l^*\bar{C}_l^* M(k_n,t)M^*(k_l,t). \tag{16}$$

III. MULTIBARRIER SURVIVAL PROBABILITY

Typically, in single- or double-barrier systems, the time evolution of the survival probability involves three time regimes:[10,19–22] (a) nonexponential decay at extremely short times, (b) exponential decay spanning over many lifetimes of the system at intermediate times, and (c) nonexponential decay as an inverse power law of time at very long times. Regime (a) depends on some general features of the Hamiltonian of the system, such as the existence of energy moments,[20] or whether the potential vanishes after a distance. The very-short-time behavior of decay depends on the very large energy spectra of the system. It is independent of whether the system possesses single, double, or multiple barriers.

There exist a number of interesting systems in which the resonance spectrum consists of groups of two overlapping resonances, such as the resonance doublets in triple-barrier systems, or more generally, groups of N overlapping resonances as the minibands in a multibarrier system with $(N+1)$ barriers. Let us consider the interesting case of an initial state $\psi(x,0)$ lying close to the lowest energy miniband of such multibarrier system. In such a case, one expects that the overlap of the initial state with each of the resonance states of this miniband be larger than with the resonance states of the rest of the minibands. This means that the sum rule given by Eq. (13) may be written to a good aproximation as

$$\mathrm{Re}\left\{\sum_{n=1}^{N} C_n\bar{C}_n\right\} \approx 1. \tag{17}$$

In writing Eq. (17), we have used the fact that $k_{-n}=-k_n^*$ and $u_{-n}(x)=u_n^*(x)$, so that $C_n\bar{C}_n=[C_n\bar{C}_n]^*$. As a consequence of the above, an expansion of $A(t)$ consisting on a finite number N of resonance terms may be sufficient to describe the time evolution of decay in multibarrier systems.

Formula of $S(t)$ for N overlapping resonances

Following the above considerations, we may write $A(t)$, using Eq. (15), as

$$A(t) = \sum_{n=1}^{N} A_n(t), \tag{18}$$

with $A_n(t)=C_n\bar{C}_n M(k_n,t) - C_n^*\bar{C}_n^* M(-k_n^*,t)$. Furthermore, since $M(k_n,t)=\exp(-i\hbar k_n^2 t/m) - M(-k_n,t)$,[15] the survival probability $S(t)=|A(t)|^2$ may then be analytically calculated and it will consist of a finite number of terms that include exponential, long-time nonexponential, and interference contributions.

In order to explicitly distinguish the exponential and nonexponential contributions of $S(t)$, each term of the survival amplitude $A_n(t)$ may be separated in the sum of a purely exponential part $A_n^{ex}(t)$ plus a purely nonexponential part $A_n^{ne}(t)$, namely,

$$A_n(t) = A_n^{ex}(t) + A_n^{ne}(t), \tag{19}$$

where $A_n^{ex}(t)$ and $A_n^{ne}(t)$ are given, respectively, by

$$A_n^{ex}(t) = C_n\bar{C}_n e^{-i\mathcal{E}_n t/\hbar} e^{-\Gamma_n t/2\hbar}, \tag{20}$$

where the resonance energy $\mathcal{E}_n$ and the decay width Γ_n follow from the definition $E_n \equiv \hbar^2 k_n^2/2m = \mathcal{E}_n - i\Gamma_n/2$, and

$$A_n^{ne}(t) = -C_n\bar{C}_n M(-y_n) + (C_n\bar{C}_n)^* M(y_{-n}). \tag{21}$$

With the above definitions, the resulting N-term approximation for the survival probability reads

$$S(t) = \sum_{\alpha}\sum_{n=1}^{N} S_n^{\alpha} + \sum_{\alpha,\beta}\sum_{m\leq n}^{N} S_{mn}^{\alpha,\beta}, \tag{22}$$

where the following notation has been adopted:

$$S_{mn}^{\alpha,\beta} \equiv 2\,\mathrm{Re}\{A_m^{\alpha^*} A_n^{\beta}\},$$

with $m \neq n$ or $\alpha \neq \beta$, and

$$S_n^{\alpha} \equiv A_n^{\alpha^*} A_n^{\alpha} = |A_n^{\alpha}|^2.$$

Indices m and n run from 1 to N, and α and β stand for *ex* and *ne*. Notice that in the restriction of the indices in $S_{m,n}^{\alpha,\beta}$, the equality of one pair is allowed but not both pairs simultaneously (e.g., we can have $m=n$ with $\alpha \neq \beta$, and vice versa). As a consequence, the second sum of Eq. (22) contains interference terms involving two different resonances ($m \neq n$) as well as terms that involve just a single resonance ($m=n$). In order to make a separation of both classes of terms, we rewrite $S(t)$ in a more convenient expression, namely,

