Kinetics of quasiparticle trapping in a Cooper-pair box

R. M. Lutchyn and L. I. Glazman

W.I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA

We study the kinetics of the quasiparticle capture and emission process in a small superconducting island (Cooper-pair box) connected by a tunnel junction to a massive superconducting lead. At low temperatures, the charge on the box fluctuates between two states, even and odd in the number of electrons. Assuming that the odd-electron state has the lowest energy, we evaluate the distribution of lifetimes of the even- and odd-electron states of the Cooper-pair box. The lifetime in the even-electron state is an exponentially distributed random variable corresponding to a homogenous Poisson process of "poisoning" the island with a quasiparticle. The distribution of lifetimes of the odd-electron state may deviate from the exponential one. The deviations come from two sources—the peculiarity of the quasiparticle density of states in a superconductor and the possibility of quasiparticle energy relaxation via phonon emission. In addition to the lifetime distribution, we also find spectral density of charge fluctuations generated by capture and emission processes. The complex statistics of the quasiparticle dwell times in the Cooper-pair box may result in strong deviations of the noise spectrum from the Lorentzian form.

I. INTRODUCTION

Properties of a mesoscopic superconducting circuit may depend crucially on the presence of quasiparticles in its elements. The operation of a superconducting charge qubit, for example, requires two-electron periodicity of its charge states.[1–5,7,8] This periodicity may be interrupted by the entrance of an unpaired electron into the Cooper-pair box (CPB) serving as an active element of a qubit. The quasiparticle changes the charge state of CPB from even to odd, and lowers the charging energy. This trapping phenomenon, commonly referred to as "quasiparticle poisoning," is well-known from the studies of the charge parity effect in superconductors.[9,10] Quasiparticle poisoning contributes to the phase relaxation in superconducting qubits.[11] For a typical CPB size and tunnel conductances of the order of unit quantum, the quasiparticle dwelling times are of the order of a few μs. This time scale is at the edge of accessibility for the modern experiments.[6] Individual quasiparticle tunneling events were resolved and the statistics of quasiparticle entrances and exits from the CPB box were investigated in Refs. 7 and 8. The observed statistics of entrances was well described by a standard Poissonian process.[7,8] For the quasiparticle exits, the results are less clear. In many cases, it may be well described by the Poissonian statistics.[7,8] However, there are indications of deviation from that simple law for some samples.[12]

In this paper, we develop a kinetic theory of quasiparticle poisoning. We find the distribution of times $N_{\text{odd}}(t)$ and $N_{\text{even}}(t)$ the CPB dwells, respectively, in odd- and even-electron states. We also find the spectrum of charge noise produced by the poisoning processes.

The conventional Poissonian statistics of the quasiparticle exits would yield an exponential distribution for odd-electron lifetime in the box. We see two reasons for the distribution function $N_{\text{odd}}(t)$ to deviate from that simple form. The first one is related to the thermalization of a quasiparticle within the CPB. If the rates of energy relaxation and of tunneling out for a quasiparticle in CPB are of the same order, then two different time scales control the short-time and long-time parts of the distribution function $N_{\text{odd}}(t)$. The shorter time scale is defined by the escape rate Γ_{out} of the unequilibrated quasiparticle from the CPB. The longer time scale is defined by the rate of activation of equilibrated quasiparticle to an energy level allowing an escape from CPB. The second reason for the deviations from the exponential distribution controlled by a single rate, comes from the singular energy dependence of the quasiparticle density of states in a superconductor. Because of it, the tunneling-out rate depends strongly on the quasiparticle energy. Thus, even in the absence of thermalization the quasiparticle escapes from CPB cannot be described by an exponential distribution.

The conventional Poissonian statistics for both entrances and exits of the quasiparticle would lead to a Lorentzian spectral density $S_Q(\omega)$ of CPB charge fluctuations.[13] The interplay of tunneling. and relaxation rates may result in deviations from the Lorentzian function. In the case of slow quasiparticle thermalization rate compared to the quasiparticle tunneling-out rate Γ_{out}, the function $S_Q(\omega)$ roughly can be viewed as a superposition of two Lorentzians. The width of the narrower one is controlled by the processes involving quasiparticle thermalization and activation by phonons, while the width of the broader one is of the order of the escape rate Γ_{out}.

The paper is organized as follows. We begin in Sec. II with the qualitative derivation and discussion of the main results. In Secs. III and IV we derive and solve the microscopic master equations for the kinetics of the quasiparticle capture and emission, and calculate the lifetime distribution functions in the even- and odd-charge states. In Sec. V we calculate charge noise spectral density $S_Q(\omega)$ for the Cooper-pair box. In Sec. VI we summarize the main results. Some technical details are relegated to the Appendix.
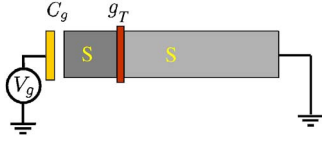
FIG. 1. (Color online) Schematic picture of the Cooper-pair box qubit. The left-hand side superconducting mesoscopic island is the Cooper-pair box connected via a tunable Josephson junction to the large superconducting lead (right-hand side). Gate bias is applied through the capacitance C_g. The junction is characterized by the dimensionless conductance g_T.

II. QUALITATIVE CONSIDERATIONS AND MAIN RESULTS

A. Relevant time scales

Dynamics of the Cooper-pair box coupled to the superconducting lead through the Josephson junction, see Fig. 1, is described by the Hamiltonian

$$H = H_C + H_{\mathrm{BCS}}^b + H_{\mathrm{BCS}}^l + H_T. \qquad (1)$$

Here H_{BCS}^b and H_{BCS}^l are BCS Hamiltonians for box and the lead; $H_C = E_c(\hat{Q}/e - N_g)^2$, with E_c, N_g, and $\hat{Q}$ being the charging energy, dimensionless gate voltage, and charge of the CPB, respectively. The tunneling Hamiltonian H_T is defined in the conventional way

$$H_T = \sum_{kp\sigma} (t_{kp} c_{k,\sigma}^{\dagger} c_{p,\sigma} + \mathrm{H.c.}), \qquad (2)$$

where t_{kp} is the tunneling matrix element, $c_{k,\sigma}$ and $c_{p,\sigma}$ are the annihilation operators for an electron in the state $|k,\sigma\rangle$ in the CPB and state $|p,\sigma\rangle$ in the superconducting lead, respectively. Here superconducting gap energy is the largest energy scale, $\Delta > E_c > E_J \gg T$. In order to distinguish between Cooper pair and quasiparticle tunneling, we present the Hamiltonian (1) in the form[14]

$$H = H_0 + V \quad \text{and} \quad V = H_T - H_J. \qquad (3)$$

Here $H_0 = H_C + H_{\mathrm{BCS}}^b + H_{\mathrm{BCS}}^l + H_J$, and H_J is the Hamiltonian describing Josephson tunneling

$$H_J = |N\rangle\langle N| H_T \frac{1}{E - H_0} H_T |N+2\rangle\langle N+2| + \mathrm{H.c.}$$

The matrix element $\langle N | H_T \frac{1}{E-H_0} H_T | N+2 \rangle$ is proportional to the Josephson energy E_J. (Here E_J is given by the Ambegaokar-Baratoff relation.) The perturbation Hamiltonian V defined in Eq. (3) is suitable for calculation of the quasiparticle tunneling rate.

Energy of the system as a function of the gate voltage is shown in Fig. 2. At $N_g = 1$ the electrostatic energy of the system is minimized when unpaired electron resides in the CPB. Thus, at $N_g = 1$ the CPB is a trap for a quasiparticle. The trap depth δE equals the ground state energy difference between the even-charge state (no quasiparticles in the CPB) and odd-charge state (an unpaired electron in the CPB). For equal gap energies in the box and the lead, $\Delta_l = \Delta_b = \Delta$, the trap is formed due to the Coulomb blockade effect. In the case $E_c \gg E_J/2$ one has
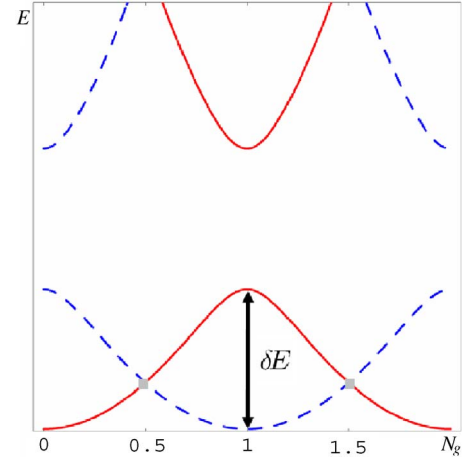

FIG. 2. (Color online). Energy of the Cooper-pair box as a function of dimensionless gate voltage N_g in units of e. Solid (red) line corresponds to even-charge state of the box, dashed (blue) line corresponds to the odd-charge state of the box. The trap depth δE is the ground state energy difference between the even-charge state (no quasiparticles in the CPB), and odd-charge state (an unpaired electron in the CPB) at $N_g = 1$. (We assume here equal gap energies in the box and the lead, $\Delta_l = \Delta_b = \Delta$.)

$$\delta E \approx E_c - \frac{E_J}{2} \gg T, \qquad (4)$$

and only two lowest charge states are important, see Fig. 2. Also, we assume that there is at most one quasiparticle in the box in the odd state.[15]

The transition probability per unit time between odd- and even-charge states $W(p,k)$ can be obtained using the Fermi golden rule ($\hbar = 1$),

$$W(p,k) = 2\pi |\langle p, \mathrm{even} | V | \mathrm{odd}, k \rangle|^2 \delta(E_p + \delta E - E_k). \qquad (5)$$

Here the state $|\mathrm{even}, p\rangle$ corresponds to even-charge state of the box and the quasiparticle in the state $|p\rangle$ in the reservoir; the state $|\mathrm{odd}, k\rangle$ corresponds to the odd-charge state of the box and quasiparticle in the state $|k\rangle$ within the box. The quasiparticle energies in the CPB and lead $E_{k/p}$ are defined as $E_{k/p} = \sqrt{\xi_{k/p}^2 + \Delta^2}$. Matrix elements $\langle p, \mathrm{even} | V | \mathrm{odd}, k \rangle$ can be calculated using the Bogoliubov transformation.[16,17] Taking into account the relation between tunneling matrix elements and the normal-state junction conductance the expression for $W(p,k)$ can be written as

$$W(p,k) = \frac{g_T \delta_l \delta_b}{8\pi} \left(1 + \frac{\xi_p \xi_k - \Delta^2}{E_p E_k} \right) \delta(E_p + \delta E - E_k) \qquad (6)$$

with $\delta_{b/l}$ being mean level-spacing in the box-lead, and g_T being dimensionless conductance of the junction.

