

# Evidence for the PSL(2|2) Wess-Zumino-Novikov-Witten model as a model for the plateau transition in the quantum Hall effect: Evaluation of numerical simulations

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In this paper, I revise arguments in favor of the PSL(2|2) Wess-Zumino-Novikov-Witten (WZNW) model as a theory of the plateau transition in integer quantum Hall effect. I show that all available numerical data (including the correlation length exponent  $\nu$ ) are consistent with the predictions of such WZNW model with the level  $k=8$ .

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## I. INTRODUCTION

This paper purports to review the evidence in favor of the Wess-Zumino-Novikov-Witten (WZNW) description of the plateau transition in integer quantum Hall effect given in Ref. 1. To avoid the issues related to a possible role of interactions in real quantum Hall systems, I will discuss the plateau transition in a model system, namely, in the Chalker-Coddington (CC) network model.<sup>2</sup> Thus the “experimental” data I discuss are numerical data taken from simulations on the CC networks.

As is well known, all attempts of rigorous derivation of the critical field theory for the plateau transition have remained unsuccessful. The attempts to arrive at such theory by means of an educated guess made in Ref. 3 and then in Ref. 1 pointed out to the WZNW model on the PSL(2|2) group. However, there have been certain disagreements about the precise form of such model which have never been resolved. The papers also did not provide a value for the correlation length exponent. Below I argue that the most recent numerical data are in very good agreement with the predictions of PSL(2|2)<sub>k=8</sub> WZNW theory. I also identify the relevant operator responsible for the correlation length exponent.

At present, we have two established models for integer quantum Hall effect: the Pruisken-Weidenmüller sigma model<sup>4,5</sup> and the Chalker-Coddington model.<sup>2</sup> One can arrive at these model descriptions by means of more or less controllable steps from the Schrödinger equation description for noninteracting electrons in a disordered potential. However, neither the continuous (the sigma model) nor the lattice (the CC model) description gave immediate access to the critical properties.

To understand the essence of the problem, it is instructive to compare the situation to the quantum critical point in a half-integer spin Heisenberg antiferromagnet. The CC model can be mapped to an antiferromagnetic superspin chain,<sup>6</sup> its analog in this story is the Heisenberg Hamiltonian:

$$H = J \sum_n \mathbf{S}_n \mathbf{S}_{n+1}. \quad (1)$$

Then the analog of the Pruisken-Weidenmüller sigma model would be the O(3) nonlinear sigma model with the action

$$A = \int d\tau dx \left\{ \frac{1}{2g} (\partial_\mu \mathbf{n})^2 + \frac{iS}{4} \epsilon_{\mu\nu} (\mathbf{n} [\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}]) \right\}, \quad (2)$$

where  $\mathbf{n}^2=1$  and  $g=1/2S$ . The latter model can be obtained from the former at  $S \gg 1$  by taking the semiclassical limit in the path integral:<sup>7</sup>

$$S_n = S [\mathbf{m}(x) + (-1)^n \mathbf{n}(x) \sqrt{1 - \mathbf{m}^2}]. \quad (3)$$

In that limit, ferromagnetic fluctuations described by field  $\mathbf{m}$  are weak and can be integrated out. In a similar fashion, one can derive the Pruisken-Weidenmüller sigma model from the CC model.<sup>6</sup>

Continuing the analogy with the Pruisken-Weidenmüller sigma model, we can identify  $2S$  with the Drude conductivity  $\sigma_0^{xx}$ . The second term in Eq. (2) is topological; its contribution to the action is  $2\pi iS$  (integer number). The coefficient at the topological term in Eq. (2) should be identified with  $\sigma_0^{xy}$  (strictly speaking, we can discuss only the case of  $\sigma^{xy}=1/2$  corresponding to half-integer  $S$  or  $\sigma^{xy}=0$  corresponding to integer  $S$ ).

Both Pruisken-Weidenmüller and O(3) nonlinear sigma models scale toward strong coupling. The topological term gives no contribution to the beta function in *any order* in coupling constant  $g$  (it gives only nonanalytic contributions). For this reason, one would not be able to notice its effect on correlation functions until very large distances. For the O(3) sigma model, the corresponding length scale is

$$\xi_{O(3)} \sim ag \exp(\pi/g), \quad g = 1/2S, \quad (4)$$

where  $a$  is the lattice spacing, and for the Pruisken-Weidenmüller model, it is

