

Loschmidt echoes in two-body random matrix ensembles

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Fidelity decay is studied for quantum many-body systems with a dominant independent particle Hamiltonian resulting, e.g., from a mean field theory with a weak two-body interaction. The diagonal terms of the interaction are included in the unperturbed Hamiltonian, while the off-diagonal terms constitute the perturbation that distorts the echo. We give the linear response solution for this problem in a random matrix framework. While the ensemble average shows no surprising behavior, we find that the typical ensemble member as represented by the median displays a very slow fidelity decay known as “freeze.” Numerical calculations confirm this result and show that the ground state even on average displays the freeze. This may contribute to explanation of the “unreasonable” success of mean field theories.

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I. INTRODUCTION

Wigner¹ proposed the use of random matrix theory (RMT), in particular, of the classical ensembles^{2,3} to explain statistical properties of nuclear spectra. Balian⁴ showed that this amounts to using a minimum information approach, where symmetry is the only characteristics taken into account. An essential element for the widespread success⁵ resides in the ergodicity of the results obtained for classical ensembles. Yet, it was soon noticed^{6,7} that the essential two-body character of the underlying interaction requires embedded ensembles, in particular, the two-body random matrix ensemble (TBRE). While the original work carried the full weight of a three dimensional few nucleon system, soon abstract models with structureless fermions were introduced⁸ to understand more easily embedded ensembles in general and the TBRE, in particular. Two problems beset these studies: one is the apparent nonergodicity of these ensembles, the other is the difficulty to obtain any analytical results (see Ref. 9 for a review). After individual unfolding of each spectrum,^{3,7} spectral statistics of the classical ensembles are recovered and this marginated the interest in embedded ensembles for some time. Interest was rekindled recently mainly by studies of interesting properties at the edge of the spectra concerning ground states.^{10–12} The question whether the ensemble is actually ergodic or not remains open,⁹ but French^{9,13} has shown that it is somewhat academic, as even the spectral density has fluctuations of the order $1/\log N$, where N is the dimension of the Hilbert space. This makes individual unfolding essential for any statistical analysis of spectra. The reason why individual unfolding does lead to the right answers is not understood to this day. The renewed interest in the TBRE has spread to various fields including applications^{9,14,15} and extensions to bosons.^{9,16}

Fidelity analysis for the stability of quantum systems under perturbation, first proposed by Peres,¹⁷ has become very fashionable since the advent of quantum information where it provides a standard criterion¹⁸ of stability; for a recent review, see Ref. 19. It seems reasonable to check how a

small residual two-body interaction affects the stability of the solution of an independent particle Hamiltonian. The diagonal part of this interaction does not affect the eigenstates and is usually included in the independent particle term, known as the mean field approximation.

We focus on the validity of the mean field approximation, given by the stability of the time-dependent solution of the mean field Hamiltonian which is measured by fidelity. This is also the main physical motivation behind defining the random matrix model in the present paper.

Fidelity has been calculated by the method of stationary phase applied to time-dependent propagator for semiclassical considerations^{20–23} and by supersymmetric techniques^{24,25} for a random matrix model presented in Ref. 26. In both contexts, most of the relevant results can be obtained by perturbative or linear response techniques, which actually do not require either model as a basis.^{19,26–30} Even in the regime where fidelity is not anymore close to 1, a simple exponentiation of the leading term yields excellent results far beyond this regime.^{19,29} Crossovers between semiclassical and perturbative considerations (the latter also known as Fermi golden rule regime) have been discussed in Refs. 31 and 32. A drastic reduction of fidelity decay termed “fidelity freeze” occurs when the perturbation is strictly off diagonal (or more generally, when its time average is zero).^{30,33,34} Such a situation can occur when perturbation breaks the antiunitary symmetry (like time reversal) of the unperturbed Hamiltonian in an optimal way,³⁰ which can perhaps be experimentally realized by some magnetic interactions.

In the case of high-level fidelity freeze, the dynamics is stable for a very long time. The question is whether we can see this effect in the situation under consideration. On the one hand, our system looks promising, due to the inclusion of the diagonal part of the interaction in the “unperturbed” independent particle Hamiltonian. On the other hand, the occurrence of the freeze in random matrix models depends heavily on level repulsion implicit in a Gaussian orthogonal ensemble or a Gaussian unitary ensemble (GUE) for the unperturbed system. Clearly, the independent particle model

does not have this repulsion and indeed we can expect results similar to those of a random spectrum where absence of a diagonal perturbation to lowest order only suppresses the quadratic decay.¹⁹

Our analysis will show that the ensemble average does not display fidelity freeze. The typical ensemble member, however, does show this freeze. This is confirmed when we consider the median of fidelity rather than its average. This fact is important by itself as it shows that quantum evolution of the full problem will follow that of the independent particle problem or mean field approximation for a very long time for most systems. It is also important because it shows that RMT models can describe physical situations qualitatively, even when the average behavior is completely off. The last point can also be reformulated as a necessity to look at the right quantities, i.e., quantities whose distributions have no long tails. If we had replaced fidelity by distortion as introduced in an elastic problem,³⁵ then the average and median would coincide. This results because distortion is defined as the logarithm of 1 minus fidelity.³⁶ Indeed, in other contexts, taking averages of the logarithm is a common remedy to eliminate exaggerated effects of tails in distributions.³⁷ Furthermore, we will see that for the ground state and the first excited state of the independent particle model, the freeze will occur even on average.

