Energy and temperature of superfluid turbulent vortex tangles

M. S. Mongiovì,[1,*] D. Jou,[2,†] and M. Sciacca[1,‡]
[1]*Dipartimento di Metodi e Modelli Matematici, Università di Palermo, c/o Facoltà di Ingegneria, Viale delle Scienze, 90128 Palermo, Italy*
[2]*Departament de Física, Universitat Autónoma de Barcelona, 08193 Bellaterra, Catalonia, Spain*
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We consider three aspects of turbulent vortex tangles in superfluids. First, we outline some contributions to the Vinen's equation for the time evolution of the vortex line density, related to the presence of pinned vortices incorporating the effects of the walls. Afterwards, we analyze some aspects of the energy balance of the vortex tangle, related to frictional dissipation and to vortex formation and destruction. Finally, we explore the concept of an effective temperature for the vortex tangle, related to the average energy of the vortex loops and to the diffusion coefficient of vortex lines. The combination of these ideas suggests some formal similarities with other kinds of driven nonequilibrium systems characterized by several different temperatures.

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I. INTRODUCTION

It is known that a disordered array of quantized vortex lines is created in counterflow superfluid turbulence,[1–4] i.e., in the turbulence which is present in liquid-helium II under a steady heat flux and no mass transfer. Usually, the vortex tangle is described by introducing a scalar quantity L, the average vortex line length per unit volume [briefly called *vortex line density* and whose dimensions are $(\text{length})^{-2}$]. The most well-known equation in this field is Vinen's equation, which describes the evolution of L, in homogeneous counterflow turbulence:[4,5]

$$\frac{dL}{dt} = \alpha_v V_{ns} L^{3/2} - \beta_v \kappa L^2, \tag{1.1}$$

where $V_{ns}=|\langle \mathbf{V}_{ns}\rangle|$ is the magnitude of the averaged counterflow velocity $\mathbf{V}_{ns}=\mathbf{v}_n-\mathbf{v}_s$ ($\mathbf{v}_n$ and $\mathbf{v}_s$ being the velocities of normal and superfluid components), α_v and β_v are dimensionless constants, and κ is the quantum of vorticity, ascribed by $\kappa=h/m$, with h the Planck constant, and m the mass of ^{4}He atom.

Equation (1.1) has been given a physical microscopic basis by Schwarz, on the basis of statistical considerations on vortex-line dynamics.[6–8] In Sec. II starting from a term neglected in previous microscopic derivations of Vinen's equation, explicitly incorporating the average of the curvature vector of the vortex lines, we obtain a generalized expression of Eq. (1.1) for dilute situations, when the contribution of pinned vortices is not negligible and which describe the effects of the walls on the superfluid.[9–11] In Sec. III we analyze some energetic aspects of the tangle, namely, the total power exerted on the system by the counterflow, and we study which part of it goes into a purely dissipative frictional contribution due to the relative motion of the normal fluid with respect to the vortices and which part is related to the formation and destruction of vortex lines.

Recently, increasing interest is being devoted to other subtler features of the tangle, as, for instance, polarization or anisotropy, because of studies on combined rotation and counterflow[12–16] and numerical simulations of the vortices,[17–20] which allow us to obtain more detailed information. At the same time, statistical descriptions of quantum vortex tangles arising in turbulent superfluids[1–4,21,22] have deserved an increasing attention. A full statistical description of such complex geometrical and topological objects as the vortex tangles would be very demanding, and it is still beyond our fundamental understanding of nonequilibrium systems.[22–24] However, some heuristic attempts to describe some particular features have been carried out. An example concerns the probability distribution of the orientation of the tangent to vortex lines with respect to one given direction in the simultaneous presence of counterflow and rotation by using a simple paramagnetic analogy describing the competition between rotation, which tends to orient the vortices along the rotation direction, and counterflow, which tends to randomize them.[14,16,19,25] Based on this analogy, in Ref. 25, we proposed to define an effective temperature of the vortex tangle, which is different from the temperature of the helium fluid, in the context of the wide current interest on effective temperatures in several kinds of systems in nonequilibrium, as glasses, shaken granular systems, or fluids under shear flow.[26–33]

In Sec. IV we explore the probability distribution of the lengths of vortex loops constituting the tangle and we define an effective temperature given by the average energy of the vortex loops, which at high enough values of L can be measured from experimental observations of the intrinsic fluctuations of the vortex-line density L. In Sec. V it is shown that this definition of effective temperature of the vortex tangle is also closely related, by means of the Einstein relation, to current results on the diffusion coefficient of vortex lines.

II. A GENERALIZED FORM OF THE VINEN'S EQUATION INCORPORATING WALL EFFECTS

A microscopic derivation of Vinen's equation was given by Schwarz[6–8] on the basis of the dynamics of the vortices. In this section we take into account a term in his derivation which was previously neglected on the basis of symmetry arguments, which are in fact valid for vortex loops but not for pinned vortices. The presence of these vortices is especially relevant for dilute turbulence, when the diameter d is of the order—or less—than the average separation between vortices, $L^{-1/2}$, in such a way that wall effects cannot be neglected in the evolution of the vortex tangle. This may be especially interesting for microfluidic flows of superfluids, i.e., in the flow of liquid helium in very narrow channels.

A few years ago, we proposed on phenomenological grounds such a kind of generalization of Vinen's equation to incorporate the effects of the walls. Our proposal had the form[10]

$$\frac{dL}{dt} = \alpha_1 V_{ns} L^{3/2}\left(1 - \omega \frac{L^{-1/2}}{d}\right) - \beta\kappa L^2\left(1 + \omega' \frac{L^{-1/2}}{d} - \omega'' \frac{L^{-1}}{d^2}\right), \quad (2.1)$$

with α_1, ω, ω', and ω'' suitable dimensionless constants, and the coefficient α_1 undergoing a step change at the transition from the TI to the TII turbulent regimes. Our equation included also a production term in V^2 which we will not discuss here for the sake of simplicity. At that time, we were not able to provide a microscopic justification of the new terms, which were proposed on the basis of dimensional analysis. We will see in this section how some of these terms may arise from microscopic considerations.

In the model by Schwarz,[5–8] a quantized vortex line is thought as a classical vortex line in the superfluid with a hollow core of radius a_0 of about 1 Å, and quantized circulation κ. The vortex line is described by a vectorial function $\mathbf{s}(\xi,t)$, ξ being the arc length measured along the vortex filament. The first two derivatives of $\mathbf{s}$ with respect to ξ, which we will denote with a prime, play an essential role in this description: $\mathbf{s}'$ is the unit vector tangent along the vortex line at a given point, and $\mathbf{s}''$ is the curvature vector. Another relevant vector is the binormal, defined by $\mathbf{s}'\times\mathbf{s}''$. All these three vectors and their relative orientations with respect to the counterflow velocity $\mathbf{V}_{ns}$ are important in the microscopic vortex dynamics.