$$\xi_Q \sim \lambda \exp[4\pi\sigma_0^{xy}], \quad (5)$$

where  $\lambda$  is the mean free path. One may suspect, however, that at large distances a difference between integer and half-integer spins (or  $\sigma^{xy}=0$  and  $\sigma^{xy}=1/2$ ) does exist. Indeed, in the first case,  $\exp[2\pi iS]=1$  and there is no contribution to the partition function, but in the second case, the contribution is nontrivial. On that grounds Haldane made his famous conjecture<sup>8</sup> that Heisenberg antiferromagnets with half-integer spins are critical. In the same way, following Pruisken,<sup>4</sup> we believe that in two dimensions noninteracting disordered electrons with  $\sigma^{xy}=1/2$  are also critical. Of course, in both cases, this belief is supported by ample nu-

merical evidence, but for the Heisenberg antiferromagnet, extra factors intervene so that for the Heisenberg magnet we are able to tell the story until the end. It was conjectured by Polyakov<sup>9</sup> that the critical point, if it exists, is the same as for the  $S=1/2$  Heisenberg antiferromagnet. This was not that evident because the sigma model derivation is valid only for large  $S$ . However, later exact solution of sigma model (2) was constructed<sup>10</sup> and it was demonstrated that the physical properties thus obtained indeed interpolate between the  $O(3)$  sigma model at small  $|x| < \xi$  and  $S=1/2$  Heisenberg at large  $|x| > \xi$  distances. The critical theory for the  $S=1/2$  Heisenberg antiferromagnet is known. The most amazing fact is that this theory is not the  $O(3)$  sigma model but the  $SU_1(2)$  WZNW model. The order parameter in the critical region is not a three-component vector but a four-component matrix  $G$ . The extra component is the dimerization operator

$$d(x) = (-1)^n \mathbf{S}_n \mathbf{S}_{n+1}, \quad (6)$$

which at the critical point has the same scaling dimension as staggered magnetization  $\mathbf{n}$ . So we have

$$\hat{G}(x) = \hat{I}d(x) + i\vec{\sigma}\mathbf{n}(x). \quad (7)$$

The lessons one takes from the Heisenberg antiferromagnet are that (i) sigma models with topological terms are not renormalizable and (ii) the resulting critical action may be defined on a manifold greater than the original one. In that spirit, a search for a suitable critical model for the plateau transition was conducted and the WZNW model on  $PSL(2|2)$  group was suggested as a candidate.<sup>3,1</sup>

$$A = \frac{1}{2g} \int d^2x \text{Str}(\partial_\mu G^{-1} \partial_\mu G) + ik\Gamma[G]. \quad (8)$$

Here  $G$  is a matrix from  $PSL(2|2)$  group,  $\Gamma[G]$  is the Wess-Zumino term (not to be confused with the topological one present in the original sigma model), and  $k$  is an integer. In Ref. 3, Zirnbauer argued for  $k=1$ ; in Ref. 1, it was suggested that  $g=1/k$ . In the latter case, the model has an extended symmetry  $PSL(2|2)_R \times PSL(2|2)_L$ .

The standard situation with WZNW models is that it is critical only at  $g=1/k$  and a deviation of  $g$  from  $1/k$  generates an irrelevant perturbation. The corresponding Renormalization Group equation is

$$\frac{dg}{d \ln(\Lambda/k)} = -a(g - 1/k), \quad (9)$$

with  $a \sim C_{adj} > 0$ , where  $C_{adj}$  is the quadratic Casimir in the adjoint representation of the group. Since for  $PSL(n|n)$  group this Casimir is zero, it was argued that Eq. (8) was critical for any value of  $g$ <sup>3</sup> though the extended (Kac-Moody) symmetry exists only at  $g=1/k$ . It was also suggested in Ref. 3 that there was no Kac-Moody symmetry at the plateau transition. One reason for that assumption was that the critical coupling  $g$  was identified with the critical value of the conductivity:

$$g = 1/\pi\sigma^{xx} = 2/\pi, \quad (10)$$

which is not integer. Since the Kac-Moody symmetry requires  $1/g=k$  to be integer, one is forced to conclude that it

is not there. The argument, however, hangs on the identification (10). However, as was pointed out in Ref. 1, since the critical action is not the original sigma model, such assumption is really difficult to justify. To this, it should be added that since at the critical point the conductivity has  $O(1)$  mesoscopic fluctuations, it is not quite clear what is meant by  $\sigma_{xx}$ . To talk about its value, one must specify how one averages over the conductance distribution.

