Construction of a paired wave function for spinless electrons at filling fraction $\nu=2/5$

Steven H. Simon,[1] E. H. Rezayi,[2] N. R. Cooper,[3] and I. Berdnikov[1,4]
[1]*Alcatel-Lucent, Bell Labs, Murray Hill, New Jersey 07974, USA*
[2]*Department of Physics, California State University, Los Angeles, California 90032, USA*
[3]*T. C. M. Group, Cavendish Laboratory, J. J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom*
[4]*Department of Physics, Rutgers University, 136 Frelinghuysen Road, Piscataway, New Jersey 08854, USA*
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We construct a wave function, generalizing the well-known Moore-Read Pfaffian, that describes spinless electrons at filling fraction $\nu=2/5$ (or bosons at filling fraction $\nu=2/3$) as the ground state of a very simple three body potential. We find, analogous to the Pfaffian, that when quasiholes are added there is a ground state degeneracy which can be identified as zero modes of the quasiholes. The zero modes are identified as having semionic statistics. We write this wave function as a correlator of the Virasoro minimal model conformal field theory $\mathcal{M}(5,3)$. Since this model is nonunitary, we conclude that this wave function is likely a quantum critical state. Nonetheless, we find that the overlaps of this wave function with exact diagonalizations in the lowest and first excited Landau level are very high, suggesting that this wave function may have experimental relevance for some transition that may occur in that regime.

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I. INTRODUCTION

The vast majority of quantum Hall states observed experimentally in the lowest Landau level (LLL) are very accurately described in terms of composite fermion[1] (or equivalently the hierarchy[2]) wave functions. Despite these successes, there are a number of quantum Hall states that remain much less well understood and may require more exotic explanations. For example, there exist a few experimentally observed quantum Hall plateaus in the LLL that do not fit into the usual hierarchy-composite-fermion framework.[3] Also, in the first excited Landau level (1LL), even the "simple" observed filling fractions (such as $\nu=2+1/3$ and $2+2/5$) appear from numerics to have strong differences from the corresponding states in the LLL.[2,4,5] Of the nonhierarchy exotic states that have been proposed, perhaps the best understood is the Moore-Read Pfaffian,[6] which is thought to describe[7] the plateau observed at $\nu=2+1/2$, and whose quasiparticle excitations have exotic non-Abelian statistics. However, for the neighboring experimentally observed plateau[8] at $\nu=2+2/5$ there are at least two competing trial states which have been proposed: the hierarchy (composite fermion) state[1,2] and the (particle-hole conjugate of the) Z_3 Read-Rezayi parafermion state,[9,10] a generalization of the Pfaffian which has an even richer non-Abelian structure. Another case where more exotic states may occur is in the quantum Hall effect of rotating bosons.[11]

In this paper we will study another type of generalization of the Pfaffian that gives a different trial state at $\nu=2/5$ (or $2+2/5$) for spinless electrons. We call this new wave function the "Gaffnian," for reasons described below. The Hamiltonian that generates this wave function as its unique highest density zero energy state is an extremely simple generalization of the Hamiltonian that similarly generates the Moore-Read Pfaffian (and is also similar in spirit to the Hamiltonians that generate the Read-Rezayi wave functions). Similar to the Pfaffian and Read-Rezayi states, when additional flux is added, there is a degeneracy of states associated with zero modes of the quasiholes. In the present case the zero modes have semionic statistics, compared to fermionic statistics in the Pfaffian case, or parafermionic statistics in the Read-Rezayi case. We then write this wave function as the correlator of a conformal field theory, known as the Virasoro $\mathcal{M}(5,3)$ minimal model. Since this field theory is nonunitary we conclude that the wave function is likely not to represent a phase, but rather a quantum critical point between phases. As such, the concept of non-Abelian statistics is not necessarily applicable. Nonetheless, exact diagonalizations show extremely high overlaps with this trial wave function. We take this to suggest that this wave function is likely relevant to a phase transition that is somehow "near" the hierarchy phase.

The outline of this paper is as follows. In Sec. II we will introduce this new wave function and explore some of its properties. We define this wave function as the unique highest density zero energy state of a simple Hamiltonian. We then consider what happens when additional flux is added to the system. As mentioned above, in the presence of quasiholes, there is a ground state degeneracy stemming from semionic zero modes. Since some of the analytic manipulations are messy, we relegate some of the details to rather lengthy appendixes. In particular, the demonstration that the Gaffnian is a unique ground state of this Hamiltonian, and the explicit counting of quasihole states is put in Appendix A. However, for the interested reader, this appendix shows explicitly the mechanism by which the semions occur. In Sec. III we construct the Gaffnian as a correlator of the $\mathcal{M}(5,3)$ Virasoro minimal model conformal field theory. In Sec. IV we examine results of exact diagonalizations. We look at low energy excitations to find evidence of criticality, and we also discover that the overlap of the Gaffnian with the hierarchy wave function is remarkably high (we also find high overlap with exact diagonalizations of systems with interactions close to Coulomb). Finally, in Sec. V we give a brief discussion of some of our results.

