Equilibrium dynamics of spin-glass systems

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We present a critical analysis of the Sompolinsky theory of equilibrium dynamics. By using the spherical $2+p$ spin-glass model we test the asymptotic static limit of the Sompolinsky solution showing that it fails to yield a thermodynamically stable solution. We then present an alternative formulation, based on the Crisanti, Hörner, and Sommers [Z. Phys. B: Condens. Matter **92**, 257 (1993)] dynamical solution of the spherical p-spin spin-glass model, reproducing a stable static limit that coincides, in the case of a one step replica symmetry breaking ansatz, with the solution at the dynamic free energy threshold at which the relaxing system gets stuck off equilibrium. We formally extend our analysis to any number of replica symmetry breakings R. In the limit $R\to\infty$, both formulations lead to the Parisi antiparabolic differential equation. This is the special case, though, where no dynamic blocking threshold occurs. The formulation does not contain the additional order parameter Δ of the Sompolinsky theory.

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I. INTRODUCTION

Recently, a great deal of work has been devoted to the study of the so-called *off-equilibrium* dynamics of glassy systems, i.e., the dynamics on time scales large enough to discard the initial condition but not to ensure equilibrium. This large amount of work has left aside the analysis of the equilibrium dynamics, i.e., the dynamics which should lead to the static properties derived from statistical mechanics.

When discussing the equilibrium dynamics of spin-glass systems, one usually refers to the Sompolinsky solution.[1] Sompolinsky assumed that the relaxation dynamics of a spin-glass system occurs via a set of large relaxation times t_x, all of which become infinite in the thermodynamic limit, reflecting the hierarchical order of free-energy barriers or states of the spin-glass phase. By incorporating explicitly this assumption into the relaxation dynamics of the Sherrington-Kirkpatrick (SK) model[2,3] he was able to construct a consistent mean-field dynamical theory that, in the limit of an infinite series of relaxation times, is described by two continuous order parameters functions: the overlap function $q(x)$, measuring the amount of correlation that has not yet decayed, and $\Delta(x)$, representing the *anomalous* contribution to the response function. As in the static calculation the variable x can be defined to vary in the interval $[0,1]$ with $x=1$ corresponding to the shortest (though infinite) time scale and $x=0$ to the longest. With this definition $q(x)$ is a nondecreasing function while $\Delta(x)$ is a nonincreasing function with boundary condition $\Delta(1)=0$.

In the static limit the Sompolinsky solution, in general, does not coincide with the Parisi static solution of the full replica symmetry breaking (FRSB) phase.[4] de Dominicis, Gabay, and Orland[5] (DGO) have, indeed, shown that the static limit of the Sompolinsky solution can be derived from a static calculation with replicas by using a replica symmetry breaking (RSB) scheme different from Parisi's. The two schemes, however, coincide in the so-called *Parisi Gauge*, i.e., choosing the function $\Delta(x)$ such that $d\Delta(x)/dx = -x dq(x)/dx$.[1]

In this work we reconsider the Sompolinsky solution, we show the instability of its static limit and we check the validity of an alternative solution, originally proposed by Crisanti, Hörner, and Sommers (CHS),[6] in the context of a generic R replica symmetry breaking scenario.

Our testing bench is the $2+p$-spin interacting spherical model whose static properties have been studied by the authors in previous works.[7,8] Such a model, for $p>3$, displays a rich phase diagram that we show in Fig. 1. It contains a replica symmetric (RS) phase (i.e., a phase in which the RS ansatz yields a thermodynamically stable solution), a one step replica symmetry breaking (1RSB) phase, an infinite steps RSB phase, and even a phase consisting of an infinite, continuous (or *full*), set of RSBs plus a separate step of RSB (we call it the 1-FRSB solution). Besides this, it has the further advantage, with respect to, e.g., the SK or the Ising p-spin models, of being exactly solvable in each one of the phases. In the same model it is, therefore, possible to analytically check the validity of the Sompolinsky solution (and any alternative proposal) both in a phase where the thermodynamics is known to be 1RSB and in one where it is FRSB. The relaxation dynamics for the present model is illustrated in Sec. II. There we also solve the equations of motion making use of two simple ansatz (the dynamic analogs of the RS ansatz and of the Sommers ansatz, respectively). This should help to fix notation and concepts and serves as a starting point for the subsequent discussion.

The line of investigation proceeds, then, along the following steps. In Sec. III, reconsidering in detail the derivation of the Sompolinsky solution, we observe that, in a Parisi 1RSB-stable phase, it tends to a static solution different from the one of Parisi as the time goes to infinity. This is not dramatic, since there is no reason preventing the dynamic limit from being different from the thermodynamic solution (corresponding to the global minimum of the free energy landscape of the system). Indeed, for 1RSB systems, it is a well-known property that in a quenching procedure from high temperature the dynamics gets stuck at a *threshold* free energy level strictly above the equilibrium one.[9]

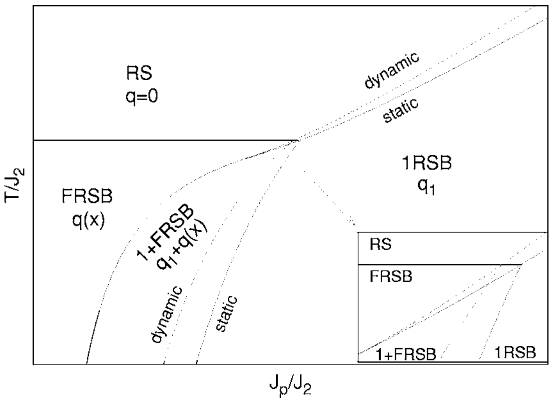

FIG. 1. Qualitative sketch of the $T-J_p$ phase diagram of the $2+p$ spin spherical model with $p>3$ in the mean field approximation. Four phases are displayed. (i) RS/paramagnetic for which the overlap order parameter is zero; (ii) 1RSB/structural glasslike, where the order parameter is the single overlap q_1; (iii) FRSB/spin glass whose order parameter is a continuous function; and (iv) 1+FRSB with an order parameter consisting of a function $q(x)$ plus a single number q_1 representing the self-overlap and such that $q_1>q(1)$. The full curves are the static transition lines, whereas the dotted ones are the dynamic ones. Notice that dynamic transitions are different from static ones only when a separate step of RSB occurs. Indeed this happens for the RS/1RSB transition, for which the dynamic transition line is rederived in the proper dynamic contest in Sec. V B, as well as for the 1RB/1-FRSB and FRSB/1-FRSB transitions (in the latter case only in a small region, see inset). For $p=3$ only the RS and the 1RSB phases are present.

In Sec. IV, always working in the 1RSB phase of the $2+p$ spherical model and using the DGO formalism, we check the stability of the Sompolinsky solution in its static limit. We find that it is thermodynamically unstable (details of the proof of the instability are reported in Appendix C). We further generalize this result to the case of a dynamics described by any finite number R of diverging relaxation times.[10] Eventually, we analyze the $R\to\infty$ limit in which the Sompolinsky static limit and the static Parisi solution coincide, provided one fixes the Parisi's gauge, and we address the reasons of the qualitative difference with the behavior at finite R.

In Sec. V, we propose an alternative formulation of the equilibrium dynamics of spin-glass systems, based on the CHS dynamical solution of the spherical p-spin model.[6] This is a solution apparently similar to Sompolinsky's, but based on slightly different assumptions, that, however, turn out to be crucial in curing the instability of the latter. As well as Sompolinsky's the CHS solution tends, as $t\to\infty$, to a solution different from Parisi's. The explicit computation of the solution on the 1RSB-stable phase of the $2+p$ spherical model shows that the infinite time limit coincides with the corresponding Parisi solution at the threshold free energy and that, unlike Sompolinsky's, it is marginally stable in that limit. The same formalism is effective for any number of steps and the limit $R\to\infty$ is considered as well. Details are reported in Appendix D.

Finally, in Appendixes A and B, we report the DGO derivation of the Sompolinsky solution and discuss its connection with the Parisi solution in the FRSB phase.

II. THE DYNAMICAL MODEL

To illustrate the equilibrium dynamics of spin-glass systems we use the spherical $2+p$ spin model defined by the Hamiltonian

$$H=\frac{r}{2}\sum_i \sigma_i^2-\sum_{i<j}^{1,N} J_{ij}^{(2)}\sigma_i\sigma_j-\sum_{i_1<\cdots<i_p}^{1,N} J_{i_1\cdots i_p}^{(p)}\sigma_{i_1}\cdots\sigma_{i_p}, \quad (1)$$

where p is an integer equal to or larger than 3 and σ_i are N continuous real spin variables which range from $-\infty$ to $+\infty$ subject to the global spherical constraint

$$\sum_{i=1}^N \sigma_i^2=N. \quad (2)$$

The coupling strengths $J_{i_1\cdots i_s}^{(s)}$ $(s=2,3,\ldots)$ are quenched independent identically distributed Gaussian variables of variance

$$\overline{(J_{i_1 i_2\cdots i_s}^{(s)})^2}=\frac{s!J_s^2}{2N^{s-1}}, \quad i_1<\cdots<i_s \quad (3)$$

and mean zero. The scaling with the system size N ensures an extensive free energy and hence a well-defined thermodynamic limit $N\to\infty$. Without losing in generality one may take either J_2 or J_p equal to 1 since this only amounts to a rescaling of the temperature T. Finally, the parameter r is a Lagrange multiplier needed to impose the spherical constraint. In the following, when discussing the FRSB and 1-FRSB phases of the model, we implicitly assume $p>3$.

The relaxation dynamics of the model is described by the Langevin equation

$$\Gamma_0^{-1}\partial_t\sigma_i(t)=\frac{\delta\beta H}{\delta\sigma_i(t)}+\xi_i(t), \quad (4)$$

where $\beta^{-1}=T$ is the temperature, Γ_0^{-1} a microscopic time scale, and the noise $\xi_i(t)$ a Gaussian variable of zero mean and variance

$$\langle\xi_i(t)\xi_j(t')\rangle_\xi=2\Gamma_0^{-1}\delta_{ij}\delta(t-t'), \quad (5)$$

which ensures the proper equilibrium distribution.

In dynamical calculations the quantities of interest are a product of spins averaged over the thermal noise and disorder. Of particular interest are the local spin correlation function

$$C(t,t')=\overline{\langle\sigma_i(t)\sigma_i(t')\rangle_\xi} \quad (6)$$

and the average local response function

$$G(t,t')=\frac{\delta\overline{\langle\sigma_i(t)\rangle_\xi}}{\delta\beta h_i(t')}, \quad t\geq t', \quad (7)$$

where $h_i(t)$ is an external magnetic field.[11]

Using the Martin-Siggia-Rose formalism[12] in the path integral formulation[13,14] the correlation and response functions can be obtained from a generating functional for dynamic correlations and response functions. The disordered average can be done directly on the generating functional without using replicas since the generating functional is normalized to one.[15] The calculation is now rather standard and we do not report it but give directly the results. The interested reader can find more details in Refs. 6, 16, and 17.

In the thermodynamic limit $N \to \infty$ the dynamics reduces to a single-spin self-consistent non-Markovian dynamics described by the equation

$$\Gamma_0^{-1}\partial_t\sigma(t) = -\beta r\sigma(t) + \int_{t_0}^{t} dt'\Sigma(t,t')\sigma(t') + \eta(t), \quad (8)$$

where t_0 is some initial time and $\eta(t)$ a Gaussian noise with zero mean and variance

$$\langle\eta(t)\eta(t')\rangle = 2\Gamma_0^{-1}\delta(t-t') + \Lambda(t,t'). \quad (9)$$

The vertex Λ and the self-energy Σ of the $2+p$ model are given by

$$\Lambda(t,t') \equiv \Lambda[C(t,t')] = \mu_2 C(t,t') + \mu_p C(t,t')^{p-1}, \quad (10)$$

$$\Sigma(t,t') \equiv \Lambda'[C(t,t')]G(t,t') = [\mu_2 + \mu_p(p-1)C(t,t')^{p-2}]G(t,t'), \quad (11)$$

where

$$\mu_2 = (\beta J_2)^2, \quad \mu_p = \frac{p}{2}(\beta J_p)^2, \quad (12)$$

and $\Lambda'(x) \equiv d\Lambda(x)/dx$.

The correlation and response functions must be evaluated self-consistently from the single-spin dynamics as

$$C(t,t') = \langle\sigma(t)\sigma(t')\rangle, \quad G(t,t') = \frac{\partial\langle\sigma(t)\rangle}{\partial\beta h(t')}, \quad (13)$$

where the average $\langle(\cdots)\rangle$ is over the random noise $\eta(t)$.

Since we are interested in the equilibrium correlation and response function we take the initial time t_0 equal to $-\infty$ so that two-times quantities become a function of the time difference only (in other words we are in a time translational invariant regime).[14] To work in Fourier space we introduce the transformed functions

$$C(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} C(t), \quad (14)$$

$$G(\omega) = \int_{0}^{+\infty} dt e^{i\omega t} G(t). \quad (15)$$

The single-spin equation of motion then reads

$$\sigma(\omega) = G(\omega)\eta(\omega), \quad (16)$$

where $G(\omega)$ obeys the Dyson equation

$$G^{-1}(\omega) = \beta r - i\Gamma_0^{-1}\omega - \Sigma(\omega) = G_0^{-1}(\omega) - \Sigma(\omega) \quad (17)$$

and $\eta(\omega)$ is a Gaussian variable of zero mean and variance

$$\langle\eta(\omega)\eta(\omega')\rangle = 2\pi\delta(\omega+\omega')[2\Gamma_0^{-1} + \Lambda(\omega)]. \quad (18)$$

In the Fourier space the correlation function $C(\omega)$ is given by

$$C(\omega) = \langle\sigma(\omega)\sigma(-\omega)\rangle, \quad (19)$$

where the average is over the noise $\eta(\omega)$. The fluctuation dissipation theorem (FDT)[11]

$$G(t) = -\theta(t)\partial_t C(t) \quad (20)$$

is recast, in Fourier space, as

$$C(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,G(\omega). \quad (21)$$

The FDT implies that the static susceptibility $G(\omega=0)$ reads

$$G(\omega=0) = C(t=0) - C(t\to\infty). \quad (22)$$

This reduces to $G(\omega=0)=1$ when the spherical constraint is imposed $[C(t=0)=1]$ and the decay to zero of $C(t)$ for large times is assumed.

A. The "replica symmetric" solution

Before introducing the Sompolinsky solution we consider the derivation of the static limit of the dynamics assuming the existence of a time persistent contribution to the correlation function. We will eventually see [Eqs. (35) and (39)] that, in the limit $\omega \to 0$ (or $t \to \infty$), it leads to a static solution equivalent to a replica symmetric (RS) one.

The strategy for constructing the solution in the spin-glass phase is the following. First one assumes that in the spin-glass phase the correlation function $C(t)$ decays to a finite value

$$\lim_{t\to\infty} C(t) = q > 0. \quad (23)$$

The parameter q is called the Edwards-Anderson order parameter and represents the time-persistent part of the correlation. This implies that $C(\omega)$ is of the form

$$C(\omega) = \tilde{C}(\omega) + 2\pi q\delta(\omega), \quad (24)$$

where $\tilde{C}(t)=C(t)-q$ is the finite-time part of $C(t)$, decaying to zero as $t\to\infty$.

For the spherical model the single-spin equation of motion (16) is linear and the self-consistent equation for q can be easily derived just substituting the equation of motion (16) into the definition (19) of $C(\omega)$ and extracting the time-persistent part. However, we derive it in the following in a more general way.