In any case, in the absence of Kac-Moody symmetry, we know nothing about the critical properties of model (8). Therefore we cannot compare the available numerical evidence with theoretical predictions which do not exist. Meanwhile, for the Kac-Moody WZNW model, we know quite a lot and can compare. Below I will argue that such comparison leads to quite a satisfactory agreement for one particular value  $k=8$ . At present, I have no clue why such value of  $k$  may be chosen by the scaling. In my defence, I can say that properties of WZNW models on supermanifolds are not well studied and future research in this direction is required to establish the soundness of my conjecture.

## II. OPERATORS, SCALING DIMENSIONS, AND THE NUMERICAL DATA

The available numerical evidence essentially consists of four sets of data. Each set contains information related to certain universal properties of the model.

(a) For eigenstates away from the critical energy  $E_c$ , we know the localization (correlation) length exponent

$$\xi(E) \sim (E - E_c)^{-\nu}, \quad \nu \approx 2.3 - 2.35. \quad (11)$$

(b) From Refs. 11 and 12, we know the statistical distribution of the two-point conductances  $P[T(x, y)]$ .

(c) From Ref. 13, we know the statistical distribution of the wave functions (participation ratios). We also have an estimate of the scaling dimension of the first irrelevant operator  $d=2.38 \pm 0.04$ .<sup>13,14</sup>

Few words about the  $PSL(2|2)$  group are in order. This group consists of  $4 \times 4$  supermatrices with unit superdeterminant. Matrix  $G$  from this group can be parametrized as follows:

$$G = e^\varphi \begin{pmatrix} I & 0 \\ \theta & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & \bar{\theta} \\ 0 & I \end{pmatrix}, \quad (12)$$

where  $\theta, \bar{\theta}$  are fermionic  $4 \times 4$  matrices,  $A$  is a matrix from  $H_+^3 = SL(2, C)/SU(2)$ , and  $B$  is a  $SU(2)$  matrix.

It turns out that field  $\varphi$  does not enter into action (8). Therefore the  $PSL(2|2)$  model can be understood as  $SL(2|2)$ , but with a certain restriction on the Hilbert space. Namely, one has to consider only operators which are invariant under *local* multiplication of field  $G$  by a number, that is, fields of the general form

$$Q_X = GXG^{-1}, \quad S \det X = 1, \quad (13)$$

with  $X$  being constant matrices.

The axioms of the conformal field theory state that one can introduce a basis in the space of fields so that every field  $\Phi(x, y)$  in the theory can be decomposed into a sum of fields

(operators) from this basis. The operators which compose the basis can be grouped into “conformal towers.” Each tower consists of a multiplet of so-called primary fields and their descendants. The latter ones are fields generated from the primaries by generators of conformal transformations. Conformal dimensions of descendants differ from dimensions of their primaries by positive integer numbers. For this reason, descendants are less relevant.

In WZNW models, primary fields compose multiplets such that fields within each multiplet transform into each other under the action of the group. The generators of group transformations are currents composing a Kac-Moody algebra. The scaling dimensions of the primary fields in the PSL(2|2) WZNW model are<sup>15</sup>

$$h^{(j_F j_B)} = \bar{h}^{(j_F j_B)} = \frac{1}{k} [j_F(j_F + 1) - j_B(j_B + 1)]. \quad (14)$$

Here  $j_F$  characterizing the compact sector run through the discrete series  $j_F=0, 1, \dots, k/2-1$  [Eq. (13) excludes half-integer  $j_F$ ]. The angular moment eigenvalue  $j_B$  may be either discrete  $j_2=-q$  [ $q$  is a positive real number  $q \leq (k+1)/2$ ] or continuous  $j_B=-1/2+ip/2$ . In the first case, the corresponding representations of  $su(1,1)$  are  $(+, j_B)$  and  $(-, j_B)$ , which denote representations with lowest and highest weight, with eigenvalues of the Cartan operator  $K_1^0$  being  $q+n$  ( $n=0, 1, 2, \dots$ ) and  $-q-n$ , respectively. In the latter case, the representations have neither highest nor lowest weight states. Eigenvalues of  $K_1^0$  take values of  $\alpha+n$ , where  $0 < \alpha < 1$ , but the Casimir is independent of  $\alpha$ .

If PSL(2|2) WZNW indeed describes the critical point of CC model, there must exist an *integer*  $k$  such that dimensions (14) describe power asymptotics of various correlation functions. This requirement stands for any WZNW model defined on a group containing compact submanifolds [such as SU(2) subgroup for the PSL(2|2)].

