Dipole-exchange spin waves in perpendicularly magnetized discs: Role of the Oersted field

R. E. Arias[1] and D. L. Mills[2]
[1]*Departamento de Fisica, FCFM, Universidad de Chile, Casilla 487-3, Santiago, Chile*
[2]*Department of Physics and Astronomy, University of California, Irvine, California 92697, USA*
(Received 5 March 2007; revised manuscript received 28 March 2007; published 4 June 2007)

We develop the theory of the exchange dipole spin waves in thin circular discs for the case where the magnetization is nominally perpendicular to the plane. Our interest is in the circumstance where a transport current is injected into the disc, with current also perpendicular to the plane of the disc. Such a current creates an azimuthal magnetic field, referred to often as the Oersted field. We develop the theory of the influence of the Oersted field on the spin-wave spectrum of the disc. This field produces a vortex state. We suggest that this vortex state is stable down to zero applied field. If the external applied field H_0 is in the $+z$ direction, perpendicular to the plane of the disc, the vortex state has magnetization at the center of the disc also parallel to $+z$ always. This is the case even when $H_0 < 4\pi M_S$, where the magnetization at the center of the disc is antiparallel to the local field $H_0 - 4\pi M_S$ there. We present calculations of the current dependence of spin-wave frequencies of several modes as a function of applied magnetic field. We also address an issue overlooked in previous studies of spin waves in thin discs. This is that for quantitative purposes, it is not sufficient to describe internal dipole fields generated by the spin motions simply by adding an effective internal field $-4\pi m_z \hat{z}$ to the equations of motion, with m_z the component of dynamic magnetization normal to the surface. For samples of present interest, we derive terms we call gradient corrections, and these play a role quantitatively comparable to exchange itself in the analysis of the spin-wave frequencies. Quantitative studies of spin dynamics in such samples thus must include the gradient corrections.

DOI: 10.1103/PhysRevB.75.214404 PACS number(s): 75.75.+a, 75.30.Ds, 76.50.+g

I. INTRODUCTION

The dynamics of the magnetization in diverse ferromagnetic nanostructures is a very active field of current research. It is the case that new physics can be explored in such structures. For instance, in objects referred to as magnetic nanopillars, torques exerted by spin-polarized current injected into one ferromagnetic constituent can excite very large amplitude spin motions.[1] Thus, we have a new laboratory for the study of highly nonlinear spin motions. In bulk materials, by virtue of the presence of the ubiquitous Suhl instabilities, the magnetization precession angle saturates with applied power at only a few degrees from equilibrium at best. A new generation of devices is envisioned which exploit one's ability to generate and manipulate large amplitude motions of the magnetization through injection of spin-polarized transport currents into nanosized ferromagnets.

It is often the case that the geometries utilized in such structures render quantitative theories of the spin motions a challenge. For example, again with magnetic nanopillars in mind, often the active ferromagnetic film is elliptical in shape, and the magnetization lies in plane. Even a description of small amplitude spin motions, the spin waves, is a challenge to the theorist for this geometry.

To explore the basic physics of such structures, it is useful to consider a configuration accessible to complete theoretical descriptions. This paper is devoted to one important class of ferromagnetic film that can be incorporated into magnetic nanopillars, the ferromagnetic circular disc. If such a disc is magnetized perpendicular to its surfaces by either an external magnetic field or by uniaxial anisotropy of sufficient strength to produce an easy axis normal to the plane of the disc, then the magnetic ground state is a very simple configuration. A fundamental issue is then the nature of the spin-wave modes of the structure. These describe its linear response to external probes. An understanding of the spin waves and the linear response characteristics provides one with a foundation for the construction of descriptions of nonlinear magnetization motions. One can find extensive discussions of the nature of the dipole-exchange spin-wave modes of discs and related structures in the earlier literature.[2]

However, a new issue has arisen in the present era. When a dc spin-polarized transport current is injected into a magnetized film, there is a magnetic field necessarily generated by the current. This magnetic field can be appreciable in magnitude, in the range of several hundred gauss in current experiments. Such a field is referred to as the Oersted field in the current literature. The Oersted field will reorient the static magnetization itself, and as a consequence, it affects the dynamic response of the structure to any external probe, including the current itself if it is spin polarized.

This paper is devoted to the study of the influence of the Oersted field on the nature of the spin waves of a disc for a particularly simple case. This is a perpendicularly magnetized disc, into which a dc electrical current is driven from below, with the current flow perpendicular to the plane of the disc. Our aim is to describe the effect of the current on the magnetization within the disc and then to discuss the nature of the spin-wave spectrum of the disc when the Oersted field is present. Of course, if the current is sufficiently strong, the magnetization can be excited into large amplitude motions. Our spin-wave description will apply to the case where the currents are subcritical, but it applies as well to the regime above the critical current so long as the precession angle is not too large. In our view, the eigenmodes and eigenvectors described here may be utilized as a basis for the description of nonlinear effects, as earlier authors have done in a different context.[3]

We shall suppose that we have a disc of radius R and thickness d, where $d \ll R$. In this limit, edge effects may be described by a suitable boundary condition applied at the rim of the disc. We thus proceed by developing a description of the structure from the equations of motion of the magnetization of a perpendicularly magnetized disc, with a spatially uniform transport current injected across one surface. We have in mind a nanopillar which incorporates such a circular film in which the magnetization is free, with spin-polarized transport current injected from a second ferromagnet separated from the first by, say, a Cu spacer layer. We describe the magnetization state produced by the dc current to find that it is the vortex state described below. We then explore the nature of the spin-wave modes associated with this vortex state.

Some technical comments are in order. First, it is the case that experiments are underway on samples with an active film such as we envision,[4] so we proceed here with these samples in mind. These discs have radii of approximately 50 nm and a thickness of 5 nm. In such small diameter discs, of course, exchange interactions influence the nature of the spin motions appreciably. A crude measure of the importance of exchange is the ratio of the exchange length $l_{ex} = (D/4\pi M_S)^{1/2}$ to the radius R of the disc. Here, D is the exchange stiffness. If we have transition metal ferromagnets in mind, then typically, D is in the range of 3×10^{-9} G cm^2, so the exchange length is typically 5 nm, and exchange contributions to the spin-wave excitation energy are in the range of 1000 G, for a film whose radius is 50 nm. This is quite comparable to typical Oersted fields, which can be several hundred gauss at the edge of such a disc, when the injected current is in the range of the critical current (typically a few milliamperes). If the magnetization is not perpendicular to the surface everywhere, as we shall see is the case when the Oersted field is present, then in the spin-wave analysis, one must take due account of the influence of magnetic poles on the film surfaces produced by the dynamic component of the precessing magnetization. The internal dipolar field produced by these poles is given by $-4\pi m_z \hat{z}$ in the simplest approximation; here, the z direction is normal to the principal surfaces of the thin disc. The effect of this field on the frequency of spin waves in films by such a term is very large, as we know from the famous Kittel formula for the ferromagnetic resonance frequency of a film, $\gamma[H_0(H_0+4\pi M_S)]^{1/2}$, with M_S the saturation magnetization. The $4\pi M_S$ in the formula has its origin in the dipolar field just mentioned, with origin in the presence of a component of the dynamic magnetization perpendicular to the surface.

