PHYSICAL REVIEW B **75**, 075318 (2007)

Generalized quantum Hall projection Hamiltonians

Steven H. Simon
Alcatel-Lucent, Bell Labs, Murray Hill, New Jersey 07974, USA

E. H. Rezayi
Department of Physics, California State University, Los Angeles, California 90032, USA

Nigel R. Cooper
T. C. M. Group, Cavendish Laboratory, J. J. Thomson Avenue, Cambridge, CB3 0HE, United Kingdom
(Received 17 August 2006; revised manuscript received 23 October 2006; published 12 February 2007)

Certain well known quantum Hall states—including the Laughlin states, the Moore-Read Pfaffian, and the Read-Rezayi Parafermion states—can be defined as the unique lowest degree symmetric analytic function that vanishes as at least p powers as some number $(g+1)$ of particles approach the same point. Analogously, these same quantum Hall states can be generated as the exact highest density zero energy state of simple angular momentum projection operators. Following this theme we determine the highest density zero energy state for many other values of p and g.

DOI: 10.1103/PhysRevB.75.075318 PACS number(s): 73.43.−f

I. INTRODUCTION

For two dimensional electron systems in very high magnetic fields, the kinetic energy becomes fully quenched, electrons become restricted to the lowest Landau level (LLL), and the effective Hamiltonian is reduced to the potential energy of the electron-electron interaction.[1] While naive intuition might suggest that a Hamiltonian with only a potential energy would result in a crystalline ground state, the analytic structure of the lowest Landau level puts enormous restrictions on the type of wave functions that can exist. It is this structure that is responsible for all the richness of the fractional quantum Hall effect.

In Laughlin's original explanation of the fractional quantum Hall effect,[1] he noticed that, due to the LLL analytic structure, his trial state could be completely defined by stating that the many particle wave function must vanish as a particular power of the distance between two electrons. In particular, for the Laughlin $\nu=1/m$ state, the wave function vanishes as $(z_1-z_2)^m$ as particles with position z_1 and z_2 approach each other. The highest density wave function with this property is precisely the Laughlin state. It was discovered soon thereafter that these Laughlin wave functions were in fact the exact unique highest density zero-energy ground state of particles interacting with particularly simple short range model potentials[2,3] that amount to projection Hamiltonians. In this paper, we focus on these two related issues—the manner in which wave functions vanish, and the existence of simple model projection Hamiltonians.

To be more explicit, let us define L_2 to be the relative angular momentum of two particles. For electrons (which are fermions), L_2 must always be odd and the minimum value of L_2 in the LLL is given by $L_2^{\min}=1$. For bosons in a magnetic field (or rotating Bose gases, which can be mapped to bosons in a magnetic field[4]), L_2 must be even and $L_2^{\min}=0$. We can then define a projection operator P_2^p to project out any state where any two particles have relative angular momentum less than $L_2^{\min}+p$. In the lowest Landau level, this projection operator is precisely the above-mentioned Hamiltonian that gives the Laughlin $\nu=1/(L_2^{\min}+p)$ state as its ground state when p is even. In other words, this projection operator, when used as a Hamiltonian, gives positive energy to any situation where the wave function vanishes as $(z_1-z_2)^m$ with $m<L_2^{\min}+p$, leaving the Laughlin state as the unique highest density zero energy (ground) state. Note that for p odd, the wave function cannot vanish as p powers, so P_2^p has the same effect as P_2^{p+1} in that both forbid relative angular momentum of $p-1$ or less.

Another very interesting set of trial wave functions have also been studied that follow very much in this spirit. The Read-Rezayi Z_g parafermionic wave functions[5] are the unique exact highest density zero energy (ground) state of simple $(g+1)$ body interactions. Correspondingly, these wave functions can be completely defined by specifying the manner in which the wave functions vanish as $g+1$ particles come to the same point. The Moore-Read Pfaffian[6] state, which is thought to be the ground state wave function for the observed $\nu=5/2$ plateau,[7] is precisely the $g=2$ member of this series. In addition, the particle hole conjugate of the $g=3$ Read-Rezayi state has been proposed to be a candidate for the observed $\nu=12/5$ fractional quantum Hall state.[8] Finally, we note that the $g=1$ element of this series is just the Laughlin state with $p=1$ or $p=2$.

Analogously to our above construction for the Laughlin series, we may define L_{g+1} to be the relative angular momentum of a cluster of $g+1$ particles. It can be shown (and we will show below) that for electrons in the LLL, the minimal value of L_{g+1} is given by $L_{g+1}^{\min}=g(g+1)/2$. For bosons, the minimal value would be $L_{g+1}^{\min}=0$. Symmetry dictates (as shown in Appendix A) that $L_{g+1}=L_{g+1}^{\min}+1$ cannot occur, although any other value of $L_{g+1}\geqslant L_{g+1}^{\min}$ can occur for $g>1$ (and L_2 must be even or odd for bosons or fermions, respectively). Again we define P_{g+1}^p to be a projection operator that projects out any state where any cluster of $g+1$ particles has relative angular momentum $L_{g+1}<L_{g+1}^{\min}+p$. The Read-Rezayi state can then be obtained[5] from using the projection opera-