$$S(t) = \sum_{n=1}^{N} S_n(t) + \sum_{m<n}^{N} S_{mn}(t), \tag{23}$$

where we now have

$$S_n(t) = S_n^{ex}(t) + S_{nn}^{ex,ne}(t) + S_n^{ne}(t) \tag{24}$$

and

$$S_{mn}(t) = \sum_{\alpha,\beta} S_{mn}^{\alpha,\beta}(t). \tag{25}$$

Let us write $A_n^{ex}(t)$ and $A_n^{ne}(t)$ in the form

$$A_n^{ex}(t) \equiv |C_n\bar{C}_n|\exp(-\Gamma_n t/2\hbar)\exp[i\varphi_n^{ex}(t)]$$

and

$$A_n^{ne}(t) \equiv |A_n^{ne}(t)|\exp[i\varphi_n^{ne}(t)],$$

where $\varphi_n^{ex}(t) = -\mathcal{E}_n t/\hbar + \eta_n$ and $C_n\bar{C}_n \equiv |C_n\bar{C}_n|\exp[i\eta_n]$. After a simple algebraic procedure, we obtain explicit analytic expressions for the different contributions to $S(t)$. The terms contained in $S_n(t)$, given by Eq. (24), are given by

$$S_n^{ex}(t) = |C_n\bar{C}_n|^2 e^{-\Gamma_n t/\hbar}, \tag{26}$$

$$S_{nn}^{ex,ne}(t) = 2|C_n\bar{C}_n A_n^{ne}(t)| e^{-\Gamma_n t/2\hbar} \cos\left[\frac{\mathcal{E}_n t}{\hbar} + \varphi_n^{ne}(t) - \eta_n\right], \tag{27}$$

$$S_n^{ne}(t) = |A_n^{ne}(t)|^2. \tag{28}$$

The interference terms involving more than one resonant state follow from Eq. (25). In the first place, we have a special interference term that arises from the purely exponential parts of $A(t)$, namely,

$$S_{mn}^{ex,ex}(t) = 2|C_m\bar{C}_m C_n\bar{C}_n| e^{-\bar{\Gamma}_{mn} t/\hbar} \cos(\Omega_{mn} t + \Delta\eta_{mn}), \tag{29}$$

where Ω_{mn} is a Rabi-type frequency defined by

$$\Omega_{mn} = \frac{\mathcal{E}_m - \mathcal{E}_n}{\hbar}, \tag{30}$$

and $\bar{\Gamma}_{mn}$ is the mean of the widths of the corresponding interacting resonances,

$$\bar{\Gamma}_{mn} \equiv (\Gamma_m + \Gamma_n)/2. \tag{31}$$

The quantity $\Delta\eta_{mn}$ is the phase difference between $C_m\bar{C}_m$ and $C_n\bar{C}_n$, that is, $\Delta\eta_{mn} \equiv (\eta_n - \eta_m)$.

We shall refer to the interference term given by Eq. (29) as *the Rabi term of the survival probability* $S_{mn}^{Rabi}(t)$. Notice that these Rabi oscillations are damped by the decreasing exponential $\exp(-\bar{\Gamma}_{mn} t/\hbar)$, which is of the same order of magnitude as the purely exponential decaying term given by Eq. (26). Consequently, the Rabi term $S_{mn}^{Rabi}(t)$ is expected to be important in the exponentially decaying time regime.

Similarly, the remaining interference terms are given by

$$S_{mn}^{ex,ne}(t) = 2|C_m\bar{C}_m||A_n^{ne}(t)| e^{-\Gamma_m t/2\hbar} \cos\left[\frac{\mathcal{E}_m t}{\hbar} + \varphi_n^{ne}(t) - \eta_m\right], \tag{32}$$

$$S_{mn}^{ne,ex}(t) = 2|C_n\bar{C}_n||A_m^{ne}(t)| e^{-\Gamma_n t/2\hbar} \cos\left[\frac{\mathcal{E}_n t}{\hbar} + \varphi_m^{ne}(t) - \eta_n\right], \tag{33}$$

$$S_{mn}^{ne,ne}(t) = 2|A_m^{ne}(t) A_n^{ne}(t)| \cos[\varphi_n^{ne}(t) - \varphi_m^{ne}(t)]. \tag{34}$$

Notice that, in general, $S_{mn}^{ex,ne} \neq S_{mn}^{ne,ex}$.

IV. EXAMPLES AND DISCUSSION OF RESULTS

In contrast to the simple trapped level within the walls of a double-barrier system, where one finds that there are only three relevant analytic contributions to the survival probability,[8] for systems with two interacting levels, as in triple-barrier systems, one deals with a more complex situation. In fact, one finds from the results of the previous section that there are ten different terms (of the form S_n^{α} and $S_{mn}^{\alpha,\beta}$) contributing to the survival probability $S(t)$ so that we can analytically explore all these contributions.