There is one more feature which must be incorporated into a quantitative theory in our view. The simple prescription that dipole fields generated by the dynamic component of the magnetization may be incorporated into the theory by only the effective field $-4\pi m_z \hat{z}$ mentioned in the previous paragraph applies only when the dynamic magnetization is uniform across the film. When the dynamic magnetization varies spatially, as in the spin-wave modes we will discuss below, a more complete description must be utilized if quantitative results are desired. We argue below that in the thin-film limit, $d \ll R$, it suffices to introduce corrections which involve the spatial gradients of the dynamic magnetization. We refer to these as gradient corrections, and we derive their form. These give corrections to the spin-wave frequencies by introducing effective fields of the order of $4\pi M_S(d/R)$ into the equations of motion. If we have Permalloy in mind, where $4\pi M_S \approx 10$ kG, then we see that the gradient corrections are as important from the quantitative point of view as exchange for samples of current interest. Thus, a quantitative theory of spin motions in discs of current interest must take both exchange and the gradient corrections into account. We note that current descriptions found in the literature overlook the gradient corrections.[5]

In this paper, Sec. II is devoted to the development of the formalism we use. In Sec. III, we present our discussion of the influence of the Oersted field on the magnetization in the disc and also our numerical studies of the dependence of the spin-wave modes on injected current and external magnetic field. Section IV is devoted to concluding remarks.

II. THEORETICAL STRUCTURE

A. Development of the formalism

We begin with a discussion of the principal assumptions we invoke to set up our formal structure. We shall imagine that our interest centers on a very thin disc of radius R whose thickness d is very small compared to R. With this limit in mind, we shall ignore edge effects with origin in the influence of the finite radius of the disc on the dipolar fields, both static and those associated with the dynamics component of the magnetization. Thus, we shall develop equations which describe a disc of infinite radius and examine solutions of these subject to boundary conditions at the outer rim of the disc. Such a procedure is valid in the limit $d \ll R$.

The basic equation we study is the simple torque equation, which we write as

$$\frac{\partial \vec{M}(\vec{\rho},t)}{\partial t} = \gamma[\vec{H}(\vec{\rho},t) \times \vec{M}(\vec{\rho},t)], \tag{1}$$

with γ the absolute value of the gyromagnetic ratio.

In the thin-film limit, the precessing magnetization will be independent of the coordinate z normal to the film surfaces, and its spatial dependence will be controlled by the vector $\vec{\rho}$, which lies in the xy plane. The magnetic field $\vec{H}(\vec{\rho},t)$ which drives the magnetization consists of three components, which we write as

$$\vec{H}(\vec{\rho},t) = \vec{H}^{(T)}(\rho) + \vec{h}^{(d)}(\vec{\rho},t) + \frac{D}{M_S}\nabla^2 \vec{M}(\vec{\rho},t). \tag{2}$$

In this expression, $\vec{H}^{(T)}(\rho)$ is the total static internal field that depends on the distance from the origin to a point in the disc. It consists of the externally applied magnetic field $\hat{z}H_0$, the internal demagnetizing field produced by the z component of the static magnetization, and the Oersted field, which we write as $\hat{\varphi}H^{(\varphi)}(\rho)$. In the presence of the Oersted field, at any point on the disc, the magnetization will lie in a plane defined by the unit vector in the z direction $\hat{z}$ and the unit vector in the azimuthal direction $\hat{\varphi}$. If we let $\psi(\rho)$ be the angle between the magnetization vector and $\hat{z}$, then in the thin-film limit, we may write

$$\vec{H}^{(T)}(\rho) = \hat{z}[H_0 - 4\pi M_S \cos\psi(\rho)] + \hat{\varphi}H^{(\varphi)}(\rho). \tag{3}$$

The field $\vec{h}^{(d)}(\vec{\rho},t)$ is the dipolar field generated by virtue of the motion of the magnetization. The form of this contribution will be discussed below. Finally, the third term in Eq. (2) is the exchange term. Here $\nabla^2=(1/\rho)(\partial/\partial\rho)\rho(\partial/\partial\rho)+(1/\rho^2)\times(\partial^2/\partial\varphi^2)$ is the two-dimensional Laplacian.

We begin by writing out the components of Eq. (1) in cylindrical coordinates, with the magnetization vector written as $\vec{M}=\hat{\rho}M_\rho+\hat{\varphi}M_\varphi+\hat{z}M_z$. One has, with $H^{(I)}=H_0-4\pi M_S\cos\psi(\rho)$,

$$\dot{M}_\rho = -[H^{(I)}+h_z^{(d)}]M_\varphi + [H^{(\varphi)}+h_\varphi^{(d)}]M_z + \frac{D}{M_S}\left[M_z\left(\nabla^2-\frac{1}{\rho^2}\right)M_\varphi - M_\varphi\nabla^2 M_z + \frac{2M_z}{\rho^2}\frac{\partial M_\rho}{\partial\varphi}\right], \tag{4a}$$

$$\dot{M}_\varphi = [H^{(I)}+h_z^{(d)}]M_\rho - h_\rho^{(d)}M_z + \frac{D}{M_S}\left[M_\rho\nabla^2 M_z - M_z\left(\nabla^2-\frac{1}{\rho^2}\right)M_\rho + \frac{2M_z}{\rho^2}\frac{\partial M_\varphi}{\partial\varphi}\right], \tag{4b}$$

and

$$\dot{M}_z = -[H^{(\varphi)}+h_\varphi^{(d)}]M_\rho + h_\rho^{(d)}M_\varphi + \frac{D}{M_S}(M_\varphi\nabla^2 M_\rho - M_\rho\nabla^2 M_\varphi) - \frac{D}{M_S\rho^2}\frac{\partial}{\partial\varphi}(M_\rho^2+M_\varphi^2). \tag{4c}$$

In Eqs. (4), we assume that we shall measure frequencies in units of magnetic field, so we have set the gyromagnetic ratio to unity in these equations and in the analysis below.

Our next step is to linearize these equations. We do this by writing $M_z=M_S\cos\psi(\rho)+m_z$, $M_\varphi=M_S\sin\psi(\rho)+m_\varphi$, and $M_\rho=m_\rho$ and then we retain terms linear in m_z, m_φ, and m_ρ. As we proceed, we recognize that the dipolar field generated by the spin motions, $\vec{h}^{(d)}$, is a linear function of these three amplitudes.

As one proceeds with the linearization, one encounters terms that are zero order in the dynamical magnetization in the equation for $\dot{m}_\rho$. A condition from which the angle $\psi(\rho)$ may be determined follows upon requiring these terms to vanish identically. This leads us to the following differential equation for the angle $\psi(\rho)$:

$$\frac{D}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\psi(\rho)}{\partial\rho}\right] = \left(\frac{D}{2\rho^2}-2\pi M_S\right)\sin 2\psi(\rho) + H_0\sin\psi(\rho) - H^{(\varphi)}(\rho)\cos\psi(\rho). \tag{5}$$

Once we impose the condition stated in Eq. (5), we are then left with linear equations for $\dot{m}_\rho$, $\dot{m}_\varphi$, and $\dot{m}_z$. There are, in fact, only two degrees of freedom in linear spin-wave theory, the two components of magnetization in the plane perpendicular to the local static magnetization. Thus, we use as our variables m_ρ, and we introduce $m_\parallel=m_z\cos\psi+m_\varphi\sin\psi$ and $m_t=m_\varphi\cos\psi-m_z\sin\psi$. When this is done, one finds that $\dot{m}_\parallel=0$, so, in fact, we may take $m_\parallel\equiv 0$. We are then left with the following two equations for the two degrees of freedom associated with the spin wave in the disc:

$$\dot{m}_\rho = -H^{(T)}(\rho)m_t - M_S\sin\psi(\rho)h_z^{(d)} + M_S\cos\psi(\rho)h_\varphi^{(d)} + D\left(\nabla^2 m_t - \left\{\frac{[\cos\psi(\rho)]^2}{\rho^2}+\left[\frac{\partial\psi(\rho)}{\partial\rho}\right]^2\right\}m_t + \frac{2\cos\psi(\rho)}{\rho^2}\frac{\partial m_\rho}{\partial\varphi}\right) \tag{6a}$$

and

$$\dot{m}_t = H^{(T)}(\rho)m_\rho - M_S h_\rho^{(d)} - D\left[\left(\nabla^2-\frac{1}{\rho^2}\right)m_\rho - \frac{2\cos\psi(\rho)}{\rho^2}\frac{\partial m_t}{\partial\varphi}\right]. \tag{6b}$$