TABLE I. Highest density zero energy ground state of bosons with Hamiltonian P^p_{g+1}. The entries in this table are "name of state" followed by filling fraction. Abbreviations are P=Pfaffian; G=Gaffnian; H=Haffnian; J^n=Jastrow Factor to the n^{th} power; R_n=Z_n Read-Rezayi state. So for example, the g=2, p=9 slot has $G\ J^2{:}\frac{2}{7}$ which means the wave function is the Gaffnian times 2 Jastrow factors which occurs at filling fraction 2/7. Note that Laughlin states are listed only as J^n. An asterisk indicates that the state is "marginal" in that there are other states competing with this state that differ at most by a finite shift. For fermions the structure of the table would be identical except that the filling fractions would be related to these bosonic filling fractions by Eq. (9).

	$p=1,2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$	$p=10$	$p=11$	$p=12$	$p=13$	…
$g=1$	$J^2{:}\frac{1}{2}$	$J^4{:}\frac{1}{4}$	$J^4{:}\frac{1}{4}$	$J^6{:}\frac{1}{6}$	$J^6{:}\frac{1}{6}$	$J^8{:}\frac{1}{8}$	$J^8{:}\frac{1}{8}$	$J^{10}{:}\frac{1}{10}$	$J^{10}{:}\frac{1}{10}$	$J^{12}{:}\frac{1}{12}$	$J^{12}{:}\frac{1}{12}$	$J^{14}{:}\frac{1}{14}$	
$g=2$	$P{:}1$	$G{:}\frac{2}{3}$	$^*H{:}\frac{1}{2}$	$J^2{:}\frac{1}{2}$	$J^2{:}\frac{1}{2}$	$PJ^2{:}\frac{1}{3}$	$PJ^2{:}\frac{1}{3}$	$GJ^2{:}\frac{2}{7}$	$^*HJ^2{:}\frac{1}{4}$	$J^4{:}\frac{1}{4}$	$J^4{:}\frac{1}{4}$	$PJ^4{:}\frac{1}{5}$	
$g=3$	$R_3{:}\frac{3}{2}$	$^*P{:}1$	$P{:}1$										
$g=4$	$R_4{:}2$	$R_3{:}\frac{3}{2}$	$R_3{:}\frac{3}{2}$										
$g=5$	$R_5{:}\frac{5}{2}$	$R_4{:}2$	$R_4{:}2$										
$g=6$	$R_6{:}3$	$R_5{:}\frac{5}{2}$	$R_5{:}\frac{5}{2}$										
⋮	⋮	⋮	⋮										

tor P^2_{g+1} as a Hamiltonian in the lowest Landau level. (Note that since $L_{g+1}=L^{\min}_{g+1}+1$ is not allowed, the effect of P^1_{g+1} and P^2_{g+1} are both the same in that they give nonzero energy to states where any cluster has relative angular momentum $L_{g+1}=L^{\min}_{g+1}$.) In this work we will consider the obvious generalization of the Read-Rezayi construction, taking the Hamiltonian in the LLL to be given by the projection operator P^p_{g+1} for general g and p.

The general restriction that the minimum relative angular momentum of $g+1$ particles be $L_{g+1}\geq L^{\min}_{g+1}+p$ can be expressed in terms of how the wave function vanishes as $g+1$ particles approach each other. For bosons, where $L^{\min}_{g+1}=0$, the wave function does not need to vanish as g particles approach a given position $\tilde{z}$ but as the $(g+1)$st particle arrives, the wave function must vanish as $(z_{g+1}-\tilde{z})^p$. The situation for fermions, however, is a bit more complicated, and will be discussed in Sec. IV below.

The purpose of this paper is to determine the highest density zero energy state of the proposed Hamiltonian P^p_{g+1} which is a natural generalization of the Laughlin, Moore-Read, Read-Rezayi, Haffnian, and Gaffnian Hamiltonians. While we will not find a solution for arbitrary g and p, we will be able to find a solution for many values of g and p that have not been previously discussed. We note that in addition to the Laughlin states (g=1 with any p) and the Read-Rezayi states (p=1 or p=2 with any g), the ground state of the g=2 and p=4 case, know as the "Haffnian" has been previously discussed by Green.[9] In addition, the ground state of g=2 and p=3 has been dubbed the "Gaffnian," and is discussed in depth in a companion paper by the current authors.[10] (The name "Gaffnian" is an alphaphonetic interpolation between the p=2 pFaffian and the p=4 Haffnian).

The outline of this paper is as follows. We begin by fixing notations and conventions in Sec. I A. In Sec. II we define the concept of a "proper" cluster wave function which is crucial to our arguments. Through much of this paper we focus on boson wave functions. In Sec. III we start filling out a table as to the highest density ground state of the Hamiltonian P^p_{g+1}. Although we do not fill in all possible values of p and g, we do determine quite a few (results are given in Table I). In Sec. IV we discuss attaching Jastrow factors to the resulting wave functions, and in particular the fermionic analog of these wave functions. We find that the structure of the table for fermions and bosons is identical.