After introducing basics about fidelity and the Hamiltonian, we will see that under specific assumptions for the independent particle spectrum, we can obtain ensemble averaged fidelity decay of a TBRE in the linear response regime. The result essentially yields the linear decay. Yet, a numerical inspection of individual Hamiltonians in the ensemble shows the existence of fidelity freeze. A more careful numerical analysis shows that the median fidelity for the ensemble indeed displays the freeze. Thus, we can consider it as typical. This result is quantitatively reconfirmed by showing that the ensemble averaged logarithm of 1 minus fidelity also displays the freeze. We relate this behavior directly to the existence of a gap in the nearest neighbor spacing distribution of the independent particle spectrum. Furthermore, we find that the fidelity decay of the ground and first excited states of the independent particle Hamiltonian displays the freeze even on average. Next, we numerically check different options for the single-particle spectrum to obtain a better feeling of the nonergodicity, which plays a central role in this context. Finally, we give some conclusions.

II. FIDELITY AMPLITUDE

The fidelity amplitude measures the overlap between two quantum states $\psi_0(t')$ and $\psi(t')$ evolving from the same initial state $\psi(0)$ but propagated with slightly different Hamiltonians H_0 and $H=H_0+\lambda V$, respectively,

$$f(t') = \langle \psi_0(t') | \psi(t') \rangle = \langle \psi(0) | e^{(i\hbar)^{-1}H_0 t'} e^{-(i\hbar)^{-1}H t'} | \psi(0) \rangle. \quad (1)$$

For small perturbation strength λ , it is convenient to use a linear response or Born expansion.^{19,27,28} We shall follow the notation established in Ref. 26 as it is well adapted to RMT. In the interaction picture, the wave function reads as $x(t')$

$= e^{(i\hbar)^{-1}H_0 t'} \psi(t')$. Time t' can be replaced by dimensionless time t measured in units of the Heisenberg time $t_H = 2\pi\hbar/d$, where d denotes the average level spacing in the spectra of H_0 which can and will be set to 1. Therefore, the time will always be given in units of t_H . Time evolution of a state $x(t)$ up to the second order $x^{(2)}(t) = X(t)x(0)$ in perturbation strength λ can be expressed as

$$X(t) = 1 - 2\pi\lambda i \int_0^t dt_1 V_I(t_1) - (2\pi\lambda)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 V_I(t_1) V_I(t_2), \quad (2)$$

where $V_I(t)$ is the abbreviation for the perturbation in the interaction picture $V_I(t) = e^{2\pi i H_0 t} V e^{-2\pi i H_0 t}$. Please note that the expansion in Eq. (2) is valid even for times much longer than the Heisenberg time ($t \gg 1$), if only the perturbation strength is small enough.

The fidelity amplitude for an initial state $|x(0)\rangle = \sum_{\alpha} x_{\alpha} |\alpha\rangle$ written as a superposition of independent particle states, i.e., state eigenstates of the unperturbed Hamiltonian $|\alpha\rangle$, then reads

$$f(t) = \langle x(0) | X(t) | x(0) \rangle = \sum_{\alpha, \beta=0}^{N-1} X_{\alpha\beta} x_{\alpha}^* x_{\beta}. \quad (3)$$

Later, we consider ensemble averages. Then, the linear term in Eq. (2) vanishes and the matrix element $X_{\alpha\beta}$ is reduced to contributions from the quadratic term only.

III. HAMILTONIAN

Although a simple RMT of full Gaussian matrices often yields impressively accurate results, it, strictly speaking, only applies to dynamical systems where all levels are coupled by the interaction. More realistic physical systems such as nuclear or atomic shell models or quantum dots do not possess that property. The interaction involves two particles only. In the framework of RMT, it should be described by TBRE.^{3,6,7,9,14} We consider a system of M “spinless”³⁸ Fermi particles in N orbitals in the presence of fermion-fermion interaction. We shall use second quantization notation with fermion creation and annihilation operators c_i^{\dagger} and c_i fulfilling the usual anticommutation relations and creating or annihilating a fermion in the i th eigenstate of the single-particle Hamiltonian defined by the mean field. The M particle eigenstate $|\alpha\rangle = (c_0^{\dagger})^{\alpha_0} \cdots (c_{N-1}^{\dagger})^{\alpha_{N-1}} |0\rangle$ (where exactly M binary digits α_i of integer α are equal to 1) of the independent particle Hamiltonian is then a product of M creation operators with distinct indices applied to a vacuum state. The Hamiltonian is written in the usual manner as

$$H = \sum_i e_i c_i^{\dagger} c_i + \lambda \sum_{i < j, k < l} V_{ijkl} c_i^{\dagger} c_j^{\dagger} c_l c_k, \quad (4)$$

where all Latin indices are running from 0 to $N-1$, and where the e_i are ordered single-particle energies, V_{ijkl} are properly antisymmetrized two-body matrix elements, and λ is the perturbation strength. The unperturbed Hamiltonian H_0

will correspond to the mean field approximation and will hence be chosen as the one-body terms $\sum_i e_i c_i^\dagger c_i$ plus the diagonal part of the interaction $\lambda \sum_{i<j} V_{ijij} c_i^\dagger c_j^\dagger c_j c_i$. A natural question is how the remaining part of V , which we shall consider as perturbation, affects the dynamics.

