Ring-diagram summations and the self-energy of the homogeneous electron gas at its weak-correlation limit

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(Received 25 July 2006; published 5 February 2007)

The small-r_s asymptotics of the self-energy $\Sigma(k,\omega)$ of the homogeneous electron gas (HEG) is studied in terms of the Feynman diagrams involving the noninteracting one-body Green's function G_0 and the static bare Coulomb repulsion v_0. The lowest-order approximation to $\Sigma(k,\omega)$ is given by the product of G_0 and v_0. The nature of the proper ring-diagram summation (equivalent to the random-phase approximation) for $\Sigma(k,\omega)$ that affords the correct small-r_s single behavior of $r_s^2 \ln r_s$ is investigated. Reexamination of ring-diagram summations for several properties of the HEG proves in a rigorous manner that the product $G_0 v_{\mathrm{r}}$, where v_{r} is the ring-diagram-summed dynamically screened repulsion, yields the correct lowest-order asymptotics, whereas $G_{\mathrm{r}} v_0$, where G_{r} is the ring-diagram-summed Green's function, contributes only to higher-order terms.

DOI: 10.1103/PhysRevB.75.085103 PACS number(s): 71.10.Ca

I. INTRODUCTION

Although an artificial construct, the homogeneous electron gas (HEG) constitutes an important model system for electronic structure theory.[1] The ground state of the spin-unpolarized HEG is characterized by only one parameter, namely, the radius r_s of the Wigner-Seitz sphere that contains one electron on average.[2] This radius determines the Fermi wave number as $k_{\mathrm{F}}=(\alpha r_s)^{-1}$ {where $\alpha=[4/(9\pi)]^{1/3}$}, and measures simultaneously the interaction strength and the particle density; high density corresponding to weak interaction and hence weak correlation (for recent papers on this limit, see Refs. 3–6). One could naively expect that at the weak-correlation limit the bare Coulomb repulsion $v_0(q)=\alpha r_s/q^2$ (where momenta and energies are measured in units of k_{F} and k_{F}^2, respectively) can be treated as perturbation. However, already in his early work on the HEG, Heisenberg[7] showed that ordinary perturbation theory fails in this case. With $e_0=3/10$ being the energy per particle of the ideal Fermi gas and $e_{\mathrm{x}}=-\frac{3}{4}\alpha r_s/\pi$ being the exchange energy in the lowest (first) order, the total energy $e=e_0+e_{\mathrm{x}}+e_{\mathrm{c}}$ defines the correlation energy $e_{\mathrm{c}}=e_2+e_3+\cdots$, where $e_n \sim (\alpha r_s)^n$ [note that $\tilde{e}=k_{\mathrm{F}}^2 e=e/(\alpha r_s)^2$ gives the energy in atomic units]. In the second order, there is a direct term $e_{2\mathrm{d}}$ and an exchange term $e_{2\mathrm{x}}$ so that $e_2=e_{2\mathrm{d}}+e_{2\mathrm{x}}$. The direct term $e_{2\mathrm{d}}$ diverges logarithmically near the Fermi surface (i.e., for the vanishing transition momenta $q\to 0$, $e_{2\mathrm{d}}\to \ln q$). This failure of perturbation theory has been remedied by Macke[8] with an appropriate partial summation of higher-order terms up to an infinite order that describes screening effects and the collective mode plasmon with a cutoff momentum $q_c=\sqrt{4\alpha r_s/\pi}$. This ring-diagram summation, which is equivalent to the random-phase approximation (RPA), yields $e_{\mathrm{c}}=(\alpha r_s)^2(a \ln r_s+b+\cdots)$, where $a=(1-\ln 2)/\pi^2\approx 0.031\,091$, for the correlation energy at the weak-correlation limit. This result has been subsequently confirmed by Gell-Mann and Brueckner.[9] The logarithmic behavior of e_{c} at the weak-correlation limit carries over to its kinetic and potential components through the virial theorem[10]

$$t_{\mathrm{c}}=-r_s^2\frac{d}{dr_s}\frac{1}{r_s}e_{\mathrm{c}}=-(\alpha r_s)^2\left[a \ln r_s+(a+b)+\cdots\right],$$

$$v_{\mathrm{c}}=r_s\frac{d}{dr_s}e_{\mathrm{c}}=(\alpha r_s)^2\left[2a \ln r_s+(a+2b)+\cdots\right]. \quad (1.1)$$