Preliminaries. We will always represent a particle's coordinate as an analytic variable z. On the plane $z=x+iy$ is simply the complex representation of the particle position $\mathbf{r}$. On the sphere, z is the stereographic projection of the position on the sphere of radius R to the plane. All distances will be measured in units of the magnetic length. In the symmetric gauge, the single particle lowest Landau level wave functions $\varphi(\mathbf{r})$ are given as analytic functions $\psi(z)$ times a measure $\mu(\mathbf{r})$:

$$\varphi(\mathbf{r}) = \mu(\mathbf{r})\psi(z). \tag{1}$$

On the disk the measure is[1]

$$\mu(\mathbf{r}) = e^{-|z|^2/4}, \tag{2}$$

whereas on the sphere (with stereographic projection) the measure is[5]

$$\mu(\mathbf{r}) = \frac{1}{[1+|z|^2/(4R^2)]^{1+N_\phi/2}} \tag{3}$$

with N_ϕ being the total number of flux penetrating the sphere. On the sphere the degree of the polynomial $\psi(z)$ ranges from z^0 to z^{N_ϕ} giving a complete basis of the $N_\phi+1$

states of the LLL. On the disk, the degree of ψ can be arbitrary.

We will write multiparticle wave functions Ψ for N particles in the lowest Landau level as an analytic functions ψ of N variables times the measure μ

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \psi(z_1, \ldots, z_N)\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{4}$$

with

$$\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i=1}^{N} \mu(\mathbf{r}_i). \tag{5}$$

On the sphere, the polynomial ψ cannot be of degree greater than N_ϕ in any variable z_i.

A quantum Hall ground state wave function will be a translationally, rotationally invariant quantum liquid. The restriction we impose on ψ is that it must be a translationally invariant homogeneous polynomial of degree N_ϕ. On the sphere, the degree N_ϕ is just the number of flux through the sphere. Conversely, given a (translationally and rotationally invariant) quantum Hall wave function on a sphere, the flux N_ϕ can be identified as the highest power of z_i that occurs. We note that so long as our interaction in the lowest Landau level is time reversal invariant, we can (and will) choose the the polynomial ψ with real coefficients of all terms. As the size of a system is extrapolated to the thermodynamic limit, we have the relation

$$N_\phi = \frac{1}{\nu}N - \mathcal{S} \tag{6}$$

with ν the filling fraction, and $\mathcal{S}$ is known as the "shift." We note that on a torus geometry there is typically no shift.[11]

For a bosonic wave function ψ must be symmetric in its arguments, whereas for a fermionic wave function it must be antisymmetric in its arguments. A well known theorem tells us that any antisymmetric function can be written as a single Vandermonde determinant times a bosonic function. In this way we can generally write

$$\psi_{\text{fermion}}(z_1, \ldots, z_N) = J\psi_{\text{boson}}(z_1, \ldots, z_N), \tag{7}$$

where

$$J = \prod_{i<j} (z_i - z_j). \tag{8}$$

Using this relation, the translation from bosons to fermions is quite easy. It is easy to see that the filling fraction ν_f for fermions is related to that of the corresponding filling fraction for bosons ν_b via

$$\nu_f = \frac{\nu_b}{\nu_b + 1}. \tag{9}$$

Throughout much of this paper we will be focused on bosonic wave functions for clarity. We will return to the issue of fermionic wave functions briefly in Sec. IV below.

II. PROPER CLUSTER WAVE FUNCTIONS

We begin by focusing on bosonic wave functions. A g-cluster wave function ψ will be defined by the analytic manner in which the wave function vanishes when the $g+1$ particles are brought to the same point $\tilde{z}$. Generally, we will write this $g+1$ particle limiting behavior

$$\lim_{z_1, \ldots, z_{g+1} \to \tilde{z}} \psi(z_1, \ldots, z_N) \sim f(z_1, \ldots, z_{g+1})\tilde{\psi}(\tilde{z}; z_{g+2} \ldots z_N), \tag{10}$$

where f is assumed to be an overall symmetric, translationally invariant, homogeneous polynomial of degree p. (By translationally invariant, we mean that we must have f invariant under shifting all $z_i \to z_i + \alpha$.) The relative angular momentum of such a $g+1$ cluster is defined to be $L_{g+1} = p$ on the disk. Thus, on the disc a group of $g+1$ particles is not allowed to have relative angular momentum less than p.

On the sphere, the notation is somewhat more complicated.[2] Each single particle state in the LLL has angular momentum $N_\phi/2$. The total angular momentum of $g+1$ bosons in the same single particle state is then $(g+1)N_\phi/2$. If the relative angular momentum of the cluster is p then the total angular momentum of the cluster is $(g+1)N_\phi/2 - p$.

On the torus, no simple concept of angular momentum exists. Indeed, the only way to describe the analog appears to be to specify the number of powers with which the wave function vanishes as particles approach each other (i.e., simply p). Thus specifying p appears to be more universal than speaking in terms of angular momentum.