Let us now consider a multibarrier system formed by N rectangular barriers [and hence $(N-1)$ rectangular wells], each barrier being characterized by a height V_0. Let the width of the mth barrier be b_m, and that of the mth well be w_m. We consider as the initial decaying state a sine pulse centered at the position x_0 of the mth well of the system, namely,

$$\psi(x,0) = \left(\frac{2}{w_m}\right)^{1/2} \sin\left[k_0(x-x_0) + \frac{\pi}{2}\right], \tag{35}$$

with $|x-x_0|<w_m/2$ and zero otherwise. Clearly, if k_0 takes the values $N\pi/w_m$ (with N an entire number), the initial state corresponds to an infinite box state. In our subsequent analysis, we choose $N=1$, so that $k_0=\pi/w_m$. A virtue of the above choice for the initial state is that it allows closed analytical expressions for the expansion coefficients defined by Eqs. (9) and (10). It is straightforward to see that each coefficient C_n is proportional to the corresponding resonant state u_n evaluated at the point x_0, namely,

$$C_n = D_n u_n(x_0), \tag{36}$$

where

$$D_n = (w_m/2)^{1/2}\left(\text{sinc}\left[\tfrac{1}{2}(k+k_n)w_m\right] + \text{sinc}\left[\tfrac{1}{2}(k-k_n)w_m\right]\right), \tag{37}$$

with $\text{sinc}(z)\equiv\sin(z)/z$. Hence, using Eqs. (36) and (37), each expansion coefficient $C_n\bar{C}_n$ may be written as the analytical expression

$$C_n\bar{C}_n = C_n^2 = [D_n u_n(x_0)]^2. \tag{38}$$

A. Detailed analysis of triple-barrier systems

In this section, we discuss in detail some features of the dynamics of decay for the simplest multibarrier system that involves a coupling of closely lying resonances, namely, the triple-barrier system, where the resonance spectra are formed by a succession of resonance doublets.

1. Decaying Rabi oscillations

Let us consider a symmetric triple-barrier (TB) system with typical potential parameters:[23,24] barrier heights $V_0=0.12$ eV, well widths $w_0=16.0$ nm, and barrier widths $b_1=b_3=3.0$ nm and $b_2=6.0$ nm. The initial state $\psi(x,0)$ consists on the sine pulse given by Eq. (35) placed on the left well of the TB system.

The dynamics of the corresponding survival probability $S(t)$ is illustrated in Fig. 1(a). A series of oscillations of regular period is clearly appreciated before the transition to an inverse power behavior with time occurs. This behavior is not present in situations where a single state governs the decay process as in double-barrier (DB) systems.[8] The decaying oscillatory behavior is produced by the term $S_{12}^{Rabi}(t)$, given explicitly by Eq. (29), which describes the exponential-exponential contribution of the resonance doublet of the TB which has a well defined frequency in terms of the two dominant resonances of the system, as shown by Eq. (30). In k space, the complex poles $k_n=\alpha_n-i\beta_n$ associated with the resonance doublet read $k_1=(0.1495-i0.001\,45)$ nm^{-1} and $k_2=(0.1540-i0.001\,59)$ nm^{-1}. The initial state has $k_0=0.1963$ nm^{-1}, a value that lies close to α_1 and α_2. Hence, it is not surprising to find that the values of $\text{Re}\{C_1^2\}=0.4656$ and $\text{Re}\{C_2^2\}=0.4732$ add up to 0.9839, which satisfies the condition given by Eq. (17). Figure 1 shows that indeed the first doublet provides an excellent description for the time evolution of the survival probability. It exhibits a comparison of calculations involving the first doublet, $N=2$, (solid line); two doublets, $N=4$, (dashed line); three doublets, $N=6$, (dotted line); and four doublets, $N=8$, (dash-dotted line); it shows that the corresponding contributions are, in fact, indistinguishable from each other. The top horizontal axis provides the time in lifetime units t/τ_1, with the lifetime $\tau_1=\hbar/\Gamma_1=1.32$ ps, to show that the decaying Rabi oscillations dominate the decay through the first few