Inserting Eq. (24) for the correlation function into the definition of the vertex function $\Lambda(\omega)$ one has

$$\Lambda(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t}\Lambda(t) = \int_{-\infty}^{+\infty} dt e^{i\omega t}[\Lambda[C(t)] - \Lambda(q) + \Lambda(q)] = \tilde{\Lambda}(\omega) + 2\pi\Lambda(q)\delta(\omega), \quad (25)$$

where $\tilde{\Lambda}(\omega)$ contains only contributions from the finite-time part of the correlation function and is, hence, nonsingular for $\omega\to 0$. Starting from this separation, and looking at Eq. (18), the noise $\eta(\omega)$ can be split into the sum of two independent Gaussian noises

$$\eta(\omega) = \phi(\omega) + z(\omega), \tag{26}$$

where $\phi(\omega)$ is defined by the finite-time part of the vertex function:

$$\langle \phi(\omega)\phi(\omega')\rangle_\phi = 2\pi\delta(\omega+\omega')[2\Gamma_0^{-1} + \tilde{\Lambda}(\omega)], \tag{27}$$

while $z(\omega)$ by the time-persistent part

$$\langle z(\omega)z(\omega')\rangle_z = 2\pi\delta(\omega+\omega')\Lambda(q)2\pi\delta(\omega). \tag{28}$$

The two noises ϕ and z represent, hence, respectively, the "fast" and "slow" parts of the noise η.

By definition, q is the remaining part of the correlation function once the correlations induced by the fast part of the noise have died out. As a consequence the self-consistent equation for q reads

$$q = \langle\langle\sigma\rangle_\phi^2\rangle_z, \tag{29}$$

where $\langle\sigma\rangle_\phi = \langle\sigma(\omega=0)\rangle_\phi$ is the static average value of $\sigma(\omega)$ induced by the noise $\phi(\omega)$ in the presence of a fixed static random noise z.

To solve the equation of motion and find $\langle\sigma\rangle_\phi$ a relation between $G(\omega)$ and $C(\omega)$ is needed. Assuming that, as in ordinary phase transitions, the effect of an external perturbation will die out on finite time scales, the full response function G is related to the finite-time part $\tilde{C}$ of C by the FDT

$$\tilde{C}(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,G(\omega), \tag{30}$$

which, in turn, implies

$$\langle\phi(\omega)\phi(-\omega)\rangle_\phi = -\frac{2}{\omega}\,\mathrm{Im}\,G^{-1}(\omega). \tag{31}$$

This relation ensures that the noise ϕ acts as a *thermal* noise and hence $\langle\sigma\rangle_\phi$ is the magnetization induced in thermal equilibrium by the static Gaussian field z,

$$\langle\sigma\rangle_\phi = \bar{m}(z) = \frac{\int_{-\infty}^{+\infty} d\sigma\,\sigma \exp\left[-\frac{1}{2}G^{-1}(\omega=0)\sigma^2 + z\sigma\right]}{\int_{-\infty}^{+\infty} d\sigma \exp\left[-\frac{1}{2}G^{-1}(\omega=0)\sigma^2 + z\sigma\right]}$$
$$= G(\omega=0)z. \tag{32}$$

Since the equation of motion of the spherical $2+p$ spin-glass model is linear in $\sigma(\omega)$, this result can be also obtained by averaging directly the equation of motion (16) over the noise $\phi(\omega)$ and taking the limit $\omega\to 0$.

Inserting this expression into Eq. (29) and using Eq. (28) one ends up with

$$q = G(\omega=0)^2\Lambda(q). \tag{33}$$

Eventually, the expression of the static susceptibility can be readily obtained with the help of the FDT relation (30) and reads

$$G(\omega=0) = \tilde{C}(t=0) - \tilde{C}(t\to\infty) = 1-q. \tag{34}$$

We then end up with the following self-consistent equation for q:

$$\Lambda(q) = \frac{q}{(1-q)^2} \tag{35}$$

that coincides with the static RS solution of the spherical $2+p$ spin-glass model.[8]

The dynamical stability of this solution requires that the $\omega\to 0$ limit of the kinetic coefficient, or generalized damping function, $\Gamma(\omega)$, must be non-negative. Its inverse is defined as

$$\Gamma^{-1}(\omega) = i\frac{\partial G^{-1}(\omega)}{\partial\omega} = \Gamma_0^{-1} - i\frac{\partial}{\partial\omega}\Sigma(\omega). \tag{36}$$

Inserting the form (24) of the correlation function into the definition of the self-energy $\Sigma(\omega)$, and using manipulations similar to those used for extracting the singular part of $\Lambda(\omega)$, we have

$$\Sigma(\omega) = \tilde{\Sigma}(\omega) + \Lambda'(q)G(\omega). \tag{37}$$

In the limit $\omega\to 0$ one then obtains[18]

$$\Gamma^{-1}(\omega=0) = \lim_{\omega\to 0}\Gamma^{-1}(\omega) = i\lim_{\omega\to 0}\frac{G^{-1}(-\omega) - G^{-1}(\omega)}{2\omega}$$
$$= \frac{\Gamma_0^{-1} + \frac{\partial}{\partial\omega}\,\mathrm{Im}\,\tilde{\Sigma}(\omega=0)}{1-\Lambda'(q)G(\omega=0)^2}. \tag{38}$$

The numerator describes the decay of the finite-time part and is, thus, positive. The requirement $\Gamma(\omega=0)\geq 0$ leads, then, to the condition

$$1-\Lambda'(q)G^2(\omega=0) = 1-\Lambda'(q)(1-q)^2 \geq 0. \tag{39}$$

In terms of the static replica formalism, the above expression exactly coincides with the de Almeida-Thouless[19] stability condition derived from the stability analysis of the RS saddle point.[8]

The replica symmetric solution is unstable everywhere in the low temperature phase, thus this solution is correct only in the paramagnetic phase up to the critical point where $\Gamma(\omega=0)=0$. Below this point a new solution is needed.

B. The *Sommers* solution

In the previous derivation of the solution the FDT was assumed to hold between the full response function G and the finite-time part $\tilde{C}$ of the correlation function. The failure of the replica symmetric solution to describe the relaxation in the spin-glass phase suggests that in this phase the presence of a time-persistent part in the correlation function must reflect itself also in the response to an external perturbation with an anomalous contribution to the response function. This extra contribution occurs, however, only in the static susceptibility, i.e., *exactly* at zero frequency:

$$G(\omega) = \tilde{G}(\omega) + \Delta\,\delta_{\omega,0}, \tag{40}$$

where $\delta_{\omega,0}$ is the Kronecker delta and Δ is the discontinuity between the static susceptibility and the $\omega \to 0$ limit of the dynamic susceptibility $G(\omega)$:

$$\Delta = G(\omega = 0) - \lim_{\omega \to 0} G(\omega). \tag{41}$$

The static limit of this solution is known as the Sommers solution.[20,21] As before, the nonsingular finite-time $\tilde{G}$ of the response function is related to the finite-time part $\tilde{C}$ of the correlation function by the FDT,

$$\tilde{C}(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,\tilde{G}(\omega). \tag{42}$$

Inserting the expressions (24) and (40) for the correlation and response functions into the Dyson equation (17), and making use of Eqs. (26) and (37), a straightforward algebra leads to the following equation of motion:

$$\sigma(\omega) = \tilde{G}(\omega)[\phi(\omega) + H(\omega)], \tag{43}$$

where $H(\omega) \equiv H(z)$ is the *effective* static noise

$$H(\omega) = z(\omega) + \Lambda'(q)\Delta\,\delta_{\omega,0}\sigma(\omega) = z(\omega) + \Lambda'(q)\Delta\,\delta_{\omega,0}\langle\sigma\rangle_\phi. \tag{44}$$

In the second expression we used the fact that, because of the Kronecker delta $\delta_{\omega,0}$, only the part of $\sigma(\omega)$ which is nonzero at $\omega=0$ may contribute to the static field $H(z)$. This part is the static magnetization $\bar{m}(z)=\langle\sigma\rangle_\phi$ induced by the static noise z. The product $\delta_{\omega,0}\langle\sigma\rangle_\phi$ is, however, ill-defined since it contains the product of the functions $\delta(\omega)$ and $\delta_{\omega,0}$, both having vanishing width. To give meaning to this product one introduces a finite-width representation of the Dirac and Kronecker delta functions, ϵ:

$$\lim_{\epsilon \to 0} \delta_\epsilon(\omega) = \delta(\omega), \quad \lim_{\epsilon \to 0} \Delta_\epsilon(\omega) = \delta_{\omega,0}. \tag{45}$$

Then $\delta_{\omega,0}\langle\sigma\rangle_\phi$ is defined as the $\epsilon \to 0$ limit of the convolution of $\delta_\epsilon(\omega)$ and $\Delta_\epsilon(\omega)$.

If the width of $\Delta_\epsilon(\omega)$ is much smaller than the one of $\delta_\epsilon(\omega)$ then the contribution of $\langle\sigma\rangle_\phi$ to the static field $H(z)$ is vanishing, and one gets back the $\Delta=0$ solution. In the opposite limit,[1] $\Delta_\epsilon(\omega)\delta_\epsilon(\omega) \to \delta(\omega)$ and the full magnetization $\bar{m}(z)=\langle\sigma\rangle_\phi$ contributes to the static field $H(z)$.

The finite-time parts $\tilde{G}$ and $\tilde{C}$ of the response and correlation functions are related by the FDT. As a consequence the fast noise ϕ is a *thermal* noise and, hence, $\langle\sigma\rangle_\phi$ is the static thermal equilibrium magnetization induced by the static field $H(z)$ [as it was in Eq. (32)]:

$$\bar{m}(z) = \tilde{G}(\omega = 0)H(z) = \tilde{G}(\omega = 0)[\sqrt{\Lambda(q)}z + \Delta\Lambda'(q)\bar{m}(z)], \tag{46}$$

where we have rescaled the Gaussian noise z to have $\langle z^2\rangle_z = 1$. Solving for $\bar{m}(z)$ and inserting the result into Eq. (29) we obtain the self-consistent equation

$$q = \left[\frac{1-q}{1-\Delta(1-q)\Lambda'(q)}\right]^2 \Lambda(q), \tag{47}$$

where we used the relation $\tilde{G}(\omega=0)=\tilde{C}(t=0)=1-q$ following from FDT.

The equation for the anomalous term Δ is obtained from the definition of the static susceptibility and reads

$$\left\langle \left.\frac{\partial \bar{m}(z)}{\partial \beta h}\right|_{h=0} \right\rangle_z = 1 - q + \Delta, \tag{48}$$

where h is a static external field. Adding the field h to the equation of motion (43) and inserting the resulting $\bar{m}(z)$ into the above equation we have, after some algebra,

$$\frac{1-q}{1-\Delta(1-q)\Lambda'(q)} = 1 - q + \Delta. \tag{49}$$

The two self-consistency equations (47) and (49) can be rewritten in the form

$$\Lambda(q) = \frac{q}{(1-q+\Delta)^2}, \tag{50}$$

$$\Lambda'(q) = \frac{1}{(1-q)(1-q+\Delta)}, \tag{51}$$

from which one readily sees that for $\Delta=0$ the solution reduces to the replica symmetric solution at criticality: $\Gamma(\omega=0)=0$ [cf. Eqs. (35) and (39)]. The dynamical stability requirement on the damping function $\Gamma(\omega=0) \geq 0$ for the Sommers solution is still given by Eq. (39). As a consequence, for any $\Delta > 0$ the solution has a positive $\Gamma^{-1}(\omega=0)$. This apparently hints that this solution (or, at least, its static limit) is physically stable. However, as noted by Hertz,[22] exactly at $\omega=0$ the time persistent part of the response function [Eq. (40)] should not change the relative static solution, thus implying a negative nonlinear susceptibility. We will reconsider this point when we will present the study of the static limit of the present dynamics in the DGO formalism in Sec. IV

III. THE SOMPOLINSKY SOLUTION

The Sommers solution assumes that there are only two relevant time scales, a short time scale related to the finite-time part of the motion, and a long time scale—actually infinite in the thermodynamic limit—related to the time-persistent part of the motion. This scenario is clearly too limitative for the description of the spin-glass phase where different time scales are involved.

The Sompolinsky solution extends the Sommers solution to the case of many different long times scales, all of which diverge in the thermodynamic limit. To be more specific one assumes that there are R different relaxation times t_r, $r=1,\ldots,R$. As $N \to \infty$ all times go to infinity with the prescription $t_r/t_s \to \infty$ if $r < s$. The short time scale relaxation time, which can be identified with t_{R+1}, is proportional to Γ_0^{-1} and remains finite for $N \to \infty$.

In each time interval, or sector, $t_{r+1} \ll t \ll t_r$. The relaxation process with characteristic times less than t_r have already

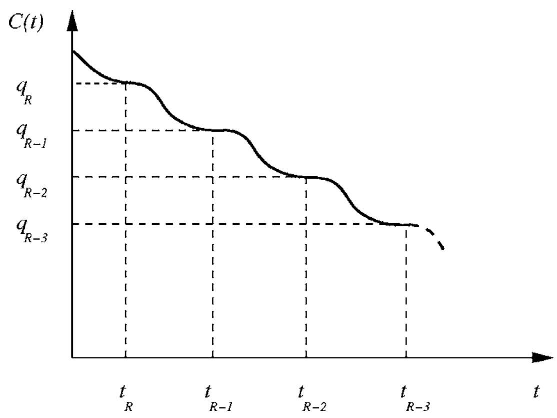

FIG. 2. Schematic form of the correlation function with many relaxation time scales.

relaxed to equilibrium, while those with longer (or equal) relaxation times have not relaxed yet. For each time interval $t_{r+1} \ll t \ll t_r$ we can introduce an order parameter

$$q_r = T_r \lim_{t\to\infty} C(t), \quad r = 0, \ldots, R, \tag{52}$$

where the "time ordered limit" $T_r \lim_{t\to\infty}$ is defined as

$$T_r \lim_{t\to\infty} \coloneqq \lim_{t_R\to\infty} \cdots \lim_{t_{r+1}\to\infty} \lim_{t\to\infty} \lim_{t_r\to\infty} \cdots \lim_{t_0\to\infty}. \tag{53}$$

The overlap q_r measures the time-persistent part of the correlation function in the interval $[t_{r+1}, t_r]$, see Fig. 2.

With this definition q_R coincides with the Edwards-Anderson order parameter previously defined. Moreover, we have introduced the additional level $r=0$ associated with the longest time scale, i.e., the equilibration time scale, of the model. The overlap q_0 represents, then, the asymptotic equilibrium value (that is equal to zero in the absence of an external magnetic field).

The next step is to split away from the full correlation and response functions the time-persistent parts as for the Sommers solution. The functions C and G thus are still of the form (24) and (40) with finite-time parts related by the FDT (42), but now[1]

$$q\,\delta(\omega) \to \sum_{r=0}^{R} (q_r - q_{r-1})\,\delta_{\epsilon_r}(\omega) \tag{54}$$

and

$$\Delta\,\delta_{\omega,0} \to -\sum_{r=0}^{R} \dot{\Delta}_r \Delta_{\epsilon_r}(\omega), \tag{55}$$

where $\dot{\Delta}_r$ is the anomalous contribution to the response functions, $\delta_\epsilon(\omega)$ and $\Delta_\epsilon(\omega)$ are finite-width representations of the Dirac and Kronecker delta functions, and $\epsilon_r = 1/t_r$. Here and in the following we use the convention that all quantities of negative subindex are zero.

As for the Sommers solution, the noise $\eta(\omega)$ is decomposed into the sum of a fast (*thermal*) noise ϕ and a slow noise z composed by the sum of independent *slow* noises z_r of zero mean and variance

$$\langle z_r(\omega) z_r(-\omega)\rangle_r = [\Lambda(q_r) - \Lambda(q_{r-1})]\,\delta_{\epsilon_r}(\omega). \tag{56}$$

From the definition of q_r and the fact that the noise z_r acts as a static noise only up to the time scale t_r, the order parameter q_r is thus given by

$$q_r = \langle \bar{m}_r^2 \rangle_z, \quad r = 0, \ldots, R, \tag{57}$$

where $\bar{m}_r \equiv \bar{m}_r(\{z\})$ is the static magnetization induced at scale r by the slow noise z and the average $\langle \cdots \rangle_z$ is over all static noises z_r. Clearly $\bar{m}_r$ is a function of $z_0, \ldots, z_r$ only since all other noises z_s with $s > r$ have died out. The magnetization $\bar{m}_r$ can be obtained from the magnetization $\bar{m}_R$ induced by the noise z on the shortest time scale by integrating out the noises z_s with $s = r+1, \ldots, R$:

$$\bar{m}_r = \int \prod_{s=r+1}^{R} Dz_s \bar{m}_R(z_0, \ldots, z_R), \tag{58}$$

where $Dz_s \equiv P(z_s) dz_s$ and $P(z_s)$ is the probability distribution of z_s.