Let us now return to the numerical evidence and see whether it supports Eq. (14) with integer  $k$ . It was suggested in Ref. 1 that the operators corresponding to the  $q$ th power of local densities of states have  $j_F=0$  and  $j_B=-q$  such that

$$h_q = \bar{h}_q = \frac{q(1-q)}{k}. \quad (15)$$

This hypothesis was checked against numerical calculations.<sup>13</sup> This work also reported that numerical simulations on CC networks are plagued by strong size effects, and to get valuable estimates, one should consider really large samples. Simulations conducted on large samples confirmed the parabolic spectrum (15) with  $2/k=0.26 \pm 0.003$ . The parabolicity of the spectrum is all important for the WZNW model interpretation. As far as the numerical value of  $k$  is concerned, it is rather close to integer value  $k=8$ .

Now let us discuss the localization length exponent  $\nu$ . In the sigma model, the corresponding perturbation is generated by deviation of  $\sigma_{xy}$  from its critical value  $1/2$ . Since  $\sigma_{xy}$  stands at the topological term and the latter term gives non-trivial contributions only when the compact sector of the theory is involved, the perturbation must have nonzero  $j_F$ . As I mentioned above,  $j_F$  must be integer. The operator with

$j_F=2$  and  $j_B=-1/2+ip/2$  (thus it is not a single operator but rather a continuum of fields) does the job. The corresponding scaling dimensions are

$$d(p) = \frac{25+p^2}{16}. \quad (16)$$

As far as physical quantities are concerned, their behavior is dominated by  $d(p)$  at  $p \ll 1$  (see the discussion in Sec. III). The minimal scaling dimension of the correlation function  $\nu=1/[2-d(0)]=16/7 \approx 2.29$  differs by 2% from the accepted value of 2.35. The fact that the perturbation consists of the entire continuum of operators must generate strong size effects. Below I will return to this matter and discuss it in more detail later.

It may well be that the first descendant of the operator with  $j_F=1$  and  $j_B=-1/2+ip/2$ , with scaling dimension

$$d_{irr}(p) = 2 + \frac{9+p^2}{16}, \quad (17)$$

also fits in the picture. If we believe that the finite-size corrections to the wave functions in Refs. 13 and 14 are dominated by the region of small  $p$ , then it gives for the observed scaling dimension the value  $d_{irr}-2 \approx 0.45$ , which is not that different from the value  $0.38 \pm 0.04$  given in Refs. 13 and 14

As we see,  $k=8$  works rather well on two pieces of numerics itemized above. In that light, I suggest looking again at earlier numerical experiments of Refs. 11 and 12. In Ref. 11, it was suggested that an average of the  $q$ th power of the two-point conductance measured between points  $(x, y)$  and  $(0, 0)$  on a cylinder is given by

$$\langle T^q(x, y) \rangle = \int_0^\infty d\mu(p) |C(q, p)|^2 \left| \frac{\pi}{W} \sinh \left[ \frac{\pi(x+iy)}{W} \right] \right|^{-4h(p)}, \quad (18)$$

where  $W$  is the circumference of the cylinder,  $h(p)$  corresponds to  $j_F=0$  and  $j_B=-1/2+ip/2$ ,

$$d\mu(p) = dp \frac{p}{2} \tanh \left( \frac{\pi p}{2} \right)$$

is the Plancherel measure for the continuous series, and  $C(q, p)$  is the Clebsh-Gordan coefficient. Comparing numerics with Eq. (18), the authors of Ref. 11 found that

(a) the conductance does scale with

$$\zeta = \left| \frac{\pi}{W} \sinh \left[ \frac{\pi(x+iy)}{W} \right] \right|,$$

which strongly supports the conformal invariance;

(b) the scaling dimension is parabolic

$$2h(p) = (1+p^2)X_t/8;$$

(c)  $X_t=0.63 \pm 0.03$ .

The numerical value of  $X_t$  was judged to be close to  $2/\pi$ , which excluded integer  $k$  and worked as an argument against the WZNW model as a possible candidate for the plateau transition. However, the later improved numerical calculations reported in Ref. 12 found  $X_t=0.54$  and  $X_t=0.57$ , which are much closer to the desired value of 0.5. It is also likely that a limited accuracy of the numerical calculations is not

the only source of deviations, and there are systematic errors. The point is that the formulas for the powers of conductance suggested by Refs. 11 and 12 interpret the fusion factor  $C(q, p)$  as the Clebsch-Gordan coefficient of the  $GL(2|2)$  group. Such choice is an approximation which can be justified only in the limit  $k \rightarrow \infty$ . For quantum field theory, structure constants characterizing the fusion of two operators do not coincide with the Clebsch-Gordan coefficients of the group theory.