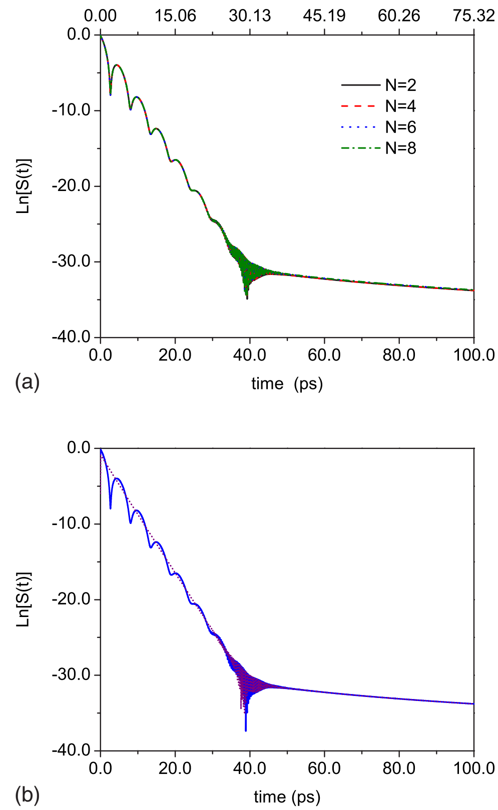

FIG. 1. (Color online) (a) Time evolution of the survival probability $S(t)$ for a TB system, where the initial state is a sine pulse placed in the first well from the left. See parameters in the text. (b) Comparison of the survival probability $S(t)$ with (solid line) and without (dashed line) the Rabi term $S_{12}^{Rabi}(t)$. See text.

lifetimes of the system and drastically modify the purely exponential regime typically observed in DB systems.[16] This is clearly illustrated in Fig. 1(b), in which we compare the resonance doublet contribution to $S(t)$, given by Eq. (23) (solid line), with a similar calculation without the Rabi contribution $S_{12}^{Rabi}(t)$ (dashed line).

The two different oscillatory patterns on the $\ln[S(t)]$ vs t curve are manifested at different time domains simply because the corresponding oscillatory functions have different time-dependent envelopes. In the Rabi term, the envelope is just the decreasing exponential $e^{-\bar{\Gamma}_{21}t/\hbar}$, while in the other oscillatory terms, the envelope goes as the product of a (slower) decreasing exponential $e^{-\Gamma_n t/2\hbar}$ by the factor $t^{-3/2}$. The above are the reasons by which the latter oscillatory pattern only appears in a small and well localized time interval around the exponential-decay–nonexponential-decay (ED-NED) transition, and also why the Rabi oscillations dominate along the exponential region, becoming damped as t approaches the ED-NED transition.

The versatility of multibarrier systems provides more options in manipulating the parameters of the potential than the simple DB structure, and hence one has more possibilities to control important physical quantities of the system, in particular, the survival probability. In order to exemplify the above, in Fig. 2 we illustrate how the Rabi oscillations can be modified without altering the rest of the structure of the $\ln[S(t)]$ vs t graph. This can be done by varying the width of the central barrier of the TB system. If we increase b_2, the energy difference between the neighboring states, $\Delta\mathcal{E}_{21}=(\mathcal{E}_2-\mathcal{E}_1)$, is reduced and so the Rabi frequency Ω_{21}. Note that for $b_2=12.0$ nm, the Rabi's period was made even longer that the exponential regime, in such a way that ED-NED transition occurs before the first oscillation can be accomplished, so that the Rabi oscillations were practically suppressed in that regime. Notice that the above variation does not affect the asymptotic value of $S(t)$ at long times; on the contrary, it lies unaltered as well as the transition time from ED to NED.

In contrast to the above variation, we can modify the asymptotic value of $S(t)$ and the transition time t_c without significantly altering the Rabi frequency. In Fig. 3, we show calculations of $\ln[S(t)]$ vs t for different values of the external barrier widths $b_1=b_3$, keeping the rest of the potential parameters fixed, namely, barrier heights $V_0=0.12$ eV, well widths $w=6.0$ nm, and barrier width $b_2=3.0$ nm. We can observe that the exponential region is dramatically shortened as b_1 and b_3 are reduced. For $b_1=b_3=0.6$ nm, the exponential region has practically disappeared, and we have a system that is practically in the regime of purely nonexponential decay, that is, the system goes directly into the nonexponential decay and never passes by an "intermediate" exponential region. As Fig. 3 also exhibits, the asymptotic value of $\ln[S(t)]$ is also changed under the mentioned variation. In fact, as we shall show later, both the transition time and the asymptotic value of $S(t)$ depend on the value of the ratio R_n.

Although the values of the resonance energies $\mathcal{E}_2$ and $\mathcal{E}_1$ remain almost unaltered for changes of the external barrier widths (and hence the Rabi frequency), the values of the decay widths are affected with this variation and so the ratios