One then proceeds as in Sec. II, inserting Eqs. (24), (40), (54), and (55) for C and G and Eq. (26) for the noise $\eta(\omega)$ into the equation of motion and looking at its static limit in order to derive the equations for the thermal equilibrium magnetizations. As we have seen in the study of the Sommers solution, in this limit one has to deal with the products $\delta_{\epsilon_r}(\omega)\Delta_{\epsilon_r}(\omega)$. Clearly one has

$$\delta_{\epsilon_r}(\omega)\Delta_{\epsilon_s}(\omega) \simeq \delta_{\epsilon_r}(\omega) \quad \text{if } r < s, \tag{59}$$

$$\delta_{\epsilon_r}(\omega)\Delta_{\epsilon_s}(\omega) = 0 \quad \text{if } r > s, \tag{60}$$

since $\epsilon_r / \epsilon_s \ll 1$ for $r < s$. Yet, for $s = r$ the product is ill-defined.

Sompolinsky solves the problem with the assumption that for $\epsilon \ll 1$ the width of the function $\delta_\epsilon(\omega)$ is much smaller than the width of $\Delta_\epsilon(\omega)$,[1] and the product goes like

$$\delta_\epsilon(\omega)\Delta_\epsilon(\omega) \simeq \delta_\epsilon(\omega), \quad \epsilon \ll 1. \tag{61}$$

With this assumption each level r contributes to the effective field $H(z)$ with the full magnetization $\bar{m}_r$. As a consequence the self-consistent equation for $\bar{m}_R$ reads [cf. Eq. (46)]

$$\bar{m}_R\{z\} = (1 - q_R) H(\{z\}), \tag{62}$$

$$H(\{z\}) = \sum_{r=0}^{R} [\sqrt{\Delta_r} z_r - \Delta'_r \bar{m}_r(\{z\})], \tag{63}$$

where we used the identity $\tilde{G}(\omega=0) = 1 - q_R$, we rescaled the Gaussian variables z_r in order to have $\langle z_r^2 \rangle_r = 1$, and we defined

$$\Delta_r = \Lambda(q_r) - \Lambda(q_{r-1}), \quad \Delta'_r = \Lambda'(q_r)\dot{\Delta}_r. \tag{64}$$

We anticipate that the cause of instability of the static limit of the Sompolinsky solution (studied in Sec. IV), is hidden right in the conjecture expressed by Eq. (61). We will come back to this problem and we will show how to overcome it in Sec. V, where we will analyze the CHS solution.

The equation for the anomalous term $\dot{\Delta}_r$ follows directly from its definition: $\dot{\Delta}_r$ represents the anomalous contribution on scale r to the static susceptibility. The total anomalous contribution to the static susceptibility from the short time scale up to scale r is then

$$\int \prod_{s\leq r} Dz_s \left. \frac{\partial \bar{m}_r}{\partial \beta h_r}\right|_{h_r=0} = 1-q_R-\sum_{s=r}^{R} \dot{\Delta}_s, \tag{65}$$

where h_r is a static external field active up to the temporal scale labeled by r, so that $\partial \bar{m}_r/\partial h_s=0$ if $r<s$. The presence of h_r just adds the term βh_r to the r contribution to $H(z)$, in Eq. (63), and this implies

$$\int \prod_{s\leq r} Dz_s \frac{1}{\sqrt{\Delta_r}} \frac{\partial \bar{m}_r}{\partial z_r} = \frac{1}{\sqrt{\Delta_r}} \left\langle \frac{\partial \bar{m}_r}{\partial z_r} \right\rangle_z = 1-q_R-\sum_{s=r}^{R} \dot{\Delta}_s, \tag{66}$$

since $\bar{m}_r$ only depends on $z_0,\ldots,z_r$.

Equations (57), (58), (63), and (66) together with the expression (62) for $\bar{m}_R$ constitute the Sompolinsky solution for the spherical $2+p$ spin-glass model. The Sommers solution is recovered by taking $R=0$.

A. Sompolinsky's functional and explicit solution of the 2+p spherical model

The Sompolinsky solution can be obtained from the Sompolinsky functional,[1] that, for the spherical $2+p$ spin-glass model, reads

$$-\beta f_S = -\beta f_0(q_R) + \frac{1}{2}\sum_{r=0}^{R} q_r \Delta_r' + \int \prod_{r=0}^{R} Dz_r \left[\frac{1}{2}\sum_{r=0}^{R} \Delta_r' \bar{m}_r^2(\{z\}) + \phi(H(\{z\})) \right], \tag{67}$$

where

$$\phi(H) = \frac{1}{2}(1-q_R)H^2 \tag{68}$$

and

$$-\beta f_0(q_R) = -\frac{1}{2}\left[g(q_R) + \Lambda(q_R)(1-q_R) - \frac{1}{1-q_R} - \log(1-q_R) \right] - \beta f_\infty. \tag{69}$$

The function g is such that

$$\frac{dg(q)}{dq} = \Lambda(q) \tag{70}$$

and the term f_∞ is the infinite temperature limit of the free energy, whose explicit form is only needed for computing thermodynamic quantities. The Sompolinsky equations follow from stationarity of f_S with respect to variations of $\bar{m}_r$, $\dot{\Delta}_r$, and $\Delta_r = \Lambda(q_r) - \Lambda(q_{r-1})$.

For the spherical $2+p$ spin-glass model equation (58) for the local magnetization $\bar{m}_r$ can be explicitly solved. After a simple algebra one gets

$$\bar{m}_r = \bar{m}_{r-1} + \frac{1-q_R}{1+(1-q_R)\sum_{s=r}^{R} \Delta_s'} \sqrt{\Delta_r} z_r \tag{71}$$

that, with Eq. (57), leads to the following equations for the order parameter q_r with $r=1,\ldots,R$:

$$q_r - q_{r-1} = \left[\frac{1-q_R}{1+(1-q_R)\sum_{s=r}^{R} \Delta_s'} \right]^2 \Delta_r \tag{72}$$

and q_0,

$$q_0 = \left[\frac{1-q_R}{1+(1-q_R)\sum_{s=r0}^{R} \Delta_s'} \right]^2 [\Lambda(q_0) - (\beta h)^2]. \tag{73}$$

In the last equation we have added an external field h to make q_0 finite. Finally from Eq. (66) we have

$$1-q_R-\sum_{s=r}^{R} \dot{\Delta}_s = \frac{1-q_R}{1+(1-q_R)\sum_{s=r}^{R} \Delta_s'}. \tag{74}$$

B. Comparison between the Parisi solution and the static limit of the Sompolinsky solution

The static solution for the spherical $2+p$ spin-glass model within the Parisi R-RSB scheme, as obtained in Ref. 8, consists of the following self-consistency equations:

$$\Lambda(q_r) - \Lambda(q_{r-1}) = \frac{q_r - q_{r-1}}{\chi_r \chi_{r+1}}, \quad r=1,\ldots,R, \tag{75}$$

$$\Lambda(q_0) = \frac{q_0}{\chi_0^2} - (\beta h)^2, \tag{76}$$

$$\Lambda'(q_r) = \frac{1}{\chi_{r+1}^2}, \tag{77}$$

where

$$\chi_r = 1-q_R+\sum_{s=r}^{R} m_s(q_s-q_{s-1}), \quad r=0,\ldots,R. \tag{78}$$

The quantities $0<m_r<1$ are the RSBs parameters, i.e., the sizes of the blocks of the Parisi R-RSB scheme in the continuation from integer to real numbers in the limit $n\to 0$, where n is the total number of replicas.

The equations yielding the infinite time limit of the Sompolinsky solution for the spherical $2+p$ spin-glass model, i.e., Eqs. (72)–(74), can be written in the equivalent form

$$\Lambda(q_r) - \Lambda(q_{r-1}) = \frac{q_r - q_{r-1}}{\chi_r^2}, \quad r=1,\ldots,R, \tag{79}$$

$$\Lambda(q_0) = \frac{q_0}{\chi_0^2} - (\beta h)^2, \tag{80}$$

$$\Lambda'(q_r) = \frac{1}{\chi_r \chi_{r+1}}, \tag{81}$$

where

$$\chi_r = 1 - q_R - \sum_{s=r}^{R} \dot{\Delta}_s, \quad r = 0, \ldots, R. \tag{82}$$

A simple inspection of the two sets of equations reveals that the Sompolinsky solution cannot be reduced to the Parisi solution, not even fixing the so-called Parisi gauge $\dot{\Delta}_r = -m_r(q_r - q_{r-1})$. This implies that, for any finite value of R, the Sompolinsky solution differs from the Parisi solution.

When the number of time sectors (or RSBs in the static counterpart) is sent to infinite, however, the static limit of the Sompolinsky solution can be formally reduced to the Parisi solution,[1] provided that the gauge $d\Delta(x) = -x dq(x)$ is set. The functions $d\Delta(x) = \dot{\Delta}(x)dx$ and $q(x)$ are the limit functions of $\dot{\Delta}_r$ and q_r as $R \to \infty$. The parameter $x \in [0,1]$ is the continuous limit of the series $\{m_r\}$.

This is more easily seen by using the replica derivation of the Sompolinsky solution introduced by DGO.[23] By using this approach it can be shown, see Appendix A, that the Sompolinsky solution can be derived from stationarity with respect to q_r and $\dot{\Delta}_r$ of the DGO functional

$$-2\beta f_{\rm DGO}^{(R)} = -g(q_R) - \sum_{r=0}^{R} \Lambda(q_r)\dot{\Delta}_r + (\beta h)^2 \chi_0 + \log(1 - q_R) + \sum_{r=0}^{R} \frac{q_r - q_{r-1}}{\chi_r}, \tag{83}$$

with χ_r given by Eq. (82). We have neglected the term f_∞, it is irrelevant to our discussion.

In the limit $R \to \infty$ in the FRSB phase we have $q_r - q_{r-1} \to dq$ while $\dot{\Delta}_r \to [d\Delta(q)/dq]dq = \dot{\Delta}(q)dq$. As a consequence, the sums can be replaced by integrals and the DGO functional for the spherical $2+p$ model becomes

$$-2\beta f_{\rm DGO}^{\infty} = -\int_0^1 dq \Lambda(q)\dot{\Delta}(q) - (\beta h)^2 \int_0^1 dq \dot{\Delta}(q) + \ln(1 - q(1)) - \int_0^{q(1)} \frac{dq}{\int_q^1 dq' \dot{\Delta}(q')}, \tag{84}$$

where $q(1) = \lim_{R\to\infty} q_R$, and we have extended the definition of $\Delta(q)$ to the whole interval [0,1] defining

$$\dot{\Delta}(q) = 0 \quad \text{if } 0 \leq q < q_0,$$

$$\dot{\Delta}(q) = -1 \quad \text{if } q(1) < q \leq 1 \tag{85}$$

to have a more compact expression.

The analogous calculation can be performed within the Parisi scheme. When the stable phase of the $2+p$ model is yielded by a FRSB solution, the Parisi functional is

$$-2\beta f_{\rm P}^{(\infty)} = \int_0^1 dq x(q)\Lambda(q) + (\beta h)^2 \int_0^1 dq x(q) + \ln(1 - q(1)) + \int_0^{q(1)} \frac{dq}{\int_q^1 dq' x(q')}, \tag{86}$$

where $x(q)$ is the inverse function of $q(x)$. It is easy to see that the two functionals, Eqs. (84) and (86), coincide in the Parisi gauge $\dot{\Delta}(q) = -x(q)$.

The fact that the two solutions differ might not be a problem. Indeed, systems whose thermodynamics are described by a 1RSB stable phase display the well-known property of having a dynamic solution—at which the system relaxation gets arrested—that is different from the static solution. This arrest is due to the presence of very many metastable states of infinite lifetime lying at a free energy level higher than the one of the global minima, selected, instead, by the static solution.

The apparent paradox of having different solutions at finite R but coinciding ones for $R \to \infty$ can be solved by inspecting the $R \to \infty$ limit of the DGO and Parisi RSB schemes. It can be shown that for any finite R the difference between the Parisi and the DGO-Sompolinsky theories is at least of order $O[(q_r - q_{r-1})^2]$, which is finite for finite R but vanishes as $R \to \infty$. The gauge invariance of the Parisi equation for the order parameter $q(x)$ which follows from the DGO-Sompolinsky theory just reflects the reparametrization invariance of the Parisi equation due to the arbitrary definition of the variable x in the Parisi scheme. The Parisi gauge $d\Delta(q)/dq = -x(q)$ is the definition of the function $x(q)$ whose q derivative is the probability density of overlaps.

IV. STABILITY IN THE REPLICA FORMALISM

The results just described raise the question of the validity of the Sompolinsky solution. Is it a different but yet acceptable solution? This question is better answered considering the phase of the spherical $2+p$ spin-glass model where the 1RSB Parisi ansatz is known to be stable.[8] In the 1RSB phase there is only one long time scale, so that the appropriate dynamical solution should be given by the Sompolinsky solution with $R=0$, i.e., the Sommers solution of Sec. II B There, we have shown that, in the dynamic limit for infinite times (i.e., $\omega \to 0$), the equation of motion appeared to be well-defined. Here, we inspect more thoroughly the Sommers solution *exactly* at $\omega = 0$ and we show that it turns out to be *unstable* everywhere in the 1RSB phase.

The present analysis is carried out in the DGO formalism (cf. Appendix A) that, for $R=0$, is analogous to the Parisi RSB formalism with $R=1$. Our approach is a straightforward generalization of the procedure adopted in Ref. 24 to study the stability of the 1RSB solution *à la* Parisi in the spherical p-spin model.

The replicated free energy density as a function of the overlap matrix $\boldsymbol{q}$ reads

$$-\beta f[\boldsymbol{q}] = \frac{1}{2n}\sum_{ab}^{1,n} g(q_{ab}) + \frac{1}{2n}\ln\det\boldsymbol{q} + s(\infty), \tag{87}$$

$$g(x) = \frac{\mu_2}{2}x^2 + \frac{\mu_p}{p}x^p, \tag{88}$$

where $\mu_p=(\beta J_p)^2 p/2$ and $s(\infty)=(1+\ln 2\pi)/2$ is the entropy per spin at infinite temperature. The parameter n is the number of replicas.

The elements of the symmetric $n\times n$ real matrix $\boldsymbol{q}$ are

$$q_{ab} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^a\sigma_i^b, \quad a,b=1,\ldots,n. \tag{89}$$

The spherical constraint, Eq. (2), implies that the diagonal elements of the matrix $\boldsymbol{q}$ are all equal to one: $q_{\alpha\alpha}=\bar{q}=1$.

The saddle point equation reads, in the $n\to 0$ limit,

$$\Lambda(q_{\alpha\beta}) + (\boldsymbol{q}^{-1})_{\alpha\beta} = 0, \quad \alpha\neq\beta. \tag{90}$$

The stability of the saddle point calculation requires that the quadratic form

$$\delta^2(-\beta f) = -\frac{1}{n}\sum_{\alpha\beta}\Lambda'(q_{\alpha\beta})(\delta q_{\alpha\beta})^2 + \frac{1}{n}\mathrm{Tr}(\boldsymbol{q}^{-1}\boldsymbol{\delta q})^2 \tag{91}$$

must be positive definite.[24] The elements of the symmetric matrix $\boldsymbol{\delta q}$ are the fluctuations δq_{ab} from the saddle point value (90).