Note that $t_0=e_0$, $t_{\mathrm{x}}=0$, $v_{\mathrm{x}}=e_{\mathrm{x}}$, and $e_{\mathrm{c}}=t_{\mathrm{c}}+v_{\mathrm{c}}$. It has been shown[5] that these small-r_s nonanalyticities result from the ring diagram summation for the momentum distribution $n(k)$[11–13] and for the static structure factor $S(q)$.[14] In the lowest order, $n(k)$ diverges near the Fermi surface, $n(k\to 1)\sim \mp(k-1)^{-2}$ for $k\lessgtr 1$, and $S(q)$ diverges for $q\to 0$, making $t_{2\mathrm{d}}$ and $v_{2\mathrm{d}}$ diverge correspondingly. The ring-diagram summations remove this unphysical behavior.[5,11,14] The chemical potential $\mu=\mu_0+\mu_{\mathrm{x}}+\mu_{\mathrm{c}}$, where $\mu_0=1/2$ and $\mu_{\mathrm{x}}=-\alpha r_s/\pi$, enters our considerations through the Seitz theorem,[15]

$$\mu_{\mathrm{c}}=\left(\frac{5}{3}-\frac{1}{3}r_s\frac{d}{dr_s}\right)e_c=(\alpha r_s)^2\left[a \ln r_s+\left(-\frac{a}{3}+b\right)+\cdots\right]. \quad (1.2)$$

In this paper, the small-r_s behavior of the self-energy $\Sigma(k,\omega)$ for $k=1$ and $\omega=1/2$ is investigated. Here and in the following, we use the term "small-r_s" with the meaning of "RPA in the lowest order," i.e., we derive and discuss only the terms containing $\ln r_s$ or those related to them. In particular, we determine which terms have to be included in the partial summation for $\Sigma(k,\omega)$ in order to ensure its correct small-r_s asymptotics. To achieve this objective, which sheds light on the mathematical complexity of a weakly correlated HEG and disproves some recent claims concerning $\Sigma(k,\omega)$, we employ several well-known theorems on the self-energy.

II. RIGOROUS THEOREMS INVOLVING THE SELF-ENERGY $\Sigma(k,\omega)$

The self-energy $\Sigma(k,\omega)$ is defined by

$$G = G_0 + G_0 \Sigma G,$$

$$G_0(k,\omega) = \frac{\theta(1-k)}{\omega - \frac{1}{2}k^2 - i\delta} + \frac{\theta(k-1)}{\omega - \frac{1}{2}k^2 + i\delta}, \quad \delta \to 0^+, \tag{2.1}$$

where G_0 and G are the Green's functions of the ideal Fermi gas and the HEG, respectively. Limiting the summation of the Feynman diagrams for $\Sigma(k,\omega)$ to those terms that afford correct results for $r_s \to 0$ allows one to apply several rigorous theorems, which yield (i) the condition for μ through the Luttinger theorem $\mathrm{Im}\,\Sigma(1,\mu)=0$,[16] (ii) the momentum distribution

$$n(k) = \int \frac{d\omega}{2\pi i} e^{i\omega\delta} G(k,\omega), \tag{2.2}$$

(iii) the quasiparticle weight (through the Luttinger-Ward formula[17])

$$z_{\mathrm{F}} = \frac{1}{1 - \mathrm{Re}\,\Sigma_{\mathrm{c}}'(1,\mu)}, \quad \Sigma_{\mathrm{c}}'(k,\omega) = \frac{\partial \Sigma_{\mathrm{c}}(k,\omega)}{\partial \omega}, \tag{2.3}$$

(iv) the potential energy (through the Galitskii-Migdal formula[18])

$$v = \frac{1}{2} \int d(k^3) \int \frac{d\omega}{2\pi i} e^{i\omega\delta} G(k,\omega)\Sigma(k,\omega). \tag{2.4}$$