We have defined

$$H^{(T)}(\rho) = \tilde{H}^{(I)}(\rho)\cos\psi(\rho) + \tilde{H}^{(\varphi)}(\rho)\sin\psi(\rho), \tag{7a}$$

where

$$\tilde{H}^{(I)}(\rho) = H^{(I)}(\rho) + D\nabla^2\cos\psi(\rho) \tag{7b}$$

and

$$\tilde{H}^{(\varphi)}(\rho) = H^{(\varphi)}(\rho) + D\left(\nabla^2-\frac{1}{\rho^2}\right)\sin\psi(\rho). \tag{7c}$$

There are two issues remaining. The first is the nature of the boundary conditions to be satisfied at the outer rim of the disc. When one envisions a real sample, this can be a complex issue to address in the absence of experimental data which provide one with information on the spin-wave spectrum. It is possible, with sufficient data in hand, to deduce the boundary condition applicable at sample surfaces or edges. With the present study in mind, the issue is the degree of spin pinning operable at the disc rim. Zaspel *et al.*, through an elegant derivation, have derived a boundary condition produced by dipolar effects at the edge of an ideal, very thin disc.[6] We note that in our own exact description of the magnetostatic modes of ferromagnetic ribbons of large aspect ratio,[7] dipolar effects at the ribbon edges led to complete pinning of the spins. In real materials, the physics may be considerably more complex at the disc edge than envisioned in both Refs. 6 and 7 Very much as in ultrathin films,[8,9] one may expect that spin-orbit anisotropy at the edge face of the disc provides pinning, possibly strong, of either easy axis or easy plane character, depending on sample details. The strength of such pinning fields can sometimes exceed the strength of the effective pinning fields provided by only the dipolar interaction by an order of magnitude.[8,9] Also, in nanomagnetic discs formed from alloys, it is possible that composition differences within the outer atomic layers near the disc edge may result in pinning, as discussed many years ago by Wigen *et al.*[10] in a very different context. We note that in recent numerical simulations of spin-wave spectra, Kruglyak *et al.*[11] have explored dipole-exchange spin waves in ferromagnetic discs with magnetic field applied in plane. The dimensions of their samples are similar, though somewhat different in detail than considered here. The agreement between theory and experiment is quite good. Unfortunately, these authors do not provide us with information on the boundary conditions they employ. Giovannini *et al.*[12] have also carried out simulations which agree nicely with data. In a very brief remark, they mention that they employed free spin boundary conditions, but they provide no discussion of the sensitivity of their simulations to the choice of boundary conditions nor do they provide a rationale for choosing the free spin form. We shall demonstrate below that the general conclusions in the present paper are not strongly affected by the choice of the boundary condition, fortunately.

To proceed within a definite framework, we shall suppose that along the edge surface of the disc, we have surface anisotropy present. Then, as discussed by Heinrich and Cochran,[8] one accounts for its presence by adding a surface energy of the form $-K_S(m_\rho/M_S)^2$ to the spin Hamiltonian, where we have easy axis anisotropy when $K_S>0$ and easy plane anisotropy when $K_S<0$. If the literature on ultrathin films is used to provide us with guidance, growth conditions, composition, and other factors control both the sign and magnitude of K_S. Following Ref. 8 we are then led to the following boundary conditions at the disc edge:

$$D\left(\frac{\partial m_\rho}{\partial \rho}\right)_R-\frac{2K_S}{M_S}m_\rho(R)=0 \tag{8a}$$

and

$$\left.\frac{\partial m_t}{\partial \rho}\right|_{\rho=R}=0. \tag{8b}$$

The boundary condition set forth in Ref. 6 can be obtained from Eq. (8a) through an appropriate choice of the parameter K_S. These authors find, as one expects on physical grounds, that the dipolar induced surface anisotropy has easy plane character. By suppressing the radial component of the magnetization at the disc edge, one eliminates magnetic poles which produce energetically costly stray fields.

In most of the calculations presented below, we shall assume that we have strong spin pinning at the disc edge, so we take $m_\rho(R)=0$. We will also see below that use of the free spin boundary condition, where $K_S=0$, leads to results quite similar to those obtained in the limit of strong spin pinning. The differences are in quantitative details and not in the qualitative features of the results. Thus, the principal qualitative conclusions of the present paper are not influenced by the choice of the boundary conditions at the edge of the disc, though, of course, quantitative details will depend on the choice of K_S. We shall see this below, when we discuss our calculations of the spin-wave spectra of the disc.

The second outstanding issue is the question of the dipolar field $\vec{h}^{(d)}(\rho)$ generated by the spin motions. The most straightforward manner of handling this is to simply suppose that $\vec{h}^{(d)}=-4\pi m_z\hat{z}=+4\pi\sin\psi m_t\hat{z}$. As noted in Sec. I, this approximation, used in earlier work,[5] assumes that the dynamic magnetization is spatially uniform over the entire sample. In fact, both m_t and m_ρ are functions of both ρ and φ, so this very simple approximation is open to question. In the Appendix, we show that one can generate corrections to this approximation which we call the gradient corrections. The discussion in Sec. I shows that these gradient corrections are as important, from the quantitative point of view, as the exchange. We remark that it is well known from earlier discussions of spin waves in ultrathin ferromagnets that this is the case.[13,14]

From the Appendix, we see that at the level of approximation described in the previous paragraph, we have $h_\rho^{(d)}=0$, and we provide approximate forms for $h_z^{(d)}$ and $h_\varphi^{(d)}$. We then seek eigensolutions of the equations of motion of the forms

$$m_\rho=f_\rho(\rho)\exp[i\nu\varphi-i\Omega t] \tag{9a}$$

and

$$m_t=if_t(\rho)\exp[i\nu\varphi-i\Omega t]. \tag{9b}$$

When these forms are inserted into Eqs. (6), for the case $\nu\neq 0$, we have our final form for the equations of motion:

$$\begin{aligned}\Omega f_\rho=&[H^{(T)}+4\pi M_S(\sin\psi)^2]f_t-\frac{2\pi M_S d}{\rho}\\&\times\left\{\left[\cos\psi\frac{d}{d\rho}(\rho f_\rho)+\frac{\rho\sin\psi}{|\nu|}\frac{d}{d\rho}f_t\sin\psi\right]\right.\\&\left.-[\nu(\cos\psi)^2+|\nu|(\sin\psi)^2]f_t\right\}\\&-D\left\{\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{(\nu^2+\cos\psi^2)}{\rho^2}-\left(\frac{d\psi}{d\rho}\right)^2\right]f_t\right.\\&\left.+\frac{2\nu\cos\psi}{\rho^2}f_\rho\right\},\end{aligned} \tag{10a}$$

$$\Omega f_t=H^{(T)}f_\rho-D\left\{\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}-\frac{(\nu^2+1)}{\rho^2}\right]f_\rho+\frac{2\nu\cos\psi}{\rho^2}f_t\right\}. \tag{10b}$$

The terms in Eq. (10a) proportional to $2\pi M_S d$ are the gradient corrections. We shall not quote the form of the equations of motion for the special case $\nu=0$. These are readily derived from Eqs. (6) above, along with Eqs. (A4b) and (A8b).

Our analysis of the spin-wave spectrum of the disc in the presence of the Oersted field will be based on the use of Eq. (5) for the description of canting in the ground state induced by the Oersted field, and then we use the equations of motion displayed in Eqs. (10) to determine the spin-wave frequencies and eigenvectors.