II. THE GAFFNIAN WAVE FUNCTION

Before constructing our new trial wave function, for motivation we review construction of Laughlin wave functions. In the lowest Landau level, two particles i and j in relative wave function $(z_i-z_j)^{L_2}$ are said to have relative angular momentum L_2. The relative angular momentum of two fermions L_2 must be odd and positive so the minimum relative angular momentum is $L_2^{\min}=1$ (for bosons, $L_2^{\min}=0$ and L_2 must be even). We define a projection operator P_2^p to project out (i.e., to keep only) states where any two particles have relative angular momentum less than $L_2^{\min}+p$ (with p even). This projection operator[12] can serve as a Hamiltonian. The Laughlin $\nu=1/(L_2^{\min}+p)$ state is the unique highest density (zero energy) ground state of P_2^p (with $L_2^{\min}+p$ odd for fermions and even for bosons). Note that the construction of projecting out states where the relative angular momentum of two particles is less than $L_2^{\min}+p$ is precisely equivalent to the statement that the resulting wave function must vanish at least as $L_2^{\min}+p$ powers, i.e., at least as quickly as $(z_i-z_j)^{L_2^{\min}+p}$, as any two particles i and j approach each other.

We now generalize this construction. Analogous to the two particle case, we can define the *three* particle relative angular momentum L_3 to be the total power (degree) of the relative wave function between the three particles. (See Refs. 14 and 15 for more precise definitions, which are not needed here.) In other words, if the wave function vanishes as L_3 powers as all three particles come to the same point, then the wave function contains no components with relative angular momentum less than L_3. In a single Landau level the relative angular momentum of three fermions L_3 has a minimum value $L_3^{\min}=3$ (for bosons, $L_3^{\min}=0$). It is not hard to show[14–17] that $L_3\neq L_3^{\min}+1$ is dictated by symmetry, but any other $L_3\geqslant L_3^{\min}$ is allowed. We analogously define a projection operator P_3^p to project out (i.e., to keep only) states where any three particles have relative angular momentum less than $L_3^{\min}+p$ which will serve as our Hamiltonian. It is well known[6,13,16] that the Pfaffian (at $\nu=1/2$ for fermions and $\nu=1$ for bosons) is the unique highest density (zero energy) ground state of the Hamiltonian P_3^2. In Ref. 17 another state, known as the "Haffnian" is shown to be the unique highest density (zero energy) ground state of P_3^4 (which is a $\nu=1/3$ state for fermions and $\nu=1/2$ state for bosons). Using the method of Refs. 16 and 17 we can show that the Hamiltonian P_3^3 also has a unique highest density (zero energy) ground state. The argument is straightforward and is given in the appendixes of this paper. This unique state occurs at $\nu=2/5$ for fermions ($\nu=2/3$ for bosons), and is the focus of this paper. Since this new $p=3$ state lies between the $p=2$ Pfaffian and the $p=4$ Haffnian, we alphaphonetically interpolate and dub this new state the "Gaffnian."

Before commencing our study of the Gaffnian, we note that several other states can be constructed analogously. By considering general k-particle angular momenta L_k, we can construct a general P_k^p. In Ref. 9 it was shown that the Hamiltonian P_k^2 generates the Z_{k-1} Read-Rezayi state (the Z_2 state being the Pfaffian). The study of several other values of p and k is given in Ref. 14 by three of the current authors. In Ref. 14 we discuss that among these possible Hamiltonians there is a class of resulting ground state wave functions (for bosons) which we name "proper," defined by the fact that they do not vanish when k particles come together but do vanish as p powers when $k+1$ come together. The filling fraction for such proper wave functions follows the formula $\nu=k/p$. The Pfaffian, Haffnian, Gaffnian, Read-Rezayi states, and Laughlin states (all for bosons) are all examples of such proper wave functions and follow this general rule for filling fraction. As usual, fermionic states can be constructed from the bosonic versions by attachment of an overall Jastrow factor, with the resulting wave functions having filling fraction $\nu_{\text{Fermi}}=\nu_{\text{Bose}}/(\nu_{\text{Bose}}+1)$.

Knowing that a unique quantum Hall ground state exists for the Hamiltonian P_3^3, we set about describing the properties of the Gaffnian. We begin by writing down the wave function explicitly.

We will represent a particle's coordinate as an analytic variable $z=x+iy$ which is simply the complex representation of the particle position $\mathbf{r}$. On the spherical geometry, z is the stereographic projection of the position on the sphere of radius R to the plane. All distances are measured in units of the magnetic length. We can write any single particle wave function as an analytic function times a measure $\mu(\mathbf{r})$. On the disk the measure is the usual Gaussian factor[2] $\mu(\mathbf{r})=e^{-|z|^2/4}$, whereas on the sphere[12] (with stereographic projection to the plane) the measure is $\mu(\mathbf{r})=[1+|z|^2/(4R^2)]^{-(1+N_\phi/2)}$ with N_ϕ ($=2R^2$ when the magnetic length is unity) being the total number of flux penetrating the sphere.[9,14] On the sphere the degree of the polynomial $\psi(z)$ ranges from z^0 to z^{N_ϕ} giving a complete basis of the $N_\phi+1$ states of the LLL. We will not write the measure explicitly, instead writing all wave functions simply as analytic functions (which must be fully symmetric for bosons and fully antisymmetric for fermions).