Note that $\Sigma = \Sigma_{\mathrm{x}} + \Sigma_{\mathrm{c}}$, $\Sigma_{\mathrm{c}} = \Sigma_2 + \Sigma_3 + \cdots$, and $\Sigma_2 = \Sigma_{2\mathrm{d}} + \Sigma_{2\mathrm{x}}$. In the lowest order, one has $\Sigma_{\mathrm{x}} = G_0 v_0$ and $v_{\mathrm{x}} = G_0 \Sigma_{\mathrm{x}}$. With $v_0(q) = \alpha r_s / q^2$ [compare Eq. (A5)], this produces

$$\Sigma_{\mathrm{x}}(k) = -\left(1 + \frac{1-k^2}{2k} \ln \left| \frac{1+k}{1-k} \right| \right) \frac{\alpha r_s}{\pi}, \quad \Sigma_{\mathrm{x}}(1) = -\frac{\alpha r_s}{\pi},$$

$$v_{\mathrm{x}} = -\frac{3}{4} \frac{\alpha r_s}{\pi}. \tag{2.5}$$

Note that $\Sigma_{\mathrm{x}}(k)$ does not depend on ω. With $G_{\mathrm{c}} = G - G_0$, the correlation part of the potential energy reads

$$v_{\mathrm{c}} = (G_0 + G_{\mathrm{c}})\Sigma_{\mathrm{c}} + G_{\mathrm{c}}\Sigma_{\mathrm{x}} = G_0 \Sigma_{\mathrm{c}} + G_{\mathrm{c}}(\Sigma_{\mathrm{x}} + \Sigma_{\mathrm{c}}). \tag{2.6}$$

However, our main interest is in the Hugenholtz–van Hove (the Luttinger-Ward) theorem,[17,19,20]

$$\mu_{\mathrm{c}} = \Sigma_{\mathrm{c}}(1,\mu), \quad \mu = \mu_0 + \mu_{\mathrm{x}} + \mu_{\mathrm{c}}, \quad \mu_0 = \frac{1}{2},$$

$$\mu_{\mathrm{x}} = -\frac{\alpha r_s}{\pi}, \quad \mu_{\mathrm{c}} = (\alpha r_s)^2 a \ln r_s + \cdots. \tag{2.7}$$

The right-hand side (RHS) of the above equation depends on r_s through both $\Sigma_{\mathrm{c}}(k,\omega)$ and μ. At the limit of $r_s \to 0$, μ can be replaced by $\mu_0 = 1/2$.

The ring-diagram summation is equivalent to setting $v_{\mathrm{r}} = v_0 + v_0 Q v_{\mathrm{r}}$, where $Q(q,\omega)$ is the polarization propagator [in the lowest order, see Eq. (A1) in the Appendix]. For the self-energy, this means that $\Sigma^{\mathrm{r}} = G_0 v_{\mathrm{r}}$. It is easy to show that employing the correlation part $\Sigma_{\mathrm{c}}^{\mathrm{r}} = G_0(v_{\mathrm{r}} - v_0)$ of Σ^{r} in conjunction with Eqs. (2.2) and (2.4) results in the RPA approximations for $n_{\mathrm{c}}^{\mathrm{r}}(k)$,[5,11] and $v_{\mathrm{c}}^{\mathrm{r}}$,[5,14] respectively.

In the following, we show that $\Sigma_{\mathrm{c}}^{\mathrm{r}}$ is also the proper RHS for Eqs. (2.3) and (2.7) at the limit of $r_s \to 0$, the remainder $\Sigma_{\mathrm{c}}^{\mathrm{nr}} = \Sigma_{\mathrm{c}} - \Sigma_{\mathrm{c}}^{\mathrm{r}}$ contributing only to the higher-order terms. We also investigate whether $\Sigma_{\mathrm{c}}^{\mathrm{HF}} = (G - G_0)v_0$, which appears in Ref. 22, is an alternative candidate for the RHS of Eq. (2.7). In fact, we find that the "remainder" $\Sigma_{\mathrm{c}}^{\mathrm{nHF}} = \Sigma_{\mathrm{c}} - \Sigma_{\mathrm{c}}^{\mathrm{HF}} = \Sigma_{\mathrm{c}}^{\mathrm{r}} + \cdots$ determines the lowest-order term and $\Sigma_{\mathrm{c}}^{\mathrm{HF}}$ contributes only to the higher-order ones, thus contradicting the conjecture that $\Sigma_{\mathrm{c}}^{\mathrm{nHF}}(1,\mu) = 0$.