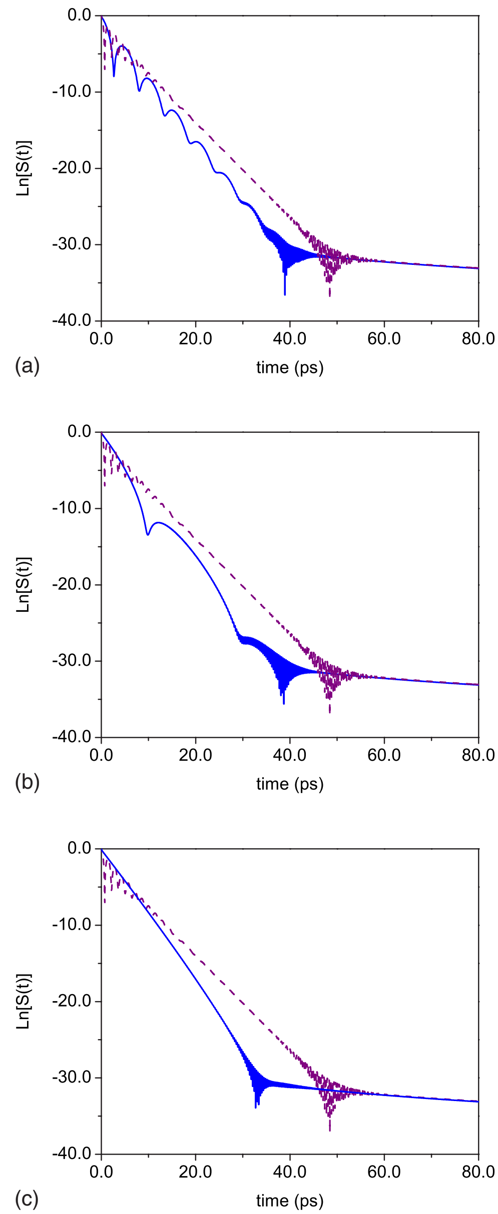

FIG. 2. (Color online) Survival probability for increasing values of the central barrier width (solid line) compared to the periodic case with $b_2=3.0$ nm (dashed line). The values of b_2 for the nonperiodic cases are (a) 6.0 nm, (b) 9.0 nm, and (c) 12.0 nm. A gradual suppression of the Rabi oscillations can be appreciated when going from (a) to (c).

R_n. This is the reason why this variation affects the time where the ED-NED transition occurs. We also observe in Fig. 3 that the Rabi oscillations become "overdamped" with the above mentioned variation, though their period remains essentially the same for all curves.

Notice that not all the interference terms are oscillatory in time, and the contribution $S_{12}^{ne,ne}$ is a monotonically decreasing function of time that goes as t^{-3} (like S_1^{ne} and S_2^{ne}), as will be shown later when we derive the corresponding asymptotic analytic expressions. See Eqs. (42) and (43). The remaining interference terms (the ones that have superscript ex,ne) are oscillatory and are responsible for the oscillatory pattern of

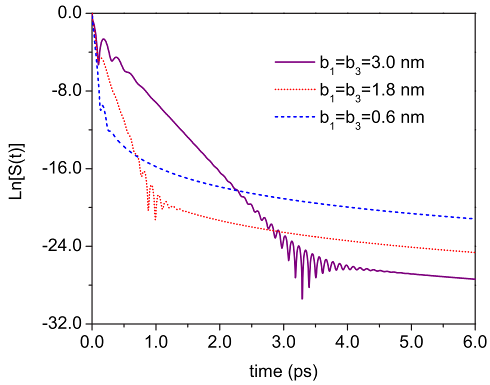

FIG. 3. (Color online) (a) Survival probability for decreasing values of the external barrier widths $b_1 = b_3$ whose values are indicated in the figure. The system parameters are given in the text. (b) Amplification of the previous plot along the short-time region. Note that the first Rabi oscillation in all curves is in phase.

the ED-NED transition. In recent work,[8,16] it is shown that, in the single resonance approximation, the interference term $S_{11}^{ex,ne}$ (in the notation of this paper) is the only term responsible for these oscillations. Although in the present two-level system, $S_{11}^{ex,ne}$ is not the only responsible for these oscillations, it certainly is the dominant contribution here in view of the fact that the decay of the system is governed predominantly by the lowest energy state, i.e., $n=1$, during the decay process.

2. Role of the exponential and nonexponential terms

The purpose of this section is to clarify the relevant contributions to $S(t)$ by analyzing each term separately. In order to illustrate the dominant contributions to $S(t)$ in the different regimes, in Fig. 4 we plot the exponential contributions S_1^{ex} (dotted line) and S_2^{ex} (dashed line) (except $S_{12}^{ex,ex}$, which was analyzed separately in the previous subsection), compared to the full survival probability $S(t)$ (solid line). It is clear from the figure that the leading contribution comes from the term S_1^{ex} and the values of S_2^{ex} are relatively small. The dominance of the lowest resonance state $n=1$ over $n=2$ is due to the fact that the former has a smaller resonance width. This is clear

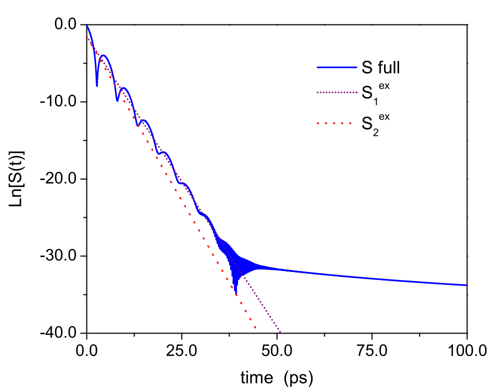

FIG. 4. (Color online) Exponential contributions S_1^e (solid line) and S_2^e (dashed line) to the survival probability $S(t)$ (dotted line). See text.