At this stage we impose the Sommers ansatz in the DGO formalism, i.e., we divide the matrix $\boldsymbol{q}$ into $n/p_0\times n/p_0$ blocks of dimension $p_0\times p_0$ and we set

$$q_{ab} = (1-q)\delta_{ab} + (q-r)\hat{\epsilon}_{ab} + r, \tag{92}$$

where the matrix $\hat{\epsilon}$ is defined as

$$\hat{\epsilon}_{ab} = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are in a diagonal block} \\ 0 & \text{otherwise.} \end{cases} \tag{93}$$

The Sommers solution is recovered by sending the block size p_0 to infinity with the constraint $p_0(q-r)\to-\dot{\Delta}$.

Details of the study of the Hessian of Eq. (87) in the present ansatz are reported in Appendix C. Here we concentrate on the results of that analysis relevant for the stability of the DGO$_{R=0}$ ansatz. We have n/p_0 clusters each composed by p_0 replicas. Different kinds of fluctuations can thus occur, e.g., between replicas in the same cluster or in different ones, or between clusters taken as a whole. In the limit of the number of replicas going to zero, the eigenvalues of the replicated Hessian matrix that might take negative values are the following.

(1) Fluctuations in the same cluster. These are the fluctuations between one replica and p_0 others, belonging to the same cluster. The correspondent eigenvalue is

$$\Lambda_1^{(1)} = -\Lambda'(q) + \frac{1}{(1-q)^2}. \tag{94}$$

This is the so-called *replicon* and describes longitudinal fluctuations in the replica space. It must be non-negative in order to ensure thermodynamic stability. It is, indeed, always positive for the Sommers solution, as long as $-\dot{\Delta}=\Delta>0$. To see this one just uses Eq. (51) to replace $\Lambda'(q)$. $\Lambda_1^{(1)}$ is the static counterpart of the dynamical stability condition $\Gamma(\omega=0)>0$ of the Sommers solution discussed at the end of Sec. II.

(2) Fluctuations between clusters. The first dangerous eigenvalue for the Sommers solution is, instead,

$$\Lambda_0^{(3)} = -\Lambda'(r) + \frac{1}{[1-q+p_0(q-r)]^2},$$
$$\overset{p_0\Rightarrow\infty}{=} -\Lambda'(q) + \frac{1}{(1-q-\dot{\Delta})^2}. \tag{95}$$

In this case we are considering contributions coming from fluctuations between clusters *as a whole*. Using Eq. (51), $\Lambda_0^{(3)}$ turns out to be always negative for $-\dot{\Delta}>0$, signaling the instability that we were mentioning at the end of Sec. II.

(3) Mixed fluctuations. Another eigenvalue indicating an instability is

$$\Lambda_1^{(2)} = \Lambda_1^{(1)} - (p_0-2)\frac{q(1-q)+p_0(q-r)^2}{(1-q)^2[1-q+p_0(q-r)]^2} \sim -p_0\frac{q(1-q)}{(1-q)^2(1-q-\dot{\Delta})^2} \quad \text{as } p_0 \gg 1. \tag{96}$$

It becomes infinitely large and negative as $p_0\to\infty$. A similar problem has been observed recently in the study of the stability of the R step DGO saddle point of the truncated model.[25]

From this analysis we can conclude that the Sompolinsky theory does not yield a physically consistent static limit.

V. THE DYNAMICAL SOLUTION

The problem with the Sompolinsky solution follows from the assumption that the width of the function $\delta_\epsilon(\omega)$, whose limit for $\epsilon\to 0$ is a Dirac delta function, is smaller than the one of the function $\Delta_\epsilon(\omega)$, whose limit is, instead, a Kronecker delta, see Eq. (61). To overcome this assumption Hertz[22] proposed a different solution that avoids the assumption Eq. (61) by using the representations

$$\delta_\epsilon(\omega) = \frac{1}{\pi}\frac{\epsilon}{\omega^2+\epsilon^2} = \frac{1}{\pi\omega}\mathrm{Im}\left[\frac{\epsilon}{\epsilon-i\omega}\right], \tag{97}$$

$$\Delta_\epsilon(\omega) = \frac{\epsilon}{\epsilon-i\omega}. \tag{98}$$

Hertz, however, assumes a standard form for the FDT and, hence, his solution is valid only at the critical point. CHS, studying the dynamics of the spherical p-spin spin-glass model, propose, instead, a solution reproducing the correct static limit in the 1RSB phase. The solution differs from the Sommers-Sompolinsky solution and is in the same spirit of Hertz,[22,26] though with a different implementation of the FDT.

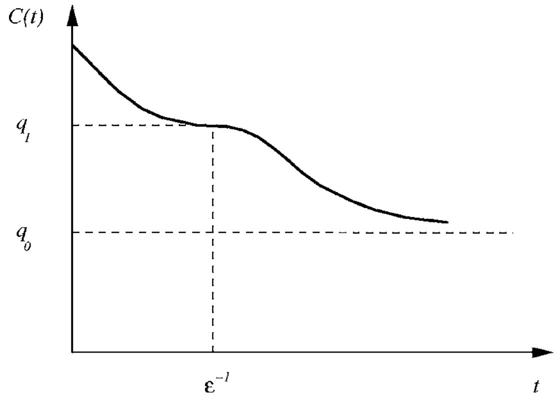

FIG. 3. Schematic form of the correlation function with two relaxation time scales.

We will first present the CHS solution for the $2+p$-spin model in the Fourier space for two time sectors and we will then generalize it to an arbitrary number of $R+1$ time sectors. The CHS solution was originally developed (for the spherical p-spin model) in the time space. The generalization of the CHS theory to R time scales in the time space, however, though feasible,[27] is quite tedious. Therefore we would rather work in the ω space.

A. The CHS solution

The CHS solution assumes, in the spirit of multiple-scale analysis, that the correlation and response functions are functions of a *fast* variable t and a *slow* variable ϵt, $\epsilon \ll 1$:

$$C(t) \Rightarrow C(t,\epsilon t), \quad G(t) \Rightarrow G(t,\epsilon t). \tag{99}$$

The *fast variable* t describes the decay of C to the *plateau* value q_1 while the *slow variable* ϵt describes the subsequent decay to the equilibrium value q_0, see Fig. 3.

If we are interested only in the leading order behavior for $\epsilon \to 0$, this is equivalent to assume time-scale separation: either the fast variable is varying while the slow variable is zero (i.e., the processes evolving on the long time scale are quenched) or the slow variable is varying and the fast variable is infinite (i.e., the processes evolving on the short time scale have already thermalized). Under this assumption the correlation and response function can be represented as the sum of two separated contributions relative to long and short time dynamics:[28]

$$C(t) = C_1(t) + C_0(\epsilon t), \tag{100}$$

$$G(t) = G_1(t) + \epsilon G_0(\epsilon t), \tag{101}$$

or

$$C(\omega) = C_1(\omega) + \epsilon^{-1} C_0(\omega/\epsilon), \tag{102}$$

$$G(\omega) = G_1(\omega) + G_0(\omega/\epsilon), \tag{103}$$

where C_1 and G_1 describe the fast part and C_0 and G_0 the slow part of C and G, see Fig. 3. Alternatively, one can employ the standard technique of multiple scale analysis, ending up again at the leading order in ϵ with the above expressions.[27]

The functions C_1 and C_0 satisfy the boundary conditions

$$C_1(t=0) = 1 - q_1, \quad C_1(t=\infty) = 0, \tag{104}$$

$$C_0(t=0) = q_1, \quad C_0(t=\infty) = q_0, \tag{105}$$

where we used the spherical constraint $C(t=0)=1$, while, as $\epsilon \to 0$,

$$G_1 \neq 0 \quad \text{iff } t \ll \epsilon^{-1} \tag{106}$$

and

$$G_0 \neq 0 \quad \text{iff } t \gg \epsilon^{-1}. \tag{107}$$

In the regime $t \ll \epsilon^{-1}$ the FDT must be satisfied and hence fast parts C_1 and G_1 are related by the FDT relation

$$C_1(\omega) = \frac{2}{\omega} \operatorname{Im} G_1(\omega). \tag{108}$$

In the long-time regime $t \gg \epsilon^{-1}$ the response to an external perturbation is given only by the degrees of freedom which *have not relaxed*, i.e., equilibrated, in the short-time regime $t \ll \epsilon^{-1}$, and hence only these degrees of freedom contribute to G_0. On the other hand *all* degrees of freedom contribute to correlation C_0. As a consequence G_0 cannot be related to the full C_0. If we introduce a parameter m, $0 \leq m \leq 1$, measuring the *fraction* of degrees of freedom which have not relaxed in the short-time regime, we have

$$G_0(\omega) = m \tilde{G}_0(\omega), \tag{109}$$

where $\tilde{G}_0$ is the response function which would be observed in the long time regime iff *all* degrees of freedom would be still active, i.e., nonequilibrated. Since all degrees of freedom contribute to $\tilde{G}_0$ this is related to the full correlation function C_0 by the FDT

$$\tilde{C}_0(\omega) = \frac{2}{\omega} \operatorname{Im} \tilde{G}_0(\omega), \tag{110}$$

where $\tilde{C}_0$ is the Fourier transform of the nonpersistent part of C_0:

$$\tilde{C}_0(t) = C_0(t) - C_0(t=\infty) = C_0(t) - q_0. \tag{111}$$

The equations for m, q_1, and q_0 are obtained by studying the dynamical equation in the static limit $\omega \to 0$ and $\epsilon \to 0$ in the two regimes $\omega \gg \epsilon$ and $\omega \ll \epsilon$.

The parameter m is related to the discontinuity of $G(\omega)$ in passing from frequencies $\omega \gg \epsilon$ to frequencies $\omega \ll \epsilon$:

$$G(\omega=0) - G(\omega_1) = m\tilde{G}_0(\omega=0) = m(q_1 - q_0), \tag{112}$$

where ω_1 is an infinitesimal frequency, $\omega_1 \ll \Gamma_0$, but goes to zero slower than ϵ: $\omega_1 \gg \epsilon \to 0$. In the last line we used the identity $\tilde{G}_0(\omega=0)=q_1-q_0$ which follows from FDT relation (110). Inserting now the Dyson equation (17) into the left-hand side of the above equation we end up with

$$m(q_1 - q_0) = G(\omega=0)G(\omega_1)[\Sigma(\omega=0) - \Sigma(\omega_1)]. \tag{113}$$

From the expression (11) for the self-energy it follows

$$\Sigma(\omega=0)-\Sigma(\omega_1)=\int_0^\infty dt(1-e^{-i\omega_1 t})\Lambda'[C(t)]G_1(t)$$
$$+\epsilon\int_0^\infty dt(1-e^{-i\omega_1 t})\Lambda'[C(t)]G_0(\epsilon t). \quad (114)$$

The first integral vanishes for $\omega_1 \ll \Gamma_0$, but $\omega_1 \gg \epsilon$ as $\epsilon \to 0$. In the second integral only the region $t \gg \epsilon^{-1}$, where G_0 is different from zero, contributes. Therefore we can replace C with C_0 in the argument of $\Lambda'[C(t)]$. Finally, by using the FDT relation (110), the leading contribution to $\Sigma(\omega=0)-\Sigma(\omega_1)$ for $\epsilon\to 0$ reads

$$\epsilon\int_0^\infty dt\Lambda'[C_0(\epsilon t)]G_0(\epsilon t)=m[\Lambda(q_1)-\Lambda(q_0)]. \quad (115)$$

Inserting this result into Eq. (113) and using the identity $G(\omega=0)=G_1(\omega=0)+G_0(\omega=0)=1-q_1+m(q_1-q_0)$ we end up with the equation

$$\Lambda(q_1)-\Lambda(q_0)=\frac{q_1-q_0}{(1-q_1)[1-q_1+m(q_1-q_0)]}. \quad (116)$$

The parameter q_0 is the time persistent part of $C(t)$ for $t\gg \epsilon^{-1}$. To study this limit we consider the infinitesimal frequency $\omega_0 \ll \epsilon \ll \Gamma_0$ and extract the part of $C(\omega_0)$,

$$C(\omega_0)=\langle\sigma(-\omega_0)\sigma(\omega_0)\rangle=G(-\omega_0)G(\omega_0)[2\Gamma_0^{-1}+\Lambda(\omega_0)], \quad (117)$$

proportional to $\delta(\omega_0)$ for $\epsilon\to 0$. From the form of the vertex function $\Lambda(t)$ we have, see also Eq. (25),

$$\Lambda(\omega_0)=\int_{-\infty}^{+\infty} dt e^{i\omega_0 t}[\Lambda[\tilde{C}_0(\epsilon t)+q_0]-\Lambda(q_0)]$$
$$+\int_{-\infty}^{+\infty} dt e^{i\omega_0 t}\Lambda(q_0). \quad (118)$$

Only the second integral contributes to the $\delta(\omega_0)$ part of $C(\omega_0)$. We then obtain the equation

$$q_0=G(\omega=0)^2\Lambda(q_0) \quad (119)$$

or, equivalently,

$$\Lambda(q_0)=\frac{q_0}{[1-q_1+m(q_1-q_0)]^2}. \quad (120)$$

Equations (120) and (116) coincide the equation for q_1 and q_0 as a function of m for the spherical $2+p$ spin-glass model obtained from the static replica calculation with the Parisi 1RSB scheme, see, e.g., Eqs. (28) and (29) of Ref. 8.

B. Stability of the CHS solution

The equation for the parameter m follows from the dynamical stability condition of the static limit which requires that the $\omega\to 0$ limit of the kinetic coefficient $\Gamma(\omega)$ has to be non-negative (simply a physical requirement). To distinguish the two regimes $\omega\gg\epsilon$ and $\omega\ll\epsilon$ we define $\Gamma^{-1}(\omega=0)$ as

$$\Gamma^{-1}(\omega=0)=\lim_{\omega\to 0}\frac{i}{2\omega}[G^{-1}(\omega)-G^{-1}(-\omega)] \quad (121)$$

and use the frequencies ω_1 and ω_0 defined previously to perform the limit in the two regimes.

Let us first consider the frequency $\omega\gg\epsilon$. In this case the limit $\omega\to 0$ must be evaluated as

$$\lim_{\omega\to 0} f(\omega) := \lim_{\omega\to 0}\lim_{\epsilon\to 0} f(\omega)=f(\omega_1). \quad (122)$$

Since $G_0(\omega_1/\epsilon)\simeq 0$, we thus have

$$G^{-1}(\omega_1)-G^{-1}(-\omega_1)=-2i\omega_1[\Gamma_0^{-1}+A_1(\omega_1)]$$
$$-\Lambda'(q_1)[G_1(\omega_1)-G_1(-\omega_1)], \quad (123)$$

where $A_1(\omega_1)$ is a finite and positive quantity. Inserting this expression into Eq. (121) we end up with[18]

$$\Gamma^{-1}(\omega_1)=\frac{\Gamma_0^{-1}+A_1(\omega_1)}{1-G_1(\omega=0)^2\Lambda'(q_1)} \quad (124)$$

so that the requirement $\Gamma(\omega_1)\geq 0$ leads to the condition

$$1-G_1(\omega=0)^2\Lambda'(q_1)=1-(1-q_1)^2\Lambda(q_1)\geq 0. \quad (125)$$

A similar calculation for $\omega\ll\epsilon$, i.e., evaluating now the limit $\omega\to 0$ as

$$\lim_{\omega\to 0} f(\omega) := \lim_{\epsilon\to 0}\lim_{\omega\to 0} f(\omega)=f(\omega_0) \quad (126)$$

leads to

$$\Gamma^{-1}(\omega_0)=\frac{\Gamma_0^{-1}+A_2(\omega_0)}{1-G(\omega=0)^2\Lambda'(q_0)}, \quad (127)$$

where $A_2(\omega_0)$ is finite and positive. We have thus the second condition for the stability:

$$1-G(\omega=0)^2\Lambda'(q_0)=1-[1-q_1+m(q_1-q_0)]^2\Lambda(q_0)\geq 0. \quad (128)$$

The dynamical stability conditions of the static limit, Eqs. (125) and (128), coincide with the stability conditions of the 1RSB saddle point computed in the static replica calculation of Ref. 8 [Eqs. (31) and (32)].