III. THE RING-DIAGRAM SELF-ENERGY $\Sigma_{\mathrm{c}}^{\mathrm{r}}(k,\omega)$

According to the diagram rules, the ring-diagram-summed self-energy is given by

$$\Sigma_{\mathrm{c}}^{\mathrm{r}}(k,\omega) = (\alpha r_s)^2 \frac{2}{\pi^3} \int \frac{d^3q}{q^2} \int \frac{d\eta}{2\pi i} \frac{Q(q,\eta)}{q^2 + q_c^2 Q(q,\eta)} \times \left(\frac{\theta(|\boldsymbol{k}+\boldsymbol{q}|-1)}{\omega + \eta - \frac{1}{2}k^2 - \boldsymbol{q}\cdot\left(\boldsymbol{k} + \frac{1}{2}\boldsymbol{q}\right) + i\delta} + \frac{\theta(1-|\boldsymbol{k}+\boldsymbol{q}|)}{\omega + \eta - \frac{1}{2}k^2 - \boldsymbol{q}\cdot\left(\boldsymbol{k} + \frac{1}{2}\boldsymbol{q}\right) - i\delta} \right). \tag{3.1}$$

If in the above equation the term $q_c^2 Q(q,\eta)$, which describes the RPA screening of the bare Coulomb repulsion of $\alpha r_s / q^2$, is deleted, $\Sigma_{\mathrm{c}}^{\mathrm{r}}(k,\omega)$ simplifies to $\Sigma_{2\mathrm{d}}(k,\omega)$. Whereas $\Sigma_{2\mathrm{d}} = \mathrm{Re}\,\Sigma_{2\mathrm{d}}(1,1/2)$ diverges with an artificial cutoff q_0 according to $(\alpha r_s)^2 \int_{q_0} dq/q$, the ring-diagram sum $\Sigma_{\mathrm{c}}^{\mathrm{r}} = \mathrm{Re}\,\Sigma_{\mathrm{c}}^{\mathrm{r}}(1,1/2)$ is nondivergent, as it effectively replaces q_0 by the "natural" cutoff $q_{\mathrm{c}} \sim \sqrt{r_s}$, producing $\Sigma_{\mathrm{c}}^{\mathrm{r}} \sim (\alpha r_s)^2 \ln r_s$. We follow the procedure of Gell-Mann and Brueckner for the correlation energy.[9] Upon the substitution $\eta = iqu$ and contour deformation from the real to the imaginary axis, one arrives at

$$\Sigma_{\mathrm{c}}^{\mathrm{r}} = -\frac{(\alpha r_s)^2}{\pi^4} \int_0^\infty du \int \frac{d^3q}{q^2} \frac{R(q,u)}{q^2 + q_c^2 R(q,u)} \frac{2(x+q/2)}{u^2 + (x+q/2)^2}$$
$$= -(\alpha r_s)^2 \frac{2}{\pi^3} \int_0^\infty du \int_0^\infty dq \frac{R(q,u)}{q^2 + q_c^2 R(q,u)} \times \ln \frac{u^2 + (q/2+1)^2}{u^2 + (q/2-1)^2}. \tag{3.2}$$

The asymptotic behavior for $r_s \to 0$ is determined by the lower integration limit of $q \to 0$, which allows for the approximate replacements of $R(q,u)$ with $R_0(u)$ [setting $R_0(u) \neq 0$ makes the Coulomb repulsion effectively screened] and $\ln[\cdots]$ with $2q/(1+u^2)$ that yield

$$\Sigma_c^r = (\alpha r_s)^2 \left[\left(\frac{2}{\pi^3} \int_0^\infty du \frac{R_0(u)}{1+u^2} \right) \ln r_s + \text{const} + \cdots \right]. \tag{3.3}$$

[see Eq. (A4) in the Appendix for the integral]. The resulting $\Sigma_c^r = (\alpha r_s)^2 (a \ln r_s + \text{const} + \cdots)$ is in full agreement with the LHS of the Hugenholtz–van Hove theorem [Eq. (2.7)].