We assume that f vanishes when all $g+1$ of its arguments coalesce at the same point. If f does not vanish when g particles coalesce, we say we have a "proper" g-clustered wave function. If f does vanish when g or fewer particles coalesce, then we say we have an "improper" g-clustered wave function.

In the proper case, the fact that f is homogeneous, translationally invariant of degree p, tells us that when g particles are put at the point $\tilde{z}$ we will have f vanishing as z^p as the $(g+1)$st particle approaches:

$$\lim_{z \to \tilde{z}} f(\tilde{z}, \tilde{z}, \ldots, \tilde{z}, z) \sim (\tilde{z} - z)^p. \tag{11}$$

The wave function ψ must vanish in this manner as *any* $(g+1)$st particle approaches. We can thus write that

$$\psi(\tilde{z}, \tilde{z}, \ldots, \tilde{z}, z_{g+1}, z_{g+2}, \ldots, z_N) \sim \left[\prod_{i=g+1}^{N} (\tilde{z} - z_i)^p\right] \tilde{\psi}_1(\tilde{z}; z_{g+1}, \ldots, z_N), \tag{12}$$

where $\tilde{\psi}_1$ is a wave function satisfying Eq. (10) for the remaining $N-g$ particles (and may have some dependence on $\tilde{z}$ as well).

Using this recursion relation, it is easy to calculate the filling fraction and shift of this wave function. We claim that for a proper f of degree p (i.e., one that does not vanish when g of its arguments come to the same point), the densest wave function satisfying condition 10 occurs at flux $N_\phi = pN/g - p$ so long as N is a multiple of g. Thus, this wave function has filling fraction and shift

$$\nu = g/p, \ \mathcal{S} = p. \tag{13}$$

To see this result more explicitly, we imagine bringing together particles into groups of g particles and using the above recursion relation (12) a total of $N/g-1$ times. Let us put the first cluster of particles at position $\tilde{z}_1$, the second at position $\tilde{z}_2$ and so forth until we have grouped the $(N/g-1)$th group at position $\tilde{z}_{N/g-1}$. The last g particles we leave ungrouped. Using the recursion law we obtain a wave function

$$\psi(\tilde{z}_1, \ldots, \tilde{z}_1, \tilde{z}_2, \ldots \tilde{z}_2, \ldots, \tilde{z}_{N/g-1}, \ldots, \tilde{z}_{N/g-1}, z_{N-g}, z_{N-g+1}, \ldots, z_N)$$
$$= \prod_{1 \le a < b \le N/g-1} (\tilde{z}_a - \tilde{z}_b)^{pg} \prod_{1 \le i \le N/g-1} \prod_{k=N-g}^{N} (\tilde{z}_i - z_k)^p \chi_{N/g-1}(\tilde{z}_1, \ldots, \tilde{z}_{N/g-1}; z_{N-g}, z_{N-g+1}, \ldots, z_N), \tag{14}$$

where $\chi_{N/g-1}$ is not allowed to vanish as any of its g remaining arguments z_j coalesce. The highest density wave function satisfying the limiting behavior Eq. (10) (i.e., the quantum Hall state with no quasiholes) could thus have χ being unity. Examining the degree of this polynomial with respect to the position of z_N we see that it is of degree $p(N/g-1)$. Thus, we have a wave function corresponding to flux $N_\phi = p(N/g-1) = (p/g)N - p$ which indicates $\nu = g/p$ and $\mathcal{S} = p$ as claimed.

For each proper function f, there exists *at most* one corresponding quantum Hall ground state wave function which would be the maximum density translationally invariant wave function for which Eq. (10) is always obeyed. Of course, just because we have constructed an appropriate f for $g+1$ particles, it is not clear how one can construct a wave function with a large number of particles N such that Eq. (10) is obeyed as any combination of $g+1$ particles approach each other. In essence we are asking how to "sew" together many functions f to form a macroscopic wave function. Sometimes no such macroscopic wave function exists. For example, in Appendix B it is shown that for odd pg no such macroscopic wave function exists. We note, however, that many proper cluster wave functions are already known. The Z_g Read-Rezayi states, for example, are proper $p=2$ states for any g (including the Pfaffian, which is $g=2, p=2$). The Laughlin states are proper for $g=1$ with even p. The Haffnian state[9] is proper with $g=2$, $p=4$ case, and recently the current authors[10] have studied the "Gaffnian" state, which is proper with $g=2$, $p=3$. Further, in the next section we will not need to know that any more proper wave functions actually exist. What is important is that *if* they exist, we know what their filling fractions are.

III. MAIN RESULTS

We now examine possible pair combinations of g and p and ask what the ground state is of the projection Hamiltonian P^p_{g+1}. Again we will consider here only the case of bosons. These results are summarized in Table I. In many of the examples below, we will use the same type of reasoning: A wave function that vanishes as $g+1$ particles come together must be either improper or proper (either it does or does not vanish as only g particles come together). We determine the densest possible zero energy state for both of the two possibilities and then compare these two with each other to find the densest of all possible zero energy states.