B. Brief discussion of a simple limiting case

The nature of the spin waves and their associated eigenfunctions may not be so familiar to the reader. Thus, to provide some orientation, we shall present a brief discussion of a simple limiting case. Suppose we assume we have the simple perpendicularly magnetized disc, with the Oersted field absent. Then the angle $\psi(\rho)$ may be set to zero everywhere. Furthermore, let us set aside the gradient corrections to the dipolar field generated by the spin motions. We may do this by ignoring the terms proportional to $2\pi M_S d$ in Eq. (10a). Notice that in this limit, $m_t = m_\varphi$. Then Eqs. (10a) and (10b) reduce to the much simpler forms

$$\Omega f_\rho = \Delta H f_t - D\left\{\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{(\nu^2+1)}{\rho^2}\right]f_t + \frac{2\nu}{\rho^2}f_\rho\right\} \tag{11a}$$

and

$$\Omega f_t = \Delta H f_\rho - D\left\{\left[\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{(\nu^2+1)}{\rho^2}\right]f_\rho + \frac{2\nu}{\rho^2}f_t\right\}. \tag{11b}$$

We have defined $\Delta H = H_0 - 4\pi M_S$. Clearly, the perpendicularly magnetized state is stable only when $\Delta H > 0$, so our attention is confined to this case.

Equations (11) admit simple analytic solutions. If we return to the variables m_ρ and m_φ, these have the forms

$$m_\rho = e^{i\nu\varphi}[A_- J_{\nu-1}(\kappa\rho) - A_+ I_{\nu+1}(\lambda\rho)] \tag{12a}$$

and

$$m_\varphi = ie^{i\nu\varphi}[A_- J_{\nu-1}(\kappa\rho) + A_+ I_{\nu+1}(\lambda\rho)]. \tag{12b}$$

Here $\kappa = (\Omega - \Delta H)^{1/2}/D^{1/2}$ and $\lambda = (\Omega + \Delta H)^{1/2}/D^{1/2}$. We assume here that we are looking for solutions with $\Omega > 0$.

It is interesting to display the Cartesian representation of the dynamic magnetization in real notation:

$$m_x = A_- J_{\nu-1}(\kappa\rho)\cos[(\nu-1)\varphi - \Omega t] - A_+ I_{\nu+1}(\lambda\rho)\cos[(\nu+1)\varphi - \Omega t] \tag{13a}$$

and

$$m_y = -A_- J_{\nu-1}(\kappa\rho)\sin[(\nu-1)\varphi - \Omega t] - A_+ I_{\nu+1}(\lambda\rho)\sin[(\nu+1)\varphi - \Omega t]. \tag{13b}$$

The simplest case to consider is the application of free spin boundary conditions, where one requires that

$$\left.\frac{\partial m_\rho}{\partial\rho}\right|_{\rho=R} = \left.\frac{\partial m_\varphi}{\partial\rho}\right|_{\rho=R} = 0. \tag{14}$$

The boundary condition may be satisfied by choosing $A_+ = 0$, and then we have eigenfrequencies $\Omega_{n\nu}$ found from

$$J'_{\nu-1}(\kappa_{n,\nu}R) = 0, \tag{15}$$

where $J'_m(x)$ is the derivative of the Bessel function with respect to its argument. The frequency is $\Omega_{n\nu} = \Delta H + D(\kappa_{n\nu})^2$. We have here spin waves in the form of simple radial standing waves, with an antinode at the rim of the disc. Since $J_{-m}(x) = (-1)^m J_m(x)$, notice that for $\nu > 0$ the modes labeled by the index ν are degenerate with the modes labeled by the index $-(\nu-2)$. Notice also that the uniform ferromagnetic resonance mode is the lowest-lying spin-wave mode with $\nu = +1$. For this mode, we have $\kappa \equiv 0$; the frequency of the mode is simply ΔH. From Eqs. (13), we see that this mode is just the uniform precession of the magnetization in the spatially uniform internal field ΔH. If we have spin pinning at the rim of the film such as those stated in Eqs. (8), then we must choose both A_- and A_+ to be nonzero. Upon noting that $I_{-m}(x) = I_m(x)$, we see that the degeneracy between the modes labeled with $\nu > 0$ and $-(\nu-2)$ is lifted by the boundary condition, since the behavior of the two eigenvectors near the rim of the disc differs.

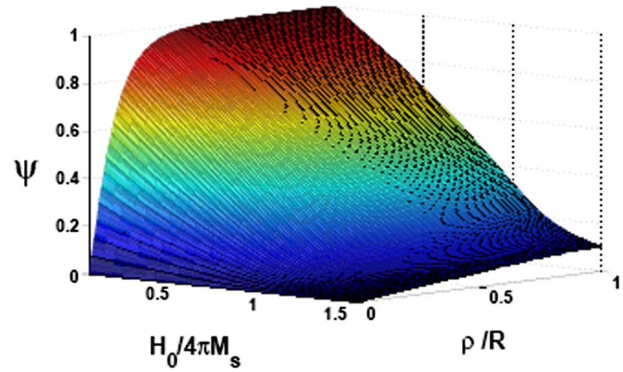

FIG. 1. (Color online) The radial variation of the angle $\psi(\rho)$ as a function of the applied external field H_0. The angle is expressed in units of $\pi/2$. The calculations assume that the dimensionless measure of Oersted field h_f assumes the value of 0.1. The ratio of the exchange length to the radius of the disc is 0.1.

We turn next to our complete studies of the linear spin dynamics of the disc with the Oersted field present. These will be based on the use of the boundary condition in Eqs. (8), a description of the canting angle $\psi(\rho)$ provided by Eq. (5), and finally, the eigenfrequencies and eigenvectors deduced from Eqs. (10). We shall also comment on the differences associated with use of the boundary conditions in Eqs. (8) and the free spin boundary conditions.

III. NUMERICAL RESULTS

In what follows, we shall assume that the Oersted field is generated by a transport current perpendicular to the disc and via a current density uniform in magnitude throughout the disc. Thus, we have $H^{(\varphi)}(\rho) = 2I\rho/cR^2$. In the results below, we use dimensionless measures of fields and frequencies (recall that we express frequencies in magnetic-field units) by dividing all fields by $4\pi M_S$. Thus, we use the dimensionless quantity $h_f = I/2\pi M_S cR$, which is the ratio of the Oersted field at the edge of the disc to $4\pi M_S$, as a measure of strength of the Oersted field. Typical values of h_f are in the range of 0.1, so we focus our attention on this region. We also have a dimensionless measure of the strength of the exchange, as discussed in Sec. I. This is the ratio l_{ex}/R where the exchange length $l_{ex} = (D/4\pi M_S)^{1/2}$. In all the calculations below, we have taken this parameter to have the value of 0.1, which is appropriate to a Permalloy disc with radius in the range of 50 nm.