It is convenient to think, for a moment, about bosons at $\nu=2/3$. Since the Hamiltonian P_3^3 puts no restriction on the two particle angular momentum, there is no restriction against two bosons being at the same point z_0. However, when a third particle approaches, it must approach the other two[14] such that the overall angular momentum of the three particles is $p\geqslant 3$, i.e., the wave function vanishes as $(z_3-z_0)^p$. (In this sense, the Gaffnian, similar to the Pfaffian and Haffnian, is a paired state in the spirit of that originally proposed in Ref. 18.) The Gaffnian wave function can be written explicitly as[19]

$$\Psi=\widetilde{S}\left[\prod_{a<b\leqslant N/2}(z_a-z_b)^{2+q}\prod_{N/2<c<d}(z_c-z_d)^{2+q}\times\prod_{e\leqslant N/2<f}(z_e-z_f)^{1+q}\prod_{g\leqslant N/2}\frac{1}{(z_g-z_{g+N/2})}\right],\qquad(1)$$

where $q=0$ corresponds to a bosonic ($\nu=2/3$) wave function and $q=1$ is a fermionic ($\nu=2/5$) wave function. We have assumed the number N of particles is even and $\widetilde{S}$ means symmetrize or antisymmetrize over all particle coordinates for bosons or fermions, respectively. One can confirm directly that the above wave function for $q=0$ correctly has the property that it does not vanish as two particles come to the same position but vanishes as three powers as the third particle arrives. Further, we show in Appendix A that this is the unique lowest density wave function (lowest degree homogeneous, translationally invariant polynomial) that has this property. As is standard in spherical geometry[2,12] we can obtain the value of the flux by looking at the maximum power of z_i that occurs. Counting powers of z we find that the Gaffnian wave function occurs on the sphere for total flux

$$N_\phi = 3N/2 - 3 + q(N-1). \tag{2}$$

This value of flux is the same as that for the standard hierarchy $\nu=2/5$ state.[12] This should not be a surprise, since some of the first trial wave functions for $\nu=2/5$ (for fermions) in the hierarchy were based on pairing.[18] In the appendix we analytically establish that this state is the unique zero energy state of the Hamiltonian P_3^3 at this flux. We have numerically confirmed this fact by explicitly diagonalizing the Hamiltonian P_3^3 with up to $N=12$ particles on the spherical geometry and up to $N=10$ particles on the torus. [We note that on the torus the Gaffnian occurs at $N_\phi=(3/2+q)N$ meaning there is no "shift," which is always the case on the torus.]

In Appendix C we consider possible generalizations of the form of the Gaffnian wave function (1). In particular, we find trial states for wave functions of the Jain series $\nu=p/(mp+1)$ (with m odd for bosons and even for fermions) with the same value of the flux as the usual Jain sequence. Since (as we will discuss below) the Gaffnian is distinct from the hierarchy (or Jain) states, we suspect that these trial states are similarly distinct from the usual Jain states. However, we leave detailed study of these wave functions for further work.

Since the Gaffnian is a paired state,[6,18] we expect that each additional flux added will correspond to two quasiholes, each with charge $e^*=e\nu/2$ with $-e$ the charge on the elementary underlying "electron" (or underlying boson for $q=0$). Generally, we define the number of extra flux added to the Gaffnian ground state to be

$$n = N_\phi - [3N/2 - 3 + q(N-1)] \tag{3}$$

[compare to Eq. (2)]. Note that n here is defined so that it is half integer if N is odd. To construct wave functions in the presence of n (integral) additional flux, we can insert a factor of

$$\prod_{a\leq N/2;j\leq n} (z_a - w_j) \prod_{N/2<b\leq N;n<k\leq 2n} (z_b - w_k) \tag{4}$$

into the above wave function inside the symmetrization where the w's indicate the quasihole positions. However, for $2n$ fixed quasihole positions, there are apparently $\binom{2n}{n}$ inequivalent ways to choose which of the positions w_j are labeled with an index $j\leq n$ and which with an index $j>n$. One might expect that the different groupings of the positions into these two groups generate equally many inequivalent quasihole wave functions. The fact that we find more than one independent quasihole wave function means that there are zero modes associated with these quasiholes.[6] Analogous to the Pfaffian,[16] however, it turns out that there are many linear dependencies between these many different wave functions.

In Appendix A, we show explicitly how to count the zero energy ground state degeneracy of the system with Hamiltonian P_3^3 at any flux (strictly speaking the appendix only addresses the case of N and $2n$ even, although the odd case proceeds similarly). We find that the degeneracy of zero energy states is given by

$$\sum_{F,(-1)^F=(-1)^N}^{F_{\max}} \binom{(N-F)/2+2n}{2n}\binom{n+F/2-1}{F}, \tag{5}$$

where the maximum value of F is given by $F_{\max}=\min(N,2n-2)$. To verify this result, we have numerically performed exact diagonalizations. For every case we have examined, we find perfect agreement between this analytic rule and the results of our exact diagonalization of the Hamiltonian P_3^3. (We have examined $N=4,6$ with $n\leq 6$, $N=8$ with $n\leq 4$, $N=10$ with $n\leq 2$, $N=5$ with $n\leq 7/2$ and $N=7$ with $n\leq 5/2$.)