$g=1$ *the Laughlin series*. The Hamiltonian P^p_2 gives positive energy to any pair of particles with relative angular momentum less than p. This leaves the highest density zero energy ground state being the $\nu=1/p$ bosonic Laughlin state for even p. For odd p, the Hamiltonian does not allow pairs to have relative angular momentum $p-1$ so the highest ground zero energy ground state is the $1/(p+1)$ bosonic Laughlin state.

$p=1, p=2$ *the Read-Rezayi series*. As discussed in the Introduction, it has been shown[5] that projecting out the minimal angular momentum of $g+1$ particles (projecting out $L_{g+1}=0$ for bosons) results in the Z_g Read-Rezayi state. Since $L_{g+1} \neq 0$ as shown in Appendix A, we then conclude that the highest density zero energy state of both P^1_{g+1} and P^2_{g+1} is the Z_g Read-Rezayi state whose filling fraction is $\nu=g/2$ for bosons. Note that this includes $g=2$ with $p=1,2$ which gives the Moore-Read state (which is just the $g=2$ member of the Read-Rezayi series).

$g=2$, $p=3$ *Gaffnian*. The case $g=2, p=3$ gives the Gaffnian state.[10] We need not go into much detail as to the physics of this state but to indicate that such a proper cluster wave function at $\nu=2/3$ for bosons exists. Detailed discussion of this wave function is given in Ref. 10. For completeness, we now consider also the possibility that the ground state is not a proper cluster wave function, but rather an "improper" wave function (meaning it vanishes as only two particles come together). However, we know that the highest density bosonic wave function that vanishes when two come together is the Laughlin $\nu=1/2$ state, which is not as dense as the Gaffnian.

$g=2$, $p=4$ *Haffnian*. Similarly, the $g=2, p=4$ case give the Haffnian.[9] Again, this proper cluster wave function for $\nu=1/2$ for bosons has been previously discussed in detail. Again, we consider the possibility that the highest density state is an improper wave function. Indeed, the highest density improper wave function is the Laughlin $\nu=1/2$ state which which vanishes even faster than the Haffnian as three particles come to the same point (so it is also a zero energy state of P^4_3). Comparing these two possibilities, the Haffnian is considered the ground since it has a shift of $\mathcal{S}=4$ whereas the Laughlin $\nu=1/2$ state has a shift of $\mathcal{S}=2$. Thus the filling fraction of the Haffnian is slightly greater by an amount order $1/N$ (with N the number of particles). Note, however, on a torus geometry, where there is no shift, the density of these two states is the same (and indeed, there are many other states with the same density too[9,13]).

The $g=2$ series for $p=5,6$. Let us start by considering the cases of $g=2$ and $p=5,6$. Suppose the highest density ground state is a proper cluster wave function. In this case, the filling fractions in these two cases would be $\nu=2/5$ and $\nu=2/6$, respectively [see Eq. (13)]. We now consider the possibility that the ground state is improper. The highest density improper state (i.e., state that vanishes as two particles come together) is the Laughlin $\nu=1/2$ state. This is denser than the proper possibilities. Furthermore the Laughlin $\nu=1/2$ state is also a zero energy state of the relevant Hamiltonians P_3^5 and P_3^6 since the Laughlin state vanishes as 6 powers when three particles come together. Thus we conclude that the Laughlin $\nu=1/2$ state is the densest zero energy state of these Hamiltonians.

The periodic $g=2$ series. For $p>6$, we proceed similarly. If the highest density ground state is proper, the filling fraction is $\nu=g/p$. Now suppose the ground state is improper. In this case, the wave function must vanish as two particles come together. It is well known that any symmetric polynomial ψ that vanishes as two particles come together can be written as two Jastrow factors [see Eq. (8)] times another symmetric polynomial ψ'

$$\psi(z_1,\ldots,z_N) = J^2\psi'(z_1,\ldots,z_N) \tag{15}$$

[cf. Eq. (7)]. The filling fraction ν of ψ is related to the filling fraction ν' of ψ' via

$$\nu = \frac{\nu'}{2+\nu'}. \tag{16}$$

This is analogous to the usual composite fermion transformation [compare also Eq. (9)]. Further, if ψ vanishes as p powers when three particles come together, then ψ' vanishes as $p'=p-6$ powers (the 6 being from the Jastrow factors). Thus, if ψ is improper with $g=2$ we are equivalently looking for a wave function ψ' that vanishes at least as $p-6$ powers when three particles come together. Thus, we discover that the highest density improper wave function for $6<p\leqslant 12$ is just two Jastrow factors times the ground state of P_2^{p-6}. For $p\leqslant 6$ we have already calculated the ground state of P_2^p (i.e., $p=1,2$ is Pfaffian, $p=3$ is Gaffnian, $p=4$ is Haffnian, and $p=5,6$ is Laughlin), thus we know the highest density improper ground state of P_2^p for $6<p\leqslant 12$. It is easy to verify that the filling fraction of this improper state is greater than the $\nu=g/p$ proper possibility. For $12<p\leqslant 18$ we can repeat the argument and find that it is again the same series but with four Jastrow factors and so forth.