In Fig. 1, as a function of the applied field H_0, we show the behavior of the canting angle $\psi(\rho)$ (normalized to $\pi/2$) as a function of distance from the center of the disc. We have

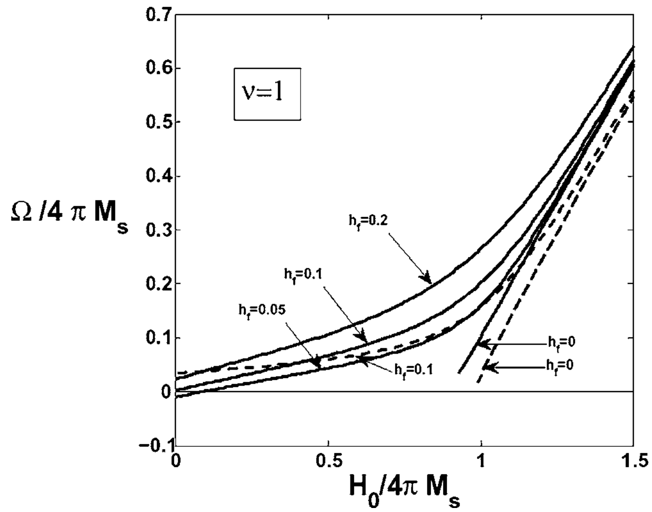

FIG. 2. The dependence on applied external magnetic field H_0 of the frequency of the low-lying spin-wave mode with $\nu=+1$ for various values of the dimensionless measure of the Oersted field h_f. The calculations here use the boundary conditions stated in Eqs. (8). This is the mode excited in ferromagnetic resonance studies. The solid curves are calculations which include the gradient corrections derived in the Appendix, and the dashed curves are calculations which ignore the gradient corrections. The ratio of the exchange length to the radius of the disc is 0.1, and the ratio of the thickness of the disc to its radius is 0.1 as well.

solved Eq. (5) to obtain these results, which are for the choice $h_f=0.1$. The boundary conditions discussed above suggests that the azimuthal component of the magnetization should have a vanishing slope at the rim of the disc, so the boundary condition we have used is $\partial\psi/\partial\rho|_R=0$. Recall that when we have an infinitely extended disc and no Oersted field present, the perpendicularly magnetized state will be stable only in the region $H_0>4\pi M_S$. The state described in Fig. 1 is a vortex state, where the magnetization at the very center of the disc is directed in the $+z$ direction always, parallel to the applied magnetic field. In the presence of the Oersted field, we find this state to exist even when $H_0<4\pi M_S$, where, in the absence of the Oersted field, the perpendicular state is unstable. Notice that when $H_0<4\pi M_S$, the magnetization right at the center of the disc is antiparallel to the nominal internal field $H_0-4\pi M_S$. As the applied external field approaches zero, we find that $\psi(R)=\pi/2$ at the outer rim of the disc, so the magnetization is parallel to the disc surfaces at the rim. For values of the applied field small compared to $4\pi M_S$, the size of the vortex core is the order of the exchange length l_{ex}. As discussed below, our spin-wave calculations suggest that down to zero applied field, the vortex state described in Fig. 1 is stable against small amplitude perturbations down to zero applied external field.

In Fig. 2, we show calculations of the magnetic-field dependence of the frequency of the low-lying spin wave with $\nu=1$ for various values of the Oersted field. The solid lines are the full calculations with the gradient corrections included, and we discuss the results obtained with these corrections first. For these calculations, the ratio of the thickness of the disc to its radius has been taken to be 0.1. When $h_f=0$, the mode shows a simple linear dependence of frequency on applied field. The frequency of this mode vanishes when H_0 approaches $4\pi M_S$ from above. In fact, if the spins are unpinned at the boundary, the frequency of this mode is zero at precisely $H_0=4\pi M_S$, with the consequence that the perpendicularly magnetized state is dynamically unstable for smaller values of the applied field. For our boundary condition, where the radial component of the dynamic magnetization is pinned at the disc edge, the ground state is dynamically stable for fields which extend a bit below $4\pi M_S$, by virtue of a bit of stiffening of the mode by exchange. If it were possible to measure the field dependence of the frequency of this mode and determine the value of the applied field for which its frequency vanishes, one could extract information about the influence of spin pinning at the rim of the disc.

The three remaining solid curves show the field dependence of the low lying $\nu=1$ mode frequency when the Oersted field is nonzero. The three solid curves are calculated for the choices $h_f=0.05$, 0.1, and 0.2 respectively. We see that the vortex state is dynamically stable down to the lowest fields, when the Oersted field is present, for sufficiently robust Oersted fields. In the figure, the frequency of this mode becomes negative at the lowest applied fields when $h_f=0.05$, suggesting that the vortex state is unstable in the low-field region for modest Oersted fields. We remark that in this low-field regime, one must question the accuracy of our gradient expansion for the $\nu=1$ mode. One may show that as $\rho\to 0$ (for $\nu>0$), the dynamic components of the magnetization vanish as $\rho^{\nu-1}$. Thus, the amplitude of the $\nu=1$ modes approaches a finite value as $\rho\to 0$; the eigenvectors have substantial amplitude in the region of the vortex core. At the same time, we see from Fig. 1 that at very small external Zeeman fields, there is a large spatial gradient in $\psi(\rho)$ within an exchange length of the center of the disc. Under such circumstances, it is not clear that the first correction term provided by the gradient expansion is quantitatively accurate. The results of Fig. 2 suggest that for sufficiently large Oersted fields, the vortex state is indeed stable down to zero applied field, but we are uncertain about the reliability of the predicted instability for low values of this field. Of course, if the gradient corrections incorporated here become large, then treatments which ignore the influence of spatial gradients in the magnetization completely[5] are surely inadequate. What is required is a full account of the dynamic dipole field; this must be expressed as an integral over the entire disc. The equations of motion for the eigenvectors then become integrodifferential equations, and their solution will prove a challenge. We have this issue under study. In addition, we have ignored edge effects associated with the static demagnetizing field. In the magnetostatic limit and in the absence of the Oersted field, such edge effects have been analyzed in a paper by Yukawa and Abe.[15]

The two dotted lines in Fig. 2 are calculations of the field dependence of the low lying $\nu=1$ mode frequencies, when the gradient corrections are set aside completely. One curve is for $h_f=0$, and the other curve is for $h_f=0.1$. There surely are substantive quantitative differences between the dotted curves and the relevant curves which incorporate the gradient corrections.

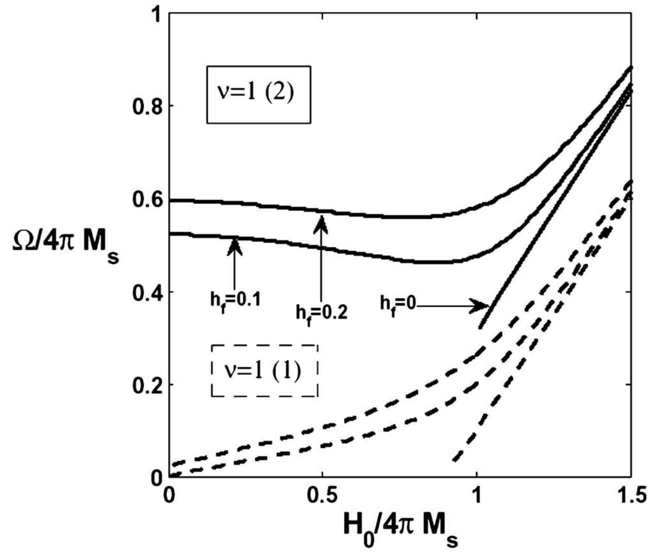

FIG. 3. We show, for various choices of the strength of the Oersted field, the field variation of the second mode in the $\nu=+1$ manifold. For comparison purposes, the frequencies displayed in Fig. 2 are shown as dotted lines. The various parameters are the same as used in Fig. 2, and the boundary conditions are those displayed in Eqs. (8).

Of course, as we appreciate from the discussion in Sec. II B, the disc displays a diverse spectrum of modes. In Fig. 2, we illustrate the influence of the Oersted field on the low-lying spin-wave mode with the azimuthal quantum number $\nu=1$. This is, as we see again from the discussion in Sec. II B, the mode excited in a ferromagnetic resonance experiment. For the case $\nu=1$, of course, there are higher-lying spin-wave modes with progressively more nodes in the radial variation of the dynamic magnetization. We show the dependence of the second mode with $\nu=+1$ on the Oersted field in Fig. 3. With no Oersted field present, we see again a nearly linear variation of frequency with externally applied field. If this curve is extrapolated down to zero field, its frequency will vanish at a lower field than that for the vanishing of the low-lying mode frequency illustrated in Fig. 2. Notice that when the Oersted field is present, the second mode remains quite "stiff" down to zero applied field. There is, in fact, a shallow minimum in the field dependence of the frequency as a function of applied external field when the Oersted field is switched on, in contrast to the monotonic dependence illustrated in Fig. 2. In Fig. 4, we show the behavior of the modes in the $\nu=0$ and $\nu=2$ manifolds. For the $\nu=2$ case, the minimum in the field variation of the mode frequency is very pronounced. This stiffening, found also for modes with larger values of ν, has its origin in the following. First of all, since the eigenfunctions for these higher modes vanish as $\rho^{\nu-1}$, the amplitude of the spin motion resides mostly in the mid- to outer portions of the disc. When $H_0 \ll 4\pi M_S$, in these regions of the disc, the angle $\psi(\rho)$ is close to $\pi/2$. The static magnetization is thus nearly parallel to the surfaces, and the spin wave is thus stiffened substantially by the dynamic dipole field which, in first approximation, is given by $-4\pi m_z$.