The first term in Eq. (5) corresponds to the positional degeneracy of the quasiholes and can be thought of as $2n$ bosons in $(N-F)/2+1$ orbitals. The second term is the degeneracy of the zero modes and can be thought of as F fermions in $n+F/2-1$ orbitals. Since the number of orbitals changes half as fast as number of particles, these zero modes are a realization of semionic exclusion statistics.[20]

The form of Eq. (5) is quite analogous to the zero-mode counting expressions found for the Pfaffian,[16] Haffnian,[17] and Read-Rezayi states.[9,21] However, in those cases the zero modes have fermionic, bosonic, and parafermionic statistics, respectively. For the fermionic (Pfaffian) case we put F fermions in a fixed number n orbitals.[16] For the bosonic (Haffnian) case,[17] we put F fermions in $n+F-2$ orbitals (which is equivalent to putting F bosons in $n-1$ orbitals). The Gaffnian case is quite naturally an interpolation between these two cases. (The Read-Rezayi parafermion case cannot be phrased in this language so easily.[21])

As with the Pfaffian,[16] Haffnian,[17] and Read-Rezayi[9,21] cases, the structure of Eq. (5) also tells us how to decompose these degenerate states into angular momentum multiplets. We simply calculate the multiplets of the $2n$ bosons in $(N-F)/2+1$ orbitals and also the multiplets of the F fermions in $n+F/2-1$ orbitals and then add these together using standard angular momentum addition rules. An explicit example of this angular momentum addition is given in Appendix B.

As discussed above, we can also look at wave functions with fixed quasiparticle positions. The number of linearly independent states should just be given by the zero-mode contribution to the above equation

$$D_n = \sum_{F,(-1)^F=(-1)^N}^{F_{\max}} \binom{n+F/2-1}{F}. \tag{6}$$

Indeed by generating wave functions [described by Eq. (4) inserted into Eq. (1)] numerically and checking for linear independence, we find precisely this number of independent states for all cases we have tried ($N=4$, $n\leq 6$; $N=6$, $n\leq 5$, $N=8$, $n\leq 3$).

It is interesting to note that in the case of $N \geqslant 2n-2$ (so $F_{\max}=2n-2$) the sum (6) gives the $2n-1$st Fibonacci number $\mathrm{Fib}(2n-1)=\mathrm{Fib}(N_{qh}-1)$. This can be proven trivially by induction on n to show that $D_n+D_{n+1/2}=D_{n+1}$. We note that the Z_3 Read-Rezayi state also has a degeneracy of $\mathrm{Fib}(N_{qh}-1)$. Another similarity we have found is that both states have a two-fold degeneracy of the ground state at zero momentum on the torus geometry (in addition to the usual center of mass degeneracy[2]). However, the two ground states (the Gaffnian and the particle-hole conjugate of the Z_3 Read-Rezayi state) occur at different values of the flux for a finite spherical system, so they are topologically different states. Also, as mentioned above, the state counting formula analgous to Eq. (5) involves parafermionic[21] zero modes for the Read-Rezayi case compared to semionic modes for the Gaffnian.

III. CONFORMAL FIELD THEORY

We now write this Gaffnian wave function as a correlator of a conformal field theory (CFT).[22] Making the connection to CFT has, in the past, been extremely powerful in understanding states with non-Abelian statistics. (See, for example, Refs. 6, 9, 21, and 23). For example, the structure of a CFT can tell us about behavior of the degenerate space under adiabatic braiding of quasiholes.[23] We note that it is certainly not the case that any analytic wave function is the correlator of a CFT, so in this respect, the Gaffnian is an example of a very select class of wave functions.[24] Further, as we will see below, the relevant CFT is one of the very simplest ones possible among an infinite set of possibilities.

A CFT describing a paired state should contain a field ψ with fusion relation $\psi \times \psi \sim \mathbf{1}$ such that it has operator product expansion

$$\psi(z)\psi(w) \sim (z-w)^{-2\Delta_\psi}[\mathbf{1}+\cdots] \tag{7}$$

with $\mathbf{1}$ the identity, Δ_ψ the conformal weight (or dimension) of ψ, and dots representing less singular terms. We can then construct a paired wave function

$$\Psi = \left\langle \prod_{i=1}^{N} \psi(z_i) \right\rangle \prod_{i<j} (z_i-z_j)^{2\Delta_\psi+q}. \tag{8}$$

Repeating the arguments which are presented in Ref. 9 it is clear that (for $q=0$, i.e., for bosons) this wave function will not vanish as two particles come to the same position since the (fractional) Jastrow factor precisely cancels the singularity of the operator product expansion. However, the wave function vanishes as $z^{4\Delta_\psi}$ powers when the third particle approaches the other two [since there are three (fractional) Jastrow factors and only one singularity]. The Moore-Read Pfaffian[6] is described in this way by the Ising CFT, also known as the $\mathcal{M}(4,3)$ minimal model,[22] which contains such a field ψ with weight $\Delta_\psi=1/2$ so the wave function vanishes as z^2 as three particles come to the same point. The Gaffnian is correspondingly described by one of the simplest generalizations of the Ising CFT, known as the minimal model $\mathcal{M}(5,3)$. This CFT has a field ψ with $\Delta_\psi=3/4$ so that the wave function vanishes as z^3 as three particles coalesce (for

	Δ
ψ	3/4
φ	1/5
σ	-1/20

$\times$	ψ	φ	σ
ψ	$\mathbf{1}$		
φ	σ	$\mathbf{1}+\varphi$	
σ	φ	$\sigma+\psi$	$\mathbf{1}+\varphi$

FIG. 1. In the Virasoro minimal model conformal field theory $\mathcal{M}(5,3)$, there are three nontrivial fields ψ, φ, and σ with dimensions Δ given in the left table and fusion algebra given in right table.