Using the transition rate (6), one can calculate the level width of the state $|\mathrm{odd}, k\rangle$ with respect to quasiparticle tunneling through the junction to the lead,
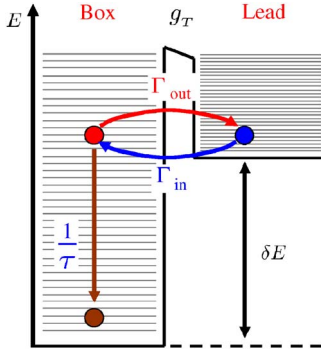
FIG. 3. (Color online). Schematic picture of the CPB-lead system showing allowed transitions for the quasiparticle injected into the excited state of the box. At $N_g = 1$ the Cooper-pair box is a trap for quasiparticle.

$$\Gamma_{\text{out}}(E_k) \equiv \sum_p W(p,k)$$
$$= \frac{g_T \delta_b}{4\pi} \frac{(E_k - \delta E)E_k - \Delta^2}{(E_k - \delta E)E_k} \nu(E_k - \delta E)\Theta(E_k - E_{\text{thd}}). \tag{7}$$

The Heaviside function $\Theta(x)$ appears in Eq. (7) because there are no states to tunnel into for a quasiparticle with energy lower than the threshold energy E_{thd}, see Fig. 3,

$$E_{\text{thd}} = \Delta + \delta E. \tag{8}$$

The quasiparticle density of states $\nu(E_k)$ (in units of the normal density of states at the Fermi level) is given by

$$\nu(E_k) = \frac{E_k}{\sqrt{E_k^2 - \Delta^2}}. \tag{9}$$

Due to the square-root singularity here, the rate $\Gamma_{\text{out}}(E_k)$ has square-root divergence at $E_k = E_{\text{thd}}$, see Fig. 4.

The quasiparticle may enter and subsequently leave the Cooper-pair box without changing its energy. For such elastic process, the excess energy of the exiting quasiparticle is equal to its initial energy, and is of the order of the temperature, i.e., $E_k - E_{\text{thd}} \sim T$. Therefore, the corresponding escape rate is
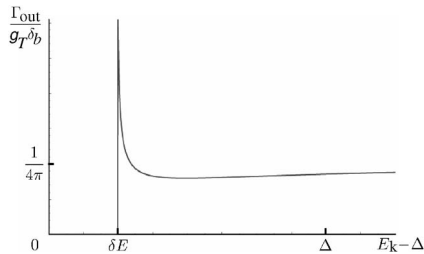

FIG. 4. The dependence of the escape rate $\Gamma_{\text{out}}(E_k)$ on energy E_k.

$$\Gamma_{\text{out}} = \frac{g_T \delta_b}{4\pi} \nu(T) \frac{\delta E}{\delta E + \Delta}. \tag{10}$$

Here for brevity we denote $\nu(T) \equiv \nu(E_k = \Delta + T)$. For the system with $g_T \lesssim 1$, volume of the CPB $V_b \lesssim 1~\mu\text{m}^3$, temperature $T \sim 50$ mK and $\delta E \sim 0.5$ K, the typical escape time Γ_{out}^{-1} is of the order of a μs.

To find the average rate Γ_{in} of quasiparticle tunneling from the lead to the CPB, we integrate the transition probability per unit time (6) with the distribution function $f(E_p)$ of quasiparticles in the lead,

$$\Gamma_{\text{in}} = \sum_{p,k} W(k,p)f(E_p). \tag{11}$$

Upon elastically tunneling into the excited state in the CPB the quasiparticle can relax to the bottom of the trap, see Fig. 3. For that, the quasiparticle needs to give away energy $\sim \delta E$. At low temperatures the dominant mechanism of quasiparticle energy relaxation is due to electron-phonon inelastic scattering rate $1/\tau(E_k)$. At low temperature quasiparticles are tunneling into the box through the energy levels just above the threshold energy $E_k \sim E_{\text{thd}}$, see Eq. (8). Assuming $\delta E \ll \Delta$, the typical quasiparticle relaxation time τ is given by[18]

$$\tau \equiv \tau(E_k \sim E_{\text{thd}}) \approx \tau_0 \left(\frac{\Delta}{T_c}\right)^{-3} \left(\frac{\delta E}{\Delta}\right)^{-7/2}. \tag{12}$$

Here τ_0 is a characteristic parameter defining the average electron-phonon scattering rate at $T = T_c$ with T_c being superconducting transition temperature. In aluminum, a typical material used for CPB, $\tau_0 \approx 0.1-0.5~\mu$s.[18–20] As one can see from Eq. (12), the quasiparticle relaxation rate is a strong function of the trap depth δE. Therefore, depending on δE there are two kinds of traps—"shallow" traps corresponding to $\tau \Gamma_{\text{out}} \gg 1$, and "deep" traps with $\tau \Gamma_{\text{out}} \ll 1$. (Note, for shallow traps we still assume $\delta E \gg T$.) The important quantity characterizing the traps is the probability P_{tr} for a quasiparticle to relax to the bottom of the trap before an escape,

$$P_{\text{tr}} = \frac{1/\tau}{1/\tau + \Gamma_{\text{out}}}. \tag{13}$$

B. Lifetime distribution function

Experimentally observable quantity,[7,8] which reveals the kinetics of quasiparticle trapping, is the lifetime distribution function $N_{\text{odd}}(t)$ of odd-charge states of the CPB. The distribution of lifetimes $N_{\text{odd}}(t)$ depends on the internal dynamics of the quasiparticle in the CPB, i.e, the ratio of $\tau \Gamma_{\text{out}}$.

We start with the discussion of the long-time asymptote of the lifetime distribution function. At $t > \tau$ the dwell-time distribution $N_{\text{odd}}(t)$ is governed by phonon-assisted activation of the thermalized quasiparticle in the trap. The phonon adsorption processes are statistically independent from each other. Hence, the lifetime distribution exponentially decays with time

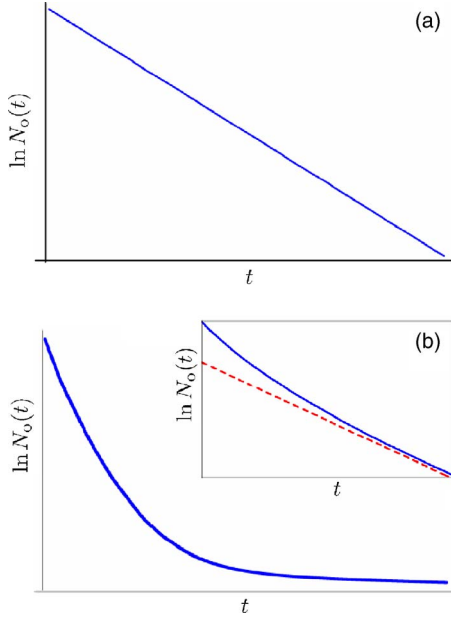$$N_{\text{odd}}(t) \propto \exp(-\gamma t) \tag{14}$$

with the rate

FIG. 5. (Color online) (a) Schematic picture of the lifetime distribution function for "deep" traps ($\tau\Gamma_{out}\ll 1$). (b) Schematic picture of the lifetime distribution function for "shallow" traps ($\tau\Gamma_{out}\gg 1$). Inset: Deviations of $N_{odd}(t)$ from exponential distribution at short times.

$$\gamma \approx \frac{1}{\tau}\frac{\nu(\delta E)}{\nu(T)}\exp\left(-\frac{\delta E}{T}\right)(1-P_{tr}). \tag{15}$$

This expression can be understood as follows. The rate of thermal activation of the quasiparticle from the bottom of the trap to the threshold energy is $\frac{1}{\tau}\frac{\nu(\delta E)}{\nu(T)}\exp(-\frac{\delta E}{T})$, for brevity we define $\nu(\delta E) \equiv \nu(E_k=E_{thd})$. The additional factor $\nu(\delta E)/\nu(T)$ here comes from the difference of the quasiparticle density of states at the bottom of the trap $\nu(T)$ and at the threshold energy $\nu(\delta E)$. The last term $(1-P_{tr})$ in Eq. (15) corresponds to the probability of the quasiparticle escape to the lead upon activation. Equation (15) allows us to consider limiting cases of $\tau\Gamma_{out}\ll 1$ and $\tau\Gamma_{out}\gg 1$.

In the case of "deep" traps ($\tau\Gamma_{out}\ll 1$) most quasiparticles upon entering the excited state in the box quickly thermalize. Therefore, the main contribution to lifetime distribution function comes from phonon-assisted escapes described by Eq. (14), see Fig. 5. The activation escape rate of Eq. (15) in this limit equals

$$\gamma_f \approx \Gamma_{out}\frac{\nu(\delta E)}{\nu(T)}\exp\left(-\frac{\delta E}{T}\right), \tag{16}$$

since $1-P_{tr}\simeq\Gamma_{out}\tau$, see Eq. (13).

In the opposite limit $\tau\Gamma_{out}\gg 1$, i.e., "shallow" traps, the probability for a quasiparticle to relax to the bottom of the trap is small $P_{tr}\ll 1$. Therefore, upon elastically tunneling into the excited state in the CPB the quasiparticles will predominantly return to the reservoir unequilibrated. Nevertheless, there is a small fraction of quasiparticles ($\sim 1/\tau\Gamma_{out}$) that do relax to the bottom of the trap, and stay in the box much longer than unequilibrated ones. Thus, at $t>\tau$ the

dwell-time distribution function $N_{odd}(t)$ has an exponentially decaying tail (14), see Fig. 5, with phonon-activated escape rate

$$\gamma_s \approx \frac{1}{\tau}\frac{\nu(\delta E)}{\nu(T)}\exp\left(-\frac{\delta E}{T}\right). \tag{17}$$

At $t\sim\tau$ the typical value of the lifetime distribution function is $N_{odd}(t\sim\tau)\sim\gamma_s/\tau\Gamma_{out}$.