The driving force which pushes the vortices is the Magnus force $\mathbf{f}_M$, generated by the relative flow of superfluid with respect to the vortex:[1–4]

$$\mathbf{f}_M = \kappa\rho_s \mathbf{s}' \times (\mathbf{v}_L - \mathbf{v}_{sl}), \quad (2.2)$$

where $\mathbf{v}_L = d\mathbf{s}/dt$ is the velocity of the vortex line element and $\mathbf{v}_{sl} = \mathbf{v}_s + \mathbf{v}_\mathbf{i}$ is the "local superfluid velocity," the sum of the superfluid velocity at large distance from any vortex line and of the "self-induced velocity," a flow due to all the other vortices including other parts of the same vortex, induced by the curvature of all these lines. In the "local induction approximation," the self-induced velocity $\mathbf{v}_\mathbf{i}$ is approximated by[1–4]

$$\mathbf{v}_\mathbf{i}^{(loc)} = \tilde{\beta}[\mathbf{s}' \times \mathbf{s}'']_{s=s_0}, \quad \text{with } \tilde{\beta} = \frac{\kappa}{4\pi}\ln\left(\frac{c}{a_0 L^{1/2}}\right), \quad (2.3)$$

with c a constant of the order of unity and a_0 the dimension of the vortex core. The intensity of $\mathbf{v}_\mathbf{i}$ is $|\mathbf{v}_\mathbf{i}| = \tilde{\beta}/R$, with R the curvature radius of the vortex line. The self-induced velocity is zero if the vortices are straight lines. The coefficient $\tilde{\beta}$ is linked to the internal energy per unit length of the vortex line by the relation $\epsilon_V = \rho_s \kappa \tilde{\beta}$.[1–4]

The vortex line moves through a gas of excitations (phonons and rotons) which react by producing a frictional force per unit length, the "mutual friction force," which can be written as[1–4]

$$\mathbf{f}_{MF} = -\alpha\rho_s\kappa\mathbf{s}' \times [\mathbf{s}' \times (\mathbf{V}_{ns} - \mathbf{v}_\mathbf{i})] - \alpha'\rho_s\kappa\mathbf{s}' \times (\mathbf{V}_{ns} - \mathbf{v}_\mathbf{i}). \quad (2.4)$$

The force $\mathbf{f}_{MF}$ is the force on a vortex line element exerted by the gas of excitations; $-\mathbf{f}_{MF}$ is the force on the normal component exerted by the vortex element; α and α' are temperature-dependent friction coefficients, linked to the Hall-Vinen coefficients B_{HV} and B'_{HV} by the relations $\alpha = B_{HV}(\rho_n/2\rho)$, $\alpha' = B'_{HV}(\rho_n/2\rho)$.

If we neglect the inertia of the core, the force balance equation on each line element, $\mathbf{f}_M + \mathbf{f}_{MF} = 0$, leads to[1–4]

$$\mathbf{v}_L = \frac{d\mathbf{s}}{dt} = \mathbf{v}_{sl} + \alpha\mathbf{s}' \times (\mathbf{V}_{ns} - \mathbf{v}_\mathbf{i}) - \alpha'\mathbf{s}' \times [\mathbf{s}' \times (\mathbf{V}_{ns} - \mathbf{v}_\mathbf{i})]. \quad (2.5)$$

This is the equation of motion of a vortex. Starting from this equation, Schwarz[8] obtained for the rate of change of the length of vortices

$$\frac{\Delta\xi}{\Delta t} = \alpha[\mathbf{V}_{ns} \cdot (\mathbf{s}' \times \mathbf{s}'') - |\mathbf{s}' \times \mathbf{s}''|^2] - \alpha'\mathbf{V}_{ns} \cdot \mathbf{s}'', \quad (2.6)$$

with $\Delta\xi$ the length of a given small segment of line and $\mathbf{s}'$ and $\mathbf{s}''$ the vector corresponding to such a segment. To obtain dL/dt this equation must be averaged over a volume containing a sufficiently high number of vortex lines. Schwarz neglected the last term, in α', because in closed vortex loops the average curvature vector $\mathbf{s}''$ is null. However, we will keep here this term, because we will be interested in the contribution of pinned vortices, which are not negligible at low vortex density and which play a role in the wall effects on vortex lines. In this way, the evolution of vortex line density becomes

$$\frac{dL}{dt} \simeq \alpha c_1 \mathbf{V}_{ns} \cdot \mathbf{I} L^{3/2} + \alpha' c_1 \mathbf{V}_{ns} \cdot \mathbf{J} L^{3/2} - \alpha\tilde{\beta} c_2 L^2. \quad (2.7)$$

Here c_1, c_2, $\mathbf{I}$, and $\mathbf{J}$ are characteristic measures of the vortex tangle introduced by Schwarz:[6]

$$c_1 = \frac{1}{\Lambda L^{3/2}}\int |\mathbf{s}''| d\xi, \quad c_2 = \frac{1}{\Lambda L^2}\int |\mathbf{s}''|^2 d\xi, \quad (2.8)$$

$$\mathbf{I}_l = \frac{1}{\Lambda L^{3/2}}\int \mathbf{s}' \times \mathbf{s}'' d\xi, \quad \mathbf{I} = \frac{\mathbf{I}_l}{c_1} = \frac{\int \mathbf{s}' \times \mathbf{s}'' d\xi}{\int |\mathbf{s}''| d\xi}, \quad (2.9)$$

$$\mathbf{J}_l = \frac{1}{\Lambda L^{3/2}} \int \mathbf{s}'' d\xi, \quad \mathbf{J} = \frac{\mathbf{J}_l}{c_1} = \frac{\int \mathbf{s}'' d\xi}{\int |\mathbf{s}''| d\xi}. \tag{2.10}$$

Vinen's equation (1.1) is obtained immediately putting $\alpha c_1 \cos\theta = \alpha_v$ and $\alpha\tilde{\beta}c_2 = \kappa\beta_v$, where θ is the angle between $\mathbf{V}_{ns}$ and $\mathbf{I}+\frac{\alpha'}{\alpha}\mathbf{J}$. In particular, if $\mathbf{J}=\mathbf{0}$, Eq. (2.7) reduces to the equation found by Schwarz, studied also by Lipniacki,[34] where $\mathbf{I}$ describes the anisotropy in the orientation of $\mathbf{s}' \times \mathbf{s}''$, which tends to orient itself parallel to $\mathbf{V}_{ns}$.

The hypothesis $\langle \mathbf{s}'' \rangle = 0$ leading to $\mathbf{J}=\mathbf{0}$ is no longer tenable when we are in the presence of pinned vortices. Consider for example a single pinned vortex. In the absence of counterflow its equilibrium configuration is approximately a straight line. Now, we consider the tangle, composed of vortex lines pinned to the walls of the container, and by a number of vortex loops. Assume, for instance, a counterflow at small values of V_{ns}, as in the laminar flows and in the so-called TI turbulence. In this case, most of the vortex lines are pinned to the surface of the walls of the container,[3,4] which will be considered a cylinder of diameter d. As shown in the numerical simulation of Schwarz,[6] the drag of the normal flow bends the vortices, thus conferring to them an average curvature $\mathbf{s}''$, to which will be superposed random fluctuations of the curvature $\Delta\mathbf{s}''$. The average curvature $\mathbf{s}''$ has nonzero component in the opposite direction of the counterflow, which in a first approximation will be of the order of the inverse of the diameter d of the tube; furthermore, this curvature could depend on the value of the counterflow velocity. Thus we assume, for $\mathbf{V}_{ns} \neq 0$, that in the laminar and in the turbulent TI regimes

$$\langle \mathbf{s}'' \rangle \approx -\gamma \hat{\mathbf{V}}_{ns}/d, \tag{2.11}$$

with $\hat{\mathbf{V}}_{ns}$ the dimensionless unit vector along $\mathbf{V}_{ns}$ and γ a dimensionless coefficient. In Eq. (2.11) we have neglected higher-order terms in V_{ns}, for instance, proportional to V_{ns}, because the latter yield terms of order V_{ns}^2 in the equation for dL/dt, which we have not considered in Eq. (2.1). Taking the mentioned terms into account would be easy but cumbersome and they will not be very relevant at low values of V_{ns}, which is the situation when pinned vortices are dominant. According to Eq. (2.11) the coefficient γ may be defined in analogous way to Eqs. (2.8)–(2.10) as

$$\gamma = -\frac{d}{\Lambda L} \int (\hat{\mathbf{V}}_{ns} \cdot \mathbf{s}'')_{pinned} d\xi. \tag{2.12}$$