The frequency derivative $\Sigma_c^{r'} = \Sigma_c^{r'}(1,1/2)$ can be treated similarly,[21]

$$\begin{aligned}\Sigma_c^{r\prime} &= \frac{(\alpha r_s)^2}{\pi^4} \int \frac{d^3q}{q^3} \int du \frac{R(q,u)}{q^2 + q_c^2 R(q,u)} \frac{\partial}{\partial u} \frac{u}{u^2 + (x+q/2)^2} \\ &= -\frac{(\alpha r_s)^2}{\pi^4} \int \frac{d^3q}{q^3} \int du \frac{R(q,u)}{q^2 + q_c^2 R(q,u)} \\ &\quad \times \frac{\partial}{\partial u} \frac{1}{2} \left(\arctan \frac{1+q/2}{u} + \arctan \frac{1-q/2}{u} \right)_\delta . \end{aligned} \tag{3.4}$$

Note that a thin layer of a vanishing thickness δ has to be deleted along $|\boldsymbol{e}+\boldsymbol{q}|$, making integration by parts possible. The small-q replacements $R(q,u)$ with $R_0(u)$ and $\arctan(1 \pm q/2)/u$ with $\arctan 1/u$ yield

$$\Sigma_c^{r\prime} = \left(\frac{\alpha}{\pi^2} \int_0^\infty du \frac{R_0'(u)}{R_0(u)} \arctan \frac{1}{u} \right) r_s + O(r_s^2) \tag{3.5}$$

[see Eq. (A4) in the Appendix for the integral]. Combining this equation with Eq. (2.3) affords the well-known RPA result of $z_F = (1 - \Sigma_c^{r'})^{-1} = 1 + \Sigma_c^{r'} + \cdots = 1 - 0.18 r_s + \cdots$.[11]

IV. THE HARTEE-FOCK SELF-ENERGY $\Sigma_c^{HF}(k)$

Since the bare Coulomb repulsion $v_0(q)$ is a static one, the Hartree-Fock (HF) self-energy $\Sigma_c^{HF} = (G - G_0) v_0$ is given by the momentum distribution $n(\boldsymbol{k})$ alone,[22]

$$\Sigma_c^{HF}(k) = \frac{\alpha r_s}{\pi} \frac{1}{k} \int_0^\infty dk' k' \ln \left| \frac{k-k'}{k+k'} \right| n_c(k'),$$

$$n_c(k) = n(k) - \theta(1-k). \tag{4.1}$$

In the above equation, the factor in front of $n_c(k)$ arises from the Coulomb repulsion. Because $\Sigma_c^{HF}(k)$ does not depend on ω, it cannot contribute to the deviations of $n(k)$ from $\theta(1-k)$ and of z_F from 1 according to Eqs. (2.2) and (2.3). Such deviations are caused by the non-HF part $\Sigma_c^{nHF} = \Sigma_c - \Sigma_c^{HF} = \Sigma_c^r + \cdots$. For $n(k)$ set to $\theta(1-k)$, the Galitskii-Migdal formula [Eq. (2.4)] yields the lowest-order exchange energy $v_x = -\frac{3}{4} \alpha r_s / \pi$, whereas for the actual $n(k)$ it produces the full exchange or Fock energy,

$$v_F = -\frac{3}{2} \frac{\alpha r_s}{\pi} \int_0^\infty dk \int_0^\infty dk' n(k) n(k') k k' \ln \left| \frac{k+k'}{k-k'} \right|, \tag{4.2}$$