From a dynamical point of view, and for the consistency of the calculation, we must require that $\Gamma(\omega_1)=0$. Indeed, if $\Gamma(\omega_1)>0$ the correlations decay exponentially fast and the system equilibrates on a time scale of order $\Gamma^{-1}(\omega_1)$. This would imply that all degrees of freedom have relaxed before entering the regime $t\gg\epsilon^{-1}$ and $m=0$, i.e., we have back the RS solution. To yield a two time scales solution, then, the condition (125) becomes the additional equation:

$$1-(1-q_1)^2\Lambda(q_1)=0, \quad (129)$$

the so-called *marginal condition*.[29] The self-consistency equations (116), (120), and (129) and the stability condition (127) yield the CHS dynamical solution of the spherical $2+p$ model in the 1RSB phase. The dynamic RS/1RSB transition line in Fig. 1 can be obtained by Eq. (129) as the curve where q_1 jumps discontinuously from zero to a finite value.

Summing up, the CHS solution presents—in the 1RSB phase—an infinite time limit different from the static solution. Indeed, it coincides with the solution known as "dynamic," where the 1RSB phase nucleates at higher free energy than the equilibrium one. The stable phase, in the sense of lower free energy, in this regime is still the RS one but a 1RSB phase exists, is locally stable, and despite a higher free energy it dominates the dynamics due to the very large degeneracy of the metastable states belonging to it. In other words, in its evolution on the free energy surface, the system will find itself with probability one in a local minimum of the 1RSB solution simply because the number of these minima is exponentially large, in the system size, with respect to the number of global minima of the RS solution. The logarithm of the number of equivalent minima is what is called the *complexity*, and hence the dynamics of the system is dominated by the solution with the largest complexity. The marginal condition Eq. (129) is, indeed, nothing else than the condition of maximal complexity in the static Parisi replica theory.[8] The static solution corresponds, instead, to the lowest minima of the free energy and has vanishing complexity. This is the reason why the two solutions differ.

To be more explicit, for the $2+p$-spin model, in Fig. 1 we noticed two lines between the RS and the 1RSB phases. The dotted one is the line at which a 1RSB solution (with a $q_1>0$) arises, even though the statics stays RS. This is the dynamic phase transition that, as we have just seen above, is also obtained from the static limit of the CHS solution. The solid line marks the thermodynamic transition to a stable 1RSB phase. In the 1RSB region an extensive complexity exists, monotonically increasing between the lowest equilibrium free energy (at which it is zero) and the threshold free energy (where it takes its maximum value). The infinite time limit of the CHS solution describes those states lying at the threshold free energy.

As Eq. (129) cannot be satisfied anymore with $0<q_1<1$ and $\Gamma(\omega_1)$ becomes negative, a different solution is needed, involving more time scales and, correspondingly, more overlap order parameters. This is the generalization that we are going to analyze in the next section.

In particular, in the $T-J_p$ diagram in Fig. 1, the limit of validity of Eq. (129) is represented by the dynamic transition line between the 1RSB and the 1-FRSB phase. In that case the number of scales required to stabilize the solution out of the region of validity of the 1RSB phase (for increasing T at fixed J_p or decreasing J_p at fixed temperature) becomes a continuous set, plus a separate step relative to the shortest time scales.

We stress that $\Gamma(\omega_0)$ remains non-negative since it describes the relaxation of the systems to equilibrium for $t \gg \epsilon^{-1}$. In Appendix D we show in detail why for the $2+p$ spherical model the marginal condition on the intermediate time scale is necessary in order to guarantee relaxation to equilibrium on the longest time scale. In the $2+p$ spherical model the instability appears on the intermediate time scale, however, in general, it may appear on the longest time scale as well, with a negative $\Gamma(\omega_0)$. In the present scenario this means that a new (infinite) time scale enters into the game and must be included in the description of the dynamics.

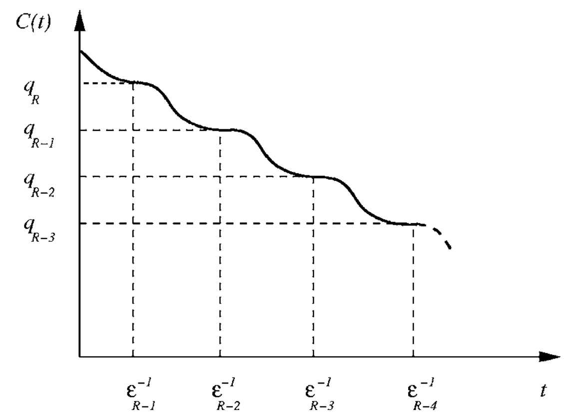

FIG. 4. Schematic form of correlation function with many relaxation time scales in CHS theory.

C. The *R*-time scale CHS solution

The extension of the CHS theory to the case of R different diverging relaxation time scales follows the same initial steps of the Sompolinsky solutions, namely one introduces R long time scales, see Fig. 4,

$$\Gamma_0^{-1} \ll \epsilon_{R-1}^{-1} \ll \epsilon_{R-2}^{-1} \ll \cdots \ll \epsilon_1^{-1} \ll \epsilon_0^{-1}, \tag{130}$$

all of which eventually diverge in the prescribed order in the thermodynamic limit.[30] In what follows we shall denote with $\boldsymbol{\epsilon}$ the set of the R frequencies ϵ_r and assume that the limit $\boldsymbol{\epsilon}\to\mathbf{0}$ is always taken in order, i.e.,

$$\lim_{\boldsymbol{\epsilon}\to 0} := \lim_{\epsilon_{R-1}\to 0} \cdots \lim_{\epsilon_0\to 0}. \tag{131}$$

A convenient way of studying the dynamics in the limit $\boldsymbol{\epsilon}\to\mathbf{0}$ is by using the multiple scale analysis. One then assumes that the correlation function $C(t)$, as well as the response $G(t)$, is a function of the fast variable $\tau_R=\epsilon_R t$ with $\epsilon_R=\Gamma_0$ and R slow variables $\tau_r=\epsilon_r t$ $(r=0,\ldots,R-1)$ describing the motion in each time sector $\epsilon_r^{-1} \ll t \ll \epsilon_{r-1}^{-1}$:

$$C(t) \Rightarrow C(\tau_R,\tau_{R-1},\ldots,\tau_1,\tau_0). \tag{132}$$

The leading behavior for $\boldsymbol{\epsilon}\to\mathbf{0}$ in the time sector $\epsilon_r^{-1} \ll t \ll \epsilon_{r-1}^{-1}$ is obtained by assuming that only processes evolving on times $t \gg \epsilon_r^{-1}$ but $t \ll \epsilon_{r-1}^{-1}$ contribute (i.e., τ_r=finite) whereas those evolving on slower time scales are quenched ($\tau_{s<r}=0$) and those evolving on faster time scale are thermalized ($\tau_{s>r}=\infty$). Under this assumption of time scale separation, $C(t)$ can be represented as the sum of $R+1$ distinct terms

$$C(t) = \sum_{r=0}^{R} C_r(\tau_r), \qquad \tau_r = \epsilon_r t, \tag{133}$$

one for each sector.

The functions C_r satisfy the normalization condition (spherical constraint)

$$C(t=0) = \sum_{r=0}^{R} C_r(\tau_r=0) = 1. \quad (134)$$

We can now split off the r-sector function C_r and take the limit $\boldsymbol{\epsilon}\to\mathbf{0}$ with τ_r finite. Taking the limit $\tau_r\to\infty$, afterward, so that $C(t)\to q_r$, cf. Eq. (52), we have the additional conditions:

$$\sum_{s=0}^{r-1} C_s(\tau_s=0) + \sum_{s=r}^{R} C_s(\tau_s=\infty) = q_r, \qquad \forall\, r=0,\ldots,R. \quad (135)$$

It is useful to define for each sector the nonpersistent part of the correlation function as

$$\tilde{C}_r(\tau) = C_r(\tau) - C_r(\tau=\infty) \quad (136)$$

so that the above conditions become

$$\tilde{C}_r(\tau=0) = q_{r+1} - q_r, \quad \tilde{C}_r(\tau=\infty)=0, \quad \forall\, r=0,\ldots,R, \quad (137)$$

with $q_{R+1}=1$, while the whole $C(t)$ reads

$$C(t) = \sum_{r=0}^{R} \tilde{C}_r(\tau_r) + q_0. \quad (138)$$

By similar arguments we obtain the following representation for the response function $G(t)$:

$$G(t) = \sum_{r=0}^{R} \epsilon_r G_r(\tau_r), \quad (139)$$

where each function G_r varies only in the corresponding sector r, where $\tau_r\sim O(1)$ for $\boldsymbol{\epsilon}\to\mathbf{0}$, and vanishes in all sectors with $s<r$. The function G_r represents the response of the system to a perturbation in the time sector labeled by r, i.e., the response due to all degrees of freedom which have not equilibrated in previous sectors. As a consequence, G_r cannot be related to the *full* correlation function $\tilde{C}_r$ since all degrees of freedom, equilibrated or not, contribute to the latter. By introducing the parameter $0\leqslant m_{r+1}\leqslant 1$ as the fraction of degrees of freedom which have *not relaxed up to sector* $r+1$, and hence do contribute to the response in the next sector r, we can write

$$G_r(\tau) = m_{r+1}\tilde{G}_r(\tau), \quad (140)$$

where $\tilde{G}_r$ is the *full* response in sector r due to *all* degrees of freedom, equilibrated and not:

$$\tilde{C}_r(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,\tilde{G}_r(\omega). \quad (141)$$

By definition $m_{R+1}=1$, since in the first sector all degrees of freedom contribute to the response, while $m_0=0$ since the system equilibrates in the last sector.

By taking the Fourier transform of Eqs. (138) and (139) we have

$$C(\omega) = \sum_{r=0}^{R} \epsilon_r^{-1}\tilde{C}_r(\omega_r/\epsilon_r) + 2\pi q_0\delta(\omega), \quad (142)$$

$$G(\omega) = \sum_{r=0}^{R} G_r(\omega_r/\epsilon_r). \quad (143)$$

As for the $\mathrm{CHS}_{R=1}$ solution, the equations for q_r and m_r are obtained by studying the static limit $\omega\to 0$ separately in each sector. We, then, introduce the set of infinitesimal frequencies ω_r, with $\omega_r\ll\epsilon_r$ but $\omega_r\gg\epsilon_{r-1}$, all of which go to zero as $\boldsymbol{\epsilon}\to\mathbf{0}$, so that the $\omega\to 0$ limit in sector r just reads

$$\lim_{\omega\to 0} f(\omega) \coloneqq \lim_{\epsilon_{R-1}\to 0}\cdots\lim_{\epsilon_r\to 0}\lim_{\omega\to 0}\lim_{\epsilon_{r-1}}\cdots\lim_{\epsilon_0\to 0} f(\omega) = f(\omega_r). \quad (144)$$

The parameter q_0 is the singular part of $C(\omega_0)$, see Eq. (142), and repeating the steps from Eqs. (117)–(119) we end up with

$$q_0 = G(\omega=0)^2\Lambda(q_0). \quad (145)$$

where $G(\omega=0)$ must be evaluated from the expression (143):

$$\begin{aligned} G(\omega=0) &= \sum_{r=0}^{R} m_{r+1}\tilde{G}_r(\omega=0) \\ &= 1 - q_R + \sum_{r=0}^{R-1} m_{r+1}(q_{r+1}-q_r) \\ &= 1 - q_R + \sum_{r=0}^{R} m_r(q_r - q_{r-1}) = \chi_0 \end{aligned} \quad (146)$$

[cf. Eq. (78)]. Here we have used the relations $\tilde{G}_r(\omega=0)=q_{r+1}-q_r$, following from the FDT relation (141), and $m_0=0$. Thus we have

$$\Lambda(q_0) = \frac{q_0}{\chi_0^2}, \quad (147)$$

coinciding with Eq. (76) obtained from the static replica calculation within the Parisi RSB scheme. The presence of an external field h would, indeed, just add a term $-(\beta h)^2$ to the right-hand side of this equation, as can be easily verified.

To find the equation for q_r with $r=1,\ldots,R$, we consider the discontinuity of $G(\omega)$ in passing from one sector to the next one: $G(\omega_{r-1})-G(\omega_r)$. By observing that $G_s(\omega_r/\epsilon_s)\simeq 0$ for $s<r$ (since $\omega_r/\epsilon_s\gg 1$) while $G_s(\omega_r/\epsilon_s)\to G_s(\omega=0)$ for $s\geqslant r$ (since $\omega_r/\epsilon_s\ll 1$), it follows that

$$\begin{aligned} G(\omega_r) &= \sum_{s=0}^{r-1} G_s(\omega_r/\epsilon_s) + \sum_{s=r}^{R} G_s(\omega_r/\epsilon_s) \\ &= \sum_{s=r}^{R} G_s(\omega=0) \\ &= \sum_{s=r}^{R} m_{s+1}(q_{s+1}-q_s). \end{aligned} \quad (148)$$

The difference $G(\omega_{r-1})-G(\omega_r)$, therefore, reads

$$G(\omega_{r-1}) - G(\omega_r) = m_r(q_r - q_{r-1}), \tag{149}$$

that, using the Dyson equation (17), can be transformed into

$$m_r(q_r - q_{r-1}) = G(\omega_r)G(\omega_{r-1})[\Sigma(\omega_{r-1}) - \Sigma(\omega_r)]. \tag{150}$$

This relation is valid for $r=1,\ldots,R$. For $R=1$ it reduces Eq. (113) of the CHS solution.

For the spherical $2+p$ model the difference $\Sigma(\omega_{r-1}) -\Sigma(\omega_r)$ can be easily evaluated: from Eq. (11) and the definition of ϵ_r we have

$$\begin{aligned}\Sigma(\omega_{r-1}) - \Sigma(\omega_r) &= \sum_{s=0}^{R} \epsilon_s \int_0^\infty dt(e^{i\omega_{r-1}t} - e^{i\omega_r t})\Lambda'[C(t)]G_s(\epsilon_s t)\\ &= \epsilon_{r-1}\int_0^\infty dt(e^{i\omega_{r-1}t} - e^{i\omega_r t})\Lambda'\\ &\quad\times[C(t)]G_{r-1}(\epsilon_{r-1}t)\\ &= \int_0^\infty d\tau\Lambda'[\tilde{C}_{r-1}(\tau) + q_{r-1}]G_{r-1}(\tau)\\ &= m_r[\Lambda(q_r) - \Lambda(q_{r-1})],\end{aligned} \tag{151}$$

where we have used the fact that only the term with $s=r-1$ yields a finite contribution for $\boldsymbol{\epsilon}\to\mathbf{0}$.

With this expression for the difference $\Sigma(\omega_{r-1})-\Sigma(\omega_r)$, and using the identity [cf. Eq. (78)]

$$\begin{aligned}G(\omega_{r-1}) &= \sum_{s=r-1}^{R} m_{s+1}\tilde{G}_s(\omega=0)\\ &= 1 - q_R + \sum_{s=r-1}^{R-1} m_{s+1}(q_{s+1} - q_s)\\ &= 1 - q_R + \sum_{s=r}^{R} m_s(q_s - q_{s-1}) = \chi_r,\end{aligned} \tag{152}$$

we finally obtain the equation for the static limit of the R time scale CHS solution:

$$\Lambda(q_r) - \Lambda(q_{r-1}) = \frac{q_r - q_{r-1}}{\chi_r\chi_{r+1}}, \quad r=1,\ldots,R \tag{153}$$

coinciding with the result from the static replica calculation, see Eq. (75).