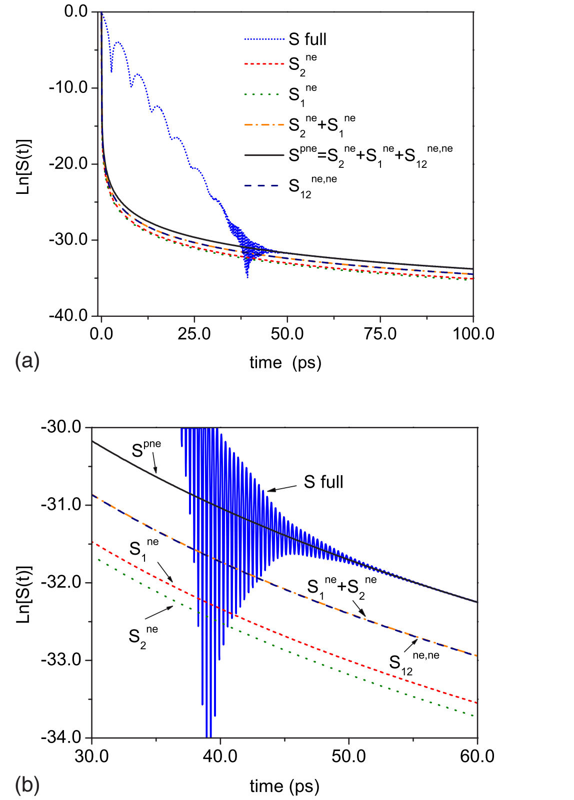

FIG. 5. (Color online) (a) Nonexponential contributions to the full survival probability $S(t)$ as indicated in the figure. (b) Same as (a), showing in more detail the region around the ED-NED transition at long times.

by comparing the slopes of the lines of the figure, whose equations are

$$\ln[S_n^{ex}(t)] = 2 \ln|C_n \bar{C}_n| - \Gamma_n t/\hbar, \tag{39}$$

for $n=1,2$. The first term on the right has essentially the same value for both resonances, and the main difference lies in the slopes $\Gamma_n/\hbar$.

In contrast to the exponential region, in which a single term (S_1^{ex}) gives the dominant contribution to $S(t)$ in the nonexponential region, instead of a single leading term, there are three relevant contributions, namely, S_1^{ne}, S_2^{ne}, and $S_{12}^{ne,ne}$, that are *all* necessary to give the correct description of the survival probability in the long-time nonexponential region. This is another important difference with a single-level system, in which there is a single leading term (S_1^{ne}, whose intersection with S_1^{ex} determines the transition from ED to NED[16]). Here such a transition is given by the intersection of S_1^{ex} and the sum $S^{pne} = S_1^{ne} + S_2^{ne} + S_{12}^{ne,ne}$ (which we call "purely nonexponential contributions" in order to distinguish it from other oscillatory contributions with superscript ex,ne).

In Fig. 5(a), we illustrate all the pure nonexponential contributions compared to the full survival probability $S(t)$ (dotted line). An amplification of the ED-NED transition shown in this graph is depicted in Fig. 5(b), in a scale that enhances the details around the transition. Here, we observe that S^{pne} successfully reproduces the values of the survival probability in the NED regime, and hence its intersection with S_1^{ex} can be used to determine such a transition (this task will be performed in the next section). The relative importance of each of the three terms involved in the S^{pne} can be clearly appreciated in that figure. We note that while S_1^{ne} and S_2^{ne} are the weaker contributions, their sum $S_1^{ne}+S_2^{ne}$ is of the same order than $S_{12}^{ne,ne}$ (actually, in the present example, they are equal). It is interesting to notice in Fig. 5(b) that twice the value of S_1^{ne} (or S_2^{ne}) gives (approximately) the value of $S_{12}^{ne,ne}$, and at the same time, twice the value of $S_{12}^{ne,ne}$ gives the total survival probability $S(t)$ in the NED regime.

3. Asymptotic expressions for S(t) and crossover from ED to NED

It is important to analyze the long-time behavior of $S(t)$. The arguments of the two M functions that appear in the analytic expression of the no-exponential part of $A(t)$, Eq. (21), are such that their corresponding phases lie in the interval $\pi/2<\arg y_n<3\pi/4$ for *proper resonance poles* (i.e., $k_n=\alpha_n-i\beta_n$, with $\alpha_n>\beta_n$), which is the case of all our examples. When this occurs, the M function at long times behave as $M(y_q)\approx -a/(k_q t^{1/2})-b/(k_q^3 t^{3/2})$, with $a\equiv(i/2\sqrt{i\pi})\times(2m/\hbar)^{1/2}$ and $b\equiv(1/2\sqrt{i\pi})(2m/\hbar)^{3/2}$. By using this approximate expression of the M in Eq. (21), we obtain the following asymptotic expression for $A_n^{ne}(t)$:

$$A_n^{ne}(t)\approx -\frac{e^{i\pi/4}}{\sqrt{\pi}}\left(\frac{2m}{\hbar}\right)^{3/2}\mathrm{Im}\left\{\frac{C_n\bar{C}_n}{k_n^3}\right\}\frac{1}{t^{3/2}}. \quad (40)$$