The modes outlined above, taken in total, constitute the complete set of spin-wave eigenmodes of the system. Thus,

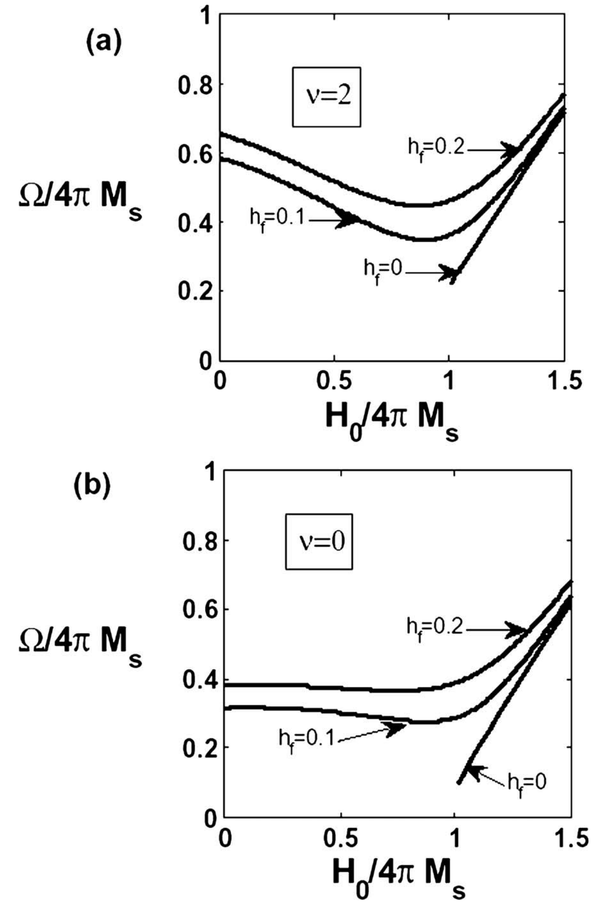

FIG. 4. For various values of the dimensionless measure of the Oersted field, we show the field dependence of the frequency of (a) the lowest-lying spin wave with $\nu=+2$ and (b) the lowest-lying spin wave with $\nu=0$. The parameters which characterize the disc are those described in the caption for Fig. 2, and the boundary conditions employed are those in Eqs. (8).

all possible excitations of the disc are contained in this set of modes. It is interesting, however, to inquire about a particular mode. This is the uniform translation mode of the vortex structure. If we imagine that the vortex is embedded in a film of infinite radius, then there will be a Goldstone mode of zero frequency associated with a rigid body translation of the vortex structure. In our finite disc, however, there will be restoring forces, and the translation mode will have a finite, nonzero frequency.

We have located this translation mode among the complex of spin-wave modes discussed above as follows. First to integrate our equations of motion, we require an initial guess for the radial functions $f_\rho(\rho)$ and $f_\varphi(\rho)$. In the vortex state, the two nonzero components of magnetization are $M_z(\rho) = M_S \cos \psi(\rho)$ and $M_\varphi(\rho) = M_S \sin \psi(\rho)$. If the vortex is displaced in the $\hat{x}$ direction, the local change of magnetization is $\delta x(\partial/\partial x)\vec{M}(\vec{x})$, which gives $m_\rho(\rho) = \delta x M_S \sin\varphi \sin\psi(\rho)/\rho = \delta x M_S \sin\psi(\rho)(e^{i\varphi}-e^{-i\varphi})/2i\rho$ and $m_t = \delta x M_S(\partial\psi/\partial\rho)\cos\varphi = \delta x M_S(\partial\psi/\partial\rho)(e^{i\varphi}+e^{-i\varphi})/2$. With these forms as a guide, we searched for an eigenmode by using the forms $f_\rho(\rho) = \sin\psi/\rho$ and $f_\varphi(\rho) = (\partial\psi/\partial\rho)$ as an initial guess for the eigenfunction, i.e., in the $\nu=1$ channel. We find after iterat-

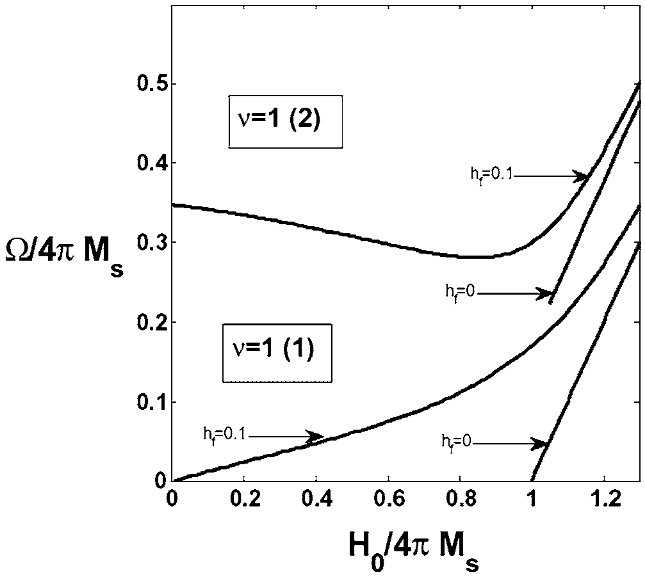

FIG. 5. For the parameters used in Fig. 2, but now with free spin boundary conditions wherein $(\partial m_{\rho,\varphi}/\partial\rho)_{\rho=R}=0$, we show the applied field dependence of the frequencies of the two low-lying modes in the $\nu=+1$ manifold. The parameters are the same as those used in Fig. 2, save for the change in boundary condition.

ing the equations of motion that this initial guess leads to a converged solution whose eigenfrequency is negative, but, in fact, agrees in absolute value precisely with that of the positive frequency of the low-lying spin-wave mode in the channel $\nu=-1$. Hence, the translation mode of the vortex core in the finite disc can be associated with this branch $\nu=-1$ of the spin-wave spectrum. A similar conclusion has been reached by Zaspel *et al.*,[6] we should remark, in studies of the vortex state of discs subject to an in-plane magnetic field. The eigenfrequency of the translational mode is rendered finite by the fact that the disc has finite radius, so there is a restoring force associated with translation of the vortex core. The eigenfunction is then also distorted in form from that provided by the rigid translation picture.

The calculations presented above all employ the boundary conditions in Eqs. (8) above with $K_S\to\infty$ to simulate strong pinning at the disc edge. It is interesting to inquire about the influence of the boundary conditions on the mode spectrum. For instance, one may consider free spin boundary conditions, wherein one has $\partial m_{\rho,\varphi}/\partial\rho|_R=0$. We have carried out studies with this boundary condition as well to find that the picture of the mode spectrum illustrated in Figs. 2–4 is altered only in quantitative detail. We illustrate this in Fig. 5, where we show the behavior of the two low-lying modes with $\nu=+1$, with free spin boundary conditions imposed in place of those stated in Eqs. (8).