D. Stability of the *R*-time CHS solution

As for the $\mathrm{CHS}_{R=1}$ solution the equation for m_r follows from the stability condition of the static limit $\omega\to 0$ in sector r. From the definition (121) of the kinetic coefficient $\Gamma(\omega)$ we have

$$\begin{aligned}\Gamma^{-1}(\omega_r) &= \frac{i}{2\omega_r}[G^{-1}(\omega_r) - G^{-1}(-\omega_r)]\\ &= \Gamma_0^{-1} - \frac{i}{2\omega_r}[\Sigma^{-1}(\omega_r) - \Sigma^{-1}(-\omega_r)].\end{aligned} \tag{154}$$

For the spherical $2+p$ model the difference $\Sigma(\omega_r)-\Sigma(-\omega_r)$ is given by

$$\begin{aligned}\Sigma(\omega_r) - \Sigma(-\omega_r) &= \int_0^\infty dt(e^{i\omega_r t} - e^{-i\omega_r t})\Lambda'[C(t)]G(t)\\ &= \tilde{\Sigma}(\omega_r) - \tilde{\Sigma}(-\omega_r) + \Lambda'(q_r)\\ &\quad\times[G(\omega_r) - G(-\omega_r)],\end{aligned} \tag{155}$$

where

$$\tilde{\Sigma}(\omega_r) = \int_0^\infty dt e^{i\omega_r t}[\Lambda'[C(t)] - \Lambda'(q_r)]G(t). \tag{156}$$

As a consequence we have

$$\Gamma^{-1}(\omega_r) = \frac{\Gamma_0 - i\dfrac{\partial}{\partial\omega_r}\tilde{\Sigma}(\omega_r)}{1 - G(\omega_r)^2\Lambda'(q_r)}. \tag{157}$$

The quantity $i(\partial/\partial\omega_r)\tilde{\Sigma}(\omega_r)$ is real and negative, therefore the requirement $\Gamma^{-1}(\omega_r)\geqslant 0$ leads to the dynamical stability condition

$$1 - G(\omega_r)^2\Lambda'(q_r) \geqslant 0 \tag{158}$$

which can be written [cf. Eq. (152)] as

$$-\Lambda'(q_r) + \frac{1}{\chi_{r+1}^2} \geqslant 0, \tag{159}$$

where χ_r is defined in Eq. (78). We recover then the condition for stability of the R-RSB saddle point in the replica calculation.[8]

Unlike the static calculation, however, the dynamical solution requires that all $\Gamma^{-1}(\omega_r)$, but the last one for $r=0$, vanish. Indeed, as discussed for the CHS solution, if it happens that $\Gamma^{-1}(\omega_r)>0$ for some $r=1,\ldots,R$, then all degrees of freedom not yet thermalized up to sector r relax in the sector r, so that $m_s=0$ for $s\leqslant r$. This, in turn, implies that the number of diverging relaxation time scale is $r<R$, and not R as initially assumed.[29] This argument does not apply to the last sector $r=0$. Indeed, by assumption, the system equilibrates in this sector and this is feasible only if $\Gamma^{-1}(\omega_0)$ is positive.[31]

If the above requirements cannot be satisfied and one or more $\Gamma(\omega_r)$ are negative, including the last sector $r=0$, then additional time scale(s) have to be included into the description. This is what happens, e.g., in the $2+p$ spherical model in the 1-FRSB and FRSB phases. Actually in these phases an infinite number of successive time scales is required in order to stabilize the dynamics. Nevertheless it can be seen that with increasing R the violation of dynamical (marginal) stability decreases and vanishes as $R\to\infty$.

We stress that similarly to what happens for the CHS solution in the 1RSB phase, also the extension of the CHS theory to R-time scales does not reproduce exactly the *same* static solution found from the static replica calculation. Indeed, while Eqs. (147) and (153) for q_0 and q_r and the condition $\Gamma(\omega_0)>0$ are the same as those found from the static replica calculation, the equations

$$\Gamma(\omega_r) = 0 \quad \text{for } r = 1, \ldots, R \tag{160}$$

differ from the corresponding ones in statics. For any finite R we have, thus, the same phenomenon already observed for systems described by the 1-RSB solution.[32,33] As in that case, the difference between the static free energy and the free energy of the static limit of the dynamical solution corresponds to the existence of an extensive complexity of the R-RSB solution, i.e., to the presence of a macroscopic number of statistically equivalent metastable states dominating the dynamics. In the free energy landscape describing the phase space of the system, such states are at a free energy level larger than the free energy of the static minimum, nevertheless they dominate the dynamics due to their macroscopic number. In the mean-field picture we are adopting here, for $\epsilon_r \to 0$, the system is stuck in these threshold states because of the consequent decoupling between processes at different time scales. Relaxing such a constraint, i.e., going beyond mean field, the evolution from the threshold states at a given step of the RSB can be, instead, allowed.[34]

The difference disappears in the FRSB phase where the complexity vanishes. Indeed defining $m_r = x(q_r)$ from either Eq. (153) or Eq. (160) we obtain in the limit $R \to \infty$

$$\frac{d}{dq}\Lambda(q) = \frac{1}{\left(1 - q_R + \int_q^{q_R} dq' x(q')\right)^2}, \tag{161}$$

which is the FRSB solution of the spherical $2+p$ model, cf. Eq. (53) of Ref. 8. It is easy to show that in the $R \to \infty$ limit the order parameter function $q(x)$ satisfies the Parisi antiparabolic differential equation. We also note that at difference with the Sompolinsky theory the R-time-scale CHS solution does not introduce the additional function $\Delta(x)$.

VI. SUMMARY AND CONCLUSIONS

We have addressed the problem of the equilibrium dynamics of spin-glass systems. One of the issues that makes equilibrium dynamics worth studying is its connection with the static properties of the systems, i.e., those obtained from statistical mechanics via the partition function. While the statistical mechanics of spin-glass systems is well-developed, the equilibrium dynamics is less known. The usual theory for equilibrium dynamics of spin-glass systems is the Sompolinsky theory that in the FRSB phase leads to a static solution in agreement with the statistical mechanic one, provided one imposes the Parisi gauge $d\Delta(x)/dx = -x dq(x)/dx$. The Sompolinsky theory has received further support from de Dominicis, Gabay, and Orland (DGO) who, using a replica symmetry breaking scheme with two order parameters (a Parisi-like overlap q and a Sompolinsky-like anomaly Δ), derived the FRSB Sompolinsky solution from equilibrium statistical mechanics. Despite these results the Sompolinsky theory was the object of criticisms and its validity is still not well-established.

In this work we have analyzed in detail the Sompolinsky solution using the spherical $2+p$ spin-glass model. The main motivations in using this model are (i) that its static solution is completely exactly known and (ii) that, besides displaying a FRSB phase, it possesses stable 1RSB and 1-FRSB phases, so that—unlike in the SK model—we can test the Sompolinsky solution in phases other than the FRSB.

The first result of our study is that if the number R of relaxation times (equivalently the number of replica symmetry breaking steps in the static equivalent DGO description) is finite then the Sompolinsky theory leads to a static solution that cannot be traced back to the static solution obtained with the Parisi RSB scheme. The two solutions can coincide only in the FRSB phase, when $R \to \infty$ and $q(x)$ becomes a continuous function, provided one fixes the Parisi gauge.

To understand the properties of the Sompolinsky solution we have studied it in the 1RSB phase of the $2+p$ model. The analysis, performed within the equivalent DGO theory, reveals that the fluctuations about the DGO saddle point yielding the static limit of the Sompolinsky solution not only have negative eigenvalues, but some of them go to minus infinite. The saddle point is, therefore, unstable and the Sompolinsky solution in the infinite time limit is not a physically consistent solution. As already noted by Hertz[22] the weak point of Sompolinsky theory is in the assumption that each time sector r is assumed to contribute to the effective static field $H(\{z\})$ in the spin equation of motion with the full magnetization $\bar{m}_r(\{z\})$ induced at that time scale by the slow noise z, mathematically expressed by Eq. (61).

In the second part of the paper we have presented an alternative theory for the equilibrium dynamics of spin-glass systems. The theory, based on the CHS solution of the spherical p-spin spin-glass model, differs from the Sompolinsky theory in that it uses a modified form of the FDT theorem to deal with the anomalous contribution to the response function, overcoming the Sompolinsky assumption Eq. (61). In this theory, in the static limit, the parameters are q_r, the time persistent part of the correlation function at (infinite) time scale t_r, and m_r, the fraction of nonequilibrated degrees of freedom at scale t_r entering into a modified FDT. No anomaly functions are introduced to represent the zero field cooled static susceptibility.

The equations for q_r have the same functional form of those derived from statistical mechanics using the Parisi RSB ansatz. For any finite R, however, the equations for m_r have a different form. The reason is that in the dynamic theory the equations for m_r follow from the condition that the dumping function must be zero (*marginal condition*) for all but the longest time scale:[31]

$$\Gamma(\omega_r) = 0, \quad r = 1, \ldots, R. \tag{162}$$

In the static replica calculation the self-consistency equations for m_r are, instead, obtained by the stationarity of the replicated free energy functional with respect to variation of m_r, i.e., from the vanishing of the derivative of the replica free energy with respect to m_r. From the replica calculation point of view, on the contrary, the dynamic marginal condition corresponds to the requirement of a maximal derivative of the free replica free energy with respect to m_r, that is maximal complexity.[6,32,33,35]

The difference between dynamic and static solutions is due to the degeneracy of the metastable excited states that yields an extensive complexity at free energies higher than the static one. Even though their weight is smaller than the one of the equilibrium state, the states at higher free energy ("threshold states") are statistically much more relevant (their number is exponentially larger as the size of the system increases) and therefore a system cooled down from high temperature will end up in one of these with probability one. Because of the mean field nature of the models considered and the consequent growth of barriers with the size, the system cannot evolve anymore out of the threshold states in a relaxation dynamics and the equilibrium states become unreachable in the thermodynamic limit. This might be possibly bypassed considering time scales that are not completely decoupled. In our notation it would amount to using nonvanishing $\boldsymbol{\epsilon}$ values and computing the first correction to the leading behavior for $\boldsymbol{\epsilon}\to\mathbf{0}$, not an easy task.

When more RSB steps are considered, the complexity depends on more breaking parameters m_r and the threshold value is obtained by maximizing the complexity with respect to all of them. In the dynamical formalism it is equivalent to impose Eq. (162). This selects the ensemble of statistically equivalent minima of the (exponentially) more numerous kind, that is, those at higher level in the free energy corrugated landscape. As the number of steps is increased, the complexity function counting the number of minima decreases, as well as the difference between the dynamic (threshold) free energy and the static (equilibrium) one.[36] In the limit where the stable phase is FRSB this difference eventually reduces to zero, as, e.g., in Ref. 37 for the case of the Ising SK model. The same effect can be detected in the Ising p-spin model[38] at zero temperature passing from a 1RSB to a 2RSB ansatz, even though both solutions are physically inconsistent even at the static point. The advantage of the spherical $2+p$ with respect to the above-mentioned models is that three separate spin-glass stable phases exist, each obtained by a different—physically consistent—RSB solution, where the above considerations have been tested.

ACKNOWLEDGMENTS

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APPENDIX A: THE DE DOMINICIS, GABAY, AND ORLAND (DGO) SOLUTION

In this appendix we sketch the derivation of the de Dominicis, Gabay, and Orland[23] (DGO) static solution for the $2+p$ spin-glass model and show its relation with the Sompolinsky solution.

In the replica approach the static solution for the spherical $2+p$ spin-glass model is given by the $n\to 0$ limit, where n is the number of replicas, of the stationary point of the replica functional $f[Q_{ab},\Lambda_{ab}]$:[24,39]

$$-n\beta f[Q_{\alpha\beta},\Lambda_{\alpha\beta}]=\frac{1}{2}\sum_{\alpha\beta}^{1,n} g(Q_{\alpha\beta})-\frac{1}{2}\sum_{ab}^{1,n}\Lambda_{\alpha\beta}Q_{\alpha\beta} + \log \mathrm{Tr}_\sigma \exp\left(\frac{1}{2}\sum_{\alpha\beta}\Lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta + b\sum_\alpha \sigma_\alpha\right), \tag{A1}$$

where $Q_{\alpha\beta}$ is the spin-overlap matrix in the replica space between replicas α and β, $\Lambda_{\alpha\beta}$ the Lagrange multiplier associated with $Q_{\alpha\beta}$, and $b=\beta h$ the external field. The function $g(x)$ is defined as $dg(x)/dx=\Lambda(x)$, with $\Lambda(x)$ given by Eq. (10). Moreover, for the spherical model

$$\Lambda_{\alpha\alpha}=\bar{\lambda} \tag{A2}$$

is the Lagrange multiplier fixing the spherical constraint $Q_{\alpha\alpha}=1$, and

$$\mathrm{Tr}_\sigma := \prod_\alpha \int_{-\infty}^{+\infty} d\sigma_\alpha. \tag{A3}$$

The R-step DGO solution is obtained by taking the $n\times n$ matrix $Q_{\alpha\beta}$ made of $(n/p_0)^2$ submatrices q_{ab} and r_{ab} of size $p_0\times p_0$,

$$Q_{\alpha\beta}=\begin{pmatrix} q_{ab} & r_{ab} & r_{ab}\\ r_{ab} & q_{ab} & r_{ab}\\ r_{ab} & r_{ab} & q_{ab}\end{pmatrix}, \tag{A4}$$

with each matrix q_{ab} and r_{ab} an R-RSB Parisi matrix:

$$q_{ab}=\sum_{t=0}^{R+1}(q_t-q_{t-1})\prod_{k=0}^{t-1}\delta_{a_k,b_k}, \tag{A5}$$

$$r_{ab}=\sum_{t=0}^{R+1}(r_t-r_{t-1})\prod_{k=0}^{t-1}\delta_{a_k,b_k}, \tag{A6}$$

where

$$a_k=0,\ldots,p_k/p_{k+1}-1 \quad \text{with}$$

$$1=p_{R+1}<p_R<\cdots<p_1<p_0,$$

$$q_{-1}=r_{-1}=0,$$

$$q_{R+1}=\bar{q}=1 \quad (\text{for the spherical constraint}),$$

$$r_{R+1}=r_R=\bar{r}. \tag{A7}$$

The matrix $\Lambda_{\alpha\beta}$ is written in a similar form with the $p_0\times p_0$ R-RSB Parisi matrices λ_{ab} and ρ_{ab}.

At difference with the Parisi RSB scheme the block sizes p_k are sent eventually to infinity in order, so that $p_k/p_{k-1}\to 0$, with the assumption that

$$p_k(q_k - r_k) \to -\dot{\Delta}_{q_k}, \quad p_k(\lambda_k - \rho_k) \to -\dot{\Delta}_{\lambda_k}, \tag{A8}$$

as $p_k \to \infty$. The limit $n\to 0$ is taken at the end.

As an example we consider

$$\sum_{\alpha\beta} \Lambda_{\alpha\beta} Q_{\alpha\beta} = n \sum_{t=0}^{R+1} p_t[(\lambda_t q_t - \rho_t r_t) - (\lambda_{t-1} q_{t-1} - \rho_{t-1} r_{t-1})] + \frac{n^2}{p_0} \sum_{t=0}^{R+1} p_t(\rho_t r_t - \rho_{t-1} r_{t-1}). \tag{A9}$$

From Eq. (A8) we have $r_t = q_t + \dot{\Delta}_{q_t}/p_t + o(1/p_t)$ and a similar expression for ρ_t. As a consequence, performing the ordered limit $p_t \to \infty$ we have

$$\sum_{\alpha\beta} \Lambda_{\alpha\beta} Q_{\alpha\beta} = n\left[\bar{\lambda} - \lambda_R q_R - \sum_{t=0}^{R} (\lambda_t \dot{\Delta}_{q_t} + q_t \dot{\Delta}_{\lambda_t})\right]. \tag{A10}$$

The evaluation of the trace is more involved. We shall give here the main steps. With the DGO form of $\Lambda_{\alpha\beta}$ we have

$$\begin{aligned}\sum_{\alpha\beta} \Lambda_{\alpha\beta} \sigma_\alpha \sigma\beta = (\bar{\lambda} - \lambda_R) \sum_\alpha \sigma_\alpha^2 &+ \sum_{t=0}^{R+1} p_t[(\lambda_t q_t - \rho_t r_t) - (\lambda_{t-1} q_{t-1} - \rho_{t-1} r_{t-1})] \\ &\times \sum_{i=1}^{p_0} \sum_{a_0\cdots a_{t-1}} \left(\sum_{a_t\cdots a_R} \sigma^i_{a_0\cdots a_R}\right)^2 \\ &+ \sum_{t=0}^{R} (\rho_t - \rho_{t-1}) \sum_{a_0\cdots a_{t-1}} \left(\sum_{i=1}^{p_0} \sum_{a_t\cdots a_R} \sigma^i_{a_0\cdots a_R}\right)^2,\end{aligned} \tag{A11}$$

where the index $i=1,\ldots,p_0$ is relative to the primary blocks of size $p_0 \times p_0$, while the index a_k is relative to the sub-blocks of the Parisi RSB scheme.