The inclusion of the diagonal part of the interaction in the unperturbed system is crucial to our argument. On one hand, it does not affect the eigenfunctions of the one-particle term and thus it seems of little importance. Yet, in the calculation of fidelity, the dephasing it produces becomes the dominant term. On the other hand, if included in the unperturbed part, it will only enter the result to the fourth order in λ because the spectral two-point function will only enter to order λ^2 and it, in turn, will only be affected to order λ^2 by the diagonal terms.

So far, we have not put any constraints on the spectral properties of H_0 nor on the perturbation V . The latter consists of independent two-body matrix elements, where the weight coefficients V_{ijkl} are chosen as independent random Gaussian complex numbers with vanishing mean and $\langle V_{ijkl} V_{mnop} \rangle_V = v^{-2} \delta_{ij,op} \delta_{kl,mn}$, where the variance v^{-2} is set to normalize the perturbation as $\langle \text{tr}[V^2] \rangle = \mathcal{N}$, where $\mathcal{N} = \binom{M}{2}$ is the dimension of full Hilbert space. The unperturbed dynamics is given by N single-particle levels e_j . We shall often choose them as the eigenvalues of the $N \times N$ Hermitian matrix chosen from the GUE, but we shall mention other options and their consequences. Similar models have recently been widely studied, e.g., in nuclear physics³⁹ and for studying chaotic quantum dots.⁴⁰

Two-body operators $c_i^\dagger c_j^\dagger c_j c_k$ can be split into three groups by inspecting indices i, j, k, l : diagonal terms with two pairs of equal indices, three-orbital terms with one pair of equal indices, and four-orbital terms without pairs of equal indices. The nonzero matrix elements $V_{\alpha\beta}$ in the full Hilbert space therefore couple many-particle states which differ by at most two single-particle states and these states are the only intermediate states in a twofold transition described by $V_I(t_1)V_I(t_2)$ in Eq. (2). After averaging over an ensemble of two-body interactions, all products of independent V_{ijkl} vanish and our interest is narrowed to transitions with the same initial and final states described by averaged matrix elements $\langle [V(t_1)V(t_2)]_{\alpha\beta} \rangle_V = \delta_{\alpha\beta} C_\alpha(t_1 - t_2)$ which only depend on the time difference and are denoted by the *correlation function* $C_\alpha(\tau)$. Using $H_0 \equiv \text{diag}(h_\alpha)$, which h_α is either single-particle Hamiltonian $h_\alpha = \langle \alpha | \sum_i \hat{n}_i e_i | \alpha \rangle$, $\hat{n}_i = c_i^\dagger c_i$, or the mean field Hamiltonian $h_\alpha = \langle \alpha | \sum_i \hat{n}_i e_i + \lambda \sum_{i<j} V_{ijij} \hat{n}_i \hat{n}_j | \alpha \rangle$, the correlation function can be written as

$$\begin{aligned} C_\alpha(\tau) &= \sum_\gamma \langle e^{2\pi i h_\alpha V_{\alpha\gamma}} e^{-2\pi i h_\gamma V_{\gamma\alpha}} \rangle_V \\ &= \sum_\gamma \langle e^{-2\pi i (h_\gamma - h_\alpha) \tau} \rangle_V \langle V_{\alpha\gamma} V_{\gamma\alpha} \rangle_V. \end{aligned} \quad (5)$$

The first average is nontrivial in the mean field case where the energy differences $h_\gamma - h_\alpha$ contain also diagonal two-body terms (mean field). The correction to $e^{-2\pi i \sum_j (\langle \gamma | \hat{n}_j | \gamma \rangle - \langle \alpha | \hat{n}_j | \alpha \rangle) e_j \tau}$ due to mean field is of order $\mathcal{O}(\lambda^2)$. As mentioned above, since the correlation function will enter the linear response

formula for fidelity²⁶ with a prefactor λ^2 , the overall correction to the fidelity amplitude is $\mathcal{O}(\lambda^4)$ and will be neglected. We split the $C_\alpha(\tau)$ into three groups according to the classification of two-body transitions

$$\begin{aligned} C_\alpha(\tau) &= \langle \alpha | v^{-2} \sum_{kl} c_k^\dagger c_l^\dagger c_l c_k c_k^\dagger c_l^\dagger c_l c_k \\ &\quad + v^{-2} \sum_{i \neq k, j} c_k^\dagger c_j^\dagger c_j c_i c_i^\dagger c_j^\dagger c_j c_k e^{-2\pi i (e_i - e_k) \tau} \\ &\quad + v^{-2} \sum_{i < j \neq k < l} c_k^\dagger c_l^\dagger c_j c_i c_i^\dagger c_j^\dagger c_l c_k e^{-2\pi i (e_i + e_j - e_k - e_l) \tau} | \alpha \rangle. \end{aligned} \quad (6)$$