At short times, $t\ll\tau$, the lifetime distribution function $N_{odd}(t)$ describes the kinetics of unequilibrated quasiparticles. Quasiparticles tunnel into the box through the energy levels $E_k=E_{thd}+\varepsilon$ (here $\varepsilon\geqslant 0$), and predominantly reside there until the escape with the rates $\Gamma_{out}(\varepsilon)$. For a given energy level ε the lifetime distribution is exponential

$$N_{odd}(\varepsilon,t)\propto\exp[-\Gamma_{out}(\varepsilon)t]. \tag{18}$$

Note that upon entering into the CPB from the reservoir the quasiparticles can populate different levels within the energy strip $\sim T$, see Eq. (36). Therefore, experimentally observable quantity $N_{odd}(t)$, obtained by the statistical averaging over a large number of the tunneling events, is given by

$$N_{odd}(t)\propto\int_0^\infty d\varepsilon\,\exp\left(-\frac{\varepsilon}{T}-\Gamma_{out}(\varepsilon)t\right). \tag{19}$$

Taking into account the singularity of $\Gamma_{out}(\varepsilon)$ at small energies $\Gamma_{out}(\varepsilon)\propto\varepsilon^{-1/2}$, we find that $N_{odd}(t)$ deviates from simple exponential distribution (see Fig. 5),

$$N_{odd}(t)\propto\exp\left[-3\left(\frac{\Gamma_{out}t}{2}\right)^{2/3}\right] \tag{20}$$

at times $t\gtrsim 1/\Gamma_{out}$. See also Sec. IV for more details.

C. Charge noise power spectrum

Anomalies in the lifetime distribution, see Fig. 5, should also lead to a specific spectrum of charge fluctuations. We define the spectral density of charge fluctuations $S_Q(\omega)$ in the Cooper-pair box as

$$S_Q(\omega)=\int_{-\infty}^\infty dt e^{i\omega t}[\langle\delta Q(t)\delta Q(0)\rangle+\langle\delta Q(0)\delta Q(t)\rangle] \tag{21}$$

with $\delta Q(t)=Q(t)-\langle Q\rangle$. The variance of the fluctuations of charge Q in the CPB,

$$\langle\delta Q^2\rangle=\int_0^\infty\frac{d\omega}{2\pi}S_Q(\omega), \tag{22}$$

is a thermodynamic, not a kinetic, quantity, and is known from statistical mechanics. The kinetics of the system is reflected in the dependence of the noise spectrum (21) on the frequency ω.

In the limit of fast relaxation $\tau\Gamma_{out}\ll 1$ the escapes from the CPB are given by one time scale (16). The quasiparticle entrances into and exits from the CPB are random, and can be described by Poisson processes. Thus, $S_Q(\omega)$ is given by the Lorentzian function corresponding to random telegraph noise,[13]

$$S_Q(\omega) \approx 4e^2 \bar{\sigma}_{\text{odd}} (1 - \bar{\sigma}_{\text{odd}}) \frac{\tau_{\text{eff}}}{(\omega\tau_{\text{eff}})^2 + 1}. \qquad (23)$$

Here $\bar{\sigma}_{\text{odd}}$ is an equilibrium average occupation of the odd-charge state in the CPB ($0 \leqslant \bar{\sigma}_{\text{odd}} \leqslant 1$), see Eq. (75) for details. At low temperature ($T \ll \delta E$) the box is predominantly in the odd-charge state, i.e., $(1 - \bar{\sigma}_{\text{odd}}) \propto \exp(-\delta E/T)$. The rate of activated quasiparticle escape processes has the same small exponent, therefore the width of the Lorentzian (23) is mainly given by the transitions from even- to odd-electron state,

$$\frac{1}{\tau_{\text{eff}}} \approx \Gamma_{\text{in}}. \qquad (24)$$

See Eq. (83) for the full result.

In the limit of slow relaxation ($\tau\Gamma_{\text{out}} \gg 1$) the charge noise power spectrum $S_Q(\omega)$ deviates significantly from the Lorentzian. These deviations stem from the fact that a quasiparticle may escape from the box before or after the equilibration, which results in two characteristic time scales for the escapes,[21] see Fig. 5. Consequently, the function $S_Q(\omega)$ can be roughly viewed as a superposition of two Lorentzians, and is similar to carrier concentration fluctuations in semiconductors due to trapping.[22] The "narrow" Lorentzian describes the dynamics of slow fluctuations due to phonon-assisted trapping of quasiparticles

$$S_Q^{(1)}(\omega) \sim e^2 \frac{\bar{\sigma}_{\text{odd}}(1 - \bar{\sigma}_{\text{odd}})\tau_{\text{eff}}}{(\omega\tau_{\text{eff}})^2 + 1}, \quad \frac{1}{\tau_{\text{eff}}} \approx \frac{1}{\tau}\frac{\Gamma_{\text{in}}}{\Gamma_{\text{in}} + \Gamma_{\text{out}}}. \qquad (25)$$

The width τ_{eff}^{-1} here is determined by the probability of quasiparticle trapping per unit time. (Like above, we assume here $T \ll \delta E$ and neglect activated escape rate.) The second (quasi) Lorentzian function $S_Q^{(2)}(\omega)$ is associated with fast charge fluctuations reflecting the kinetics of unequilibrated quasiparticles. Assuming $\omega \gg \Gamma_{\text{out}} \gg \Gamma_{\text{in}}$ the asymptote of $S_Q^{(2)}(\omega)$ is

$$S_Q^{(2)}(\omega) \sim e^2 \frac{\bar{\sigma}_{\text{odd}}}{\Gamma_{\text{out}}} \exp(-\delta E/T) \left(\frac{\Gamma_{\text{out}}}{\omega}\right)^2. \qquad (26)$$

The width of $S_Q^{(2)}(\omega)$ is determined by the typical escape rate of unequilibrated quasiparticles from the box Γ_{out} defined in Eq. (10). Similar to the lifetime distribution, see Fig. 5, we predict deviations of $S_Q^{(2)}(\omega)$ from the Lorentzian function at $\omega \sim \Gamma_{\text{out}}$ due to the peculiarity of the quasiparticle density of states.

The high-frequency tail of $S_Q(\omega)$ is provided by Eq. (26). However, the contribution of $S_Q^{(2)}(\omega)$ to the sum rule (22) is much smaller than that from $S_Q^{(1)}(\omega)$. In other words, the main contribution to the noise power comes from slow fluctuations. It resembles the case of the current noise in superconducting detectors.[23]

In the rest of the paper, we provide detailed derivation of the results discussed qualitatively in this section.

III. LIFETIME DISTRIBUTION OF THE EVEN-CHARGE STATE

Let us assume that the system switched to the even state at $t=0$, and introduce the probability density $N_{\text{even}}(E_k,t)$ for a quasiparticle to enter the CPB for the first time through the state E_k. Then, the probability density for the CPB to reside in the even state until time t is

$$N_{\text{even}}(t) = \sum_k N_{\text{even}}(E_k,t). \qquad (27)$$

$N_{\text{even}}(E_k,t)$ is given by the conditional probability of quasiparticle entering the CPB into an empty state E_k during the interval $(t,t+dt)$ times the probability that any quasiparticle has not entered into any state in the CPB during the preceding interval $(0,t)$,

$$N_{\text{even}}(E_k,t)dt = \sum_p W(k,p) f(E_p)$$
$$\times \left(1 - \sum_{k'} \int_0^t dt' N_{\text{even}}(E_{k'},t')\right) dt. \qquad (28)$$

Summing Eq. (28) over states k and solving for $N_{\text{even}}(t)$ one finds

$$N_{\text{even}}(t) = \Gamma_{\text{in}} \exp(-\Gamma_{\text{in}}t), \qquad (29)$$

which corresponds to a homogenous Poisson process. The quasiparticle tunneling rate from the lead to the CPB Γ_{in} is given by Eq. (11).

Recent experiments by Aumentado *et al.*[2,7] indicate that the density of quasiparticles n_{qp}^l in the lead exceeds the equilibrium one at the temperature of the cryostat. The origin of nonequilibrium quasiparticles is not clear, but it is plausible to assume that quasiparticle distribution function in the lead $f(E_p)$ is given by the Boltzmann function

$$f(E_p) = \exp\left(-\frac{E_p - \mu_l}{T}\right) \qquad (30)$$

with some effective chemical potential and temperature, μ_l and T, respectively. The chemical potential μ_l is related to the quasiparticle density by the equation

$$n_{\text{qp}}^l = \frac{1}{V_l} \sum_p f(E_p). \qquad (31)$$

Here V_l is the volume of the lead. We consider the density of quasiparticles n_{qp}^l and their effective temperature as input parameters here, which can be estimated from the experimental data.[2,7,8] Taking into account Eq. (30) we can evaluate the right-hand side (rhs) of Eq. (11) to obtain

$$\Gamma_{\text{in}} = \frac{g_T n_{\text{qp}}^l}{4\pi\nu_F} \nu(\delta E) \frac{\delta E}{\Delta + \delta E}. \qquad (32)$$

Here ν_F is the normal density of states at the Fermi level.

The average waiting time in the even-charge state is

$$\langle T_e \rangle = \int_0^\infty N_{\text{even}}(t) t \, dt = \Gamma_{\text{in}}^{-1}. \tag{33}$$

This result is expected for the conventional Poisson process.

