Comment on "Negative refractive index in gyrotropically magnetoelectric media"

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A recent paper [Shen, Phys. Rev. B **73**, 045113 (2006)] studied the possibility of realizing negative refraction with gyrotropically chiral media. Formulations have been provided to prove that the magnetoelectric coupling, either isotropic or anisotropic, would favor the realization of negative refraction. We show that the formulations are incorrect in either case, while the conclusion of realizing negative refraction by gyrotropic chiral media can still be drawn.

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In Ref. 1, Shen studied a gyrotropic chiral medium which is generalized from a bi-isotropic medium. Due to the magnetoelectric coupling of chiral materials, negative refraction can be anticipated.[2] In particular, Shen discussed the role of the nonreciprocity parameter (Tellegen parameter) in realizing negative-index materials. In naturally occurring chiral materials, the Tellegen parameter χ is normally quite small. However, by the proper manipulation of mixing rules, the Tellegen parameter can be moderately large.[3,4] Thus, the refractive indices for the eigenmodes can be reduced by χ as pointed out by Shen.[1] However, based on the wave-field theory,[3] Shen ended up with incorrect results for the equivalent medium for both nonreciprocal chiral and gyrotropic chiral media. Since the conclusion of realizing negative refraction is impressive, it certainly deserves further clarification and investigation. In addition to the discussion of the errors in the formulation process, the corrected results under specific conditions are presented here.

Similar notations (the Tellgen formalism) are employed for the convenience of comparison. The constitutive relations of general nonreciprocal bi-isotropic media were introduced by Sihvola *et al.*,[5]

$$\boldsymbol{D} = \epsilon\epsilon_0\boldsymbol{E} + (\chi + j\kappa)\boldsymbol{H}, \tag{1}$$

$$\boldsymbol{B} = \mu\mu_0\boldsymbol{H} + (\chi - j\kappa)\boldsymbol{E}. \tag{2}$$

It is a special case of gyrotropic chiral media, in which the permittivity and permeability are characterized by gyrotropic tensors

$$\bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \epsilon_1 & -j\epsilon_2 & 0 \\ j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \tag{3}$$

$$\bar{\boldsymbol{\mu}} = \begin{bmatrix} \mu_1 & -j\mu_2 & 0 \\ j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}. \tag{4}$$

Apparently, if the gyroelectric and gyromagnetic parameters are zero (i.e., $\epsilon_2=\mu_2=0$), the material becomes bi-isotropic. According to the results obtained by the equivalent medium theory,[3] the equivalent parameters for bi-isotropic materials can be deduced from Eq. 6 in Ref. 1 by assuming $\epsilon_2=\mu_2=0$,

$$\epsilon_\pm = \sqrt{\frac{\epsilon_1}{\mu_1}}\left(\sqrt{\epsilon_1\mu_1 - \frac{\chi^2}{\epsilon_0\mu_0}} \pm \frac{\kappa}{\sqrt{\epsilon_0\mu_0}}\right), \tag{5}$$

$$\mu_\pm = \sqrt{\frac{\mu_1}{\epsilon_1}}\left(\sqrt{\epsilon_1\mu_1 - \frac{\chi^2}{\epsilon_0\mu_0}} \pm \frac{\kappa}{\sqrt{\epsilon_0\mu_0}}\right). \tag{6}$$

The wave impedance is then

$$\eta_\pm = \sqrt{\mu_\pm/\epsilon_\pm} = \sqrt{\mu_1/\epsilon_1}. \tag{7}$$

Thus, only one impedance is obtained for the two eigenwaves and it is independent of the nonreciprocity parameter χ or chirality κ. This is quite problematic for a nonreciprocal medium. Our further derivation shows that it is certainly not the case as stated in Ref. 1.

We still consider the simplest situation (i.e., $\epsilon_2=\mu_2=0$, $\epsilon_3=\epsilon_1$, and $\mu_3=\mu_1$) as a proof. Hence, the bi-isotropic medium can be expressed using Shen's notation,

$$\boldsymbol{D} = \epsilon_1\epsilon_0\boldsymbol{E} + (\chi + j\kappa)\boldsymbol{H}, \tag{8}$$

$$\boldsymbol{B} = \mu_1\mu_0\boldsymbol{H} + (\chi - j\kappa)\boldsymbol{E}. \tag{9}$$

Substituting the above equations into Maxwell equations, we have

$$\nabla\times\nabla\times\boldsymbol{E} - 2\omega\kappa\nabla\times\boldsymbol{E} - \omega^2[\epsilon_1\mu_1\epsilon_0\mu_0 - (\chi^2+\kappa^2)] = 0. \tag{10}$$

By using Bohren's method of decomposition,[6] electromagnetic waves can be expressed linearly in terms of two right- and left-circularly polarized (RCP and LCP) waves. In accordance with Shen's notation, the subscripts + and − denote RCP and LCP, respectively. Therefore, the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are defined as

$$\begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix} = \bar{A}\begin{bmatrix} \boldsymbol{Q}_- \\ \boldsymbol{Q}_+ \end{bmatrix}, \tag{11}$$

where $Q_\pm$ satisfy the Helmholtz equation

$$\nabla^2\boldsymbol{Q}_- + k_-^2\boldsymbol{Q}_- = 0, \tag{12}$$

$$\nabla^2 \boldsymbol{Q}_+ + k_+^2 \boldsymbol{Q}_+ = 0. \tag{13}$$

By modifying the results of diagonalization, one can end up with

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 1 & -j\eta_+ \\ -j/\eta_- & 1 \end{bmatrix}, \tag{14}$$

from which the electromagnetic fields can be presented finally as

$$\boldsymbol{E} = \boldsymbol{Q}_- - j\eta_+ \boldsymbol{Q}_+, \tag{15}$$

$$\boldsymbol{H} = \frac{-j}{\eta_-}\boldsymbol{Q}_- + \boldsymbol{Q}_+. \tag{16}$$