In this case, the second term in Eq. (2.7) will take the form

$$\alpha' c_1 \mathbf{V}_{ns} \cdot \mathbf{J} L^{3/2} = -\gamma\alpha' \frac{V_{ns}}{d} L. \tag{2.13}$$

In fact, only the pinned vortices are expected to contribute to the curvature term in Eq. (2.12), and this is the reason we have written the subscript "pinned." Thus in a strict sense, one should specify L_{pinned} and $L_{unpinned}$. However, the correction terms obtained here are only relevant when L is small, in which case L is mainly composed of pinned vortices. A more general possibility would be to write two different evolution equations, one for L_{pinned} and another for $L_{unpinned}$, but this is too detailed as far as the present experimental possibilities and theoretical needs.

Furthermore, the third term in Eq. (2.7) will also change its form. Neglecting the contribution to $\mathbf{s}''$ of the component orthogonal to $\mathbf{V}_{ns}$, which is expected to vanish on the average in axially symmetric tubes, we have

$$c_2 L^2 = \frac{1}{\Lambda} \int |\Delta\mathbf{s}'' + \langle \mathbf{s}'' \rangle|^2 d\xi = \frac{1}{\Lambda} \int \left| \Delta\mathbf{s}'' - \frac{\gamma}{d}\hat{\mathbf{V}}_{ns} \right|^2 d\xi$$
$$= c_{20} L^2 + \frac{\gamma^2}{d^2} L, \tag{2.14}$$

where c_{20} is the value of c_2 for negligible values of $\mathbf{V}_{ns}$. Thus substituting Eqs. (2.13) and (2.14) in Eq. (2.7), we are led to a generalized Vinen's equation of the form

$$\frac{dL}{dt} = \alpha\gamma_1 V_{ns} L^{3/2} - \alpha' \gamma \frac{V_{ns}}{d} L - \alpha\tilde{\beta}c_{20}\left(1 + \frac{\gamma^2}{c_{20}} \frac{L^{-1}}{d^2}\right) L^2, \tag{2.15}$$

where γ_1 is defined in a similar way to γ, i.e.,

$$\gamma_1 = \frac{1}{\Lambda L^{3/2}} \int \hat{\mathbf{V}}_{ns} \cdot (\mathbf{s}' \times \mathbf{s}'') d\xi. \tag{2.16}$$

This coefficient takes into account the averaged angle between the counterflow velocity direction $\hat{\mathbf{V}}_{ns}$ and binormal vector $\mathbf{s}' \times \mathbf{s}''$, and it is related to the coefficient α_1 of Eq. (2.1) through $\alpha_1 = \alpha\gamma_1$.

Thus the term in $\mathbf{J}$ in Eq. (2.7), neglected up to now in the standard derivation of Vinen's equation, may play a physically interesting role in situations where L is relatively low, and the influence of the pinned vortices is relatively relevant, as for instance in the transition from laminar flow to TI turbulence or in the last phase of the decay of the vortex tangle after that V_{ns} has been cancelled.[9,10] The second term in Eq. (2.15) is analogous to the term with coefficient ω in Eq. (2.1), and it was also postulated by Vinen on some occasions on phenomenological grounds. The last term in Eq. (2.15) is analogous to the term in ω'' in Eq. (2.1), although with a different sign, which was partially compensated in Eq. (2.1) by the term in ω'.

Now, it is also possible to provide a justification for the step in α_1 at the TI-TII transition. Indeed, as we have said, in the TI turbulent regime the pinned vortices have the mean curvature prevalently in the direction of the counterflow velocity $\mathbf{V}_{ns}$; as a consequence, the prevalent component of the binormal vector $\mathbf{I}$ is orthogonal to $\mathbf{V}_{ns}$. Consequently, in the TI regime the scalar product $\mathbf{I} \cdot \mathbf{V_{ns}}$, which is related to the coefficient γ_1 in Eq. (2.15), is much lower than in the TII regime and the step change at the transition can be understood observing that at this transition several vortices unpin.

Another situation in which the term in $\mathbf{J}$ could yield additional contributions to the Vinen's equation is in the simultaneous presence of counterflow and rotation, but this would require a more complex analysis, going maybe to an equation more general than Eq. (2.7).

III. ENERGETIC ASPECTS OF THE VORTEX TANGLE

The energy aspects of the vortex tangle are especially interesting because it is a nonequilibrium system sustained by the external heat flow, related to the counterflow velocity. Macroscopic analysis of energy dissipation and entropy production in vortex tangle have been pointed to on several occasions.[4,15] In order to emphasize that the energetic aspects indeed deserve attention, we first present a short evaluation of the total power delivered to the whole system (superfluid, normal fluid, and vortex tangle) and the power released by the destruction of the vortices in the tangle—which must be compensated by the formation of an equivalent quantity of new vortices in the steady state.

To carry out this illustration we use the data from the paper of Griswold *et al.*,[35] where the rate of total heat input to the system in a counterflow experiment in a tube of 132 μm diameter and 1 cm length and the corresponding vortex line density are measured. For example, when $\dot{Q}=137$ μW are supplied to liquid helium at 1.6 K, they measure the average vortex-line density to be $L=1.11\times 10^7$ cm^{-2}. The energy per unit length at this temperature is of the order of $\epsilon_V=1.85\times 10^{-7}$ erg/cm, so that the contribution of the tangle to the energy density of the system will be $E/V=\epsilon_V L=2.053$ erg/cm^3.

The power per unit volume supplied to the system is approximately equal to 1 W/cm^3. On the other side, according to the destruction term of the Vinen equation (1.1) one has for the power per unit volume delivered by the destruction of the vortices (which in the steady situation is the same as the power spent in vortex formation)

$$(P/V)_{vortex\ destruction}=(P/V)_{vd}=\epsilon_V\beta_v\kappa L^2. \tag{3.1}$$

Recalling that at 1.6 K, coefficient β_v assumes the value 1.04,[10] we obtain

$$(P/V)_{vd}=(1.85\times 10^{-7}\ \text{erg/cm})(1.04)(9.97\times 10^{-4}\ \text{cm}^2/\text{s})\times(1.11\times 10^7\ \text{cm}^{-2})^2=2.36\times 10^4\ \frac{\text{erg}}{\text{s cm}^3}. \tag{3.2}$$

This value is much less than the power supplied to the total system, which is of the order of 10^7 erg/s cm^3. Thus one should not identify the total power with the power spent on the growth of vortex lines. For instance, steady vortices without vortex production or destruction would also dissipate energy of the incoming counterflow through pure friction.