IV. LIFETIME DISTRIBUTION OF THE ODD-CHARGE STATE

A. Master equation for survival probability

The distribution of dwell times for odd-charge state is more complicated than for even-charge state due to the internal dynamics of the quasiparticle in the CPB. Upon tunneling elastically into the box the quasiparticle enters into the excited state with typical excess energy δE above the gap in the island. The dwell time of the quasiparticle in the box depends whether upon tunneling into the excited state it relaxes to the bottom of the trap or tunnels out unequilibrated, see Fig. 2. In order to describe the physics of quasiparticle tunneling we develop a formalism similar to the rate equations theory. We will include electron-phonon collision integral into our equations to account for the internal dynamics of the quasiparticle inside the CPB. The experimentally accessible quantity is the probability density $N_{\text{odd}}(t)$ of leaving an odd state in the time interval $(t, t+dt)$ assuming that the quasiparticle resided continuously in the box during the time interval $(0, t)$. The object convenient for evaluation is the conditional probability $S_{\text{odd}}(t)$ (or survival probability) for a quasiparticle to occupy a given level, under the condition that the unpaired electron continuously resided in the box over the time interval $(0, t)$. The lifetime distribution $N_{\text{odd}}(t)$ can be easily obtained from $S_{\text{odd}}(t)$,

$$N_{\text{odd}}(t) = \frac{d}{dt}[1 - S_{\text{odd}}(t)] = -\frac{dS_{\text{odd}}(t)}{dt}. \tag{34}$$

Probability $S_{\text{odd}}(t)$ is simply related to the conditional probability $S(E_k, t)$ for a quasiparticle to occupy level E_k at the moment t in the box assuming that a quasiparticle entered CPB at $t=0$ and resided continuously in the box during the time interval $(0, t)$,

$$S_{\text{odd}}(t) = \sum_k S_{\text{odd}}(E_k, t). \tag{35}$$

We assume that in the initial moment of time the quasiparticle has just entered the state E_k in the box. Therefore, the initial probability $S_{\text{odd}}(E_k, 0)$ of the occupation of the level E_k in the box is determined by the tunneling rate into the state E_k,

$$S_{\text{odd}}(E_k, 0) = \frac{1}{\Gamma_{\text{in}}} \sum_p W(p, k) f(E_p). \tag{36}$$

The normalization of $S_{\text{odd}}(E_k, 0)$ is chosen to satisfy $S_{\text{odd}}(0) = \Sigma_k S_{\text{odd}}(E_k, 0) = 1$. According to Eq. (36) the initial conditional probability $S_{\text{odd}}(E_k, 0)$ is zero below the threshold energy $E_k < E_{\text{thd}}$, and is proportional to Gibbs factor above the threshold $E_k > E_{\text{thd}}$. This reflects out-of-equilibrium quasiparticle distribution at $t=0$.

The conditional probability $S_{\text{odd}}(E_k, t)$ consistent with initial conditions (36) satisfies the following master equation:

$$\dot{S}_{\text{odd}}(E_k, t) + \Gamma_{\text{out}}(E_k) S_{\text{odd}}(E_k, t) = -\frac{S_{\text{odd}}(E_k, t) - S_{\text{odd}}^{\text{eq}}(E_k, t)}{\tau}. \tag{37}$$

The second term on the left-hand side (lhs) corresponds to the loss from the state E_k due to the tunneling through the junction to the lead with the rate $\Gamma_{\text{out}}(E_k)$ of Eq. (7). Note that unlike in the theory of the rate equations there is no "gain" term in Eq. (37). This is due to the condition that the box is occupied at $t=0$ and remains occupied continuously until some time t. The rhs of Eq. (37) corresponds to the electron-phonon collision integral in the relaxation time approximation with τ of Eq. (12) and

$$S_{\text{odd}}^{\text{eq}}(E_k, t) = S_{\text{odd}}(t) \rho_{\text{odd}}^b(E_k).$$

Note that Eq. (37) is nonlocal in E_k due the dependence of the collision integral on $S_{\text{odd}}(t)$ [see Eq. (35)]. The collision integral in Eq. (37) describes the phonon-induced relaxation of the trapped quasiparticle to an equilibrium,

$$\rho_{\text{odd}}^b(E_k) = \frac{\exp(-E_k/T)}{Z_{\text{odd}}}. \tag{38}$$

Here T is the quasiparticle temperature in the box. (For simplicity, we assume that the effective quasiparticle temperature in the lead is the same as in the box, $T_l = T_b = T$.) The normalization factor Z_{odd} at $T \ll T^*$ is given by

$$Z_{\text{odd}} = \sqrt{\frac{\pi}{2}} \frac{\Delta}{\delta_b} \sqrt{\frac{T}{\Delta}} \exp\left(-\frac{\Delta}{T}\right). \tag{39}$$

B. General solution for $S_{\text{odd}}(t)$

Using Laplace transform,

$$S_{\text{odd}}(E_k, s) = \int_0^\infty dt \, S_{\text{odd}}(E_k, t) e^{-st}, \tag{40}$$

we reduce the differential equation (37) supplied with the initial conditions (36) to an algebraic one

$$\left(s + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}\right) S_{\text{odd}}(E_k, s) = \frac{S_{\text{odd}}^{\text{eq}}(E_k, s)}{\tau} + S_{\text{odd}}(E_k, 0). \tag{41}$$

Equation (41) can be solved for $S_{\text{odd}}(E_k, s)$. Then, by summing that solution over momenta k and utilizing Eqs. (35) and (36) we find the survival probability $S_{\text{odd}}(s)$

$$S_{\text{odd}}(s) = \frac{B(s)}{1 - A(s)}. \tag{42}$$

Here functions $B(s)$ and $A(s)$ are defined as

$$B(s) = \frac{1}{\Gamma_{\text{in}}} \sum_k \frac{f(E_k - \delta E) \Gamma_{\text{out}}(E_k)}{s + 1/\tau + \Gamma_{\text{out}}(E_k)},$$
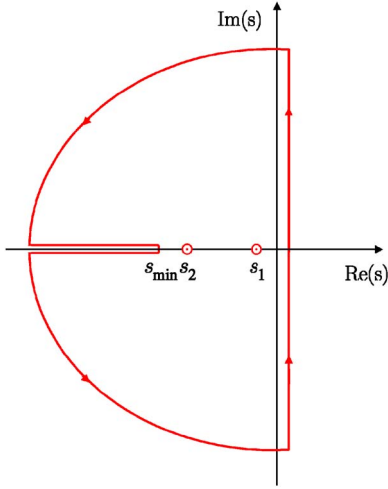
FIG. 6. (Color online) Contour of integration (red line) chosen to calculate inverse Laplace transform Eq. (46). Points of nonanalytic behavior of $\sigma_{++}(s)$ are shown. Poles at s_1, s_2, and a cut $s \in (-\infty, -s_{\min})$.

$$A(s) = \frac{1}{\tau} \sum_k \frac{\rho_{\text{odd}}^b(E_k)}{s + 1/\tau + \Gamma_{\text{out}}(E_k)}. \tag{43}$$

At $T \gg \delta_b$ one can take the thermodynamic limit and replace the sums with the integrals in Eq. (43). Further simplification of the denominator in Eq. (42) is possible if one splits the integral in $A(s)$ into the intervals (Δ, E_{thd}), where $\Gamma_{\text{out}}(E_k) = 0$, and (E_{thd}, ∞). Then, Eq. (42) becomes

$$S_{\text{odd}}(s) = \left(s + \frac{1}{\tau}\right) \frac{B(s)}{s + X(s)} \tag{44}$$

with the functions $B(s)$ and $X(s)$ defined as

$$B(s) = \frac{1}{\Gamma_{\text{in}}} \int_{E_{\text{thd}}}^{\infty} \frac{dE_k}{\delta_b} \nu(E_k) \frac{f(E_k - \delta E)\Gamma_{\text{odd}}(E_k)}{s + 1/\tau + \Gamma_{\text{odd}}(E_k)},$$

$$X(s) = \frac{1}{\tau} \int_{E_{\text{thd}}}^{\infty} \frac{dE_k}{\delta_b} \nu(E_k) \frac{\rho_{\text{odd}}^b(E_k)\Gamma_{\text{odd}}(E_k)}{s + 1/\tau + \Gamma_{\text{odd}}(E_k)}. \tag{45}$$

The inverse Laplace transform is given by

$$S_{\text{odd}}(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds\, S_{\text{odd}}(s) e^{st}, \tag{46}$$

where ϵ is chosen in such way that $S_{\text{odd}}(s)$ is analytic at $\text{Re}[s] > \epsilon$. The integral (46) can be calculated using complex variable calculus by closing the contour of integration as shown in Fig. 6 and analyzing the enclosed points of nonanalytic behavior of $S_{\text{odd}}(s)$. In general, the singularities of $S_{\text{odd}}(s)$ consist of two poles and a cut. The latter is due to the singularities of the function $B(s)$ causing $S_{\text{odd}}(s)$ to be nonanalytic along the cut $s \in (-\infty, -s_{\min})$, where

$$s_{\min} = \frac{1}{\tau} + \min[\Gamma_{\text{odd}}(E_k)]. \tag{47}$$

The plot of $\Gamma_{\text{odd}}(E_k)$ is shown in Fig. 4. The function $\Gamma_{\text{odd}}(E_k)$ has a minimum at $E_k^{\min} = E_{\text{thd}} + \delta E/2$. (For the estimate of the minimum we assumed $\delta E \ll \Delta$.) In addition to the cut, $S_{\text{odd}}(s)$ has two poles. The poles s_1 and s_2 are the solutions of the following equation in the region of analyticity of $B(s)$:

$$s + X(s) = 0. \tag{48}$$

We now analyze the singularities $S_{\text{odd}}(s)$ and find their contribution to the integral (46).

The contribution from the cut to Eq. (46) corresponds to the kinetics of unequilibrated quasiparticles. Formally it comes from the nonanalyticity of $S_{\text{odd}}(s)$ due to the singularities of the function $B(s)$ itself. The proper contribution to Eq. (46) can be calculated by integrating along the contour enclosing the cut,

$$I_{\text{cut}} = \frac{-1}{2\pi i} \int_{s_{\min}}^{\infty} ds e^{st} [S_{\text{odd}}(s + i\epsilon) - S_{\text{odd}}(s - i\epsilon)]. \tag{49}$$

At low temperature $T \ll \delta E$, the discontinuity of the imaginary part of $S_{\text{odd}}(s)$ at the cut is

$$S_{\text{odd}}(s + i\epsilon) - S_{\text{odd}}(s - i\epsilon) = 2i\left(s + \frac{1}{\tau}\right) \frac{\text{Im}\, B(s + i\epsilon)}{s}. \tag{50}$$

Substitution of this expression into Eq. (49) yields

$$I_{\text{cut}} = \frac{1}{\Gamma_{\text{in}}} \int_{E_{\text{thd}}}^{\infty} \frac{dE_k}{\delta_b} \nu(E_k) f(E_k - \delta E)\Gamma_{\text{odd}}(E_k)$$

$$\times \frac{\tau\Gamma_{\text{odd}}(E_k)}{1 + \tau\Gamma_{\text{odd}}(E_k)} \exp\left(-\frac{t}{\tau} - \Gamma_{\text{odd}}(E_k)t\right). \tag{51}$$

To simplify the above expression we introduce the dimensionless variable z,

$$z = \frac{E_k - E_{\text{thd}}}{T}, \tag{52}$$

and write the integral in I_{cut} in terms of z,

$$I_{\text{cut}} = \frac{1}{\sqrt{\pi}\Gamma_{\text{out}}\nu(\delta E)} \int_0^{\infty} dz\,\nu(z)\Gamma_{\text{odd}}(z) \frac{\tau\Gamma_{\text{odd}}(z)}{1 + \tau\Gamma_{\text{odd}}(z)}$$

$$\times \exp[-z - \Gamma_{\text{odd}}(z)t - t/\tau]. \tag{53}$$

Here and thereafter $\Gamma_{\text{odd}}(z)$ and $\nu(z)$ are given by Eqs. (7) and (9), respectively, with $E_k = E_{\text{thd}} + Tz$.