During the process, the wave numbers and wave impedances are found as

$$k_\pm = \omega(\sqrt{\epsilon_1\mu_1\epsilon_0\mu_0 - \chi^2} \pm \kappa), \tag{17}$$

$$\eta_\pm = \frac{\mu_1\mu_0}{\sqrt{\epsilon_1\mu_1\epsilon_0\mu_0 - \chi^2} \mp j\chi}. \tag{18}$$

Since $k_\pm = \omega\epsilon_\pm\mu_\pm$ and $\eta_\pm = \sqrt{\mu_\pm/\epsilon_\pm}$, the equivalent medium parameters can be determined as shown below,

$$\epsilon_\pm = \frac{1}{\mu_1}(\sqrt{\epsilon_1\mu_1 - \chi^2/\epsilon_0\mu_0} \pm \kappa/\sqrt{\epsilon_0\mu_0}) \times (\sqrt{\epsilon_1\mu_1 - \chi^2/\epsilon_0\mu_0} \mp j\chi/\sqrt{\epsilon_0\mu_0}), \tag{19}$$

$$\mu_\pm = \mu_1 \frac{\sqrt{\epsilon_1\mu_1 - \chi^2/\epsilon_0\mu_0} \pm \kappa/\sqrt{\epsilon_0\mu_0}}{\sqrt{\epsilon_1\mu_1 - \chi^2/\epsilon_0\mu_0} \mp j\chi/\sqrt{\epsilon_0\mu_0}}. \tag{20}$$

Comparing our results in Eqs. (19) and (20) with Shen's results in Eqs. (5) and (6), it is obvious that, if the medium is lossless, the equivalent parameters will still possess imaginary parts, contrary to Shen's results. On the other hand, we have obtained two independent wave impedances which are dependent on the nonreciprocity parameter χ [see Eq. (18)] although the wave number is identical with the result obtained by Shen.

Furthermore, Shen[1] discussed negative refraction in a magnetoelectrically anisotropic material. Actually, it is just a case of a uniaxial Ω material, which has been well developed by Tretyakov *et al.*[7] If we assume $\chi_{12} = -\chi_{21} = -jK$ (which is certainly one of the kinds of Shen's "magnetoelectrically anisotropic material"), the wave numbers of two mutually perpendicular polarized eigenmodes in Ref. 1 would be

$$k_{a,b}^\pm = \omega(\pm\sqrt{\epsilon\mu} - jK), \tag{21}$$

where K represents the magnetoelectric coupling effect of Ω-shaped particles.

Following all the assumptions made in Ref. 1 (i.e., propagation parallel to the z direction, $\bar{\boldsymbol{\epsilon}} = \epsilon\bar{\boldsymbol{I}}$, and $\bar{\boldsymbol{\mu}} = \mu\bar{\boldsymbol{I}}$), we finally find that the wave numbers of the eigenmodes can be expressed by taking the Fourier transform of the Maxwell equations:

$$k = \omega\sqrt{\epsilon\mu - K^2}, \tag{22}$$

which agrees with the results in Ref. 7.

Therefore, it is proved that Shen's results in Ref. 1 are incorrect because the assumption made by Shen in Sec. IV will simplify the analysis of uniaxial Ω materials inappropriately, and will lose much information. Instead, if one still wants to study this case, one has to start from the wave splitting first by imposing

$$\boldsymbol{E} = E_z\hat{\boldsymbol{z}} + \boldsymbol{E}_t, \quad \boldsymbol{H} = H_z\hat{\boldsymbol{z}} + \boldsymbol{H}_t, \tag{23}$$

use the Fourier transform in the Maxwell equations in the transverse plane, and eliminate the normal fields. After obtaining the transverse components, the normal fields can thus be expressed by these transverse fields. Finally, the propagation constants for polarized eigenmodes will be found. Only after this stage can one assume that the wave is traveling along the z direction (i.e., the transverse wave number $k_t = 0$), and the proper solutions for the normal propagation can be obtained as in Eq. (22).

In summary, when the Beltrami[8] or wave-field theory[3] is involved, one has to be cautious. For instance, let us consider the most generalized form of materials,

$$\boldsymbol{D} = \bar{\boldsymbol{\epsilon}}\boldsymbol{E} + \bar{\boldsymbol{\xi}}\boldsymbol{H}, \tag{24}$$

$$\boldsymbol{B} = \bar{\boldsymbol{\zeta}}\boldsymbol{E} + \bar{\boldsymbol{\mu}}\boldsymbol{H}. \tag{25}$$

If all of the parameters in the above equations are scalars, the Beltrami or wave-field theory can be employed with no restriction, i.e.,

$$\boldsymbol{D}_\pm = \epsilon_\pm \boldsymbol{E}_\pm, \tag{26}$$

$$\boldsymbol{B}_\pm = \mu_\pm \boldsymbol{H}_\pm, \tag{27}$$

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_-, \tag{28}$$

$$\boldsymbol{H} = \boldsymbol{H}_+ + \boldsymbol{H}_-. \tag{29}$$

However, if any of the parameters is a gyrotropy tensor, the wave-field theory cannot be implemented directly. All the formulation has to be started from solving the Maxwell equations first.

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