The abbreviation in the last sum means that indices i, j, k, l are all different. The first term of Eq. (6) is actually only present if we would be interested in considering the full two-body term as perturbation and thus consider the corresponding dephasing.⁴¹ Partially summing the expression yields

$$\begin{aligned} C_\alpha(\tau) &= v^{-2} \binom{M}{2} + v^{-2} (M-1) \sum_{ik} n_k \bar{n}_i e^{-2\pi i (e_i - e_k) \tau} \\ &\quad + v^{-2} \frac{1}{4} \sum_{ijkl} \bar{n}_i \bar{n}_j n_k n_l e^{-2\pi i (e_i + e_j - e_k - e_l) \tau}. \end{aligned} \quad (7)$$

Here, $n_i = \langle \alpha | c_i^\dagger c_i | \alpha \rangle$ indicates the Fermion occupation number and we use the convention $\bar{n}_i = 1 - n_i$. The relation between the fidelity amplitude and the correlation function after averaging over the interaction reads

$$\langle f(t) \rangle_V = \sum_\alpha |x_\alpha|^2 \left[1 - 4\pi^2 \lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 C_\alpha(t_1 - t_2) \right]. \quad (8)$$

If the spectrum of H_0 is known, this allows us to determine the fidelity amplitude by integrating the correlation function in Eq. (6) twice for each eigenstate α and evaluate the sum in Eq. (8). Actually, we are more interested in ensemble behavior which we shall study in the next two sections.

IV. ENSEMBLE AVERAGED FIDELITY AMPLITUDE

Relation (8) can be simplified if we average over initial states or select a random initial state by choosing Gaussian random coefficients x_α normalized by the requirement $\langle |x_\alpha|^2 \rangle_0 = 1/\mathcal{N}$. The correlation function can be averaged as $C(\tau) = \frac{1}{\mathcal{N}} \sum_\alpha \langle C_\alpha(\tau) \rangle_0$ which implies averaging over all distributions of M particles on N orbitals. We find that

$$\begin{aligned} C(\tau) &= \frac{v^{-2}}{\mathcal{N}} (M-1) \binom{N-2}{M-1} \sum_{m \neq n} \langle e^{-2\pi i (e_m - e_n) \tau} \rangle_0 \\ &\quad + \frac{v^{-2}}{\mathcal{N}} \binom{N-4}{M-2} \sum_{m \neq n \neq p \neq q} \langle e^{-2\pi i (e_m + e_n - e_p - e_q) \tau} \rangle_0. \end{aligned} \quad (9)$$

A closer inspection of first term in Eq. (9) shows that it is connected with the probability distribution of pairs of energies of two *arbitrary* orbitals which is given by Dyson's

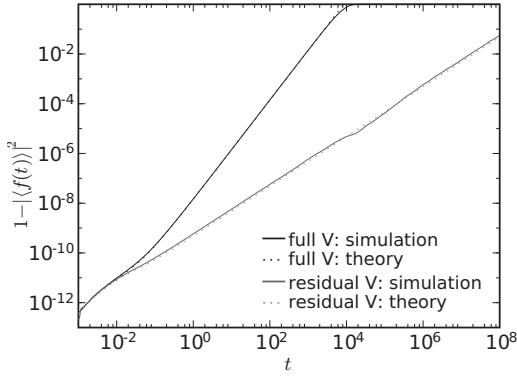


FIG. 1. Fidelity amplitude decay for 6 particles in 12 orbitals averaged over 2×10^5 realizations of V , H_0 , and initial states, with the perturbation strength $\lambda = 10^{-4}$. For sufficiently long times, the decay is quadratic in the case of full perturbation (upper) and linear in the residual case (lower). Initial states are taken from the center of spectra.

two-point correlation function R_2 following the definition in Ref. 42. The average can then be obtained by integration. A similar argument connects the second average with the four-point correlation function R_4 . Finally, using the normalization factor $v^2 = \binom{M}{2} [2(N-M) + \binom{N-M}{2}]$, we rewrite the correlation function in compact form (using $\eta = \frac{N-M-1}{4}$)

$$C(t) = \frac{1}{1+\eta} \mathcal{F}[R_2] \left(\frac{2\pi\sqrt{2t}}{d} \right) + \frac{1}{1+\eta^{-1}} \mathcal{F}[R_4] \left(\frac{2\pi\sqrt{2t}}{d} \right). \quad (10)$$

The functions $\mathcal{F}[R_n](t)$ are obtained by Fourier transformation of the correlation functions R_n (see Appendix for precise definitions). In the case of GUE single-particle spectra, the functions $\mathcal{F}[R_n](t)$ can be calculated analytically and are given as products of finite polynomials in t and Gaussians (see Appendix). They are normalized such that $\mathcal{F}[R_{2,4}](0) = 1$.