which constitutes only one component of the exact potential energy v [see Eq. (43) of Ref. 12]. Consider the (dimensionless) pair density $g(r)$ and its cumulant partitioning $g(r) = 1 - \frac{1}{2} f^2(r) - h(r)^3$,[3] where $f(r)$ is the (dimensionless) one-body reduced density matrix [i.e., the Fourier transform of $n(k)$] and $h(r)$ is the cumulant pair density [i.e., the diagonal part of the cumulant (nonreducible) two-body density matrix]. The potential energy $v = v_F + v_{cum}$ follows from the full pair density $g(r)$. The Hartree term $g_0(r) = 1$ is compensated by the positive background, whereas $g_x(r) = -\frac{1}{2} f^2(r)$ and $g_{cum}(r) = -h(r)$ give rise to v_F of Eq. (4.2) and v_{cum}, respectively. Consequently, the knowledge of the non-HF part $\Sigma_c^{nHF} = \Sigma_c^r + \cdots$ is essential for proper evaluation of $n(k)$, z_F, and v. One may enquire whether it is nevertheless possible to employ the expression (4.1) in Eq. (2.7). Within perturbation theory, the leading term of $n_c(k)$ is proportional to r_s^2, requiring that $\Sigma_c^{HF}(1) \sim r_s^3$, which contradicts the scaling of $\mu_c \sim r_s^2$. The following analysis demonstrates that this contradiction remains after the ring-diagram summation, which turns out to yield, respectively, $r_s^2 \ln r_s$ and $r_s^3 (\ln r_s)^2$ as the leading terms for the LHS and RHS of Eq. (2.7).

Because of the availability of exact $n_c(k)$,[11] the RHS of

$$\Sigma_c^{HF}(1) = \frac{\alpha r_s}{\pi} I, \quad I = \int_0^\infty dk\, n_c(k) f(k), \quad f(k) = k \ln \left| \frac{1-k}{1+k} \right|, \tag{4.3}$$

can be readily computed at the weak-correlation limit of $r_s \to 0$. In the following, the approach previously employed in relating the small-r_s nonanalyticities of t_c and v_c to the peculiarities of $n_c(k)$ and the static structure factor $S_c(q)$ at the limit of $r_s \to 0$ (Ref. 5) is used.

The small-r_s behavior of the RHS of Eq. (4.1) is determined by the behavior of $n_c(k)$ near the Fermi surface. As shown by Daniel, Vosko, and Kulik,[11] and reiterated in later works,[4,5] two functions are needed to describe this behavior, namely, $F(k)$ with the properties

$$F(k \to 0) = 4.112\,34 + O(k^2),$$

$$F(k \to \infty) = \frac{8\pi^2}{9} \frac{1}{k^8} + O\left(\frac{1}{k^{10}}\right),$$

$$F(k \to 1) = \frac{\pi^2}{3} \frac{1 - \ln 2}{k^2 (1-k)^2}, \tag{4.4}$$

and $G(x)$ with the asymptotics

$$G(0) = 3.353\,34, \quad G(x \gg 1) = \frac{\pi}{6} \frac{1 - \ln 2}{x^2} + O\left(\frac{1}{x^4}\right). \tag{4.5}$$

Near $k=1$, $n_c(k)$ is given by

$$n_c(k) = \left(\frac{q_c^2}{4\pi}\right)^2 \times \begin{cases} -F(k), & 0<k<1-\xi, \\ -\frac{2\pi}{q_c^2}\frac{1}{k^2}G\left(\frac{1-k}{q_c}\right), & 1-\xi<k<1, \\ +\frac{2\pi}{q_c^2}\frac{1}{k^2}G\left(\frac{k-1}{q_c}\right), & 1<k<1+\xi, \\ +F(k), & 1+\xi<k, \end{cases} \quad (4.6)$$

where $1 \gg \xi \gg q_c$.[5] The function $F(k)$ contributes to $I = I_F + I_G$ through the expression

$$I_F = I_F^> + I_F^<,$$

$$I_F^> = \left(\frac{q_c^2}{4\pi}\right)^2 \int_{1+\xi}^{\infty} dk\, F(k) f(k),$$

$$I_F^< = -\left(\frac{q_c^2}{4\pi}\right)^2 \int_0^{1-\xi} dk\, F(k) f(k). \quad (4.7)$$

With a fixed positive number A sufficiently small to assure that $F(k)$ can be replaced by its asymptotics (4.4), one obtains

$$I_F^> \approx \left(\frac{q_c^2}{4\pi}\right)^2 \left(\int_{1+A}^{\infty} dk\, F(k) f(k) + \frac{\pi^2}{3}(1-\ln 2)\int_{1+\xi}^{1+A} dk \frac{f(k)}{k^2(1-k)^2}\right)$$
$$= O(r_s^2) + q_c^4 \frac{1-\ln 2}{48}\int_{\xi}^{A} dk \frac{f(1+k)}{(1+k)^2 k^2}. \quad (4.8)$$