Using Eq. (40), we can explicitly write asymptotic expressions for all the time-dependent terms of $S(t)$ that contain the factor $|A_m^{ne}(t)|$. These terms are

$$S_{mn}^{ex,ne}(t)\approx -\frac{2}{\sqrt{\pi}}\left(\frac{2m}{\hbar}\right)^{3/2}|C_m\bar{C}_m|\mathrm{Im}\left\{\frac{C_n\bar{C}_n}{k_n^3}\right\}e^{-\Gamma_m t/2\hbar}\times\cos\left[\frac{\mathcal{E}_m t}{\hbar}-\eta_m+\frac{\pi}{4}\right]\frac{1}{t^{3/2}}, \quad (41)$$

and similarly for $S_{mn}^{ne,ex}(t)$,

$$S_n^{ne}(t)\approx\frac{1}{\pi}\left(\frac{2m}{\hbar}\right)^3\mathrm{Im}\left\{\frac{C_n\bar{C}_n}{k_n^3}\right\}^2\frac{1}{t^3}, \quad (42)$$

$$S_{mn}^{ne,ne}(t)\approx\frac{2}{\pi}\left(\frac{2m}{\hbar}\right)^3\mathrm{Im}\left\{\frac{C_m\bar{C}_m}{k_m^3}\right\}\mathrm{Im}\left\{\frac{C_n\bar{C}_n}{k_n^3}\right\}\frac{1}{t^3}. \quad (43)$$

Notice that, as anticipated in the previous sections, there are oscillatory terms whose envelope goes as $t^{-3/2}$, and others that decrease monotonically as t^{-3}. The former dominate in the transition (and rapidly oscillating) region, and the latter dictate the long-time nonexponential behavior of $S(t)$. The numerical findings of the previous section that $S_n^{ne}\approx S_m^{ne}\approx(1/2)S_{mn}^{ne,ne}$ can easily be understood from the above asymptotic expressions, if we consider the fact that in systems where the nth and mth resonances are too close, the factors $\mathrm{Im}[C_m\bar{C}_m/k_m^3]$ and $\mathrm{Im}[C_n\bar{C}_n/k_n^3]$ have very close values.

As discussed above, the ED-NED transition may be determined by the crossover of S^{pne} and S_1^{ex}. Using the asymptotic expressions (42) and (43), the quantity S^{pne} can be written as

$$S^{pne}(t)\approx\frac{1}{\pi}\left(\frac{2m}{\hbar}\right)^3\left[\mathrm{Im}\left\{\frac{C_m\bar{C}_m}{k_m^3}\right\}+\mathrm{Im}\left\{\frac{C_n\bar{C}_n}{k_n^3}\right\}\right]^2\frac{1}{t^3}. \quad (44)$$

An alternative expression for S^{pne} can be written in terms of the complex energy pole $E_n=(\mathcal{E}_n-i\Gamma_n/2)$, namely,

$$S^{pne}(t)\approx\frac{\hbar^3}{\pi}\left[\frac{|C_m\bar{C}_m|}{|E_m|^{3/2}}\cos(y_m)+\frac{|C_n\bar{C}_n|}{|E_n|^{3/2}}\cos(y_n)\right]^2\frac{1}{t^3}, \quad (45)$$

where $y_i=(\eta_i-3\theta_i)$, with $(i=m,n)$, and $\theta_n\equiv\arg k_n$. For very close resonances, the above simplifies to

$$S^{pne}(t)\approx\frac{\hbar^3}{\pi}\frac{4|C_n\bar{C}_n|^2}{|E_n|^3}\cos^2(\eta_n-3\theta_n)t^{-3}, \quad (46)$$

which can be rewritten in lifetime units ($\tau\equiv t/\tau_n\equiv t\Gamma_n/\hbar$) as

$$S^{pne}(\tau)\approx\frac{4|C_n\bar{C}_n|^2}{\pi}\frac{\cos^2(\eta_n-3\theta_n)}{[R_n^2+1/4]^{3/2}}\tau^{-3}, \quad (47)$$

where $R_n\equiv\mathcal{E}_n/\Gamma_n$.