By inserting this expression into the exponent of the exponential in the trace one ends up after a straightforward algebra with

$$\begin{aligned}\mathrm{Tr}_\sigma \exp\left(\frac{1}{2}\sum_{\alpha\beta} \Lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta + b\sum_{\alpha\beta}\sigma_\alpha\right) &= \prod_{t=0}^{R}\left\{\prod_{a_0\cdots a_{t-1}}\left[\int Dz_t \prod_i \int Dy_{i,t}\right]\right\} \\ &\times \prod_{a_0\cdots a_{R-1}} \prod_i \exp[p_R \phi_R(H_R)],\end{aligned} \tag{A12}$$

where $z_t \equiv z(a_0\cdots a_{t-1})$ and $y_{i,t} \equiv y_i(a_0\cdots a_{t-1})$ are the auxiliary Gaussian variables used to linearize the squares in Eq. (A11), and we have used the short-hand notation

$$Dz_t \coloneqq \frac{dz_t}{\sqrt{2\pi}} e^{-z_t^2/2}, \quad Dy_{i,t} \coloneqq \sqrt{\frac{p_t}{2\pi}} dy_{i,t} e^{-p_t y_{i,t}^2/2}. \tag{A13}$$

The function $\phi_R(H)$ is defined as

$$\exp[\phi_R(H)] = \mathrm{Tr}_\sigma \exp\left[\frac{\bar{\lambda} - \lambda_R}{2}\sigma^2 + H\sigma\right] = \sqrt{\frac{2\pi}{\lambda_R - \bar{\lambda}}} \exp\left[\frac{H^2}{2(\lambda_R - \bar{\lambda})}\right], \tag{A14}$$

while H_R is given by

$$H_R = \sum_{t=0}^{R} [\sqrt{\Delta\lambda_t} z_t + \sqrt{-\dot{\Delta}_{\lambda_t}} y_{i,t}] + b \tag{A15}$$

with $\Delta\lambda_t = \lambda_t - \lambda_{t-1}$. In Eq. (A12) we used the fact that H_R does not depend on a_R.

In the limit $p_R \gg 1$ the integral over $y_{i,R}$ can be evaluated at the saddle point

$$y_{i,R} = \sqrt{-\dot{\Delta}_{\lambda_R}} m_R, \tag{A16}$$

where

$$\bar{m}_R = \left.\frac{d}{dH}\phi_R(H)\right|_{H=H_R} = \phi_R'(H_R). \tag{A17}$$

Noticing that $\Sigma_i[\cdots] = O(n)$ and $n \ll 1$, we then have

$$\int Dz_R \prod_{i=1}^{p_0} \int Dy_{i,R} e^{p_R\phi_R(H_R)} = \prod_i \exp[p_{R-1}\phi_{R-1}(H_{R-1})], \tag{A18}$$

where H_{R-1} is given by Eq. (A15) with the replacement $R \to R-1$, while

$$\phi_{R-1}(H) = \int Dz_R\left[\frac{1}{2}\dot{\Delta}_{\lambda_R}\bar{m}_R^2 + \phi_R(\sqrt{\Delta\lambda_R} z_R - \dot{\Delta}_{\lambda_R}\bar{m}_R + H)\right]. \tag{A19}$$

The procedure can be repeated integrating over $y_{i,R-1}$, then over $y_{i,R-2}$, and so on. After having integrated out all $y_{i,t}$ we end up with

$$\mathrm{Tr}_\sigma \exp\left(\frac{1}{2}\sum_{\alpha\beta}\Lambda_{\alpha\beta}\sigma_\alpha\sigma_\beta + b\sum_{\alpha\beta}\sigma_\alpha\right) = \exp[n\phi_{-1}(b)], \tag{A20}$$

where

$$\phi_{-1}(b) = \int \prod_{t=0}^{R} Dz_t\left[\frac{1}{2}\sum_{t=0}^{R}\dot{\Delta}_{\lambda_t}\bar{m}_t^2 + \phi_R(H\{z\})\right] \tag{A21}$$

with

$$H(\{z\}) = \sum_{t=0}^{R}[\sqrt{\Delta\lambda_t} z_t - \dot{\Delta}_{\lambda_t}\bar{m}_t] + b \tag{A22}$$

and

$$\bar{m}_t \equiv \bar{m}_t(z_0, \ldots, z_t) = \int Dz_{t+1}\bar{m}_{t+1}(z_0, \ldots, z_{t+1})$$
$$= \int \prod_{r=t+1}^{R} Dz_r \bar{m}_R(z_0, \ldots, z_R) \tag{A23}$$

with $\bar{m}_R(z_0, \ldots, z_R)$ defined by Eq. (A17).

Collecting all terms, and redefining $\phi_R(H)$ as

$$\phi_R(H) = \frac{1}{2}\frac{H^2}{\lambda_R - \bar{\lambda}} \tag{A24}$$

to extract trivial factors, we obtain the Sompolinsky functional[1,23] for the spherical $2+p$ model

$$-\beta f_S = \frac{1}{2}[g(1) - q(q_R) - \bar{\lambda} + \lambda_R q_R] - \frac{1}{2}\left[\sum_{t=0}^{R} \Lambda(q_t)\dot{\Delta}_{q_t} - \sum_{t=0}^{R}(\lambda_t \dot{\Delta}_{q_t} + q_t \dot{\Delta}_{\lambda_t})\right] + \int \prod_{t=0}^{R} Dz_t \left[\frac{1}{2}\sum_{t=0}^{R} \dot{\Delta}_{\lambda_t}\bar{m}_t^2 + \phi_R(H(z))\right] + \frac{1}{2}\log\left(\frac{2\pi}{\lambda_R - \bar{\lambda}}\right). \tag{A25}$$

The Sompolinsky solution follows from stationarity of f_S with respect to variations of $\bar{m}_t$, $\dot{\Delta}_{\lambda_t}$, $\Delta\lambda_t = \lambda_t - \lambda_{t-1}\dot{\Delta}_{q_t}$, q_t, and $\bar{\lambda}$ leading to, respectively,

$$\bar{m}_r = \int Dz_{r+1}\bar{m}_{r+1} \tag{A26}$$

with $\bar{m}_R = \phi_R'(H\{z\})$,

$$q_r = \int \prod_{t=0}^{R} Dz_t \bar{m}_r^2, \tag{A27}$$

$$\frac{1}{\lambda_R - \bar{\lambda}} - \sum_{t=r}^{R} \dot{\Delta}_{q_t} = \frac{1}{\sqrt{\Delta\lambda_r}} \int \prod_{t=0}^{R} Dz_t \frac{\partial \bar{m}_r}{\partial z_r}, \tag{A28}$$

$$\lambda_r = \Lambda(q_r), \tag{A29}$$

$$\dot{\Delta}_{\lambda_r} = \Lambda'(q_r)\dot{\Delta}_{q_r}, \tag{A30}$$

and

$$(\lambda_R - \bar{\lambda})^{-1} = 1 - q_R \tag{A31}$$

that is the spherical constraint.

By using the stationary equations (A29)–(A31) to eliminate $\bar{\lambda}$, λ_t, and $\dot{\Delta}_{\lambda_t}$ from f_S, and changing the notation as $\dot{\Delta}_{\lambda_t} \to \Delta_t'$, $\dot{\Delta}_{q_t} \to \dot{\Delta}_t$, and $\Delta\lambda_t \to \Delta_t = \Lambda(q_t) - \Lambda(q_{t-1})$ it is easy to see that the functional (A25) reduces to the Sompolinsky functional (67).

APPENDIX B: $R \to \infty$ DGO THEORY

To compare the DGO theory with the Parisi theory in the limit $R \to \infty$ we first eliminate the local magnetization $\bar{m}_r$ using the the stationary equation (A26). For the the spherical $2+p$ spin-glass model the equations can be easily solved obtaining

$$\bar{m}_r = \sum_{t=0}^{r} \frac{\sqrt{\Delta\lambda_t}}{F_t} z_t + \frac{b}{F_0}, \tag{B1}$$

where

$$F_r = \lambda_R - \bar{\lambda} + \sum_{t=r}^{R} \dot{\Delta}_{\lambda_t}. \tag{B2}$$

As a consequence

$$\int \prod_{t=0}^{R} Dz_t \sum_{t=0}^{R} \dot{\Delta}_{\lambda_t}\bar{m}_t^2 = \sum_{r=0}^{R} \frac{\Delta\lambda_r}{F_r^2} \sum_{t=r}^{R} \dot{\Delta}_{\lambda_t} + \frac{b^2}{F_0^2}\sum_{t=0}^{R} \dot{\Delta}_{\lambda_t} \tag{B3}$$

and

$$\int \prod_{t=0}^{R} Dz_t H(\{z\})^2 = (\lambda_R - \bar{\lambda})^2 \left[\sum_{t=0}^{R} \frac{\Delta\lambda_t}{F_t^2} + \frac{b^2}{F_0^2}\right]. \tag{B4}$$

Collecting all terms one finally has

$$-\beta f_{DGO} = \frac{1}{2}[g(1) - g(q_R) - \bar{\lambda} + \lambda_R q_R] - \frac{1}{2}\left[\sum_{t=0}^{R} \Lambda(q_t)\dot{\Delta}_{q_t} - \sum_{t=0}^{R}(\lambda_t \dot{\Delta}_{q_t} + q_t \dot{\Delta}_{\lambda_t})\right] + \frac{1}{2}\sum_{t=0}^{R} \frac{\lambda_t - \lambda_{t-1}}{F_t} + \frac{1}{2}\frac{b^2}{F_0} + \frac{1}{2}\log\left(\frac{2\pi}{\lambda_R - \bar{\lambda}}\right), \tag{B5}$$

which is the more usual form of the DGO functional. Again the equations for order parameters follow from stationarity of f_{DGO}. It can be checked that by eliminating the order parameters λ_t and $\dot{\Delta}_{\lambda_t}$ and $\bar{\lambda}$ with the corresponding stationary equations the DGO functional (B5) reduces to the DGO functional (83) given in the main text.

In the limit $R \to \infty$, and assuming that we are in a FRSB phase, the DGO functional (B5) of the spherical $2+p$ spin-glass model becomes

$$-\beta f_{DGO} = \frac{1}{2}[g(1) - g(q_1) - \bar{\lambda} + \lambda_1 q_1] - \frac{1}{2}\int_0^1 dx \Lambda[q(x)]\dot{\Delta}_q(x) + \frac{1}{2}\int_0^1 dx[\lambda(x)\dot{\Delta}_q(x) + q(x)\dot{\Delta}_\lambda(x)] + \frac{1}{2}\int_0^1 dx \frac{\dot{\lambda}(x)}{F(x)} + \frac{1}{2}\frac{\lambda(0) + b^2}{F(0)}$$

$$+\frac{1}{2}\log\left(\frac{2\pi}{\lambda_1-\bar{\lambda}}\right), \tag{B6}$$

where $\dot{\lambda}(x)=d\lambda(x)/dx$, $\lambda_0=\lambda(0)$, and

$$F(x)=\lambda_1-\bar{\lambda}+\int_x^1 dx'\dot{\Delta}_\lambda(x'). \tag{B7}$$

The expression (B6) is specific of the spherical $2+p$ model, however, it can be written in the more usual form for the FRSB phase:[4,5,39]

$$\begin{aligned}-\beta f_{\mathrm{DGO}}=&\frac{1}{2}[g(1)-g(q_1)-\bar{\lambda}+\lambda_1 q_1]-\frac{1}{2}\int_0^1 dx\Lambda[q(x)]\dot{\Delta}_q(x)\\&+\frac{1}{2}\int_0^1 dx[\lambda(x)\dot{\Delta}_q(x)+q(x)\dot{\Delta}_\lambda(x)]\\&+\int_{-\infty}^{+\infty}\frac{dy}{\sqrt{2\pi\lambda(0)}}\exp\left[-\frac{(y-b)^2}{2\lambda(0)}\right]\phi(0,y)\\&+\frac{1}{2}\log\left(\frac{2\pi}{\lambda_1-\bar{\lambda}}\right),\end{aligned} \tag{B8}$$

where

$$\phi(x,y)=\frac{1}{2}\left[\frac{y^2}{F(x)}+\int_x^1 dx'\frac{\dot{\lambda}(x')}{F(x')}\right] \tag{B9}$$

is the solution of the Parisi antiparabolic differential equation

$$\dot{\phi}(x,y)=-\frac{\dot{\lambda}(x)}{2}\phi''(x,y)+\frac{\dot{\Delta}_\lambda(x)}{2}\phi'(x,y)^2 \tag{B10}$$

with the boundary condition

$$\phi(1,y)=\frac{1}{2}\frac{y^2}{\lambda_1-\bar{\lambda}}. \tag{B11}$$

As usual a "dot" in the Parisi equation denotes the derivative with respect to x while a "prime" the derivative with respect to y.

The Parisi solution is recovered by setting $\dot{\Delta}_\lambda=-x\dot{\lambda}(x)$, $\dot{\Delta}_q=-x\dot{q}(x)$, see, e.g., Ref. 8.

APPENDIX C: STABILITY OF THE DGO-SOMMERS SOLUTION

In the $\mathrm{DGO}_{R=0}$ ansatz the free energy fluctuations in the replica space, cf. Eq. (91), become

$$\begin{aligned}\delta^2[-\beta f(r,q,m)]=&-\frac{1}{n}\sum_{ab}\{\Lambda'(r)\\&+\hat{\epsilon}_{ab}[\Lambda'(q)-\Lambda'(r)]\}(\delta q_{ab})^2\\&+\frac{A^2}{n}\sum_{ab}(\delta q_{ab})^2+\frac{B^2}{n}\mathrm{Tr}(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})^2\\&+C^2\left(\sum_{ab}\delta q_{ab}\right)^2+2AB\,\mathrm{Tr}\,\boldsymbol{\delta q}\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q}\\&+2AC\sum_{ab}(\boldsymbol{\delta q}\,\boldsymbol{\delta q})_{ab}+2BC\sum_{ab}(\boldsymbol{\delta q}\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}\end{aligned} \tag{C1}$$

with

$$A=\frac{1}{1-q}, \tag{C2}$$

$$B=-\frac{q-r}{(1-q)\chi_1}, \tag{C3}$$

$$C=-\frac{r}{\chi_1}, \tag{C4}$$

$$\chi_1=1-q-\dot{\Delta}, \tag{C5}$$

where $\dot{\Delta}=-p_0(q-r)$ by definition. The eigenvalue equation is

$$\begin{aligned}&[A^2-\Lambda'(r)]\delta q_{ab}-[\Lambda'(q)-\Lambda\tilde{A}(r)]\hat{\epsilon}_{ab}\delta q_{ab}\\&+B^2(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q}\hat{\boldsymbol{\epsilon}})_{ab}+C^2\left(\sum_{cd}\delta q_{cd}\right)\delta q_{ab}\\&+AB[(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}+(\boldsymbol{\delta q}\hat{\boldsymbol{\epsilon}})_{ab}]\\&+AC\sum_c(\delta q_{ac}+\delta q_{bc})\\&+BC\sum_c[(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ac}+(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{bc}]=\lambda\,\delta q_{ab}.\end{aligned} \tag{C6}$$

The above equation is valid for $a\neq b$. The diagonal elements δq_{aa} are all zero because of the spherical constraint. In the present ansatz we have n/p_0 blocks each containing p_0 elements. The diagonal blocks contain q elements, whereas the off-diagonal ones contain r elements. q is the overlap value of replicas belonging to the same cluster, r the overlap between replicas of different clusters. The different eigenvalues, solutions of Eq. (C6), can be grouped in three different sets each one corresponding to a given subspace of the replica space. One subspace involves fluctuations of the overlaps of one replica with other p_0 replicas (both belonging to the same cluster and different clusters). Another one involves fluctuations of the overlaps of groups of p_0 replicas with other p_0 replicas. The third one consists of the eigenvalues determining the stability of the fluctuations between clusters as a whole (roughly speaking). We look in detail at the eigenvalues and at their behavior as $n\to 0$.