We now analyze the contribution to Eq. (46) from the poles. The pole at s_1 may be found from the iterative solution of Eq. (48) at small s ($s \ll s_{\min}$),

$$s_1 \approx -X(s = 0), \tag{54}$$

with $X(s)$ given by Eq. (45). The contribution from the pole at s_1 can be easily calculated using residue calculus yielding

$$I_1 = Y(0)\exp[-X(0)t].\tag{55}$$

Equation (55) describes the kinetics of thermalized quasiparticles. At low temperature $X(0) \propto \exp(-\delta E/T)$, which justifies the approximation used in Eq. (54), see also next section. The function $Y(0)$ depends on $\tau\Gamma_{\text{out}}$, and is approximately given by

$$Y(0) \approx \frac{1}{\sqrt{\pi}} \int_0^\infty dz \frac{\exp(-z)}{\sqrt{z} + \tau\Gamma_{\text{out}}}.\tag{56}$$

Here we used small z asymptote $\left(z \ll \frac{\delta E}{2T}\right)$ for the escape rate,

$$\Gamma_{\text{out}}(z) \approx \frac{\Gamma_{\text{out}}}{\sqrt{z}}.\tag{57}$$

The second pole s_2 is given by the solution of Eq. (48) at large s. At small temperature $T \ll \delta E$ one can show that the contribution of the second pole s_2 to Eq. (46) is small, and thus can be neglected. (For details, see Appendix in Ref. 11).

C. Results and discussions

Combining all relevant contributions to the inverse Laplace transform, Eqs. (53) and (55), we obtain the solution for the survival probability

$$S_{\text{odd}}(t) = Y(0)\exp(-\gamma t) + F(t).\tag{58}$$

The first term here corresponds to the kinetics of the quasiparticle that relaxed to the bottom of the trap. The thermally activated decay rate γ, found with the help of Eqs. (54) and (52), is

$$\gamma = \frac{1}{\tau} \frac{\nu(\delta E)}{\nu(T)} \exp\left(-\frac{\delta E}{T}\right)\left(1 - \int_0^\infty dz \frac{e^{-z}/\tau}{1/\tau + \Gamma_{\text{out}}/\sqrt{z}}\right).\tag{59}$$

The integral in Eq. (59) reflects the probability for a quasiparticle to relax to the bottom of the trap [cf. Eq. (13)]. The second term in Eq. (58) describes the kinetics of unequilibrated quasiparticles with $F(t)$ given by

$$F(t) = \frac{1}{\sqrt{\pi}\Gamma_{\text{out}}\nu(\delta E)} \int_0^\infty dz\,\nu(z)\Gamma_{\text{odd}}(z) \frac{\tau\Gamma_{\text{odd}}(z)}{1 + \tau\Gamma_{\text{odd}}(z)}$$
$$\times \exp\left(-z - t\Gamma_{\text{odd}}(z) - \frac{t}{\tau}\right).\tag{60}$$

In the next paragraphs we will analyze $S_{\text{odd}}(t)$ for fast and slow relaxation limits.

In the fast relaxation limit $\tau\Gamma_{\text{out}} \ll 1$ ("deep" trap), the leading contribution to the survival probability $S_{\text{odd}}(t)$ comes from the first term in Eq. (58), the second term in Eq. (58) is proportional to $\tau\Gamma_{\text{out}}$, and can be neglected. Consequently, the survival probability is given by

$$S_{\text{odd}}(t) \approx \exp(-\gamma_f t).\tag{61}$$

Using Eq. (34) we find the lifetime distribution function

$$N_{\text{odd}}(t) = \gamma_f \exp(-\gamma_f t),\tag{62}$$

cf. Eqs. (14) and (16). As discussed qualitatively in Sec. II in the fast relaxation limit the majority of quasiparticles enter-

ing the CPB into the excited state $E_k \sim E_{\text{thd}}$ relax to the bottom of the trap and stay in the box until they are thermally excited out of the trap by phonons with the rate γ_f of Eq. (16). Finally, using Eq. (61) we find the average lifetime of the odd-charge state

$$\langle T_{\text{odd}} \rangle = \int_0^\infty S_{\text{odd}}(t)dt = 1/\gamma_f.\tag{63}$$

In the opposite limit of the "shallow" trap, $\tau\Gamma_{\text{out}} \gg 1$, the majority of quasiparticles tunnel out unequilibrated to the lead ($P_{\text{tr}} \approx 1/\tau\Gamma_{\text{out}}$). The expression for the survival probability (58) in this limit becomes

$$S_{\text{odd}}(t) = F(t) + \frac{1}{\sqrt{\pi}\tau\Gamma_{\text{out}}} \exp(-\gamma_s t).\tag{64}$$

Note that in addition to the first term describing the kinetics of unequilibrated quasiparticles the survival probability has a tail corresponding to the small fraction of quasiparticles that do relax to the bottom of the trap. These quasiparticles reside in the box until they are thermally excited by phonons. In the slow relaxation limit the activation exponent (59) can be reduced to

$$\gamma_s \approx \frac{1}{\sqrt{\pi}\tau} \frac{\nu(\delta E)}{\nu(T)} \exp\left(-\frac{\delta E}{T}\right).\tag{65}$$

[Rigorous evaluation produces a difference in the numerical factor here compared to Eq. (17).] The tail of the distribution function (64) describes the processes that are much slower than $1/\Gamma_{\text{out}}$, thus it must be retained despite its small amplitude, see also Eq. (71).

The function $F(t)$ defined in Eq. (60) can be evaluated using small z asymptote of $\Gamma_{\text{odd}}(z)$, see Eq. (57). This approximation substantially simplifies $F(t)$,

$$F(t) \approx \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dz}{\sqrt{z}} \frac{\tau\Gamma_{\text{out}}}{\sqrt{z} + \tau\Gamma_{\text{out}}} \exp\left(-z - \frac{t\Gamma_{\text{out}}}{\sqrt{z}} - \frac{t}{\tau}\right).\tag{66}$$

Here we assumed that the main contribution to the $F(t)$ comes from the small z region, $z \ll \delta E/2T$, which limits the applicability of Eq. (66) to $t \ll \Gamma_{\text{out}}^{-1}\left(\frac{\delta E}{2T}\right)^{3/2}$. The asymptotic expression for $F(t)$ in Eq. (64) can be obtained using the saddle-point approximation

$$F(t) \approx \frac{2}{\sqrt{3}} \frac{\tau\Gamma_{\text{out}}}{\tau\Gamma_{\text{out}} + \left(\frac{1}{2}\Gamma_{\text{out}}t\right)^{1/3}} \exp\left[-3\left(\frac{\Gamma_{\text{out}}t}{2}\right)^{2/3} - \frac{t}{\tau}\right].\tag{67}$$

The integral (66) can be also expressed in the analytic form in terms of the Meijer's G-function.[28] As one can see from Fig. 7 at low temperature $T \ll \delta E$ there is a time window

$$\frac{1}{\Gamma_{\text{out}}} \lesssim t \ll \frac{1}{\Gamma_{\text{out}}}\left(\frac{\delta E}{2T}\right)^{3/2},\tag{68}$$

in which the survival probability deviates from the exponentially decaying function. We assumed in Eq. (68) that the
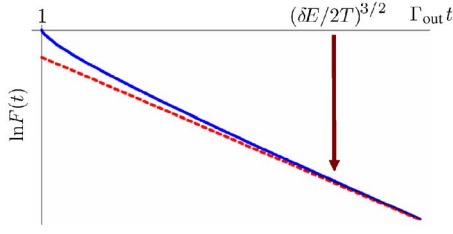
FIG. 7. (Color online) Deviation of $F(t)$ (solid blue line) defined in Eq. (60) from the exponentially decaying function at $\Gamma_{\text{out}} t \gtrsim 1$. (We assumed $\tau = \infty$ here.)

upper limit is more restrictive than $t \ll \frac{1}{\Gamma_{\text{out}}} (\Gamma_{\text{out}} \tau)^3$ so that the τ-dependent term in the exponent of Eq. (67) can be neglected.

The fractional power $2/3$ in Eq. (67) stems from the peculiarity of superconducting density of states at low energies. Assuming the quasiparticle distribution in the lead is given by Eq. (30), every time a quasiparticle tunnels into the box it may occupy a different energy level, which is reflected in the initial conditions, Eq. (36). However, due to the singularity of the escape rate $\Gamma_{\text{out}}(E_k)$ at $E_k \sim E_{\text{thd}}$, this results in a strong energy dependence of the dwell time of a quasiparticle. Therefore, averaging over many such events leads to the deviation of $F(t)$ from the simple exponential function, as shown in Fig. 7.

At $t \gtrsim \frac{1}{\Gamma_{\text{out}}} \left(\frac{\delta E}{2T}\right)^{3/2}$ the minimum of the exponent in (66) is beyond the limit of applicability of the small z approximation for the rate $\Gamma_{\text{out}}(z)$ given by Eq. (57), and instead of Eq. (66) one should use Eq. (60). Since at $z \sim \delta E / 2T$ the escape rate $\Gamma_{\text{out}}(z)$ is a smooth function, $F(t)$ decays exponentially,

$$F(t) \propto \exp\left(-\frac{\delta E}{2T} - \Gamma_{\text{out}}(z_{\min})t - \frac{t}{\tau}\right). \qquad (69)$$

Here $\Gamma_{\text{out}}(z_{\min}) \approx \frac{g_T \delta_b}{2\pi} \sqrt{\frac{\delta E}{\Delta}}$.