Here, we want to identify in a microscopic way the dissipation due to pure friction and that associated to vortex destruction and formation. The total power per unit volume P/V delivered to the system in the superfluid reference frame will be

$$(P/V)_{total}=\langle \mathbf{V}_{ns}\cdot\mathbf{f}_{MF}\rangle. \tag{3.3}$$

We will split the relative velocity of the normal fluid with respect to the superfluid as $\mathbf{V}_{ns}=\mathbf{V}_{ns}-\mathbf{v}_L+\mathbf{v}_L$, which is a suggestive splitting because the speed of a vortex element is $\mathbf{v}_L$, and we will try to identify the physical meaning of both contributions to the power, namely

$$(P/V)_{total}=\langle \mathbf{v}_L\cdot\mathbf{f}_{MF}\rangle+\langle(\mathbf{V}_{ns}-\mathbf{v}_L)\cdot\mathbf{f}_{MF}\rangle. \tag{3.4}$$

We study first the contribution

$$(P/V)_1\equiv\langle \mathbf{v}_L\cdot\mathbf{f}_{MF}\rangle=\frac{1}{\Lambda}\int \mathbf{v}_L\cdot\mathbf{f}_{MF}d\xi, \tag{3.5}$$

where Λ denotes the volume of the system. To determine Eq. (3.5), we calculate the power of the friction force exerted on a single vortex element. It is

$$\mathbf{f}_{MF}\cdot\mathbf{v}_L=\{-\alpha\rho_s\kappa\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V}_{ns}-\mathbf{v_i})]-\alpha'\rho_s\kappa\mathbf{s}'\times(\mathbf{V}_{ns}-\mathbf{v_i})\}\cdot\mathbf{v}_L. \tag{3.6}$$

In the superfluid velocity reference frame, where $\mathbf{v}_{sl}$ is equal to $\mathbf{v_i}$, the product of the second and third terms of the vortex velocity $\mathbf{v}_L$ (2.5) with $\mathbf{f}_{MF}$ vanish and it only remains the product $\mathbf{f}_{MF}\cdot\mathbf{v_i}=\mathbf{f}_{MF}\cdot\tilde{\beta}(\mathbf{s}'\times\mathbf{s}'')$, in the local-induction approximation. Then, we obtain

$$\mathbf{f}_{MF}\cdot\mathbf{v}_L=\rho_s\kappa[\alpha\tilde{\beta}\mathbf{V}_{ns}\cdot(\mathbf{s}'\times\mathbf{s}'')+\alpha'\tilde{\beta}\mathbf{V}_{ns}\cdot\mathbf{s}''-\alpha\tilde{\beta}^2(\mathbf{s}'\times\mathbf{s}'')\cdot(\mathbf{s}'\times\mathbf{s}'')]. \tag{3.7}$$

Integrating over the tangle, recalling that $\mathbf{s}'$ is a unit vector and that $\mathbf{s}''$ is perpendicular to $\mathbf{s}'$, we get

$$(P/V)_1=\rho_s\kappa\tilde{\beta}[\alpha\mathbf{V}_{ns}\cdot\langle\mathbf{s}'\times\mathbf{s}''\rangle+\alpha'\mathbf{V}_{ns}\cdot\langle\mathbf{s}''\rangle-\alpha\tilde{\beta}\langle|\mathbf{s}''|^2\rangle], \tag{3.8}$$

where angular brackets stand for the average over the length of the vortex lines contained in the unit volume.

Using definitions (2.8)–(2.10), we obtain

$$(P/V)_1=\rho_s\kappa\tilde{\beta}(\alpha c_1\mathbf{V}_{ns}\cdot\mathbf{I}L^{3/2}+\alpha' c_1\mathbf{V}_{ns}\cdot\mathbf{J}L^{3/2}-\alpha\tilde{\beta}c_2L^2). \tag{3.9}$$

The term within brackets is precisely dL/dt, as given by Eq. (2.7). Thus this contribution has the form

$$(P/V)_1=\rho_s\kappa\tilde{\beta}\frac{dL}{dt}. \tag{3.10}$$

Recalling now that the energy per unit volume related to the vortices is given by $\epsilon_V L$, with $\epsilon_V=\rho_s\kappa\tilde{\beta}$ and $\tilde{\beta}$ given in Eq. (2.3), and neglecting the small logarithmic variation of $\tilde{\beta}$ with L, we have

$$(P/V)_1=\rho_s\kappa\tilde{\beta}\frac{dL}{dt}=\epsilon_V\frac{dL}{dt}\simeq\frac{d(\epsilon_V L)}{dt}. \tag{3.11}$$

Therefore this is the contribution to the vortex line formation and destruction.

Second, we evaluate the power per unit volume $(P/V)_2$, which would represent a dissipation due to the friction between normal fluid and vortices:

$$(P/V)_2 \equiv \langle(\mathbf{V}_{ns}-\mathbf{v}_L)\cdot\mathbf{f}_{MF}\rangle L = \frac{1}{\Lambda}\int(\mathbf{V}_{ns}-\mathbf{v}_L)\cdot\mathbf{f}_{MF}d\xi. \tag{3.12}$$

As before, in the superfluid velocity reference frame, the scalar product of the mutual friction force $\mathbf{f}_{MF}$ and the two terms in α and α' of the vortex velocity $\mathbf{v}_L$ (2.5) vanishes, namely

$$(\mathbf{V}_{ns}-\mathbf{v}_L)\cdot\mathbf{f}_{MF} \equiv (\mathbf{V}_{ns}-\mathbf{v}_\mathbf{i})\cdot\mathbf{f}_{MF}, \tag{3.13}$$

where $\mathbf{v}_\mathbf{i} \equiv \tilde{\beta}(\mathbf{s}'\times\mathbf{s}'')$ in the local-induction approximation.

Recalling that $-\alpha\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V}_{ns}-\mathbf{v}_\mathbf{i})]=\alpha(\mathbf{V}_{ns}-\mathbf{v}_\mathbf{i})_\perp$, where $\perp$ stands for orthogonal component to the unit vector $\mathbf{s}'$, the following value for the power $(P/V)_2$ is obtained:

$$(P/V)_2 = \alpha\rho_s\kappa\langle|(\mathbf{V}_{ns}-\mathbf{v}_\mathbf{i})_\perp|^2\rangle L. \tag{3.14}$$

It is interesting to note that only the term related to α in the expression of the friction force $\mathbf{f}_{MF}$ (2.4) leads to an effective dissipation of the energy, whereas the term related to α' contributes, according to Eq. (3.9), to the destruction and formation of vortices. From the previous expression of the power $(P/V)_2$, we note also that when the vortex is straight, which means $\mathbf{v}_\mathbf{i}=\mathbf{0}$, and the counterflow velocity has the direction of the vortex, namely parallel to $\mathbf{s}'$, then the power $(P/V)_2=0$ as it is well known, because according to Eq. (2.4) there is no longitudinal friction when the normal fluid flows along the vortex lines. This term will also vanish when $\mathbf{V}_{ns} \equiv \mathbf{v}_\mathbf{i}$. Expression (3.14) is analogous to that of the friction of a usual viscous fluid along a line of length L having friction coefficient $\alpha\rho_s\kappa$.