Read-Rezayi series again for $p=3,4$. We now consider the case of $p=3,4$ for arbitrary g. If the highest density state is a proper g-cluster wave function then the filling fraction will be $\nu=g/p$ as usual. If the wave function is improper, then it must vanish as only g particles come together. However, we already know that the highest density state that vanishes as g particles come together is the Z_{g-1} Read-Rezayi state whose filling fraction is $\nu=(g-1)/2$. Furthermore, as shown in Appendix C the Z_{g-1} Read-Rezayi wave function vanishes as four powers when $g+1$ particles come together (for $g>1$). Thus, so long as $(g-1)/2>g/p$, the Read-Rezayi Z_{g-1} state is the highest density zero energy state of P_{g+1}^3 and P_{g+1}^4. Note that this inequality is satisfied for $g>2,p=4$ and $g>3,p=3$.

The $g=3,p=3$ Pfaffian. For the $g=3,p=3$ case, the above inequality $[(g-1)/2>g/p]$ is instead an equality. Thus, this case is marginal. Here, the putative proper state occurs at $\nu=1$, and the improper state is the Z_2 Read-Rezayi state (the Moore-Read Pfaffian) which is also $\nu=1$. The shift of the Pfaffian is $\mathcal{S}=2$, whereas the shift of a $p=3$ proper state should be $\mathcal{S}=3$. Thus, we would expect that the proper state is denser. However, in Appendix B we show that, by symmetry, no proper state can exist for pg odd as we have in this case. So there is no wave function at $\nu=1$ with shift $\mathcal{S}=3$. Thus, the Pfaffian is the densest possible zero energy state of P_3^3. In this case, we do not eliminate the possibility that another zero energy state may exist with exactly this filling fraction (and perhaps the same shift). An otherwise "proper" state where a term has been added to fix the symmetry could occur. Indeed, exact diagonalization on the torus has revealed at least one other zero energy state at the same filling fraction.

The $g=3,p=5,6$ states: Gaffnian conjecture. We again consider first the possibility that the ground state of P_4^5 and P_4^6 are proper. These wave functions would have filling fractions 3/5 and 3/6, respectively. The other possibility is that the highest density ground state is improper (i.e., it vanishes as only three particles come together). Now consider the Gaffnian wave function. This has filling fraction 2/3, and from the explicit form of the wave function given in Ref. 10 it can be seen that it vanishes as six powers when four particles come together. Hence, the highest density ground states of P_4^5 and P_4^6 must be improper. However, there could be another (improper) zero energy state that also vanishes as three particles come together which is higher density than the Gaffnian. We conjecture that the Gaffnian is indeed the highest density zero energy state in these cases. However, we have not been able to prove this conjecture.

IV. ADDING JASTROW FACTORS

So far we have only considered bosonic wave functions. Given any bosonic wave function ψ_0 such as any of those discussed above, we can construct wave functions

$$\psi = \psi_0 \prod_{i<j}(z_i-z_j)^M = J^M\psi_0. \tag{17}$$

For even M this would then be another bosonic wave function, whereas for odd M this would be a fermionic wave function. Of particular interest is the $M=1$ case which was also discussed above in Eq. (7). Here, more generally, the filling fraction ν of ψ in terms of the filling fraction ν_0 of ψ_0 as

$$\nu = \frac{\nu_0}{M+\nu_0}. \tag{18}$$

There is, of course, a one to one mapping between the possible space of wave functions ψ_0 and those in the space of ψ. The defining limiting behavior of the wave function ψ is now given by [cf. Eq. (10)]

$$\lim_{z_1,\ldots,z_{g+1}\to\tilde{z}} \psi(z_1,\ldots,z_N) \sim f(z_1,\ldots,z_{g+1})\left[\prod_{1\le i<j\le g+1}(z_i-z_j)^M\right]\tilde{\psi}_0(\tilde{z};z_{g+2}\ldots z_N) \quad (19)$$

when $g+1$ particles come together and (for f "proper")

$$\lim_{z_1,\ldots,z_k\to\tilde{z}} \psi(z_1,\ldots,z_N) \sim \left[\prod_{1\le i<j\le k}(z_i-z_j)^M\right]\tilde{\psi}_{0k}(\tilde{z};z_{k+1}\ldots z_N) \quad (20)$$

when $k<g+1$ particles come together. In other words, the wave function vanishes as the Jastrow factor only when less than $g+1$ particles come together, and vanishes increasingly quickly (as defined by the function f) when $g+1$ come together. Thus, if f vanishes as p powers when $g+1$ particles come together, the wave function ψ vanishes as $Mg(g+1)/2+p$ powers when $g+1$ particles come together.