IV. RESULTS AND DISCUSSION

We have developed the theory which describes the influence of an Oersted field generated by a transport current injected into a thin circular ferromagnetic disc, with magnetization oriented perpendicular to its surfaces. As discussed in Sec. I, this is a geometry where one may expect direct comparisons between theory and experimental data to be possible, in principle. Alternate geometries, such as elliptical samples magnetized in plane, present a challenge to theory. Our spin-wave theory describes linear excitations out of the ground state, and the modes we discuss can be used to describe the dynamics of the magnetization even above the threshold where the transport current drives the magnetization of the disc into motion, providing one is sufficiently near threshold that the precession angle is not large. Well above threshold, of course, linear theory such as ours is inapplicable and one must resort to numerical simulations such as those presented in Ref. 5. It is our view that eigenfunctions generated from our linear theory may be used as a basis for the description of the nonlinear regime, as in earlier discussions.[3]

The vortex state we explore here owes its existence to the presence of the Oersted field, and quite clearly in our case, the externally applied magnetic field is perpendicular to the plane of the disc. There is an extensive literature on vortex states and the related spin dynamics in ferromagnetic discs, wherein the external magnetic field is in plane. In such a geometry, for high magnetic fields, the disc is in the saturated state with magnetization in plane. As the applied field is reduced, a vortex can enter the disc, and this serves as a nucleation center for the reversal of the magnetization as the field is lowered further. The reversal of the magnetization is driven by translation of the vortex across the disc. In contrast to the configuration explored here, the presence of the vortex is not related to the presence of an Oersted field generated by a transport current, so the physics of these systems is very different from that considered here. A review of these interesting states and the spin dynamics associated with them is found in a recent review by Back and *et al.*.[16]

The fundamental assumption we have made to develop the theory is that the disc is very thin compared to its radius. This allows the various simplifications described above. We find that when the transport current, and thus the Oersted field is present, a vortex state is formed wherein the magnetization at the center of the disc is parallel to the externally applied magnetic field. When the Oersted field is strong, in the sense that our dimensionless parameter h_f is in the range of 0.1 or so, the vortex state appears stable with respect to small amplitude disturbances even as the applied Zeeman field approaches zero. At low external fields and in the low range of applied Oersted fields, our full calculations suggest that the vortex state is stable, though, as discussed in Sec. III, it is not clear at this writing if the predictions here are reliable for the reasons stated. This issue is under study at present.

It is our hope that the results presented here will stimulate further experimental activity in the area.

ACKNOWLEDGMENTS

We have enjoyed stimulating discussions with I. Krivorotov. This research was supported by the U.S. Army through Contract No. CS000128. R.E.A. also acknowledges support from FONDECYT, No. 1061106 (Chile).

APPENDIX: GRADIENT CORRECTIONS TO THE DIPOLAR FIELD

In this appendix, we develop corrections we call gradient corrections to the simple approximation $\vec{h}^{(d)}=-4\pi m_z\hat{z}$ for the dipolar fields generated by the motion of the magnetization. The spirit of this discussion is very similar to that used earlier, in our derivation of the dispersion relation of long-wavelength spin waves in extended ultrathin films with magnetization in plane.[8]

There are two sources of the dipolar field. First, when the dynamic magnetization varies in space, there are magnetic poles in the interior of the film that generate dipolar fields. Second, when the magnetization is canted by the Oersted field and $m_z\neq 0$, we have magnetic poles distributed over the surface of the film. We consider each of these in turn.

1. Dipolar fields from effective magnetic charges within the film

We begin here by considering a film of infinite extent and thickness d. We shall allow d to become very small, and we calculate the contribution to the dipolar field from charges within the film to first order in the thickness d. This is in keeping with the discussion in Sec. II, which considers a very thin film. The dipolar field from magnetic charges within the film will be denoted by $\vec{h}^{(da)}$, and we may write $\vec{h}^{(da)}=-\nabla\Phi_M^{(a)}$, where $\nabla^2\Phi_M^{(a)}=4\pi\nabla\cdot\vec{m}$ within the film (in the statement just made, ∇^2 is the three-dimensional Laplacian). When m_ρ, m_φ, and m_z are independent of z, we may write

$$\Phi_M^{(a)}(\rho,\varphi,z)=-\int\frac{d^2\rho'\,dz'}{[(\vec{\rho}-\vec{\rho}')^2+(z-z')^2]^{1/2}}\times\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}(\rho' m_\rho)+\frac{1}{\rho'}\frac{\partial m_\varphi}{\partial\varphi'}\right].\qquad\text{(A1)}$$

We shall evaluate all fields within the film at the center of the film, $z=0$. It is easy to see that $h_z^{(da)}$ is an odd function of z and vanishes at the midpoint of the film. Thus, for our purposes, we have $h_z^{(da)}=0$.

As in the main text, we shall seek eigensolutions of the equation of motion in the forms

$$m_\rho(\rho,\varphi)=f_\rho(\rho)e^{i\nu\varphi}\qquad\text{(A2a)}$$

and

$$m_t(\rho,\varphi)=if_t(\rho)e^{i\nu\varphi},\qquad\text{(A2a}')$$

where ν is an integer. One may insert these forms into Eq. (A1), set $z=0$, and then the integral over z' from $-d/2$ *to* $+d/2$ may be performed in closed form. As noted above, we then assume d to be very small and generate an expression for $\Phi_M^{(a)}(\rho,\varphi,0)\equiv\Phi_M^{(a)}(\rho,\varphi)$ to lowest order in d. The result may be cast into the form, for the case $\nu\neq 0$,

$$\Phi_M^{(a)}(\rho,\varphi)=-\frac{\rho d}{\nu}e^{i\nu\varphi}\int_0^\infty d\rho'\rho'\left\{\frac{\partial}{\partial\rho'}[\rho' f_\rho(\rho')]-\nu f_\varphi(\rho')\right\}\times\int_0^{2\pi}\frac{d\varphi''\sin(\nu\varphi'')\sin(\varphi'')}{[\rho^2+(\rho')^2-2\rho\rho'\cos\varphi'']^{3/2}}.\qquad\text{(A3)}$$

As one proceeds with the derivation that leads to Eq. (A3), one may show that for the special case that $\nu=0$, $\Phi_M^{(a)}(\rho,\varphi)$ is independent of φ as one expects, so when $\nu=0$, one has $h_\varphi^{(2a)}=0$. We shall keep this in mind, and as one can see from Eq. (A3), in what follows, we suppose that $\nu\neq 0$.

We may next form expressions for the two dipole field components $h_\rho^{(da)}$ and $h_\varphi^{(da)}$ from the expression in Eq. (A3). When this is done, if we assume that the quantity in square brackets in the integration over ρ' varies slowly with ρ', then we may evaluate this at $\rho'=\rho$ and remove it from the integral. We are then left with two integrals which, after considerable labor, may be evaluated analytically. We then find that to first order in the film thickness d, we have for $\nu=0$,

$$h_\varphi^{(da)}=\frac{2\pi id}{\rho}\left[\frac{\partial}{\partial\rho}(\rho f_\rho)-\nu f_\varphi\right]e^{i\nu\varphi}\quad(\nu\neq 0),\qquad\text{(A4a)}$$

while as noted above,

$$h_\varphi^{(da)}=0\ (\nu=0).\qquad\text{(A4b)}$$

To first order in the film thickness d, we find $h_\rho^{(da)}=0$, and we have already seen that $h_z^{(da)}=0$, as well.