The simulations have been performed both including and excluding the diagonal terms of the two-body interaction and coincide with the theoretical prediction in Eq. (10) (or the corresponding full calculation) if the random states are chosen from the whole energy spectrum or from its center. The ensemble and state averaged fidelity amplitude can now be obtained by integrating the averaged correlation function in Eq. (10) twice

$$\langle f(t) \rangle = 1 - 4\pi^2 \lambda^2 \int_0^t dt_1 \int_0^{t_1} dt_2 C(t_1 - t_2). \quad (11)$$

In the *residual case* where the perturbation V has no diagonal terms (i.e., in the case of mean field unperturbed Hamiltonian), the correlation function has no constant term. The correlation integral can then be approximated by a linear function for fairly long times as long as the linear response formalism is justified. The simulation (Fig. 1) is done with random initial states from the center of the spectra and is in agreement with the full-spectra prediction. The residual case thus coincides qualitatively with the result of a general

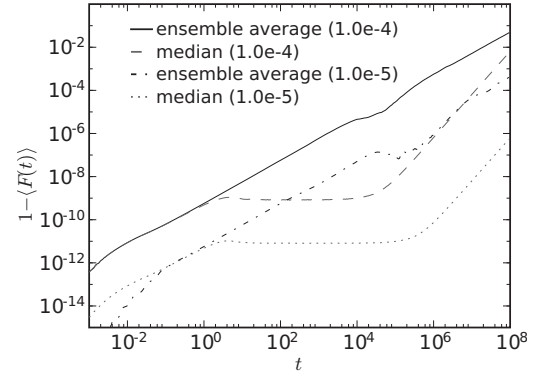


FIG. 2. Average fidelity in 10^5 realizations of H_0 and V compared to the median fidelity for 6 particles in 12 orbitals and with two values of perturbation strengths $\lambda = 10^{-4}$ and $\lambda = 10^{-5}$. The median fidelity decay (long-dashed, dotted) shows a plateau practically absent in the ensemble average (full, short-dashed curves).

residual perturbation of a Hamiltonian with a random spectrum^{19,30} also known as Poissonian orthogonal ensemble.⁴³ Fidelity freeze is not found and we could easily conclude that this exercise is rather disappointing. Yet, considering the known nonergodicity of the TBRE, it seems worthwhile to check whether the average behavior reflects the typical behavior.

V. FIDELITY DECAY FOR A TYPICAL ENSEMBLE MEMBER

We therefore proceed to analyze the median of fidelity decay. For this purpose, we use fidelity $F(t) = |f(t)|^2$, which is a real quantity, instead of fidelity amplitude $f(t)$. (We note, however, that the results are practically the same for the real part of fidelity amplitude.) We define *median fidelity* $F_m(t)$ such that at any time, $F_m(t)$ is lower than fidelity $F(t)$ in half of the realizations. For a randomly chosen member of the ensemble, there is thus a 0.5 probability that a plateau in fidelity decay will last longer than the median fidelity plateau (Fig. 2).

A useful alternative in such situation is to consider the average of the logarithm of the quantity under consideration—in our case, $\langle \ln[1 - F(t)] \rangle$. This quantity has been used in elastodynamics under the name distortion as we have discussed in the Introduction. The distortion shown in Fig. 4 in the following section has essentially the same feature as the median. Note that it is not obvious how to obtain analytical results for either of these quantities but the latter may be slightly more accessible.

It remains to be understood how the freeze, which is present in most realizations, comes about and is eventually completely averaged out. To elucidate this problem, fidelity of a few individual ensemble members is presented (Fig. 3). However, the cases shown in the figure are not all equally probable; the upper one occurs rarely, whereas the lower two curves truly represent the majority of cases. The upper left inset of Fig. 3 shows the beginning of the cumulative level spacing distribution of H_0 corresponding to the fidelity curves shown in the main figure. Interestingly, the plateau

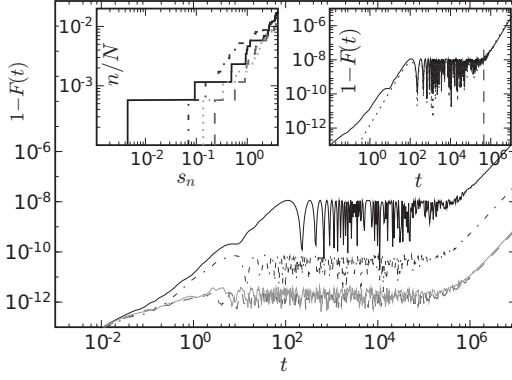


FIG. 3. Fidelity decay in a few individual ensemble members for the residual case with 6 particles and 12 orbitals. The upper left inset shows a small region of the corresponding cumulative level spacing distribution of the unperturbed spectra H_0 . (We plot a relative number of level spacing n/N below s_n .) The agreement with the minimum spacing only theory (dotted) is shown in the upper right inset; dashed line denotes the ending time t_e [Eq. (16)].

properties of the exceptional large deviation cases with high plateaus are determined by the minimum energy spacing alone and the conjecture is that the plateau vanishes together with the smallest level spacing.