Enforcing the presence of Jastrow factors is a well known procedure. For bosons, $M=2$ is obtained by forbidding any two particles to have relative angular momentum of zero. In other words, adding a term P_2^2 to the Hamiltonian will assure that any zero energy wave function contains an overall $M=2$ Jastrow factor. This term, P_2^2 is usually known as a V_0 interaction[2] since it projects out pairs of particles with relative angular momentum zero. Similarly, to enforce an $M=4$ Jastrow factor, one adds P_2^4 to the Hamiltonian. (In the usually nomenclature this is a V_0 term and a V_2 term). So, for example, if a given wave function ψ_0 is the highest density zero energy ground state of P_{g+1}^p then $\psi=J^M\psi_0$ should be the highest density zero energy ground state of

$$P_2^M + P_{g+1}^{Mg(g+1)/2+p} \quad (21)$$

with M even. It is interesting to note that in cases listed in Table I above, the term enforcing the Jastrow factors is not needed. For example, the highest density zero energy state of P_3^3 is the Gaffnian. Thus, choosing any even M we would expect that the highest density zero energy state of $P_2^M+P_3^{3M+p}$ should be J^M times the Gaffnian. It is interesting that in this particular case the highest density zero energy state of P_3^{3M+p} is already J^M times the Gaffnian without including the Jastrow forcing term P_2^M. This is an intriguing phenomenon, and we do not know if it is general.

We now return to the case of fermions. As mentioned above in the Introduction [see Eq. (7)] any Fermi wave function can be written as a Bose wave function times a single Jastrow factor. Thus, by simply using a system of fermions, an $M=1$ Jastrow factor is automatically obtained. We also note that this immediately tells us that the minimum angular momentum of $g+1$ fermions in the LLL is

$$L_{g+1}^{\text{min;fermion}} = g(g+1)/2. \quad (22)$$

Since we have defined P_{g+1}^p to project out relative angular momenta $L<L^{\text{min}}+p$, the table generated as the highest density zero energy state of P_{g+1}^p is the same for fermions as it is for bosons only the resulting fermion wave functions have an overall Jastrow factor attached $(M=1)$.

To add further Jastrow factors to a fermionic wave function, we follow the analogous scheme to the bosonic case, projecting out any pairs of fermions with the minimal angular momenta. Thus, for fermions, our operator P_2^2 is defined to project out any pair with minimum angular momentum less than $L=L_2^{\text{min;fermion}}+2=3$. Thus, a zero energy state of P_2^2 for fermions must have at least $M=3$ Jastrow factors in the wave function. Conventionally such a term is known as a V_1 term of the Hamiltonian. Similarly, a zero energy state of P_2^4 for fermions must have at least $M=5$ Jastrow factors in the wave function. Note that, by construction, this again follows the rule that the resulting wave functions will always be the bosonic analog times a single Jastrow factor.

V. DISCUSSION

The wave functions we have constructed in this paper all stem from reasonably simple Hamiltonians, which involve projecting out clusters of particles with given angular momenta. The simplicity of this construction is, of course, much of the attraction of our theory. It is interesting that the only fundamentally "new" state that has appeared on our table of states so far is the Gaffnian, which will be discussed in depth in a companion to this paper.[10] It would be interesting to fill in the rest of Table I to see if any other new states might appear.

Some of the states that fit in our scheme are of course well known and well established to occur in nature. For example, the Laughlin states are certainly seen in the lowest Landau level.[1] Also among the states that fit in our construction is the Moore-Read Pfaffian,[6] which is believed to be the origin of the plateau seen in the first excited Landau level[7] at $\nu=2+1/2$. In addition, there are several states in our scheme that seem likely to be seen in nature, although there remains some level of uncertainty. For example, there is some evidence[5] that the particle-hole conjugate of the $g=3$ Read-Rezayi state is a good trial state for $\nu=2+2/5$, which has been observed recently.[8] A detailed discussion of the Gaffnian wave function is given in a companion to this paper.[10] Although the Gaffnian has extremely high overlap with $\nu=2/5$ there is reason to believe that the Gaffnian is a critical state rather than a phase.

It is interesting to note that in the lowest Landau level, most of the known physics appears to be outside of the general scheme set out in this paper. Instead, it appears that most of the states seen in the LLL are most easily explained within a composite fermion theory.[12] In contrast to the current work, the composite fermion wave functions (with the exception of the Laughlin states) are not the exact ground state of any known simple Hamiltonian—even though they are extremely accurate wave functions for Coulomb (and similar) interactions in the LLL. There are also possibilities that some of these states might be observed in systems of cold atoms. Rotating Bose condensates can be thought of as Bosons in a magentic field and thus (if sufficiently two dimensional) become quantum Hall systems.[4] In cold atom systems, the great freedom to tune parameters experimentally allows Hamiltonians with desired interactions to be designed. Indeed, a scheme has been devised[15] which essentially generate exactly the type of $(g+1)$ particle interaction necessary to yield the Read-Rezayi cluster series. Another approach to generating the Pfaffian in cold atoms have also been proposed[16] which does not rely on rotation. Since the Hamiltonians we are proposing in this paper are relatively simple, we might hope that experimentalists will be able to devise systems in which these Hamiltonians are realized.

It is important to note, however, that even in the absence of a physical realization of these multiparticle interactions, the classification scheme laid out here remains relevant to real systems with realizable physical interactions. As discussed, the Pfaffian and Read-Rezayi states are exact ground states of nonphysical many body Hamiltonians. Nonetheless, these states (more precisely these "phases of matter") also appear to be realized for the more realistic and experimentally relevant interactions. The use of the nonphysical multiparticle interaction is simply a way to get a more analytically tractable handle on a state of matter that is hard to study for more realistic interactions. Similarly in this work, we do not actually suspect that multiparticle interactions will be directly relevant anytime soon. Nonetheless, the phases of matter in our "periodic table" may be realized in much more realistic situations.