The result for $I_F^<$ is similar, the above integrand being replaced by $-f(1-k)/(1-k)^2 k^2$. Therefore

$$I_F \approx O(r_s^2) + q_c^4 \frac{1-\ln 2}{48}\int_{\xi}^{A} \frac{dk}{k^2} w(k),$$

$$w(k) = \frac{f(1+k)}{(1+k)^2} - \frac{f(1-k)}{(1-k)^2}. \quad (4.9)$$

The contribution of $G(x)$ to $I = I_F + I_G$ is treated analogously,

$$I_G = I_G^> + I_G^<,$$

$$I_G^> = \frac{q_c^2}{8\pi}\int_1^{1+\xi} dk \frac{1}{k^2} G\left(\frac{|k-1|}{q_c}\right) f(k),$$

$$I_G^< = -\frac{q_c^2}{8\pi}\int_{1-\xi}^{1} dk \frac{1}{k^2} G\left(\frac{|k-1|}{q_c}\right) f(k). \quad (4.10)$$

With a fixed positive number B sufficiently large to assure that $G(x)$ can be replaced by its asymptotics (4.5), it follows that

$$I_G^> = \frac{q_c^2}{8\pi}\left(\int_1^{1+q_c B} + \int_{1+q_c B}^{1+\xi}\right)\frac{dk}{k^2} G\left(\frac{|k-1|}{q_c}\right) f(k)$$
$$\approx \frac{q_c^3}{8\pi}\int_0^B dx\, G(x) \frac{f(1+q_c x)}{(1+q_c x)^2} + q_c^4 \frac{1-\ln 2}{48}\int_{q_c B}^{\xi} \frac{dk}{k^2}\frac{f(1+k)}{(1+k)^2}. \quad (4.11)$$

The result for $I_G^<$ is similar, the respective parts of the first and second integrands being replaced by $-f(1-q_c x)/(1-q_c x)^2$ and $-f(1-k)/(1-k)^2$. Therefore,

$$I_G \approx \frac{q_c^3}{8\pi}\int_0^B dx\, G(x) w(q_c x) + q_c^4 \frac{1-\ln 2}{48}\int_{q_c B}^{\xi} \frac{dk}{k^2} w(k). \quad (4.12)$$

Combining the above estimates, one obtains

$$I \approx O(r_s^2) + \frac{q_c^3}{8\pi}\int_0^B dx\, G(x) w(q_c x) + q_c^4 \frac{1-\ln 2}{48}\int_{q_c B}^{A} \frac{dk}{k^2} w(k). \quad (4.13)$$

Since for a sufficiently small positive k

$$w(k) = \frac{1}{1+k}\ln\left|\frac{k}{2+k}\right| - \frac{1}{1-k}\ln\left|\frac{k}{1-k}\right| \approx -2k \ln k, \quad (4.14)$$

the integrals of Eq. (4.13) yield the leading terms of

$$\frac{q_c^3}{8\pi}\int_0^B dx\, G(x)(-2q_c x)\ln(q_c x)$$
$$= -q_c^4 \frac{1-\ln 2}{24}\left(\ln q_c \ln B + C_0 \ln q_c + \frac{1}{2}(\ln B)^2\right), \quad (4.15)$$

where the constant C_0 does not depend on B, and

$$q_c^4 \frac{1-\ln 2}{48}\int_{q_c B}^{A} \frac{dk}{k^2}(-2k)\ln k$$
$$= q_c^4 \frac{1-\ln 2}{48}[(\ln q_c + \ln B)^2 - (\ln A)^2] \quad (4.16)$$

(note the cancellation of the terms dependent on B in the combined integrals). Thus $\Sigma_c^{\mathrm{HF}}(1) = (\alpha r_s/\pi)^3[(1-\ln 2)/12] \times [(\ln r_s)^2 - 4C_0 \ln r_s] + \cdots$, which clearly demonstrates that for $r_s \to 0$ the non-HF term $\Sigma_c^{\mathrm{nHF}}(1,1/2) = \Sigma_c^{\mathrm{r}}(1,1/2) + \cdots$ has to be used in the RHS of Eq. (2.7). In summary, the terms that correctly describe the small-r_s behavior are contained in $\Sigma_c^{\mathrm{nHF}}(k,\omega) = \Sigma_c^{\mathrm{r}}(k,\omega) + \cdots$.[23] However, together with $v_F - v_x$, $\Sigma_c^{\mathrm{HF}}(1)$ can serve as a measure of the correlation strength; see also Refs. 12 and 24 for entropy measures of electron correlation.