Another peculiarity results from the two-body nature of interaction which only connects certain levels, as shown in Sec. IV. Transitions involving three orbitals will very likely have big energy difference because of the level repulsion in single-particle spectra; this contrasts with the transitions involving four orbitals that can easily have small energy differences if one particle lowers the energy by roughly the same amount by which the other one raises it. The effect of a such two-body operator will be greater than the effect of any other operator.

Considering the established connection between the level spacing of four-orbital two-body operators and the fidelity amplitude plateau, an illustration is made where all three-orbital terms are omitted. Each four-orbital two-body operator $c_i^\dagger c_j^\dagger c_l c_k$ is in a unique way connected with the energy difference $\mu_{ijkl} := e_i + e_j - e_k - e_l$ in the spectrum of H_0 and the four-orbital part of the correlation function (6) averaged over the perturbation becomes a sum over all possible positive spacings

$$C_\alpha(t) = v^{-2} \sum_{i < j \neq k < l} [\langle \alpha | \bar{n}_i \bar{n}_j n_l n_k e^{-2\pi i |\mu_{ijkl}|} | \alpha \rangle + \langle \alpha | \bar{n}_k \bar{n}_l n_j n_i e^{2\pi i |\mu_{ijkl}|} | \alpha \rangle]. \quad (12)$$

Assuming random initial states, averaging can be performed,

$$\frac{1}{N} \sum_\alpha C_\alpha(t) = \frac{8(N-4)!}{N! \left(1 + \frac{4}{N-M-1}\right)^q} \sum_q \cos 2\pi \mu_q t, \quad (13)$$

and the fidelity amplitude is obtained by integration (11) as

$$f(t) = 1 - \lambda^2 \frac{8(N-4)!}{N! \left(1 + \frac{4}{N-M-1}\right)^q} \sum_q \frac{1 - \cos 2\pi \mu_q t}{\mu_q^2}. \quad (14)$$

If the smallest degenerate spacing μ_0 is orders of magnitude smaller than any other spacing, the sum above can be approximated by the smallest spacing term alone. All other terms are smaller by a factor $(\mu_0/\mu_q)^2$. This approximation is very illustrative and can be used to estimate the beginning time t_b and the shift of the plateau by the condition $\mu_0 t_b = 1$ and by removing the time-dependent part, respectively. Estimation of the ending time t_e of the plateau is a little more tedious. There, we need the fourth-order terms in the linear response formula (2). Following the same principles and keeping the minimum spacing term only, we obtain the fourth-order correction to fidelity of highest order in time (we are interested in very long times) as

$$f^{(4)}(t) = \frac{128\pi^2(N-4)!(N-M-1)}{N!(N-M)M(M-1)(N-M+3)^2} \frac{\lambda^4 t^2}{\mu_0^2}. \quad (15)$$

The ending time t_e can then be estimated by comparing amplitudes of the second- and fourth-order terms which gives the well known^{33,34} λ^{-1} dependence

$$t_e = \frac{1}{2\pi\lambda} \sqrt{\binom{M}{2} (N-M)(N-M+3)}. \quad (16)$$

The ending time does not depend on level spacings, which is again in agreement with the data shown in Fig. 3. If $t_b \sim t_e$, the plateau, which is a pure second-order phenomenon, would begin in the region where the second-order approximation is no longer valid and hence cannot exist. The upper right inset on Fig. 3 shows that the second- and fourth-order terms together describe all important features of the fidelity amplitude decay for the uppermost realization in Fig. 3 with an extremely small level spacing.

We now understand why the plateau, nearly always present in individual realizations, does not appear in the ensemble average. Realizations of the unperturbed spectrum with an extremely small level spacing occur rarely but when averaging the fidelity amplitude, they eventually dominate the behavior and the plateaus are averaged out.

VI. COMPARISON TO OTHER MODELS

The theory we have developed to describe the behavior of the model under consideration can be applied to other physical models and used to understand the differences between them. The key features of the model (4) are (1) level repulsion in single-particle spectra and (2) sparse perturbation. By comparing to models with full random matrix spectra which correspond to chaotic dynamics on the one hand, and independent Poissonian spectra which correspond to regular dynamics on the other hand, we find that our model lies somewhere in between as we can see from Fig. 4. Besides the median fidelity which corresponds to the typical behavior, an alternative measure, namely, the average of the logarithm of