IV. EFFECTIVE TEMPERATURE FOR A SUPERFLUID VORTEX TANGLE

It is known that rotation of the superfluid tends to orient the vortex lines along it, whereas the counterflow tends to randomize their orientation. This situation is analogous to paramagnetic systems, in which the external magnetic field $\mathbf{H}$ tends to orient along it the microscopic magnetic moments $\vec{\mu}$ of the particles constituting the system, whereas the thermal agitation, as measured by absolute temperature T, tends to randomize their orientation. Several authors have used this analogy to characterize the degree of orientation of the tangent unit vector $\mathbf{s}'$ to the vortex lines in order to evaluate the polarization of rotating vortex tangles.[14,16,19]

In Ref. 25, we have examined the concept of nonequilibrium effective temperature in turbulent vortex tangles under the simultaneous presence of rotation and thermal counterflow. Since the form of the distribution function proposed in Refs. 14, 16, and 19 is similar to the canonical one, we proposed to define an effective temperature of the vortex tangle, different from the temperature of the helium fluid. The aim of this section is to further explore this concept in homogeneous vortex tangles in pure counterflow, for high values of L, when the turbulence is completely developed, and to relate it to the effective temperature proposed in Ref. 25.

The reader should be warned about the complexities in defining temperature in nonequilibrium steady states, a topic of much current interest which has been reviewed at length in Ref. 26. It turns out that several different nonequilibrium effective temperatures may be defined by extrapolating several well-known equilibrium expressions containing temperature. However, the effective temperatures defined in this way, which coincide all of them at equilibrium, are in general different from each other out of equilibrium, because they are reflecting different degrees of freedom, which out of equilibrium have different temperatures. Here, we will stress a definition of effective nonequilibrium temperature related to the average energy associated to the length of vortex loops. It must be noted that in equilibrium, i.e., in the absence of the counterflow, the vortex filaments—very short—have been shown to be in equilibrium with the Bose condensate[37] in a weakly imperfect Bose gas, in thermal equilibrium and when no other random actions exist. The situation we are dealing with here is different, as there is a continuous energy flow through the system, due to the nonvanishing counterflow velocity, which is a source of vortex filaments.

Since the vortex tangle is composed of many closed vortex loops, one may inquire the probability distribution function of their respective lengths. Since the energy of the loops is approximately proportional to the loop length, we will study the energy distribution function. It must be emphasized that the length of the vortex loops is only one of the aspects necessary to describe the tangle. Indeed, the vortices are linked to each other, they suffer very frequent reconnections, they are not smooth curves but curled and coiled, so that many other features should be included for a satisfactory description.[3–5] The present analysis is only an exploratory attempt to assess the usefulness of the idea of this nonequilibrium temperature.

A relatively immediate way to define the temperature of the vortices would be by relating it to their average energy. Indeed, the energy E_l of a single vortex line is approximately proportional to its length l as

$$E_l = \epsilon_V l, \tag{4.1}$$

with ϵ_V defined below Eq. (2.3). In fact, it has been shown that it gives a good estimate of the energy of the tangle, despite the geometrical complexity of the vortex lines (for a detailed expression of the energy, including vortex interaction and polarization, see Nemirovskii and Nedoboiko in Refs. 37–39, which show that the corresponding contributions are logarithmically small). On practical grounds, the logarithmic term in Eq. (2.3) has only a mild variation and $\frac{1}{4\pi}\ln\left(\frac{1}{aL^{1/2}}\right)$ is of order 1. Since the vortex lines form a very entangled structure, which probably possesses fractal properties, it could be, in principle, that E_l is not proportional to l but to some power of l. However, here we will stick to the usual assumption (4.1).

We could define T_{eff}, an effective temperature of the vortex loops, as

$$k_B T_{eff} = \langle E_l\rangle = \langle l\rangle\epsilon_V, \tag{4.2}$$

$\langle l\rangle$ being the average length of the vortex loops and k_B the Boltzmann's constant. Equation (4.2) is analogous to the expression $U=ak_BT$ for ideal gases (with U the average energy of the particles). In ideal gases the constant a takes the value $a=3/2$ or $5/2$ for monatomic or diatomic ideal gases, respectively, but here we will take $a=1$ for the sake of simplicity. From the purely formal point of view, Eq. (4.2) is comparable to the definition of the kinetic temperature in nonequilibrium gases as $k_BT=\frac{2}{3}\langle\frac{1}{2}mv^2\rangle$, which is strictly justified only in equilibrium, but which is much used even in situations far from equilibrium.[40] In order that T_{eff} has a deeper meaning and could be associated to an equilibrium-like distribution, the time constant for internal equilibration of the tangle should be much less than the time constant of the energy loss of the system to the surroundings. In fact, the decay of a tangle after the counterflow has been suddenly eliminated is relatively slow, whereas the typical vortex recombination time is very short because at high vortex densities, vortices are crossing themselves at a high rate. A condition for the consistency of the attribution to $\epsilon_V \langle l\rangle$ of the meaning of a temperature is that $\langle l\rangle$ should be the same in different regions of the vortex tangle, in the steady state. This is currently an open question.

This effective temperature T_{eff} provides a measure of the average energy of the vortex loops, and would be much higher than the helium temperature (which is typically less than 2 K). In fact, the temperature for vortex lines with length of the order of millimeters would be of the order of million degrees because, when expressed in K, the value of ϵ_V is of the order of 13.3 K/Å.[1] This may seem astonishing at the first sight, but this high temperature affects only a small number of atoms; in fact, this high value results from attributing to the very thin vortex line the rotation energy of many particles of the superfluid up to a typical distance of the order of $L^{-1/2}$, which is the average separation amongst vortices; thus the collective energy of many particles is attributed to a reduced number of particles around the hollow core of the vortex line, which becomes in this way a convenient representation of a collective excitation. This situation with two different temperatures is analogous to that found in glasses, where the temperature of the fast relaxing vibrational degrees of freedom is equal to the room temperature, whereas the slowly relaxing configurational degrees may have temperatures of hundreds of thousands of degrees.[28] However, this high temperature is not directly perceptible, because of the extremely slow heat exchange between these degrees of freedom and the surroundings. On the other side, the fact that the average energy of vortex loops under counterflow is so much higher than that of the background Bose condensate is an indication that the global system is rather far from equilibrium, and that long loops require a considerable nonequilibrium forcing by the counterflow. The system as a whole could be considered as a two-temperature system: the helium background and the vortex tangle, where the difference in temperatures is sustained by the external forcing. Similar two-temperature situations are found, as we have said, in glasses (vibrational temperature and configurational temperature) and also in plasma physics (ion temperature and electron temperature), in fluids under shear (kinetic temperatures different along the different axes), in shaken granular mixtures, and other systems.[26,27]