V. CONCLUSIONS

The correct small-r_s behavior of the correlation contribution $\Sigma_c(k,\omega)$ to the self-energy is given by the ring-diagram-summed $\Sigma_c^r(k,\omega)$. The summation eliminates the divergence of $\Sigma_{2d}(1,1/2) \sim r_s^2 \int_0 dq/q$ and of $n_{2d}(k)$ at the Fermi surface. Upon application of the Galitskii-Midgal formula, the correct potential energy $v_c = 2a(\alpha r_s)^2 \ln r_s + \cdots$ results. The derivative $\partial\Sigma_c^r(1,\omega)/\partial\omega|_{\omega=1/2}$ used in conjunction with the Luttinger-Ward formula affords the correct $z_F = 1 - 0.18 r_s + \cdots$ for $r_s \to 0$. Finally, $\Sigma_c^r(1,1/2) = (\alpha r_s)^2(a \ln r_s + \text{const} + \cdots)$ is in full agreement with the Hugenholtz–van Hove formula $\mu_c = \Sigma_c(1,\mu)$ with $\mu \to 1/2$ at the limit of $r_s \to 0$, contradicting the previously published conjecture that $\Sigma_c^{\text{nHF}}(1,\mu) = 0$.[22]

ACKNOWLEDGMENTS

The research described in this publication has been supported by Komitet Badan Naukowych under the Grant No. 4 T09A 031 25. One of the authors (P.Z.) acknowledges P. Fulde for support of this work and the other (J.C.) extends his thanks to MPI PKS Dresden for hospitality and financial support.

APPENDIX: THE POLARIZATION PROPAGATOR

In the lowest order, the polarization propagator is given by

$$Q(q,\eta) = \int \frac{d^3k}{4\pi}\left[\frac{1}{\boldsymbol{q}\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\right) - \eta - i\delta} + \frac{1}{\boldsymbol{q}\left(\boldsymbol{k}+\frac{1}{2}\boldsymbol{q}\right) + \eta - i\delta}\right]\theta(1-k)\,\theta(|\boldsymbol{k}+\boldsymbol{q}|-1). \tag{A1}$$

For $\eta = iqu$, the real function $R(q,u)$ arises,[11]

$$R(q,u) = Q(q,iqu) = \frac{1}{2}\left[1 + \frac{1+u^2-q^2/4}{2q}\ln\frac{(q/2+1)^2+u^2}{(q/2-1)^2+u^2} - u\left(\arctan\frac{1+q/2}{u} + \arctan\frac{1-q/2}{u}\right)\right], \tag{A2}$$

which is even in u. The function $R(q,u)$ has the small-q expansion $R(q,u) = R_0(u) + O(q^2)$ with

$$R_0(u) = 1 - u\arctan\frac{1}{u}. \tag{A3}$$

The integrals

$$\int_0^\infty du \frac{R_0(u)}{1+u^2} = \frac{\pi}{2}(1-\ln 2) \approx 0.482\,003$$

and

$$\int_0^\infty du \frac{R_0'(u)}{R_0(u)}\arctan\frac{1}{u} \approx -3.353\,337 \tag{A4}$$

appear in Sec. II of this paper. The integrals

$$\int_0^1 dk' k' \ln\left|\frac{k+k'}{k-k'}\right| = k + \frac{1-k^2}{2}\ln\left|\frac{1+k}{1-k}\right|$$

and

$$\int_0^1 dk \int_0^1 dk' kk' \ln\left|\frac{k+k'}{k-k'}\right| = \frac{1}{2} \tag{A5}$$

appear in Secs. I and III.

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