ACKNOWLEDGMENTS

E.H.R. acknowledges support from DOE under Contract No. DE-FG03-02ER-45981. N.R.C. acknowledges support from EPSRC Grant No. GR/S61263/01 and ICAM. The authors acknowledge conversations with F. D. M. Haldane, N. Read, and I. Berdnikov.

APPENDIX A: $L_{g+1} \neq L_{g+1}^{\min}+1$

The statement that $g+1$ bosons have relative angular momentum p is equivalent to saying that as the particles all approach the same point, the wave function vanishes as a pth degree polynomial f in the sense of Eq. (10). The function f must be a translationally invariant symmetric polynomial. We claim that no such polynomial exists of degree one. To see this we note that there is only a single symmetric polynomial in $g+1$ variable of degree one

$$\sum_{i=1}^{g+1} z_i \tag{A1}$$

and under translation $z_i \rightarrow z_i + a$ this is not invariant. Thus we conclude that $g+1$ bosons cannot have relative angular momentum 1. Writing any fermion wave function as an overall Jastrow factor times a boson wave function [See Eq. (7)] one can then show that generally L_{g+1} cannot be $L_{g+1}^{\min}+1$.

APPENDIX B: ODD pg PROPER BOSON WAVE FUNCTIONS DO NOT EXIST

Here, we claim that when both g and p are odd no macroscopic bosonic wave function exists with shift of p for that g and p. To see this, we use the recursion relation Eq. (12) (which is true as long as the wave function does not vanish as g particles coalesce, i.e., as long as it is proper) and group the particles into groups of g at positions $\tilde{z}_i$. The wave function of the clustered superparticles is given by

$$\psi = \prod_{i<j} (\tilde{z}_i - \tilde{z}_j)^{pg}. \tag{B1}$$

However, a cluster of g bosons must remain a bosonic object (i.e., the wave function is symmetric under interchange), whereas pg is odd. This tells us immediately that no such wave function can exist.

APPENDIX C: THE READ-REZAYI WAVE FUNCTION

As shown by Ref. 14, the bosonic Read-Rezayi wave function can be written by dividing the particles into g groups, giving Jastrow factors only between particles in the same group, and then symmetrizing over all choices of which particle is in which group. We will assume the total number of particles N is divisible by g and define the first group to be particles $1,\ldots,N/g$ the second group to be $N/g+1,\ldots,2N/g$ and so forth. We thus write the Z_g Read-Rezayi bosonic wave function as

$$\psi = S_N\Bigg[\prod_{0<i_1<j_1\leqslant N/g} (z_{i_1}-z_{j_1})^2 \prod_{N/g<i_2<j_2\leqslant 2N/g} (z_{i_2}-z_{j_2})^2 \cdots \times \prod_{(g-1)N/g<i_g<j_g\leqslant N} (z_{i_g}-z_{j_g})^2\Bigg], \tag{C1}$$

where S_N represents symmetrization over all particle coordinates. It is simple to establish that the filling fraction is $\nu = g/2$ and the shift is $\mathcal{S}=2$. When g bosons come together, one can be in each group so the wave function does not vanish. When $g+1$ bosons come together, at least two of them must be in the same group and the wave function vanishes as $p=2$ powers. Similarly when $g+2$ particles come together (for $g>1$), at least two groups have two bosons in them, meaning the wave function vanishes as $p=4$ powers.

[1] For a classic review of quantum Hall physics, see *The Quantum Hall Effect*, edited by R. Prange and S. M. Girvin, (Springer-Verlag, New York, 1987).
[2] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[3] S. A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).
[4] See, for example, N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, Phys. Rev. Lett. **87**, 120405 (2001).
[5] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).

[6] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[7] R. H. Morf, Phys. Rev. Lett. **80**, 1505 (1998); E. H. Rezayi and F. D. M. Haldane, *ibid.* **84**, 4685 (2000).
[8] J. S. Xia, W. Pan, C. L. Vincente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **93**, 176809 (2004).
[9] Dmitri Green, Ph.D. thesis, Yale University, New Haven, 2001, cond-mat/0202455.
[10] S. H. Simon, E. H. Rezayi, N. R. Cooper, and I. Berdnikov, preceding paper, Phys. Rev. B **75**, 075317 (2007).
[11] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).
[12] See *Composite Fermions*, edited by O. Heinonen (World Scientific, Singapore, 1998), and references therein.
[13] E. H. Rezayi (unpublished).
[14] This is particularly clear using the form of the Read-Rezayi wave function written down by A. Cappelli, L. S. Georgiev, and I. T. Todorov, Nucl. Phys. B **599**, 499 (2001).
[15] N. R. Cooper Phys. Rev. Lett. **92**, 220405 (2004).
[16] V. Gurarie, L. Radzihovsky, and A. V. Andreev, Phys. Rev. Lett. **94**, 230403 (2005).