The definition (4.2) does not rely on any particular distribution, as well as the kinetic definition of temperature, $\langle\frac{1}{2}mv^2\rangle=\frac{3}{2}k_BT$, is used in kinetic theory even when the velocity distribution function is very different from the Maxwell-Boltzmann one. However, to go into deeper detail, following our analogy with the kinetic theory of gases, we could tentatively assume that the distribution of vortex loops with respect to their energy $P(E_l)$ is proportional to $\exp[-\beta E_l]$, so that $P(E_l)dl$ is the average number of loops with energy between E_l and E_l+dE. We assume this quasi-equilibrium expression because we are considering regions with high vortex densities L, where the internal dynamics of vortices is very high, with very frequent breakings and reconnections, probably leading to an internal equilibration. Because the energy of a vortex loop is proportional to its length, the distribution of vortex loops with respect to their length, $f(l)$, will be given in this hypothesis by

$$f(l) \sim \exp[-\beta E_l] = \exp[-\beta\epsilon_V l], \tag{4.3}$$

where $f(l)dl$ is the average numbers of loops with length between l and $l+dl$. Here β is a parameter which in the usual canonical distribution in equilibrium would be interpreted as $\beta=(k_BT)^{-1}$. We do not pretend that this relation is valid for T identified as the helium equilibrium temperature, but as a more general parameter. Note that we are applying this idea to counterflow vortex tangles; in other situations, as in the turbulence generated by a towing or vibrating grid or by means of a pressure gradient or a propeller, the quantum turbulence becomes analogous in several aspects to the classical one, in which case it could be expected that $f(l)$ follows some potential law;[36] for instance, Nemirovskii[38] has proposed that $f(l)$ is proportional to $l^{-5/2}$, working on an evolution equation for $f(l)$ and assuming some expressions for the transition probabilities of breaking and recombination of vortices and neglecting thermal effects; other authors[41] obtained from similar arguments a distribution $f(l)$ proportional to $l^{-5/2}\exp(-\beta l)$, dealing with cosmic strings, but without identifying explicitly β. The topic of the vortex length distribution is currently an open problem.

We insist on the fact that Eq. (4.3) does not imply that the whole system is in global equilibrium, because the temperature of the vortices is different from that of the liquid helium, due to the external energy supply. Here we will tentatively consider the vortex tangle as an ideal gas of closed loops, for the sake of illustration. Indeed, since the energy of the full tangle is given within a logarithmic precision by the same expression (4.1) as a single vortex line,[38] this may be interpreted in some way as if the interactions could be neglected, at least from the energetic point of view.

Since the number of loops is continuously changing, because of frequent reconnections and growing, it would be more realistic to use, instead of the canonical distribution (4.3), a macrocanonical distribution including a chemical potential for the loops. In this case, the number of vortices with length l, namely N_l [proportional to $f(l)$], would also fluctuate. In the macrocanonical distribution, the probability expression would be

$$P(E_l,N_l) \propto \exp(-\beta E_l - \alpha N_l), \tag{4.4}$$

with $\beta=1/k_BT$ and $\alpha=-\mu/k_BT$, μ being the vortex chemical potential. According to standard results for the macrocanonical distribution, the second moments of the fluctuations of N_l around their average value $\langle N_l \rangle$ would be

$$\langle (\delta N_l)^2 \rangle = k_B T \frac{\partial N_l}{\partial \mu} = \langle N_l \rangle. \tag{4.5}$$

However, the detailed discussion of this topic goes beyond the aims of the present discussion. We will briefly comment on it when dealing with the fluctuations in the vortex line length.

Another way leading to a distribution function of the form (4.3) may be obtained from a proposal by Nemirovskii.[22] By analogy with the theory of polymer chains, he divides the single vortex line into a set of discrete points and uses a Gaussian distribution function of the tangent vector $\mathbf{s}'$ to the vortex lines, namely[22]

$$Pr(\mathbf{s}') \sim \exp[-\mathbf{s}' \cdot \mathbf{\Lambda} \cdot \mathbf{s}']. \tag{4.6}$$

In the Nemirowskii model $\mathbf{s}'$ is not equal to the unit but is smeared out, due to the fact that, the vortices being in motion, some parts of them shrink and some other ones stretch. From Eq. (4.6), the relative probability of a value of $\mathbf{s}' \cdot \mathbf{\Lambda} \cdot s'$ integrated along the length of a loop would be $Pr \sim \exp[-\int \mathbf{s}' \cdot \mathbf{\Lambda} \cdot \mathbf{s}' d\xi]$. One could obtain from here the probability to have a vortex line of length l by integrating over the parameter ξ (see Sec. III B of Ref. 22). In an isotropic system the matrix $\mathbf{\Lambda}$ characterizing Eq. (4.6) would be proportional to the unit tensor, let us say $\mathbf{\Lambda} = \Lambda \mathbf{U}$. In this case, one obtains

$$Pr(l) \sim \exp\left[-\Lambda \int |\mathbf{s}'|^2 d\xi \right] = \exp[-\Lambda l], \tag{4.7}$$

which has the form (4.3) provided Λ is identified as $\Lambda = \beta \epsilon_V$.

The distribution function (4.3) allows us to find β by taking into account that

$$N\beta\epsilon_V \int l \exp[-\beta\epsilon_V l] dl = \frac{N}{\beta\epsilon_V} = N\langle l \rangle, \tag{4.8}$$

where the prefactor $\beta\epsilon_V$ in front of the left-hand side is the normalization factor of $f(l)$. Thus if we interpret β as $(k_B T_{eff})^{-1}$, we are led to our previous definition (4.2). Furthermore, one may also obtain from Eq. (4.3) the second moments of the fluctuations of $\mathcal{L}$, the total length of the vortex tangle $\mathcal{L} = VL$, defined as $\langle (\delta \mathcal{L})^2 \rangle = \frac{\langle (l - \langle l \rangle)^2 \rangle}{N}$. This yields

$$\frac{\langle (\delta \mathcal{L})^2 \rangle}{\mathcal{L}^2} = \frac{\langle l \rangle}{\mathcal{L}}. \tag{4.9}$$

Since the total vortex length $\mathcal{L}$ is known, the relation (4.9) allows one to measure $\langle l \rangle$ from experimental observations of the intrinsic fluctuations of the vortex-line density.[35] In other words, it may be said that the fluctuations of $\mathcal{L}$ could provide a "thermometer" of the vortex tangle.

The actual situation is somewhat more complicated, because Eq. (4.9) is valid for the canonical distribution. Indeed, Eq. (4.9) is a particular case of the standard result of the energy fluctuations in the canonical distribution, namely

$$\langle (\delta U)^2 \rangle = k_B T^2 \left(\frac{\partial U}{\partial T} \right)_{V,N}, \tag{4.10}$$

using that $U = \epsilon_V \mathcal{L}$ and $k_B T = \epsilon_V \langle l \rangle$, and $\mathcal{L} = N \langle l \rangle$. In the macrocanonical distribution including fluctuations in the number of vortices N_l, Eq. (4.10) should be changed to

$$\langle (\delta U)^2 \rangle = k_B T^2 \left(\frac{\partial U}{\partial T} \right)_{V,\mu}, \tag{4.11}$$

i.e., the differentiation is carried out at constant μ rather than at constant N. However, according to Eq. (4.5) the corrections to Eq. (4.9) will be small if N_l is high, i.e., when there are many vortices, a situation found for high values of the counterflow $\mathbf{V}_{ns}$ because $\langle (N_l)^2 \rangle^{1/2} / \langle N_l \rangle = \langle N_l \rangle^{-1/2}$. This condition is also needed for a fast internal equilibration of the vortex tangle, and therefore we restrict our analysis to this situation.