$q=0$). The dimensions and fusion rules for the three primary fields (ψ, σ, and φ) in this model are given in Fig. 1. The fusion of the field σ with the field ψ gives us the operator product expansion[22]

$$\psi(z)\sigma(w) \sim (z-w)^{-1/2}\varphi(w)+\cdots, \tag{9}$$

where here the exponent $-1/2$ is determined by the conformal weights in Fig. 1 as $\Delta_\varphi-\Delta_\psi-\Delta_\sigma$. As described in Ref. 9 this power of $1/2$ means that the quasihole created by the field σ must have charge $e^*=e\nu/2$ consistent with our expectation for a paired state. To see how this happens we write a general wave function in the presence of $2n$ quasiholes as

$$\Psi = \left\langle \prod_{j=1}^{2n} \sigma(w_j) \prod_{i=1}^{N} \psi(z_i) \right\rangle \times \prod_{i<j} (z_i-z_j)^{3/2+q} \prod_{i=1}^{N}\prod_{j=1}^{2n} (z_i-w_j)^{1/2}. \tag{10}$$

Given the operator product expansion (9), the final exponent in Eq. (9) must have power $1/2$ so that the wave function is single valued in the z's. This Jastrow factor then pushes precisely a charge $e\nu/2$ away from each quasihole.

We can also use the fusion rules to count the degeneracy of the $2n$ quasihole state. The degeneracy is given by the number of ways the σ fields in Eq. (10) can fuse together to

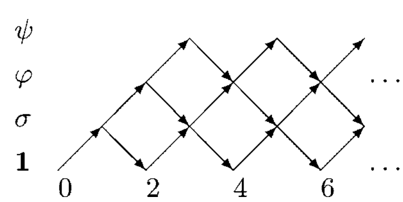

FIG. 2. The Bratteli diagram shows how the $2n$ quasihole fields σ fuse together. This is just a graphical representation of the fusion rules (Table I) where at each horizontal step, the states at the previous horizontal position are fused with one more σ field. The number of conformal blocks—which gives the non-Abelian degeneracy—is seen graphically by the number of paths through the diagram starting and ending at the bottom when N (and $2n$) is even. When N (and $2n$) is odd, the path needs to start at the bottom but end at the top to fuse with the one unpaired ψ field. By straightforward counting, the number of such paths with $2n$ steps can be seen to be the $2n-1$st Fibonacci number.

form the identity. This is illustrated graphically as the number of paths through the Bratteli diagram[23] shown in Fig. 2. The number of paths is $\mathrm{Fib}(2n-1)$, which is consistent with the result of our above counting formula. If the number of particles N is even, then we pair the ψ fields to form identities, and the σ fields must also pair to form the identity. However, if N is odd, we can only form the identity if the σ fields fuse to form one more ψ that can then fuse to form the identity with the one remaining ψ field.

One may ask how we know that we have the correct conformal field theory (particularly in light of the fact that classification of all conformal field theories is an ongoing research field). The fact that we have a paired state at filling fraction $\nu=2/3$ for bosons (i.e., the fact that the wave function does not vanish when two particles come together) means we must have a field ψ which fuses with itself to form the identity. The fact that the Hamiltonian forces the wave function to vanish as three powers when three particles come together further fixes the dimension Δ_ψ. It is easy enough to show that the only Virasoro minimal model conformal field theory with such a field is $\mathcal{M}(5,3)$. If we further insist that the charge of the quasihole should be $e\nu/2$, as is expected for a paired state, this fixes the exponent of the final factor in Eq. (10), and this in turn fixes $\Delta_\varphi-\Delta_\sigma=1/4$. We must also insist that the fusion relations for fusing many quasiparticles with each other have the form of the Bratteli diagram in Fig. 2. Finally, one can look at the subleading behavior of the wave function as particles approach each other to extract the central charge of the theory, which again is consistent with $\mathcal{M}(5,3)$ (we do not perform this calculation here). These restrictions place serious constraint on any possible conformal field theory we would like to use to represent the Gaffnian state. Certainly there is no "simple" (i.e., minimal model) theory other than $\mathcal{M}(5,3)$ with the required properties. However, we have not proven that no other theory exists.

The conformal field theory $\mathcal{M}(5,3)$ is nonunitary.[22] This highly suggests that the Gaffnian wave function does not represent a true phase, but rather represents a quantum critical point. The argument for this goes as follows: The edge state theory in 1+1 dimensions of a quantum Hall state should be described by the same conformal field theory as the bulk two-dimensional theory.[6] However, since the edge state theory is a dynamical theory, it must be unitary. If we have a trial wave function that is generated by a nonunitary theory, apparently the only way out of this conundrum is that the edge state theory does not exist; i.e., edge excitations do not stay on the edge, but leak into the bulk. This could indeed be the case if the ground state has arbitrarily low energy excitations in the thermodynamic limit. This could, in turn, occur if the wave function represents a quantum critical point. Indeed, there have been past examples of critical quantum Hall states which are described by nonunitary CFTs.[16,17,26] While there is no strict proof that a nonunitary conformal field theory necessarily implies a critical state, there is also no understanding of how anything else could occur.