The lifetime distribution function $N_{\text{odd}}(t)$ for the odd-charge state can be obtained from $S_{\text{odd}}(t)$ by substituting Eq. (64) into Eq. (34). Under conditions of Eq. (68) the lifetime distribution function $N_{\text{odd}}(t)$ will deviate from the exponential distribution

$$N_{\text{odd}}(t) \approx \frac{2^{4/3}}{\sqrt{3}} \frac{\Gamma_{\text{out}}}{(\Gamma_{\text{out}} t)^{1/3}} \exp\left[-3\left(\frac{\Gamma_{\text{out}} t}{2}\right)^{2/3}\right]. \qquad (70)$$

The average lifetime of the odd-charge state $\langle T_{\text{odd}} \rangle$ in the slow relaxation case is

$$\langle T_{\text{odd}} \rangle = \int_0^\infty S_{\text{odd}}(t) dt \approx \frac{1}{\sqrt{\pi} \tau \Gamma_{\text{out}} \gamma_s} = \frac{1}{\gamma_f}. \qquad (71)$$

Despite the quasiparticle having small probability of relaxing to the bottom of the trap, the main contribution to the average dwell time $\langle T_{\text{odd}} \rangle$ is given by the tail of $S_{\text{odd}}(t)$. This is because once the quasiparticle is trapped in the CPB it stays there for an exponentially long time, see Eq. (65). As expected $\langle T_{\text{odd}} \rangle$ is the same for fast and slow relaxation cases since the average lifetime determines the thermodynamic probability to occupy a given charge state.

V. CHARGE NOISE

The complex statistics of capture and emission processes discussed in the preceding section also manifest itself in the spectral density of charge fluctuations of the Cooper-pair box. In this section we study the charge noise power spectrum for "deep" ($\tau \Gamma_{\text{out}} \ll 1$) and "shallow" ($\tau \Gamma_{\text{out}} \gg 1$) traps.

The kinetic equations for occupational probabilities of odd- and even-charge state have the form[11]

$$\dot{P}_{\text{even}}(E_p, t) + \sum_k W(p, k)[P_{\text{even}}(E_p, t) - P_{\text{odd}}(E_k, t)] = 0,$$

$$\dot{P}_{\text{odd}}(E_k, t) + \sum_p W(p, k)[P_{\text{odd}}(E_k, t) - P_{\text{even}}(E_p, t)]$$

$$= -\frac{1}{\tau}[P_{\text{odd}}(E_k, t) - P_{\text{odd}}^{\text{eq}}(E_k, t)]. \qquad (72)$$

Here $P_{\text{odd}}^{\text{eq}}(E_k, t) = \rho_{\text{odd}}^b(E_k) \sigma_{\text{odd}}(t)$ with $\sigma_{\text{odd}}(t) = \Sigma_k P_{\text{odd}}(E_k, t)$, and the quasiparticle transition rate $W(p, k)$ is defined in Eq. (6). Assuming that the lead is a heat bath of quasiparticles we can write even-charge occupational probability as $P_{\text{even}}(E_p, t) = f(E_p) \sigma_{\text{even}}(t)$ with $f(E_p)$ being the distribution function of the quasiparticles in the lead, and $\sigma_{\text{even}}(t) = \Sigma_p P_{\text{even}}(E_p, t)$ being the occupational probability of the even state. This allows us to reduce Eqs. (72) to

$$\dot{\sigma}_{\text{even}}(t) + \sum_{k,p} W(p, k)[f(E_p) \sigma_{\text{even}}(t) - P_{\text{odd}}(E_k, t)] = 0,$$

$$\dot{P}_{\text{odd}}(E_k, t) + \sum_p W(p, k)[P_{\text{odd}}(E_k, t) - f(E_p) \sigma_{\text{even}}(t)]$$

$$= -\frac{1}{\tau}[P_{\text{odd}}(E_k, t) - P_{\text{odd}}^{\text{eq}}(E_k, t)]. \qquad (73)$$

One can see that Eqs. (73) satisfy the normalization condition

$$\sigma_{\text{even}}(t) + \sigma_{\text{odd}}(t) = 1. \qquad (74)$$

The stationary occupational probabilities $\bar{\sigma}_{\text{even}}$ and $\bar{\sigma}_{\text{odd}}$ are given by the Gibbs equilibrium state. Assuming that $f(E_p)$ is given by Eq. (30), we obtain

$$\bar{\sigma}_{\text{even}} = \frac{1}{1 + n_{\text{qp}}^l V_b \exp\left(\frac{\delta E}{T}\right)}, \quad \bar{\sigma}_{\text{odd}} = 1 - \bar{\sigma}_{\text{even}}. \qquad (75)$$

Here n_{qp}^l is the quasiparticle density in the lead, see Eq. (31), and V_b is the volume of the CPB.

The fluctuations around this equilibrium state can be taken into account within the Boltzmann-Langevin approach, which assumes that the occupational probabilities fluctuate around the stationary solution (75) due to the randomness of the tunneling and scattering events as well as partial occupations of the quasiparticle states.[25] The kinetic equations for the charge fluctuations can be derived by properly varying Eqs. (73) and adding Langevin sources corresponding to the relevant random events,[24]

$$\left(\frac{d}{dt} + \Gamma_{\text{in}}\right)\delta\sigma_{\text{even}}(t) = \sum_{k,p} W(p,k)\delta P_{\text{odd}}(E_k,t) + \sum_p \xi_p^T(t),$$

$$\left(\partial_t + \sum_p W(p,k) + \frac{1}{\tau}\right)\delta P_{\text{odd}}(E_k,t)$$

$$= -\frac{\delta\sigma_{\text{even}}(t)}{\tau}\rho_{\text{odd}}^b(E_k) + \sum_p W(p,k)f(E_p)\delta\sigma_{\text{even}}(t) + \xi_k^T(t)$$

$$+ \xi_k^{ph}(t). \tag{76}$$

Here the relation $\delta\sigma_{\text{even}}(t) = -\delta\sigma_{\text{odd}}(t)$ was taken into account. The Langevin sources $\xi_{p(k)}^T(t)$ and $\xi_k^{ph}(t)$ correspond to quasiparticle tunneling from/to the state $|p\rangle/|k\rangle$ through the junction, and inelastic electron-phonon scattering, respectively. [Note that $\Sigma_p \xi_p^T(t) = -\Sigma_k \xi_k^T(t)$ and $\Sigma_k \xi_k^{ph}(t) = 0$.] These random processes are considered to be Poissonian with the following correlation functions:

$$\langle \xi_k^T(t)\xi_{k'}^T(t')\rangle = 2\delta(t-t')\delta_{k,k'}\sum_p W(p,k)f(E_p)\bar{\sigma}_{\text{even}}$$

$$= 2\delta(t-t')\delta_{k,k'}\Gamma_{\text{out}}(E_k)f(E_k - \delta E)\bar{\sigma}_{\text{even}},$$

$$\langle \xi_k^{ph}(t)\xi_{k'}^{ph}(t')\rangle = \delta(t-t')\frac{2P_{\text{odd}}^{\text{eq}}(E_k)}{\tau}\left(\delta_{k,k'} - \frac{P_{\text{odd}}^{\text{eq}}(E_{k'})}{\sigma_{\text{odd}}}\right)$$

$$= \delta(t-t')\frac{2\bar{\sigma}_{\text{odd}}\rho_{\text{odd}}^b(E_k)}{\tau}[\delta_{k,k'} - \rho_{\text{odd}}^b(E_{k'})]. \tag{77}$$

The latter expression is consistent with the collision integral in the relaxation time approximation and conservation of the probability $\sigma_{\text{odd}}(t)$ by the electron-phonon scattering.[26,27]

The spectral density of charge fluctuations in the CPB is defined as

$$S_Q(\omega) = 2e^2\langle\delta\sigma_{\text{even}}(\omega)\delta\sigma_{\text{even}}(-\omega)\rangle, \tag{78}$$

and can be obtained from Eqs. (76) and (77). The solution of the second equation of the system (76) in the frequency domain is

$$\delta P_{\text{odd}}(E_k,\omega) = \frac{\Gamma_{\text{out}}(E_k)f(E_k - \delta E) - \frac{1}{\tau}\rho_{\text{odd}}^b(E_k)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}\delta\sigma_{\text{even}}(\omega)$$

$$+ \frac{\xi_k^T(\omega) + \xi_k^{ph}(\omega)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}. \tag{79}$$

Substituting this expression into the equation for $\delta\sigma_{\text{even}}(\omega)$ we find

$$\mathcal{L}(\omega)\delta\sigma_{\text{even}}(\omega) = \sum_k \frac{\left(i\omega - \frac{1}{\tau}\right)\xi_k^T(\omega) + \Gamma_{\text{out}}(E_k)\xi_k^{ph}(\omega)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}, \tag{80}$$

where the function $\mathcal{L}(\omega)$ is given by

$$\mathcal{L}(\omega) = -i\omega + \frac{1}{\tau}\sum_k \frac{\rho_{\text{odd}}^b(E_k)\Gamma_{\text{out}}(E_k)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}$$

$$+ \sum_k f(E_k - \delta E)\frac{\Gamma_{\text{out}}(E_k)\left(-i\omega + \frac{1}{\tau}\right)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}. \tag{81}$$

Finally, using Eqs. (78) and (80) we can find the correlation function $\langle\delta\sigma_{\text{even}}(\omega)\delta\sigma_{\text{even}}(-\omega)\rangle$, and obtain charge noise power spectrum $S_Q(\omega)$,

$$S_Q(\omega) = \frac{2e^2}{\mathcal{L}(\omega)\mathcal{L}(-\omega)}\sum_{k,k'} \frac{\left(\omega^2 + \frac{1}{\tau^2}\right)\langle\xi_k^T(\omega)\xi_{k'}^T(-\omega)\rangle + \Gamma_{\text{out}}(E_k)\Gamma_{\text{out}}(E_{k'})\langle\xi_k^{ph}(\omega)\xi_{k'}^{ph}(-\omega)\rangle}{\left(-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}\right)\left(i\omega + \Gamma_{\text{out}}(E_{k'}) + \frac{1}{\tau}\right)}. \tag{82}$$

Upon substituting correlation functions (77) into Eq. (82) the general solution for $S_Q(\omega)$ can be obtained (after cumbersome but straightforward calculations, see the Appendix). Rather than going through the full derivation, we study here $S_Q(\omega)$ in the limiting cases $\tau\Gamma_{\text{out}} \ll 1$ and $\tau\Gamma_{\text{out}} \gg 1$, which can be derived from Eqs. (77), (81), and (82).