Another nonequilibrium temperature was obtained in the analysis of the orientation distribution function of the unit vectors $\mathbf{s}'$ tangent to the vortex lines, with respect to the direction set out by the rotation vector $\mathbf{\Omega}$, the angle between $\mathbf{s}'$ and $\mathbf{\Omega}$ being denoted as θ, in the simultaneous presence of counterflow and rotation.[14–16,19,25] In that case, one assumed for the probability of cos θ an expression of the form

$$f(\theta) \sim \exp[-\tilde{\alpha} \cos \theta]. \tag{4.12}$$

In equilibrium paramagnetic systems, $\tilde{\alpha}$ would have the form $\tilde{\alpha} = \bar{\mu} H / k_B T$ with H the magnetic field and $\bar{\mu}$ the magnetic moment of the particles. In rotating counterflow, the rotation Ω orients the vortices along its direction, in an analogous way to H, whereas the counterflow V_{ns} plays a disordering role, analogous to thermal agitation. In that case, it was found in Ref. 19 that $\tilde{\alpha} \sim \frac{\Omega \kappa}{V_{ns}^2}$ or, in more explicit terms

$$\tilde{\alpha} = \frac{22}{\gamma_H^2} \frac{\Omega \kappa}{V_{ns}^2} = 11 \frac{L_R}{L_H} = 11 \frac{\mathcal{L}_R}{\mathcal{L}_H}, \tag{4.13}$$

where L_H is the vortex length density corresponding to the pure counterflow $L_H = \gamma_H^2 V_{ns}^2$, with $\gamma_H = 98.2$ s/cm^2, and L_R is the vortex length density under pure rotation $L_R = 2\Omega / \kappa$. Thus L_H was found to have some analogy with temperature; of course, in Eq. (4.13) one could divide the numerator and denominator by the number of vortex loops N_H and obtain Eq. (4.2) in the denominator, but this was not done at that moment. However, here the problem is more complicated than in paramagnetic systems because the number of vortex segments depends on energy, and it is difficult to find, whereas the number of magnetic moments in paramagnetism does not depend on the magnetic field H nor on temperature.

V. DIFFUSION COEFFICIENT FOR A SUPERFLUID VORTEX TANGLE

Another usual way of defining an effective nonequilibrium temperature in nonequilibrium situations is by starting from the Einstein relation for the diffusion coefficient D, which states that

$$D = \frac{k_B T}{\zeta}, \tag{5.1}$$

with ζ the friction coefficient of the entities being diffused (usually molecules, but vortex loops in our case). This expression has been used[26,30,31] to define an effective nonequilibrium temperature T'_{eff} in granular systems, amorphous semiconductors, colloidal fluids in shear flow, and some other kinds of systems as

$$k_B T'_{eff} = D\zeta. \tag{5.2}$$

In these systems, both D and ζ may be measured or obtained from molecular-dynamics simulations, and therefore the definition (5.2) is operationally meaningful. In fact, the definition based on an averaged kinetic energy and on the Einstein relation are especially useful in shaken granular systems, where standard thermometers are not useful.

The analysis of the vortex diffusion coefficient D in inhomogeneous tangles is a topic of much recent interest, as the local vortex density may be measured and the relative motion of vortices is simulated with much detail,[20] but rigorous information concerning D is still scarce.

Here, previous to the discussion of the effective temperature (5.2) we will briefly derive an approximate expression for D and for ζ. In inhomogeneous counterflow superfluid turbulence, the evolution of L cannot be described by Eq. (1.1), but additional terms must be considered. In Ref. 42, a thermodynamical model of inhomogeneous superfluid turbulence has been built up, which chooses as fundamental fields the density, the velocity, the energy density, the heat flux, and the averaged vortex line length per unit volume. In particular, the following evolution equations for $\mathbf{q}$ and L were obtained:[42]

$$\dot{\mathbf{q}} + \zeta_0 \nabla T + \chi_0 \nabla L = \sigma^{\mathbf{q}} = -KL\mathbf{q}, \tag{5.3a}$$

$$\dot{L} + L \nabla \cdot \mathbf{v} + \nabla \cdot (\nu_0 \mathbf{q}) = \sigma^L, \tag{5.3b}$$

where σ_L, the vortex net production rate, is the right-hand side of Eq. (1.1), the coefficient K is related to the friction coefficient α and to the Hall-Vinen coefficient B_{HV} as $K = \frac{1}{3}B_{HV}\kappa = \frac{2}{3}\frac{\rho}{\rho_n}\kappa\alpha$; ζ_0 is a coefficient linked to second sound velocity [as $V_2^2 = \zeta_0/(\rho c_V)$ in the absence of the vortex tangle], and ν_0 and χ_0 are coefficients which take into account the interaction between second sound and vortex tangle. A further term proportional to $L^{3/2}\hat{\mathbf{q}}$ may be added to the right-hand side of Eq. (5.3a), describing a dry friction term. Here, we will omit it for the sake of simplicity.

Introducing Eq. (5.3a) in Eq. (5.3b), and assuming for the sake of simplicity an isothermal perturbation, we find

$$\frac{1}{KL_0}\ddot{L} + \dot{L} = \frac{\nu_0\chi_0}{KL_0}\nabla^2 L + \sigma_L + \frac{\dot{\sigma}_L}{KL_0}. \tag{5.4}$$

For $\ddot{L}/(KL_0) \ll \dot{L}$, Eq. (5.4) yields a reaction-diffusion equation for L, with a diffusion coefficient given by

$$D = \frac{\nu_0\chi_0}{KL_0} = \frac{3}{2}\frac{\rho_n}{\rho}\frac{\nu_0\chi_0}{\kappa\alpha L_0} \geqslant 0. \tag{5.5}$$

At high frequencies, i.e., for $\omega \gg KL_0$ and much higher than the inverse of the characteristic time of the vortex destruction and formation as described by σ_L, namely $\omega \gg 2BL_0 - \frac{3}{2}AqL_0^{1/2}$, we get $\ddot{L} \approx \nu_0\chi_0\nabla^2 L$, that is, we get vortex density waves with speed:

$$v_L^2 = \nu_0\chi_0. \tag{5.6}$$

A microscopic estimation of the velocity of the vortex density waves may be obtained in the following way.[43] The line density L and the vortex velocity $\mathbf{v}_L$, which may be identified in Eq. (5.3b) as $\nu_0\mathbf{q}/L$, must satisfy evolution equations of the type

$$\dot{L} + \nabla \cdot (L\mathbf{v}_L) = \sigma^L,$$

$$\rho_{eff}\frac{\partial \mathbf{v}_L}{\partial t} + \nabla p_V = \sigma^J, \tag{5.7}$$

where p_V is the vortex contribution to the pressure, which is found to be[42] $p_V = L\epsilon_V$ and ρ_{eff} is an effective density of the vortices. Though the vortex lines themselves do not have mass, as their core is hollow, they have indirectly associated an effective mass which is the superfluid mass rotating around the vortex itself. This effective vortex density will be of the order $\rho_{eff} = \rho_s L r^2$, with r a characteristic radius of the zone in which the superfluid is affected by the motion of the vortices; this will be of the order of the average vortex separation $\delta = L^{-1/2}$. From this, we deduce that $\rho_{eff} = b\rho_s$, the proportionality coefficient b being unknown but of the order unit. In fact, ρ_{eff} could be safely identified with ρ_s, because it is the superfluid component which participates in the rotation around the vortex core.