IV. EXACT DIAGONALIZATIONS

We now turn to exact diagonalizations. Strictly speaking, the Hamiltonian P_3^3 has been defined to be a projection operator that acts on the full wave function (to keep any states where any three particles have relative angular momentum less than three). As such, this Hamiltonian has eigenvalues that are either zero (for the zero energy space) or unity. A more physical version of this Hamiltonian can be written as

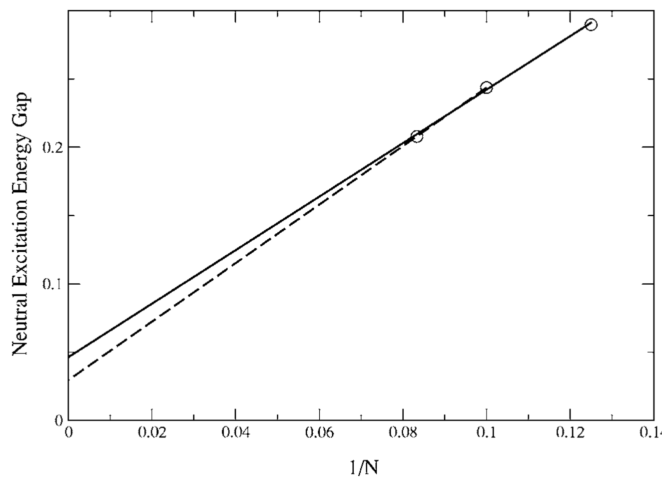

FIG. 3. Lowest neutral gap excitation of the Gaffnian as a function of system size [using the Hamiltonian in Eq. (11)] in units of $V_{3,0}=V_{3,2}$. Data is shown for $N=8,10,12$ particles. The solid is a linear fit of all three data points. The dashed line is a fit of the two larger systems only (suggesting that if we could access even larger systems, the extrapolations might be even closer to zero). This data suggests the possibility that the gap may extrapolate to zero in the thermodynamic limit, as would be expected for a critical state. However, from the available numerical data, we cannot exclude the possibility that it extrapolates to a finite value.

$$H=\sum_{i<j<k}(V_{3,0}P_{ijk}^0+V_{3,2}P_{ijk}^2), \tag{11}$$

where we have defined a general three body operator P_{ijk}^p which projects out (i.e., keeps) any component of the wave function where the three particles i, j, and k have relative angular momentum $L_{\min}+p$. [On the sphere,[12,14] one defines P_{ijk}^p to project out (i.e., keep) any cluster of three particles with total angular momentum $3N_\phi/2-p$.]

Note that three particles cannot[14] have relative angular momentum of $L_{\min}+1$, so this Hamiltonian gives energy to any case where the relative angular momentum of any cluster of three particles is less than three. [Some readers may have assumed that the form of Eq. (11) is what we meant all along when we have been writing P_3^3, as we were not very explicit about what we meant.] Since the Hamiltonian (11) gives energy to any cluster of three particles with relative angular momentum less than 3, it has precisely the same zero energy space as P_3^3. However, the excitation spectrum here is different, and is dependent on the values of $V_{3,0}$ and $V_{3,2}$.

Let us first examine the issue of criticality. In Fig. 3, we show the lowest energy neutral excitation of H [from Eq. (11)] as a function of system size for $N=8,10,12$ on a spherical geometry with $V_{3,0}=V_{3,2}$ (We have chosen to look at bosons on a sphere because we can go to larger systems.) As can be seen in the figure it appears that the gap extrapolates to a positive value, but it is not possible to rule out

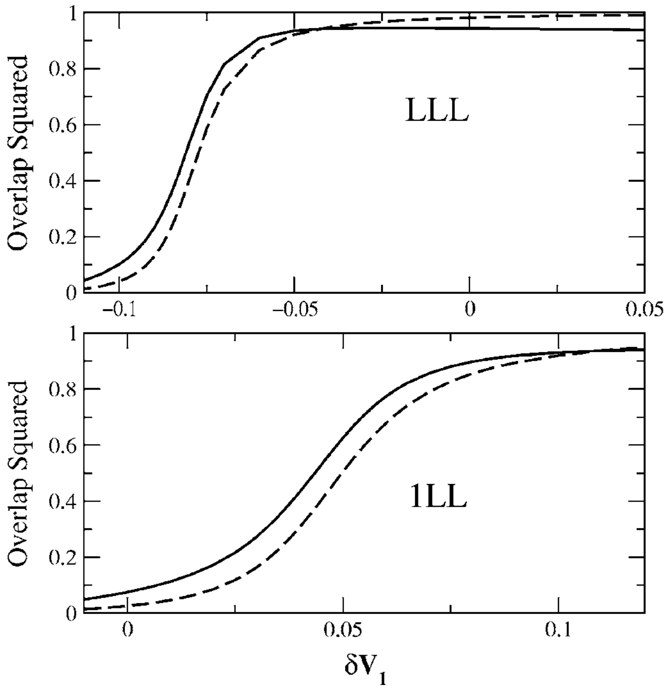

FIG. 4. Squared overlaps of trial states with the exact ground state at $\nu=2/5$ on a sphere with 10 electrons, as we vary the interaction. Solid line is the overlap of the Gaffnian wave function with the exact ground state. The dashed line is the hierarchy 2/5 state (Ref. 25) with the exact ground state. The top is results for the lowest Landau level, the bottom is the first excited Landau level. In the horizontal direction the interaction is varied around the Coulomb interaction by adding an additional δV_1 Haldane pseudopotential.