We first consider fast relaxation limit $\tau\Gamma_{\text{out}} \ll 1$. In this case one can neglect the second term in the numerator of Eq. (82). For $\omega\tau \ll 1$ one can simplify Eqs. (81) and (82) further.

After straightforward manipulations one finds that the leading contribution to the noise is given by Eq. (23) with the rate

$$\frac{1}{\tau_{\text{eff}}} = \gamma_f + \Gamma_{\text{in}}, \tag{83}$$

which includes all processes changing the population $\bar{\sigma}_{\text{even}}$. The first term in Eq. (83) corresponds to the rate of thermal activation of the quasiparticles by phonons from the bottom
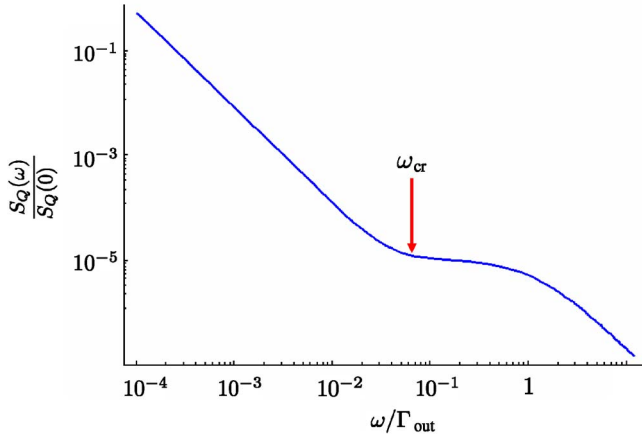
FIG. 8. (Color online) Spectral density of charge fluctuations generated by quasiparticle capture and emission processes in the Cooper-pair box for the slow relaxation case ($\tau\Gamma_{\rm out}=10^3$). Here $\omega_{\rm cr}\approx\sqrt{\Gamma_{\rm out}/\tau}$ is a crossover frequency between two different regimes governed by Eqs. (84) and (87).

of the trap to the lead, see Eq. (16); the second term is the rate of quasiparticle tunneling from the lead to the box given by Eq. (11), also cf. Eq. (24).

In the opposite limit $\tau\Gamma_{\rm out}\gg1$, the charge noise power spectrum $S_Q(\omega)$ can be roughly viewed as the superposition of two Lorentzians, see Fig. 8. The first one corresponds to the processes involving quasiparticle thermalization and activation by phonons, and is dominant at low frequencies. The second (quasi) Lorentzian describes the fast processes ($\omega \sim \Gamma_{\rm out}$) associated with the escape of unequilibrated quasiparticles from the box.

At low frequencies $\omega \ll \omega_{\rm cr}$, see Fig. 8, the noise power spectrum is well approximated by the Lorentzian function. This can be obtained by neglecting the first term in the numerator of Eq. (82), and keeping the leading terms in $1/\tau\Gamma_{\rm out}$ and $\omega/\Gamma_{\rm out}$ in Eqs. (81) and (82), see the Appendix. After straightforward manipulations one finds

$$S_Q(\omega) \approx 4e^2\bar{\sigma}_{\rm odd}(1-\bar{\sigma}_{\rm odd})\frac{1-D}{1+C}\frac{\tau_{\rm eff}}{(\omega\tau_{\rm eff})^2+1}. \quad (84)$$

The constants C and D here are defined as

$$C=\frac{1}{\sqrt{\pi}}\frac{\Gamma_{\rm in}}{\Gamma_{\rm out}} \quad \text{and} \quad D=\frac{1}{\sqrt{\pi}}\frac{\nu(\delta E)}{\nu(T)}\exp\left(-\frac{\delta E}{T}\right). \quad (85)$$

The width of the Lorentzian (84) is given by

$$\frac{1}{\tau_{\rm eff}}=\frac{1}{\tau}\frac{\Gamma_{\rm in}}{\Gamma_{\rm in}+\sqrt{\pi}\Gamma_{\rm out}}+\gamma_s\frac{\sqrt{\pi}\Gamma_{\rm out}}{\Gamma_{\rm in}+\sqrt{\pi}\Gamma_{\rm out}}. \quad (86)$$

The first term here corresponds to the transitions from even-to odd-charge state involving the relaxation of a quasiparticle to the bottom of the trap [cf. Eq. (25); difference in the numerical coefficients comes from rigorous solution of Eqs. (77), (81), and (82)]. It is determined by the quasiparticle relaxation rate $1/\tau$ times the portion of the time the unequilibrated quasiparticle spends in the box. The second term in Eq. (86) describes the transitions odd to even state involv-

ing the escapes of a thermalized quasiparticle from the CPB by phonon activation. It is proportional to the phonon-assisted quasiparticle escape rate from the box to the lead γ_s times the probability to find an empty trap upon the escape of the thermalized quasiparticle. This probability is determined by the portion of the time the trap spends in the even state upon the escape of the thermalized quasiparticle, and is determined by the fast processes involving $\Gamma_{\rm out}$ and $\Gamma_{\rm in}$.

At high frequencies, $\omega\gg\omega_{\rm cr}$, the dominant is the first term in the numerator of Eq. (82). Then, in the leading order in $1/\tau\Gamma_{\rm out}$ the power spectrum becomes

$$S_Q(\omega) \approx \frac{4e^2}{\Gamma_{\rm out}}\cdot\frac{CZ_1(\omega)\bar{\sigma}_{\rm even}}{[1+CZ_2(\omega)]^2+\left(\dfrac{\omega}{\Gamma_{\rm out}}\right)^2[CZ_1(\omega)]^2}. \quad (87)$$

Here the sums over momentum k in Eq. (82) are replaced with the integrals ($T\gg\delta_b$). In terms of the dimensionless variable z (52) these integrals are denoted as $Z_1(\omega)$ and $Z_2(\omega)$ (see Appendix),

$$Z_1(\omega) \approx \int_0^\infty dz\frac{e^{-z}\sqrt{z}}{(\omega/\Gamma_{\rm out})^2z+1},$$

$$Z_2(\omega) \approx \int_0^\infty dz\frac{e^{-z}}{(\omega/\Gamma_{\rm out})^2z+1}. \quad (88)$$

As shown in Fig. 8, in the frequency window $\omega_{\rm cr}\ll\omega\ll\Gamma_{\rm out}$ the noise power $S_Q(\omega)$ becomes flat with the amplitude

$$S_Q(\omega) \approx \frac{2\sqrt{\pi}e^2}{\Gamma_{\rm out}}\frac{C\bar{\sigma}_{\rm even}}{(1+C)^2}. \quad (89)$$

At higher frequencies $\omega\gtrsim\Gamma_{\rm out}$ and $C\ll1$ the noise power spectrum (87) can be approximated by

$$S_Q(\omega) \approx \frac{4e^2}{\Gamma_{\rm out}}CZ_1(\omega)\bar{\sigma}_{\rm even} \quad (90)$$

with $Z_1(\omega)$ given by Eq. (88). At these frequencies the charge noise power spectrum $S_Q(\omega)$ describes charge fluctuations due to the tunneling of the unequilibrated quasiparticles from the box to the lead. By taking a Fourier transform of Eq. (90), one can notice that the noise power spectrum in time domain has the same functional form as $F(t)$ of Eq. (67). Therefore, charge noise power spectrum also reveals the deviations from the conventional Poisson statistics due to the singularity of the quasiparticle density of states at low energies. The deviations of the charge noise power spectrum (90) from the Lorentzian function at $\omega\sim\Gamma_{\rm out}$ are illustrated in Fig. 9. At higher frequencies $\omega\gg\Gamma_{\rm out}$ charge noise power spectrum $S_Q(\omega)$ decays as $1/\omega^2$, see Eq. (26).

VI. CONCLUSIONS

In this work we studied the kinetics of the quasiparticle trapping and releasing in the mesoscopic superconducting island (Cooper-pair box). We found the lifetime distribution
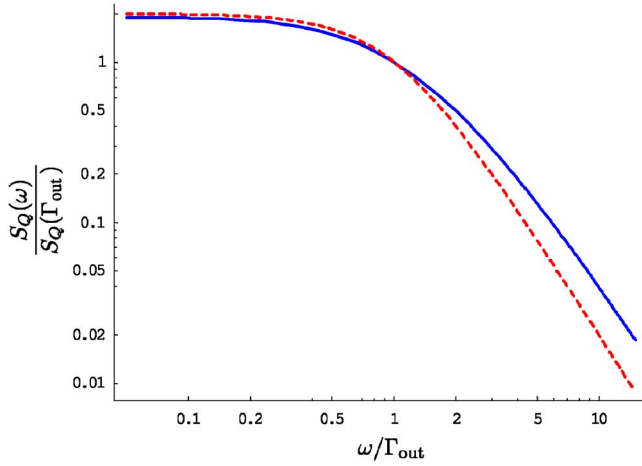
FIG. 9. (Color online) The deviations of the charge noise power spectrum $S_Q(\omega)$ from the Lorentzian function at high frequencies $\omega \sim \Gamma_{\text{out}}$. Blue solid line corresponds to $S_Q(\omega)$ given by Eq. (90), red dashed line is the normalized Lorentzian function with the width Γ_{out}.

of even- and odd-charge states of the Cooper-pair box. We also calculate charge noise power spectrum generated by quasiparticle capture and emission processes.

The lifetime of the even-charge state is an exponentially distributed random variable corresponding to the homogenous Poisson process. However, the lifetime distribution of the odd-charge state may deviate from the exponential one. The deviations come from two sources—the peculiarity of the quasiparticle density of states in a superconductor and the possibility of quasiparticle energy relaxation via phonon emission. The odd-charge-state lifetime distribution function depends on the ratio of the escape rate of the unequilibrated quasiparticle from the box Γ_{out} and quasiparticle energy relaxation rate $1/\tau$.