Combination of both equations (5.7), neglecting the productions σ, yields, in the linear approximation,

$$\frac{\partial^2 L}{\partial t^2} = \left(\frac{L_0}{\rho_{eff}}\epsilon_V\right)\nabla^2 L. \tag{5.8}$$

Then, the speed of the vortex density wave is

$$v_L^2 = \frac{L_0}{b\rho_s}\epsilon_V = \frac{\kappa^2}{4\pi}\ln\left(\frac{c}{a_0 L^{1/2}}\right)\frac{L_0}{b}. \tag{5.9}$$

Now, the diffusion coefficient D may be found by combining Eqs. (5.5), (5.6), and (5.9). We obtain

$$D = \frac{\nu_0\chi_0}{KL_0} = \frac{1}{b}\frac{3\rho_n}{2\rho\rho_s}\frac{\epsilon_V}{\kappa\alpha} = \frac{3}{2b}\frac{\rho_s\rho_n}{\rho}\frac{\kappa}{4\pi\alpha}\ln\left(\frac{c}{a_0 L^{1/2}}\right). \tag{5.10}$$

On the other side, we may obtain the friction coefficient per unit length for vortex lines ζ starting from the expression of the friction force (2.4) which, in counterflow ($\rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n = 0$) and neglecting the self-induced velocity $\mathbf{v}_i$, assumes the expression

$$\mathbf{f}_{MF} = \alpha\rho_s\kappa\mathbf{s}' \times \left[\mathbf{s}' \times \left(\frac{\rho}{\rho_n}\mathbf{v}_s\right)\right] + \alpha'\rho_s\kappa\mathbf{s}' \times \left(\frac{\rho}{\rho_n}\mathbf{v}_s\right). \tag{5.11}$$

Thus the friction coefficient, for a vortex loop with the average length $\langle l \rangle$, in isotropic conditions, is

$$\zeta = \frac{2}{3}\frac{\rho\rho_s}{\rho_n}\alpha\kappa\langle l\rangle. \tag{5.12}$$

The factor 2/3 takes into account that in isotropic situations the loop will not be plane, but 2/3 of its total length will be orthogonal to $\mathbf{V}_{ns}$ and 1/3 parallel to $\mathbf{V}_{ns}$; this latter part will not experience friction. Note also that crossings and reconnections of vortex lines could also influence the actual friction, but we are not aware of an estimation of this feature.

Introducing Eqs. (5.10) and (5.12) into Eq. (5.2) we obtain that

$$kT'_{eff} = \frac{1}{b}\epsilon_V\langle l\rangle. \tag{5.13}$$

Thus the effective temperature defined in Eq. (4.2) and that defined in Eq. (5.2) are closely related. In fact, if $b=1$ (i.e., $\rho_{eff}=\rho_s$), one has $T'_{eff}=T_{eff}$. It must be observed that in the definition of T_{eff} based on an analogy with kinetic theory of gases, we have put the coefficient a equal to 1, a multiplicative coefficient, linked to the degree of freedom of the molecules of the gas. Here we see that this coefficient may be interpreted as the ratio between the effective mass density of the vortices and the density ρ_s of the superfluid component in the flow as $b=\frac{\rho_{eff}}{\rho_s}$. This reinforces the idea that the quantity $\epsilon_V\langle l\rangle$ has indeed an interesting physical meaning worthy of deeper analysis.

It must be recognized that a rigorous microscopic derivation of Eq. (5.10) for the nonequilibrium situation we are dealing with has not been done. Expression (5.1) is valid in the classical Langevin theory of Brownian motion, or in the linear-response theory. Admittedly, our use for vortex loops is tentative, but is seems fruitful. In the case of vortex loops, Nemirovskii[37] has studied a Langevin equation with an infinite number of degrees of freedom, near equilibrium. The corresponding Fokker-Planck equation does not describe the motion of vortices in real space, but in an abstract space of vortices, i.e., it describes the stochastic aspects of the change of length of vortices. To our knowledge, for the moment there is not a definitive and clear microscopic model for the spatial vortex diffusion. The complexity of such a task is understandable, because the vortices continuously cross each other, and recombine in multiple ways. However, vortex diffusion is receiving an increasing interest, because of recent observations and numerical simulations, and therefore it seems reasonable to initiate a deeper analysis of the diffusion coefficient. It seems that the effective temperature related to the average vortex length may play an interesting role in this search, but this will not be rigorously confirmed until a detailed understanding of vortex diffusion in space is understood.

Indeed it is known from dimensional arguments and from numerical simulations[20] that D is of the order of κ and dependent on T—recall that κ has dimensions length^2/time, which are the dimensions of D. Expression (5.10) seems qualitatively reasonable, because it is logical that the higher the friction coefficient α, the lower will be D. Furthermore, D vanishes when the average separation of vortex lines $L^{-1/2}$ becomes equal to the vortex radius a_0, because there is not space for independent vortex motion. Equation (5.10) generalizes that proposed in Ref. 43, where the logarithmic term in L was not found.

VI. CONCLUSIONS

In summary, in Sec. II we have shown that some terms which had been up to now neglected in the derivation of Vinen's equation from microscopic grounds may lead to additional terms in the presence of pinned vortices, at low values of L and which support a previous generalization of Vinen's equation incorporating the effects of the walls, as in Eq. (2.1). In Sec. III we explore some energetic aspects of the tangle; concerning purely frictional dissipation (3.14) and the contribution to vortex formation (3.11), which turns out to have considerably different values in some observed situations.

In Sec. IV we have suggested that the idea of an effective temperature for the vortex tangle is a concept which deserves attention. In fact, it is a particular case of effective temperatures for nonequilibrium systems, explored in Refs. 26–28 and 33. This effective temperature is defined in Eq. (4.2) as the average energy of the vortex loops or, in other terms, it is related to the average length of the vortex loops, and is consistent with a previous proposal related to the paramagnetic analogy for the vortex orientation distribution function in rotating counterflow tangles.[14,15,19,25]

Even in the case when the form (4.3) was not reliable, Eq. (4.2) could be taken as a direct definition of the effective temperature T_{eff}. We have also studied another effective temperature defined in terms of the Einstein relation and the friction coefficient in Eq. (5.2). The use of a value of D previously obtained from a mechanical reasoning leads to an effective temperature which is closely related to the effective temperature defined from the average energy of the vortex lines. Anyway, the microscopic aspects of vortex diffusion in the real space under a counterflow are not still sufficiently known to make this effective temperature a rigorous concept but it certainly has an appealing meaning. Future research will clarify whether this proposal is actually relevant or is only a theoretical curiosity.

Besides the interest of the vortex length distribution, another topic of much interest would be the distribution of the local curvature radius. Since the vortex loops are very complex curves, each of them may have a complex distribution of the values of the local curvature radius. Taking one vortex loop, one could study its spatial Fourier transform, which will include many different values of the wave vector, related to many different values of the curvature—as each wave vector corresponds to a curvature. Thus it must be emphasized that the energy distribution in terms of the vortex length may be very different from the energy distribution expressed in terms of the wave vector, which, according to some authors, has a potential form, similar to that of the Kolmogorov expression for classical turbulence. We cannot deal here with this complex topic.

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*Corresponding author. Electronic address: mongiovi@unipa.it

†Electronic address: david.jou@uab.es

‡Electronic address: msciacca@unipa.it

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