extrapolation to a zero value which would be a sign of criticality. Furthermore, changing the ratio of $V_{3,0}/V_{3,2}$ (data not shown) does not appear to substantially affect the ratio of the extrapolated energy to the reference energy of the gap for $N=10$.

We now turn to the question of whether the Gaffnian is physically relevant to the physics of 2D electron systems. We have performed exact diagonalization studies on a spherical geometry for ten electrons in the lowest Landau level (LLL) and first excited Landau Level (1LL), and we have varied the electron-electron interaction in the neighborhood of the Coulomb interaction by varying the Haldane pseudopotential[2,12] coefficient V_1. In Fig. 4, we show the overlap of the exact ground state with our trial wave function. Results are shown for the Gaffnian (solid) and the hierarchy 2/5 state[2,25] (dashed). Over a range of V_1 both trial states have quite good overlaps with the ground state considering that the zero angular momentum Hilbert space has 52 dimensions. (Note that for many of the well known numerical cases[1,2] where extremely large overlaps have been reported, the dimension of the available Hilbert space is much smaller than this.) Near the regime of V_1 where the overlaps drop, we believe the system is in the Read-Rezayi phase[10] (although at a different value of flux on the sphere). Since both the Gaffnian and hierarchy states have such large overlaps with the ground state, they necessarily have large overlaps with each other, although in the thermodynamic limit they become orthogonal.

We have also performed exact diagonalization on the torus geometry. Here, the Gaffnian ground state is found to be doubly degenerate (in addition to the usual center of mass degeneracy). The two zero energy ground states are distinguished by a parity quantum number. The state with positive parity again has extremely high overlap with the hierarchy state, similar to the overlaps on the sphere. As on the sphere, both of these have a high overlap with the exact ground state for a wide range of interactions. However, we do not find that the exact ground state has an even approximate double degeneracy in the regimes where the overlaps of the Gaffnian and the hierarchy are large. Approximate double degeneracy of the ground state is found where we believe the Read-Rezayi state is the proper ground state.[10]

V. DISCUSSION

If the Gaffnian does turn out to be a critical state, as suggested here, this then raises the question as to what the neighboring phases are. It is reasonable that one would be a "strong pairing" phase (albeit one that cannot be easily described within BCS theory[26]) which may correspond to the hierarchy wave function itself.[18] This would be quite natural considering the high mutual overlaps of the hierarchy and Gaffnian.

The nature of the state on the opposite side of the transition is a bit harder to guess at. One possibility is that it is the Read-Rezayi state. This would make some sense because of the similar ground state degeneracy. Here, we imagine that as we approach the transition from the hierarchy side, the putative zero energy state would drop continuously and hit zero energy at the Gaffnian critical point. It would then stay at zero energy through the Read-Rezayi phase. On the other hand, we should note that there is a notable topological difference between the Read-Rezayi and Gaffnian state, which is more evident on the sphere as they occur at different values of flux.

Yet another possible candidate for a state that might occur nearby is a charge density wave state. We leave the project of sorting out the details of this transition for future work.

If the Gaffnian is in fact a critical state, this means that the concept of "non-Abelian" statistics[6] may not be well defined. Indeed, the idea of statistics describes what happens to a system when particles are adiabatically exchanged. Since the definition of adiabatic usually requires any perturbation to the system to be on a time scale slower than $\hbar/\Delta$ with Δ the minimum gap in the system, if the system has gapless excitations, there is generally no way to have adiabaticity. One might ask whether any remnant of the idea of non-Abelian statistics still remain. This is a question that is hard to answer without knowing the details of what these "critical" low energy excitations are.

We now turn to the question of actual experiments. If we believe the Gaffnian to be a critical point, it would have to be observed as a (compressible) transition point between two phases. Certainly the macroscopic degeneracy in the presence of quasiparticles would be one clear experimental signature (which in principle should show up in, say, the specific heat). However, as discussed above, there may be other low energy critical modes in the system which could make it hard to pick out this contribution from other degrees of freedom. Further study of the critical modes will certainly be required before any detailed prediction can be made.

In this paper we have discussed in detail the Gaffnian wave function which is the exact ground state of a particular three-body interaction. One might be concerned that such an interaction is not particularly physical. (Although should fractional quantum Hall effect ever be realized in cold rotating atoms, it appears possible, at least in principle, to engineer multiparticle interactions.[27]) Nonetheless, we remind the reader that this strange interaction is simply a way to make the analytic study of this wave function tractable. We believe that very similar physics may be observed even for quite different (and more realistic) interactions. This technique has been used successfully in the past to study states such as the Pfaffian[16] and the Read-Rezayi[9] states which, although exact ground states of many-body interactions, also appear to properly describe the same phase realized for the two body Coulomb interaction. In other words, the nonphysical interaction represents a wide range of possible interactions that show the same physics. Indeed, our numerical work seems to suggest something similar may be at play here.