The conventional Poissonian statistics for both quasiparticle entrances to and exits from the Cooper-pair box would lead to a Lorentzian spectral density $S_Q(\omega)$ of CPB charge fluctuations.[13] The interplay of tunneling and relaxation rates in the exit events may result in deviations from the Lorentzian function. In the case of slow quasiparticle thermalization rate compared to the quasiparticle tunneling out rate Γ_{out}, the function $S_Q(\omega)$ roughly can be viewed as a superposition of two Lorentzians. The width of the first one is controlled by the processes involving quasiparticle thermalization and activation by phonons, while the width of the broader one is of the order of the escape rate Γ_{out}.

ACKNOWLEDGMENTS

APPENDIX A: POWER SPECTRUM OF CHARGE NOISE

Combining Eqs. (77), (80), and (81) we obtain the expression for the charge noise power spectrum,

$$S_Q(\omega) = \frac{4e^2}{\mathcal{L}(\omega)\mathcal{L}(-\omega)}\left(\sum_k \frac{\left(\omega^2 + \frac{1}{\tau^2}\right)\Gamma_{\text{out}}(E_k)f(E_k - \delta E)\bar{\sigma}_{\text{even}} + \Gamma_{\text{out}}(E_k)^2 \rho^b_{\text{odd}}(E_k)\frac{\bar{\sigma}_{\text{odd}}}{\tau}}{(\omega^2 + (\Gamma_{\text{out}}(E_k) + 1/\tau)^2)} + \frac{\bar{\sigma}_{\text{odd}}}{\tau}\left|\sum_k \frac{\Gamma_{\text{out}}(E_k)\rho^b_{\text{odd}}(E_k)}{-i\omega + \Gamma_{\text{out}}(E_k) + \frac{1}{\tau}}\right|^2\right).$$

(A1)

Here the product $\mathcal{L}(\omega)\mathcal{L}(-\omega)$ is given by

$$\mathcal{L}(\omega)\mathcal{L}(-\omega) = \omega^2\left(1 - \frac{1}{\tau}\sum_k \frac{\rho^b_{\text{odd}}(E_k)\Gamma_{\text{out}}(E_k)}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2} + \sum_k \frac{f(E_k - \delta E)\Gamma_{\text{out}}(E_k)^2}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2}\right)^2 + \left(\frac{1}{\tau}\sum_k \frac{\rho^b_{\text{odd}}(E_k)\Gamma_{\text{out}}(E_k)[\Gamma_{\text{out}}(E_k) + 1/\tau]}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2}\right.$$

$$+ \sum_k \frac{f(E_k - \delta E)\Gamma_{\text{out}}(E_k)[\omega^2 + 1/\tau^2 + \Gamma_{\text{out}}(E_k)/\tau]}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2}\right)^2.$$

(A2)

Equation (A1) can be simplified in the thermodynamic limit by introducing functions $Z_1(\omega)$ and $Z_2(\omega)$,

$$Z_1(\omega) = \frac{\Gamma_{\text{out}}}{D}\sum_k \frac{\rho^b_{\text{odd}}(E_k)\Gamma_{\text{out}}(E_k)}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2}$$

(A3)

and

$$Z_2(\omega) = \frac{1}{D}\sum_k \frac{\rho^b_{\text{odd}}(E_k)\Gamma_{\text{out}}(E_k)^2}{\omega^2 + [\Gamma_{\text{out}}(E_k) + 1/\tau]^2}.$$

(A4)

Here C and D are given by Eqs. (85). Substituting Eqs. (A2)–(A4) into Eq. (A1) one can obtain the general expression for $S_Q(\omega)$,

$$S_Q(\omega) = \frac{4e^2}{\Gamma_{\text{out}}} \frac{\left[\left(\frac{\omega}{\Gamma_{\text{out}}}\right)^2 + \left(\frac{1}{\tau\Gamma_{\text{out}}}\right)^2\right] CZ_1(\omega)\bar{\sigma}_{\text{even}} + DZ_2(\omega)\frac{\bar{\sigma}_{\text{odd}}}{\tau\Gamma_{\text{out}}} - D^2\frac{\bar{\sigma}_{\text{odd}}}{\tau\Gamma_{\text{out}}}\left[\left(\frac{Z_1(\omega)}{\tau\Gamma_{\text{out}}} + Z_2(\omega)\right)^2 + \left(\frac{\omega}{\Gamma_{\text{out}}}\right)^2 Z_2^2(\omega)\right]}{\left(\frac{\omega}{\Gamma_{\text{out}}}\right)^2\left(1 - \frac{D}{\tau\Gamma_{\text{out}}}Z_1(\omega) + CZ_2(\omega)\right)^2 + \left\{\frac{D+C}{\tau\Gamma_{\text{out}}}Z_2(\omega) + \left[C\left(\frac{\omega}{\Gamma_{\text{out}}}\right)^2 + \frac{C+D}{(\tau\Gamma_{\text{out}})^2}\right]Z_1(\omega)\right\}^2}. \quad \text{(A5)}$$

The functions $Z_1(\omega)$ and $Z_2(\omega)$ can be written in the form of the dimensionless integrals

$$Z_1(\omega) = \frac{\Gamma_{\text{odd}}}{\nu(\delta E)}\int_0^\infty dz \frac{e^{-z}\nu(z)\Gamma_{\text{odd}}(z)}{\omega^2 + [\Gamma_{\text{odd}}(z) + 1/\tau]^2}, \quad \text{(A6)}$$

and

$$Z_2(\omega) = \frac{1}{\nu(\delta E)}\int_0^\infty dz \frac{e^{-z}\nu(z)\Gamma_{\text{odd}}^2(z)}{\omega^2 + [\Gamma_{\text{odd}}(z) + 1/\tau]^2}. \quad \text{(A7)}$$

The dimensionless variable z here is defined in Eq. (52). Assuming that at low temperature the main contribution to the integrals (A6) and (A7) comes from the small z region,

$z \ll \delta E / 2\ T$, one can simplify $Z_1(\omega)$ and $Z_2(\omega)$ using Eq. (57) to obtain

$$Z_1(\omega) \approx \int_0^\infty dz \frac{e^{-z}\sqrt{z}}{(\omega/\Gamma_{\text{odd}})^2 z + (1 + \sqrt{z}/\tau\Gamma_{\text{odd}})^2}$$

and

$$Z_2(\omega) \approx \int_0^\infty dz \frac{e^{-z}}{(\omega/\Gamma_{\text{odd}})^2 z + (1 + \sqrt{z}/\tau\Gamma_{\text{odd}})^2}.$$

In the slow relaxation case $\tau\Gamma_{\text{odd}} \gg 1$ functions $Z_1(\omega)$ and $Z_2(\omega)$ are approximately given by Eqs. (88).

Finally, by taking the appropriate limits in Eq. (A5) one can recover Eq. (23) for "deep" and Eqs. (84) and (87) for "shallow" traps, respectively.

[1] J. Mannik and J. E. Lukens, Phys. Rev. Lett. **92**, 057004 (2004).

[2] J. Aumentado, M. W. Keller, J. M. Martinis, and M. H. Devoret, Phys. Rev. Lett. **92**, 066802 (2004).

[3] A. Guillaume, J. F. Schneiderman, P. Delsing, H. M. Bozler, and P. M. Echternach, Phys. Rev. B **69**, 132504 (2004).

[4] J. F. Schneiderman, P. Delsing, G. Johansson, M. D. Shaw, H. M. Bozler, and P. M. Echternach (unpublished).

[5] B. A. Turek, K. W. Lehnert, A. Clerk, D. Gunnarsson, K. Bladh, P. Delsing, and R. J. Schoelkopf, Phys. Rev. B **71**, 193304 (2005).

[6] W. Lu, Z. Ji, L. Pfeiffer, K. W. West, and A. J. Rimberg, Nature (London) **423**, 422 (2003).

[7] O. Naaman and J. Aumentado, Phys. Rev. B **73**, 172504 (2006).

[8] A. J. Ferguson, N. A. Court, F. E. Hudson, and R. G. Clark, Phys. Rev. Lett. **97**, 106603 (2006).

[9] T. M. Eiles and J. M. Martinis, Phys. Rev. B **50**, 627 (1994).

[10] K. A. Matveev, M. Gisselfalt, L. I. Glazman, M. Jonson, and R. I. Shekhter, Phys. Rev. Lett. **70**, 2940 (1993).

[11] R. M. Lutchyn, L. I. Glazman, and A. I. Larkin, Phys. Rev. B **74**, 064515 (2006).

[12] O. Naaman and J. Aumentado, APS March Meeting 2005 (X16.00002); (private communication).

[13] S. Machlup, J. Appl. Phys. **25**, 341 (1954).

[14] R. Lutchyn, L. Glazman, and A. Larkin, Phys. Rev. B **72**, 014517 (2005).

[15] We assume the system is at low temperature $T \ll T_b^*$, where T_b^* is a characteristic temperature at which thermal quasiparticles appear, $T_b^* = \frac{\Delta}{\ln(\Delta/\delta_b)}$.

[16] J. R. Schrieffer, *Theory of Superconductivity* (Perseus, Oxford, 1999).

[17] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996), p. 81.

[18] S. B. Kaplan *et al.*, Phys. Rev. B **14**, 4854 (1976).

[19] C. C. Chi and J. Clarke, Phys. Rev. B **19**, 4495 (1979).

[20] P. Santhanam and D. E. Prober, Phys. Rev. B **29**, 3733 (1984).

[21] M. J. Uren, M. J. Kirton, and S. Collins, Phys. Rev. B **37**, 8346 (1988).

[22] M. Lax and P. Mengert, J. Phys. Chem. Solids **14**, 248 (1960).

[23] C. M. Wilson, L. Frunzio, and D. E. Prober, Phys. Rev. Lett. **87**, 067004 (2001).

[24] Sh. Kogan, *Electronic Noise and Fluctuations in Solids* (Cambridge University Press, Cambridge, England, 1996); Sh. M. Kogan and A. Ya. Shul'man, Zh. Eksp. Teor. Fiz. **56**, 862 (1969) [Sov. Phys. JETP **29**, 467 (1969)].

[25] Boltzmann-Langevin approach is adequate for calculating noise at low frequencies $\omega \ll T$ (classical limit). This frequency domain is broad enough to include the rates of the relevant processes affecting the noise spectrum, see Eqs. (10)–(12).

[26] M. Lax, Rev. Mod. Phys. **32**, 25 (1960).

[27] M. Bixon and R. Zwanzig, Phys. Rev. **187**, 267 (1969).

[28] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).