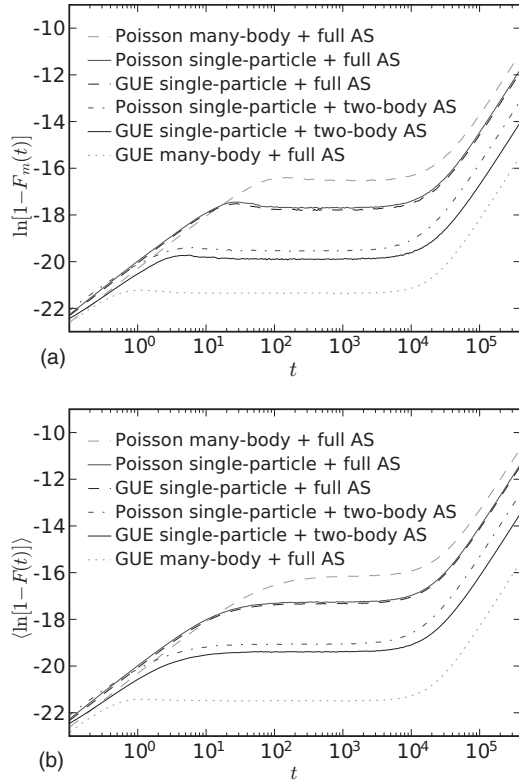


FIG. 4. Median fidelity decay (a) and average natural logarithm of $1-F(t)$ (b) for various models (described in text and labeled in the figure—ordered from top to bottom) for five particles in ten orbitals and perturbation strength $\lambda=10^{-4}$. Perturbation is always imaginary antisymmetric. The results are averaged over 2×10^4 realizations.

$1-F(t)$ (also known as distortion), is considered in parallel. Although not completely identical, they both show essentially the same features and to some extent even quantitative agreement. In order to eliminate any potential effect of diagonal perturbation terms, we consider in the following models always purely imaginary antisymmetric perturbation, individually normalized in the usual way ($\text{tr}[V^2]=\mathcal{N}$). However, we will be switching between the cases of a *full* and a *two-body* perturbation matrix.

The first model under consideration (uppermost) has a random spectrum of H_0 such that the level spacing distribution is Poissonian. The perturbation is chosen from an ensemble of full antisymmetric matrices. The Poissonian level spacing distribution favors small spacings between spectral levels which contribute to fast (initially linear) fidelity decay. In this case the type of perturbation has no significant effect as small spacings are probable for any pair of levels. This is not true for H_0 constructed from independent single-particle spectra which we take as Poissonian or GUE as in our original model (4). For the latter, small level spacing for *some* transitions (e.g., three-orbital two-body transitions) is very improbable. If the perturbation is a full matrix, the effect of such transitions is relatively small and there is no significant difference between the two cases. If, on the other hand, only a few levels are coupled as in the case of two-body interactions, three-orbital transitions in the case of Poissonian

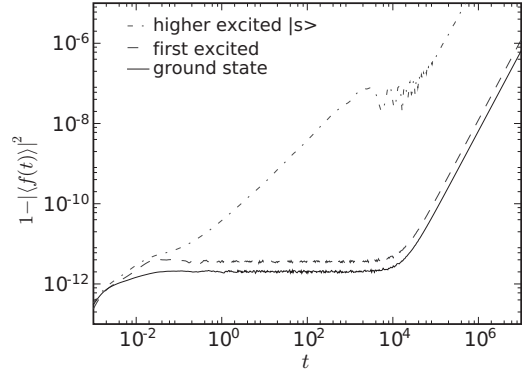


FIG. 5. Fidelity amplitude decay for the ground state and first excited state compared to higher excited state $|s\rangle$ [Eq. (17)] for 6 particles in 12 orbitals and perturbation strength $\lambda=10^{-4}$. The results are averaged over 10^4 realizations of perturbation and spectra.

single-particle spacings *can* and in the case of GUE spacings *cannot* involve small spacings. Thus, in the former case, they cannot be neglected. Typical decay of fidelity is in both cases slower than in the case of full perturbation. For illustration, also a completely different physical situation with random matrix many-body spectrum of H_0 is shown. Because of the level repulsion in many-body spectra, *any* transition will very unlikely have small level spacing. Only then, the plateau in fidelity decay is present not only in a typical case but also on average.

VII. GROUND STATE FIDELITY AMPLITUDE

We have up to now considered only random initial states, but the behavior of the ground state of the independent particle model H_0 , i.e., the Hartree-Fock ground state, under a perturbation formed by residual interactions is also of interest. The correlation function can now no longer be simplified by state averaging. The main problem is to determine the matrix element $[V(t)V(0)]_{gg} = C_g(t)$, where $|g\rangle = c_0^\dagger \cdots c_{M-1}^\dagger |0\rangle$ is the ground state occupying the lowest M single-particle levels. Regardless of the realization of the spectra of H_0 , any nondiagonal two-body operator raises the energy for at least the minimum level spacing in the single-particle spectra where levels are unlikely to be close together. The same incidentally holds in the first excited state with the only difference that there the energy can also be lowered by the same amount. Note that more freedom does exist for higher excited states such as

$$|s\rangle = c_0^\dagger \cdots c_{M-3}^\dagger c_{M-1}^\dagger c_M^\dagger |0\rangle, \quad (17)$$

where the operator $c_{M-2}^\dagger c_{M+1}^\dagger c_M c_{M-1}$ or its Hermitian conjugate increases or decreases energy by $e_{M-2} + e_{M+1} - e_{M-1} - e_M = (e_{M+1} - e_M) - (e_{M-1} - e_{M-2})$ which indeed can be very small.