Let me illustrate this argument by an example. Consider some well familiar group, say, the  $SU(2)$  one. The representations of this group are labeled by spin  $j$ . A tensor product of two operators (matrices) belonging to representations  $j_1$  and  $j_2$  can be decomposed into the sum

$$\Phi^{(j_1)}\Phi^{(j_2)} = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} C_{j_3}^{(j_1, j_2)} \Phi^{(j_3)}. \quad (19)$$

Now let us consider the WZNW model on the same group. The operators carry the group indices and transform according to representations of the group:

$$\Phi_{m, m'}^j(z, \bar{z}).$$

The crucial difference is that now they depend on a space point, so they are not simply matrices from  $SU(2)$  group but fields. Studying correlation functions of the corresponding WZNW model instead of Eq. (19), one arrives at more complicated fusion rules:

$$\begin{aligned} \Phi^{(j_1)}(z_1, \bar{z}_1)\Phi^{(j_2)}(z_2, \bar{z}_2) &= \sum_{j_3=|j_1-j_2|}^{j_1+j_2} |z_{12}|^{(d_3-d_1-d_2)} C_{j_3}^{(j_1, j_2)} \\ &\times [\Phi^{(j_3)}(z_2, \bar{z}_2) + z_{12}^n \bar{z}_{12}^m A_{n, m} \\ &\times \Phi^{(j_3; n, m)}(z_2, \bar{z}_2)], \end{aligned} \quad (20)$$

where  $\Phi^{(j_3; n)}$  are operators with conformal dimensions  $h(j_3)+n$  and  $h(j_3)+m$  (descendants), and  $A_{nm}$  are numerical coefficients. The quantum Clebsch-Gordan coefficients  $\mathcal{C}$  do not coincide with the group coefficients except for  $k \rightarrow \infty$ .<sup>16</sup>

### III. PREDICTIONS FOR THE OFF-CRITICAL SCALING

I suggest that at  $E \neq E_c$ , the effective action has the form

$$\begin{aligned} S &= S^* + a(E - E_c) \int dz d\bar{z} \\ &\times \int d\mu(p) \text{Str}[\Phi^{(j_F=2, j_B=-1/2+ip/2)}(z, \bar{z})], \end{aligned} \quad (21)$$

where  $S^*$  is the critical action. As a consequence, the corrections to physical quantities in a system of size  $L$  will include integrals as

$$\begin{aligned} \tau \int d\mu(p) L^{[2-d(p)]} &= \tau L^{2-d(0)} \int dp p \tanh\left(\frac{\pi p}{2}\right) \\ &\times \exp\left[-\frac{\ln L}{16} p^2\right]. \end{aligned} \quad (22)$$

From this expression, one can conclude that the dominant contribution to scaling will come from  $d(0)$ , but there will be significant logarithmic corrections. These corrections will be a source of errors in one-parameter scaling fits of numerical data obtained for finite samples. Since logarithm is a slow function, fits made in the limited range of sample sizes will produce a perfect illusion of a single power law for a given physical quantity. The corrections to power laws will survive even for very large samples when the integral (22) is dominated by  $p \ll 1$  (this corresponds to  $L > e^{16} \sim 10^4$ ). Indeed, the estimate of Eq. (22) shows us that at large  $L$  the parameter  $\tau = a(E - E_c)$  enters in combination

$$\tau L^{7/16} [\ln L]^{-3/2}, \quad (23)$$

so that the correlation length behaves as

$$\xi \sim \tau^{16/7} [\ln \tau]^{24/7}. \quad (24)$$

The described effects set a very tough standard for the size of the critical region and may explain the large magnitude of the size effects observed in Ref. 13.

### IV. CONCLUSIONS

The review of numerical results on plateau transition in the CC model demonstrates that  $PSL(2|2)$  WZNW model with  $k=8$  stands as a very good candidate to model this transition. It goes without saying that this identification leaves many questions unanswered. One may wonder, for instance, what makes  $k=8$  so special or why the correlation length exponent is controlled by the operator with  $j_F=2$  and not with  $j_F=1$ . All these questions can be answered only when detailed information about operator expansion in  $PSL$  WZNW models are available.

As a way forward, I can suggest two things. First, one should study the operator algebra for the  $PSL(2|2)$  WZNW models and see whether it possesses some peculiar properties which allow the selection of certain values of  $k$ . From this algebra, one will also be able to extract quantum Clebsch-Gordan coefficients to use in Eq. (18). In this way, we will obtain more accurate theoretical prediction for the distribution of conductances. Second, one may undertake to study numerically the deviations from a simple scaling predicted in the last section. In this context, it should be emphasized that the continuous spectra of scaling dimensions, being a generic feature of critical sigma models on noncompact manifolds, are not a unique feature of the  $PSL(2|2)$  WZNW theory.

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