2. Dipolar fields from magnetic effective charges on the film surfaces

If the magnetization of the disc is canted by the presence of the Oersted field, the presence of a nonzero z component of the dynamic magnetization m_z will generate magnetic charges on the two surfaces of the film. In the limit of a very thin disc, we will have the effective magnetic charge density of $+m_z(\rho,\varphi)$ on the upper surface of the film and an effective magnetic charge density of $-m_z(\rho,\varphi)$ on the lower surface. If we call $\vec{h}^{(db)}$ the dipolar field generated by this array of poles, then one may see that both $h_\rho^{(db)}$ and $h_\varphi^{(db)}$ are odd functions of the coordinate z and thus vanish at the film center. We thus are concerned here with only $h_z^{(db)}$, which in the thin-film limit is independent of z. It is straightforward to show that in the thin-film limit, we have

$$h_z^{(db)}(\rho,\varphi)=-d\int\frac{d^2\rho' m_z(\rho',\varphi')}{[(\vec{\rho}-\vec{\rho}')^2+(d/2)^2]^{3/2}}.\qquad\text{(A5)}$$

In the integrand in Eq. (A5), assuming that $m_z(\rho',\varphi')$ varies slowly with ρ', we replace this quantity by $m_z(\rho,\varphi)+[m_z(\rho',\varphi')-m_z(\rho,\varphi)]$. The integral which multiplies $m_z(\rho,\varphi)$ may be evaluated in closed form, and this yields $-4\pi m_z(\rho,\varphi)\hat{z}$ as the lowest-order contribution to the dynamic dipole field. We then write in the remaining integral $m_z(\rho,\varphi)=f_z(\rho)e^{i\nu\varphi}$. We then have

$$h_z^{(db)}=-4\pi m_z(\rho,\varphi)+\Delta h_z^{(db)}(\rho,\varphi),\qquad\text{(A6a)}$$

where

$$\Delta h_z^{(db)}(\rho,\varphi)=-de^{i\nu\varphi}\int\frac{d\rho'\rho' d\varphi'[f_z(\rho')\cos\nu\varphi'-f_z(\rho)]}{[\rho^2+(\rho')^2-2\rho\rho'\cos\varphi']^{3/2}}.\qquad\text{(A6b)}$$

We proceed next by assuming that $f_z(\rho')$ is slowly varying, so we write $f_z(\rho')\approx f_z(\rho)+(\rho'-\rho)(\partial f_z/\partial\rho)$ to obtain

$$\Delta h_z^{(db)} = de^{i\nu\varphi}\left\{ f_z(\rho) \int \frac{d\rho' \rho' d\varphi' (1-\cos \nu\varphi')}{[\rho^2 + (\rho')^2 - 2\rho\rho' \cos \varphi']^{3/2}} + \frac{\partial f_z}{\partial \rho} \int \frac{d\rho' \rho' d\varphi' (\rho' - \rho)\cos(\nu\varphi')}{[\rho^2 + (\rho')^2 - 2\rho\rho' \cos \varphi']^{3/2}} \right\}. \quad \text{(A7)}$$

An examination of the second integral in Eq. (A7) shows that for the special case $\nu=0$, the integral on ρ' diverges logarithmically if the upper limit is taken to be infinity. For the moment, we proceed by assuming $\nu \neq 0$, and we return to comment on the case where $\nu=0$ later.

It is possible to evaluate the integrals which appear in Eq. (A7) in closed form. When this is done for the case $\nu \neq 0$, we are led to

$$h_z^{(db)} = -4\pi f_z(\rho) e^{i\nu\varphi} + 2\pi d e^{i\nu\varphi} \left[\frac{|\nu|}{\rho} f_z(\rho) + \frac{1}{|\nu|} \frac{\partial f_z(\rho)}{\partial \rho} \right] \quad (\nu \neq 0). \quad \text{(A8a)}$$

We require a special procedure for the case $\nu=0$. Here we proceed as above, but we set the upper limit on the ρ' integration to be $\rho'=R$, and we keep R finite but suppose $R \gg \rho$. This then leads us to the prescription

$$h_z^{(db)} = -4\pi f_z(\rho) + 2\pi d \left[\frac{\partial f_z(\rho)}{\partial \rho} \right] \left[\ln\left(\frac{4R}{\rho} \right) - 2 \right] \quad (\nu = 0). \quad \text{(A8b)}$$

[1] I. N. Krivorotov, N. C. Emley, J. C. Sankey, S. I. Kiselev, R. C. Ralph, and R. A. Burhman, Science **307**, 228 (2005); J. C. Sankey, P. M. Braganca, A. G. F. Garcia, I. N. Krivorotov, R. A. Buhrman, and D. C. Ralph, Phys. Rev. Lett. **96**, 227601 (2006); S. Kaka, M. R. Pufall, W. H. Rippard, T. J. Silva, S. E. Russek, and J. A. Katine, Nature (London) **437**, 389 (2005); S. Kaka, M. R. Pufall, W. H. Rippard, T. J. Silva, S. E. Russek, J. A. Katine, and M. Carry, J. Magn. Magn. Mater. **286**, 375 (2005); Q. Mistral, Joo Von Kim, T. Devolder, P. Crozat, C. Chappert, J. A. Katrine, M. J. Carey, and K. Ito, Appl. Phys. Lett. **88**, 192507 (2006). For a review of the area and further references, see the article by M. D. Stiles and Jacques Miltat, in *Spin Dynamics in Confined Magnetic Strucutures III*, Topics in Appl. Physics Vol. 101, edited by B. Hillebrands and A Thiaville (Springer Verlag, Heidelberg, 2006), p. 101.

[2] See, for example, M. Sparks, Phys. Rev. B **1**, 3831 (1970). This paper uses a variational method to explore eigenfrequencies and eigenvectors. It has been pointed out that, unfortunately, the variational principle used in this and related papers is inappropriate for the class of problems studied. Thus, quantitative conclusions must be treated with care. For a discussion of the structure of the variational principle, see H. J. Benson and D. L. Mills, Phys. Rev. **188**, 849 (1969).

[3] See the discussion by P. E. Wigen, in *Nonlinear Phenomena and Chaos in Magnetic Materials*, edited by P. E. Wigen (World Scientific, Singapore, 1994), Chap. 1.

[4] I. Krivorotov (private communication).

[5] M. A. Hoefer, M. J. Ablowitz, B. Ilan, M. R. Pufall, and T. J. Silva, Phys. Rev. Lett. **95**, 267206 (2005).

[6] C. E. Zaspel, B. A. Ivanov, J. P. Park, and P. A. Crowell, Phys. Rev. B **72**, 024427 (2005).

[7] R. Arias and D. L. Mills, Phys. Rev. B **72**, 104418 (2005).

[8] B. Heinrich and J. Cochran, Adv. Phys. **42**, 523 (1993).

[9] B. Heinrich, in *Ultrathin Magnetic Structures II*, edited by B. Heinrich and J. A. C. Bland (Springer, Heidelberg, 1994), p. 195.

[10] P. E. Wigen, C. F. Kooi, M. R. Shanabarger, and Thomas D. Rosing, Phys. Rev. Lett. **9**, 206 (1962).

[11] V. V. Kruglyak, A. Barman, R. J. Hicken, J. R. Childress, and J. A. Katine, Phys. Rev. B **71**, 220409(R) (2005).

[12] L. Giovannini, F. Montoncello, F. Nizzoli, G. Gubbiotti, G. Carlotti, T. Okuno, T. Shinjo, and M. Grimsditch, Phys. Rev. B **70**, 172404 (2004).

[13] R. Arias and D. L. Mills, Phys. Rev. B **60**, 7395 (1999).

[14] D. L. Mills and S. M. Rezende, *Spin Dynamics in Confined Magnetic Structures II* (Springer, Heidelberg, 2003), Chap. 2.

[15] T. Yukawa and K. Abe, J. Appl. Phys. **45**, 3146 (1974).

[16] Christian H. Back, Daniel Pescia, and Mattias Buess, in *Spin Dynamics in Confined Structures III*, Topics in Applied Physics Vol. 101, edited by B. Hillebrands and A. Thiaville (Springer Verlag, Heidelberg, 2006), p. 137.