1. Fluctuations of the overlaps of one replica with p_0 other replicas

The first subspace is determined by the condition

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}=0,\quad\forall\, a,b. \tag{C7}$$

Two eigenvalues are associated to this subspace. One corresponds to fluctuations of the overlap between replicas in two different clusters (off-diagonal elements), for which all diagonal blocks are zero:

$$\hat{\epsilon}_{ab}\delta q_{ab}=0, \quad \forall\, a,b. \tag{C8}$$

The eigenvalue and its degeneracy are

$$\Lambda_0^{(1)}=-\Lambda'(r)+A^2,$$

$$n_0^{(1)}=\frac{n(n-p_0)(p_0-1)^2}{2p_0^2}. \tag{C9}$$

The other one controls fluctuations of the q overlaps, i.e., the off-diagonal blocks are zero:

$$(1-\hat{\epsilon}_{ab})\delta q_{ab}=0, \quad \forall\, a,b. \tag{C10}$$

Its expression and its degeneracy are

$$\Lambda_1^{(1)}=-\Lambda'(q)+A^2,$$

$$n_1^{(1)}=\frac{n(p_0-3)}{2}. \tag{C11}$$

2. Fluctuations of the overlaps of p_0 replicas with other p_0 replicas

We now look at the fluctuations in the subspace

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q}\hat{\boldsymbol{\epsilon}})_{ab}=0, \quad \forall\, a,b \tag{C12}$$

with $(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}\neq 0$ [Eq. (C8) not satisfied].

The first eigenvalue can be addressed as the one related to fluctuations between different clusters as a whole, that is the subspace given by the further condition

$$\hat{\epsilon}_{ab}(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}=0, \quad \forall\, a,b. \tag{C13}$$

Eigenvalue and degeneracy are

$$\Lambda_0^{(2)}=-\Lambda'(r)+A^2+p_0AB,$$

$$n_0^{(2)}=\frac{n(n-p_0)(p_0-1)}{p_0^2}. \tag{C14}$$

The second eigenvalue deals with the subspace orthogonal to Eq. (C13), i.e., with fluctuations between replicas in the same cluster:

$$(1-\hat{\epsilon}_{ab})(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab}=0, \quad \forall\, a,b. \tag{C15}$$

Its form and degeneracy are

$$\Lambda_1^{(2)}=-\Lambda'(q)+A^2+(p_0-2)A(B+C),$$

$$n_1^{(2)}=\frac{n(p_0-1)}{p_0}. \tag{C16}$$

3. Fluctuations of the overlap of one cluster with other clusters

Here we consider the clusters as single elements and the relative fluctuations. The subspace we look at is orthogonal to the first two subspaces and in order to express the condition defining it we introduce the cluster matrix

$$\boldsymbol{C}_{\alpha\beta}=(\hat{\boldsymbol{\epsilon}}\boldsymbol{\delta q})_{ab} \quad \text{with } a\in\alpha, \quad b\in\beta, \tag{C17}$$

α,β are cluster indexes. In terms of this matrix one identifies a first subsubspace associated with purely off-diagonal fluctuations (i.e., between different clusters):

$$\boldsymbol{C}_{\alpha\alpha}=0, \quad \sum_\beta \boldsymbol{C}_{\alpha\beta}=0, \quad \forall\,\alpha. \tag{C18}$$

The eigenvalue and its degeneracy are

$$\Lambda_0^{(3)}=-\Lambda'(r)+(A+p_0B)^2, \tag{C19}$$

$$n_0^{(3)}=\frac{n(n-3p_0)}{2p_0^2}. \tag{C20}$$

There are, then, two other subspaces (for finite n), whose physical meaning is less clear since mixed fluctuations are involved.

One subsubspace is determined by the eigenvectors for which

$$\sum_\alpha \boldsymbol{C}_{\alpha\alpha}=0, \quad \sum_{\alpha\neq\beta}\boldsymbol{C}_{\alpha\beta}=0. \tag{C21}$$

Defining

$$U=-\Lambda'(r)-\Lambda'(q)+2(A+p_0B)^2-B(2A+p_0B)+W+Z, \tag{C22}$$

$$V=-WZ+[-\Lambda'(r)+(A+p_0B)^2+W]\times[-\Lambda'(q)+(A+p_0B)^2-B(2A+p_0B)+Z], \tag{C23}$$

$$W=(n-2p_0)C(A+p_0B), \tag{C24}$$

$$Z=2(p_0-1)C(A+p_0B), \tag{C25}$$

the two eigenvalues are

$$\Lambda_{1,2}^{(3)}=\frac{U}{2}\left[1\pm\sqrt{1-4\frac{U}{V}}\right], \tag{C26}$$

$$n_{1,2}^{(3)}=\frac{n-p_0}{p_0}. \tag{C27}$$

The last subspace is set by the eigenvectors orthogonal to Eq. (C21):

$$\sum_\alpha \boldsymbol{C}_{\alpha\alpha}\neq 0, \quad \sum_{\alpha\neq\beta}\boldsymbol{C}_{\alpha\beta}\neq 0. \tag{C28}$$

Also in this case there are two different eigenvalues, whose expression is identical to Eq. (C26) provided that $U=2(n-p_0)C(A+p_0B)+n(n-p_0)(C^2+b^2)$. Their degeneracy is 1.

APPENDIX D: DYNAMICAL SOLUTION FOR THE 2+P SPHERICAL MODEL

In this appendix we show that the CHS dynamical solution of the spherical $2+p$ spin-glass model requires margin-

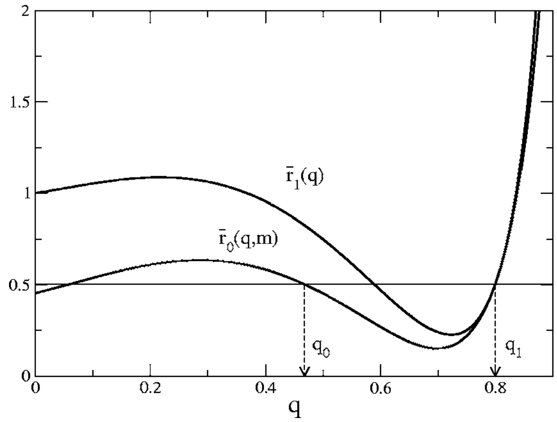

FIG. 5. Schematic form of $\bar{r}_1(q)$ and $\bar{r}_0(q,m)$ in the 1RSB phase. The horizontal line shows the value of $\bar{r}$. In the plot the slope of $\bar{r}_1(q)$ at q_1 is strictly positive, implying that the slope of $\bar{r}_0(q,m)$ at q_0 [the largest solution of the equation $\bar{r}_0(q,m)=\bar{r}$ below q_1] cannot be positive.

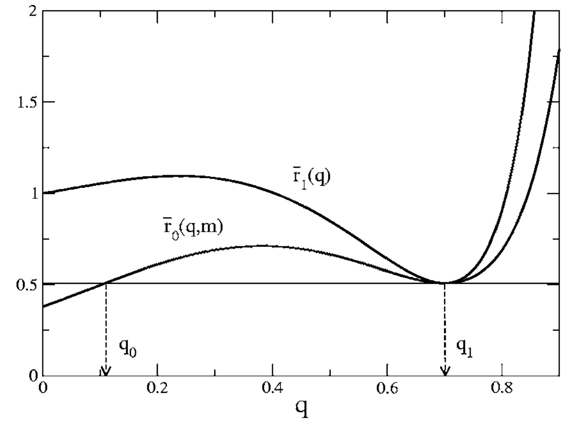

FIG. 6. Schematic form of $\bar{r}_1(q)$ and $\bar{r}_0(q,m)$ in the 1RSB phase. The horizontal line shows the value of $\bar{r}$. Here the slope of $\bar{r}_1(q)$ at q_1 is zero, implying a positive slope of the function $\bar{r}_0(q,m)$ at q_0.

ality of the dynamics in the intermediate time scales. To keep the notation simple and to refer to a physically well-known system, we shall consider the case of two time scales, appropriate for the 1RSB-type phase. With minor changes the derivation can be generalized to any number of time scales.

By inserting the forms (94) and (95) for the correlation and response function into the Dyson equation (17) and separating out the short time behavior $\omega \gg \epsilon$ as $\epsilon \to 0$ and the long time behavior $\omega = \epsilon \Omega$ as $\epsilon \to 0$, one obtains the following equations of motion for $G_1(\omega)$ and $G_0(\Omega)$:

$$\left(r - \frac{i\omega}{\Gamma_0}\right) G_1(\omega) - \Sigma_1(\omega) G_1(\omega) = 1, \tag{D1}$$

$$\left(\bar{r} + \Lambda(q_1) - \epsilon \frac{i\Omega}{\Gamma_0}\right) G_0(\Omega) - (1-q_1)\Sigma_0(\Omega) - \Sigma_0(\Omega) G_0(\Omega) = 0, \tag{D2}$$

where $\Sigma_1(\omega)$ and $\Sigma_0(\Omega)$ are the short and long time part of the self-energy $\Sigma(\omega)$, and

$$\bar{r} \equiv r - \Lambda(1) = -\Lambda(q_1) + \frac{1}{1-q_1} \tag{D3}$$

to ensure the correct static limit $\omega \to 0$ of Eq. (D1). The static limit $\Omega \to 0$ of Eq. (D2) gives the equation for q_0.

$$\Lambda(q_1) - \Lambda(q_0) = [\bar{r} + \Lambda(q_1)] \frac{q_1 - q_0}{1 - q_1 + m(q_1 - q_0)}. \tag{D4}$$

The parameter $\bar{r}$ can be eliminated from these equations with the help of the spherical constraint

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} C(\omega) = 2(1-q_1) - \bar{r}(1-q_1)^2 + 2m(1-q_1)(q_1-q_0)\Lambda(q_1) - m[\bar{r} + (1-m)\Lambda(q_1)](q_1-q_0)^2 = 1. \tag{D5}$$

One, then, recovers Eqs. (116) and (120) of the main text.

The equations for the correlation functions C_1 and C_0 are obtained from Eqs. (D1) and (D2) by using the relations:

$$G_1(\omega) = (1-q_1) + i\omega \int_0^\infty dt e^{i\omega t} C_1(t) = (1-q_1) + i\omega \hat{C}_1(\omega), \tag{D6}$$

$$G_0(\omega) = m(1-q_1) + mi\omega \int_0^\infty dt e^{i\omega t}[C_0(t) - q_0] = m(q_1 - q_0) + mi\omega \hat{C}_0(\omega), \tag{D7}$$

and

$$\Sigma_1(\omega) = \Lambda(1) - \Lambda(q_1) + i\omega \int_0^\infty dt e^{i\omega t}(\Lambda[C_1(t) + q_1] - \Lambda(q_1)) = \Lambda(1) - \Lambda(q_1) + i\omega \hat{\Lambda}_1(\omega), \tag{D8}$$

$$\Sigma_0(\omega) = m[\Lambda(q_1) - \Lambda(q_0)] + mi\omega \int_0^\infty dt e^{i\omega t}(\Lambda[C_0(t)] - \Lambda(q_0)) = m[\Lambda(1) - \Lambda(q_1)] + mi\omega \hat{\Lambda}_0(\omega) \tag{D9}$$

that follow from FDT. A simple algebra leads to the equations

$$\left(\bar{r} + \Lambda(q_1) - \frac{i\omega}{\Gamma_0}\right) \hat{C}_1(\omega) - \hat{\Lambda}_1(\omega) G_1(\omega) - \frac{1}{\Gamma_0}(1-q_1) = 0 \tag{D10}$$

and

$$-\frac{\epsilon}{\Gamma_0}[q_1-q_0+i\Omega\hat{C}_0(\Omega)] + \{\bar{r}+\Lambda(q_1)-m[\Lambda(q_1)-\Lambda(q_0)]\}\hat{C}_0(\Omega) - (1-q_1)\hat{\Lambda}_0(\Omega)-\hat{\Lambda}_0(\Omega)G_0(\Omega)=0. \quad \text{(D11)}$$

To study the stability of the static limits it is useful to rewrite these equation in the time space in the following equivalent form:

$$\Gamma_0^{-1}\partial_t C_1(t)+\{\bar{r}_1[C_1(t)+q_1]-\bar{r}\}[1-q_1-C_1(t)] + \int_0^t dt'\{\Lambda[C_1(t-t')+q_1]-\Lambda[C_1(t)+q_1]\}\partial_{t'}C_1(t')=0, \quad \text{(D12)}$$

$$\epsilon\Gamma_0^{-1}\partial_t C_0(t)+\{\bar{r}_0[C_0(t),m]-\bar{r}\}\{1-q_1+m[q_1-C_0(t)]\} + m\int_0^t dt'\{\Lambda[C_0(t-t')]-\Lambda[C_0(t)]\}\partial_{t'}C_0(t')=\delta, \quad \text{(D13)}$$

where $\delta=\lim_{t\to 0^+}\epsilon\Gamma_0^{-1}\partial_t C_0(t)$ and $\bar{r}_1(q)$ and $\bar{r}_0(q,m)$ are the functions

$$\bar{r}_1(q)=-\Lambda(q)+\frac{1}{1-q}, \quad \text{(D14)}$$

$$\bar{r}_0(q,m)=\bar{r}_1(q)-\frac{(1-m)(q_1-q)^2}{(1-q_1)(1-q)[1-q_1+m(q_1-q)]}. \quad \text{(D15)}$$

In terms of these equations the physical values of q_1,q_0 $(0\leqslant q_0\leqslant q_1\leqslant 1)$ yielding the *plateau* are the largest solution of the equation[6]

$$\bar{r}_1(q_1)=\bar{r}_0(q_0,m)=\bar{r} \quad \text{(D16)}$$

with $\bar{r}$ fixed by the spherical constraint.

Expanding Eqs. (D12) and (D13), near the plateau to the first order in the deviation one obtains the dynamic stability conditions

$$\left.\frac{\partial}{\partial q}\bar{r}_1(q)\right|_{q=q_1}\geqslant 0, \quad \text{(D17)}$$

$$\left.\frac{\partial}{\partial q}\bar{r}_0(q,m)\right|_{q=q_0}\geqslant 0. \quad \text{(D18)}$$

It is easy to check that these coincide with dynamical stability conditions (125) and (128) given in the main text.

In the 1RSB phase $\bar{r}_1(q)$ and $\bar{r}_0(q,m)$ have the shape depicted in Figs. 5 and 6, while $\bar{r}<1$.[6] A simple analysis of these figures shows that in order to satisfy Eq. (D18) to have $(\partial/\partial q)\bar{r}_0(q_0,m)>0$ it is necessary that

$$\left.\frac{\partial}{\partial q}\bar{r}_1(q)\right|_{q=q_1}=0, \quad \text{(D19)}$$

i.e., the solution at shorter time scales must be marginally stable.

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[30] We changed the enumeration with respect to Sompolinsky for convenience.

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