To find the critical time τ_c for the transition from ED to NED, the above expression for S^{pne} can be combined with the expression for S_n^{ex} in lifetime units, namely,

$$S_n^{ex}(\tau)=|C_n\bar{C}_n|^2e^{-\tau}. \quad (48)$$

It is straightforward to see that their intersection is given by the formula

$$3\ln(\tau_c)=\tau_c+b(R_n), \quad (49)$$

where

$$b(R_n)=\ln\left\{\frac{4}{\pi}\frac{\cos^2(\eta_n-3\theta_n)}{[R_n^2+1/4]^{3/2}}\right\}. \quad (50)$$

Equations (49) and (50) generalize a result for the long-time exponential-nonexponential transition involving a single resonance.[25,26] According to the obtained formula, for different values of the ratio R_n, the transitions from ED to NED are given by the crossover of a simple logarithmic function with a family of parallel straight lines with unity slope.

Consistent with our numerical results, the shift of both the asymptotic value of $S(t)$ and the transition ED-NED observed in Fig. 3 under variations of R_n is explained by the R_n dependence of both the value of S_n^{pne} and of the transcendental equation found above that gives the transition time τ_c.

B. Decaying Rabi oscillations in other multibarrier systems

In this section, we give a couple of further examples for the behavior of the survival probability for other multibarrier

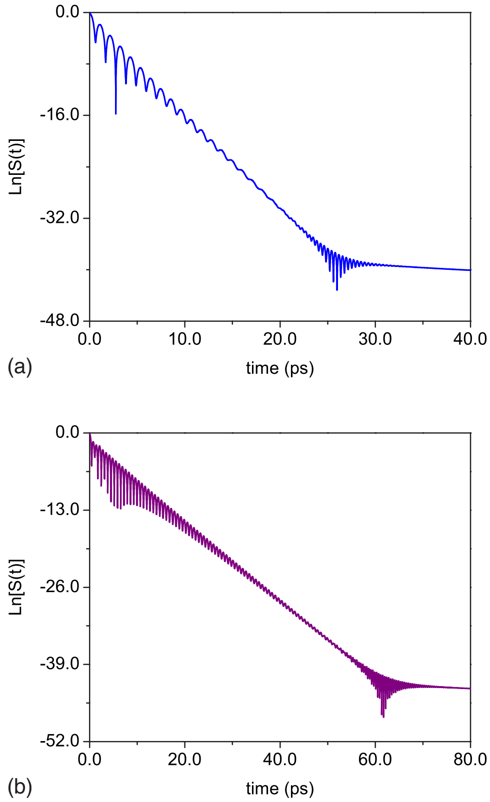

FIG. 6. (Color online) (a) Time evolution of the survival probability $S(t)$ for a multibarrier system with (a) four barriers and (b) eight barriers. See text for the corresponding parameters.

systems, namely, a system with four barriers and another with six barriers, to show that the decaying Rabi oscillations are indeed a distinctive feature of decay in multibarrier systems.

Figure 6(a) shows the natural logarithm of survival probability versus time for a system with four barriers. The two external barrier widths have the values $b_1 = b_4 = 16.0$ nm, the two internal barrier widths have $b_2 = b_3 = 3.0$ nm, all the three well widths possess the same value $w_0 = 1.6$ nm, and the barrier heights have $V_0 = 0.12$ eV. The initial state consists also of a sine pulse, given by Eq. (35), seated on the central well. Figure 6(b) also shows a plot of the natural logarithm of survival probability as a function of time for a multibarrier system with six barriers. Using a similar notation for the potential parameters as in the previous example, the parameters for the widths of barriers and wells and barrier heights are, respectively, $b_1 = b_6 = 1.4$ nm, $b_2 = b_3 = b_4 = b_5 = 2.5$ nm, $w = 16.0$ nm, and $V_0 = 0.12$ eV. One notices in both figures that the decaying Rabi oscillations dominate over purely exponential decay, particularly along the early stages of decay.

V. CONCLUDING REMARKS

Using a resonance formalism, we have derived analytical expressions for the exponential and nonexponential contributions to the survival probability in multibarrier resonance systems. It is worth emphasizing the result for the exponential-nonexponential transition at long times, given by Eqs. (49) and (50), that generalizes a result for isolated resonances. The main result of this work is, however, the modification of exponential decay into a nonexponential behavior due to the contribution of decaying Rabi oscillations. As follows from inspection of Eq. (29), the decaying Rabi oscillations arise from the exponential-exponential interference of closely lying resonances in multibarrier systems. We stress that this contribution differs from the nonexponential behavior of the survival probability at very short times, which depends on the very high components of the energy spectra of the system. From our analysis, we may conclude that decaying Rabi oscillations are a distinctive feature of decay in multibarrier systems. We have also shown that by appropriately varying the distinct parameters of the multibarrier system, one may design systems with specific values for the Rabi frequency and/or the exponential-nonexponential transition. Although the examples presented here were inspired by parameters typical of finite superlattices, we would like to emphasize that our results hold, in general, for decaying multibarrier quantum systems.

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