This leads to the suspicion that the freeze might exist on average for these two lowest independent particle initial states. The numerical results indeed show (Fig. 5) the plateau in the ensemble averaged fidelity amplitude for the ground and the first excited state.

VIII. CONCLUSION

We have analyzed a very elementary many-fermion model in the context of echo dynamics. The model describes spinless fermions whose underlying single-particle dynamics is chaotic and is perturbed by random two-body interactions. Our interest was focused on the stability of a mean field approximation where the diagonal terms of the interaction have been included in the unperturbed Hamiltonian. For weak perturbations, the decay of the fidelity amplitude in this case typically displays a high-level plateau also known as freeze of fidelity. This freeze lasts for times long on the scale of the Heisenberg time. The unexpected point is that the freeze typically present in most realizations of members of the ensemble will vanish on average giving a very dramatic example of the nonergodicity of the TBRE. To see the effect which should dominate most experiments and should hence be observable, we have to consider median behavior or consider logarithmic averages. This fact beyond the specific interest for the TBRE could pave the road for interesting new applications of random matrix theory analyzing nonergodic situations.

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APPENDIX: ANALYTICAL CALCULATION OF THE CORRELATION FUNCTION

The final compact expression for the averaged correlation function $C(\tau)$ [see Eq. (10)] involves Fourier integrals of Dyson's correlation functions $R_n(\mathbf{x})$,

$$\mathcal{F}[R_n](k) = \frac{(N-n)!}{N!} \int_{-\infty}^{\infty} R_n(\mathbf{x}) e^{-ik \sum_{j=0}^{n-1} (-1)^j x_j} d^n \mathbf{x}. \quad (\text{A1})$$

If the correlation functions R_n correspond to GUE, such integrals can be analytically integrated and presented in terms of functions $J_{nm}(t)$:

$$J_{nm}(t) = \int_{-\infty}^{\infty} H_n(x) e^{-x^2} H_m(x) e^{-itx} dx, \quad (\text{A2})$$

where $H_n(x)$ are standard Hermite polynomials. It can easily be shown using recursion that a function $J_{mn}(t)$ can be written as finite series

$$J_{mn}(t) = \frac{e^{-t^2/4} (-it)^{n+m} \min(m,n)}{\sqrt{2^{n+m} n! m!}} \sum_{j=0}^{\min(m,n)} (-2)^j j! \binom{m}{j} \binom{n}{j} \frac{1}{t^{2j}}. \quad (\text{A3})$$

The simplest case is the transformation of $R_1(x) = \sum_{n=0}^{N-1} \frac{1}{2^n n! \sqrt{\pi}} e^{-x^2} H_n(x)^2$ which gives a simple result

$$\mathcal{F}[R_1](t) = \sum_{n=0}^{N-1} J_{nn}(t). \quad (\text{A4})$$

Higher-level correlation functions R_n are expressible by lower level R_n and cluster functions T_2 .⁴² Because \mathcal{F} is a linear transformation, it can also be expressed by lower order transformations where transformations of cluster functions involve terms without the sign factors $(-1)^j$ in the exponential in Eq. (A1), denoted by $\mathcal{F}[T_n]^+$. Straightforward calculation gives

$$\mathcal{F}[T_2]^+(t) = \sum_{nm} J_{nm}(t) J_{nm}(\mp t), \quad (\text{A5})$$

$$\mathcal{F}[T_3]^+(t) = \sum_{nmk} J_{mn}(t) J_{mk}(-t) J_{nk}(-t), \quad (\text{A6})$$

$$\mathcal{F}[T_4]^+(t) = \sum_{mnlk} J_{ml}(t) J_{mn}(t) J_{kl}(-t) J_{nk}(-t). \quad (\text{A7})$$

Finally, following the equality $R_2(x, y) = R_1(x) R_1(y) - T_2(x, y)$ and similarly for R_4 , it finally holds that

$$\mathcal{F}[R_2](t) = \mathcal{F}[R_1](t)^2 - \mathcal{F}[T_2]^-(t), \quad (\text{A8})$$

$$\begin{aligned} \mathcal{F}[R_4](t) &= \mathcal{F}[R_1](t)^4 + \mathcal{F}[T_2]^+(t)^2 + 2\mathcal{F}[T_2]^-(t)^2 \\ &\quad - 2\mathcal{F}[R_1](t)^2 \mathcal{F}[T_2]^+(t) - 4\mathcal{F}[R_1](t)^2 \mathcal{F}[T_2]^-(t) \\ &\quad + 8\mathcal{F}[R_1](t) \mathcal{F}[T_3](t) - 6\mathcal{F}[T_4](t). \end{aligned} \quad (\text{A9})$$

Hence, the averaged correlation function, although not as a compact expression, can be analytically evaluated using the above equations.

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