ACKNOWLEDGMENTS

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APPENDIX A: ANALYTIC COUNTING OF ZERO ENERGY STATES

In this appendix we will enumerate all possible zero energy states of the Hamiltonian P_3^3 on the sphere with any number of particles and at any given flux. Our approach will be in two parts. In Sec. A 1 we will write down a linearly independent set of zero energy states (and we will count them). It is this section that shows most clearly how the semionic zero modes arise. Then in Sec. A 2 we will show that these wave functions are indeed zero energy states of P_3^3. Finally, we will show in Appendix A 3 that these states indeed form a complete set of the zero energy states. The arguments here are quite similar to those given in Refs. 16 and 17. However, here the situation is more complicated as our zero modes are not simple fermions and bosons as in those two references. Note that throughout this appendix we will focus on the case where N (and therefore $2n$) is even. The N odd case is a relatively simple generalization.

1. Counting states

We start with the requirement that ψ vanishes as three powers as any three particles approach each other. The wave function (1) clearly provides one such solution (at a given value of flux). We will call this the Gaffnian ground state. In this section we propose a more general form for wave functions when there are some arbitrary number $2n$ quasiholes (or n additional flux) added to the ground state, and we will count the number of such states that are linearly independent.

Inspired by the work of Refs. 16 and 17 and analogous to the Pfaffian, Haffnian, and Haldane-Rezayi states, we write our proposed wave function in a form with broken and unbroken pairs. Let us declare that F of the N total particles are unpaired. Restrictions on F will be determined later. We then propose the following form for our wave functions:

$$\psi_G = \tilde{S}\Bigg[\prod_{1\le a<b\le N/2}(z_a-z_b)^{2+q}\prod_{N/2+1\le c<d\le N}(z_c-z_d)^{2+q}\prod_{1\le e\le N/2<f\le N}(z_e-z_f)^{1+q}$$
$$\times\prod_{1\le g\le (N-F)/2}\frac{\Phi(z_{g+F/2},z_{g+(N+F)/2};w_1,\ldots,w_{2n})}{z_{g+F/2}-z_{g+(N+F)/2}}\left(\prod_{i=1}^{F/2}\prod_{j=N/2+1}^{(N+F)/2}(z_i-z_j)^{1+q}\Omega(z_1,\ldots,z_{F/2};z_{N/2+1},\ldots,z_{(N+F)/2})\right)\Bigg], \quad \text{(A1)}$$

where as above $\tilde{S}$ either symmetrizes (for bosons, even q) or antisymmetrizes (for fermions, odd q) over all of the z coordinates. Here we have defined Φ to be the Read-Rezayi quasihole insertion[16]

$$\Phi(z_1,z_2;w_1,\ldots,w_{2n}) = \frac{1}{(n!)^2}\sum_{\tau\in S_{2n}}\prod_{i=1}^{n}(z_1-w_{\tau(2i-1)})(z_2-w_{\tau(2i)}) \quad \text{(A2)}$$

and Ω is a wave function for the zero modes to be determined later. (The sum over $\tau\in S_{2n}$ is the sum over permutations of the $2n$ variables w.) We now specialize to the case of $q=0$ (bosons) for simplicity. Since the particles must be at the flux $N_\phi=3\left(\frac{N}{2}-1\right)+n$ we will deduce that the highest degree of the unpaired particle coordinates is $n-1$. To see how this is deduced we start by defining $A\equiv\{z_1,\ldots,z_{N/2}\}$, and $B\equiv\{z_{N/2+1},\ldots,z_N\}$). We then simply count up powers of z_k appearing in ψ_G

$$\psi_G = S\Big[\underbrace{\underbrace{\prod_{i<j}(z_{A_i}-z_{A_j})^2}_{2(\frac{N}{2}-1)\text{ for A}}\underbrace{\prod_{i<j}(z_{B_i}-z_{B_j})^2}_{2(\frac{N}{2}-1)\text{ for B}}\underbrace{\prod_{i,j}(z_{A_i}-z_{B_j})}_{\frac{N}{2}\text{ for A\&B}}}_{\text{both paired \& unpaired}}\underbrace{\underbrace{\prod_i\frac{\Phi(z_{A_i},z_{B_i})}{(z_{A_i}-z_{B_i})}}_{-1+n\text{ for A\&B}}}_{\text{paired only}}\underbrace{\underbrace{\prod_{i,j}(z_{A_i}-z_{B_j})}_{\frac{F}{2}\text{ for A\&B}}\quad\underbrace{\Omega(\ldots)}_{???\text{ for A\&B}}}_{\text{unpaired only}}\Big]. \tag{A3}$$

Here S symmetrizes over all coordinates z. In this equation, we have written beneath each term the number of powers of z_k occurring. Thus adding up the powers, we conclude that Ω is some polynomial in unpaired coordinates of degree $m_i: 0 \leq m_i \leq n-1-\frac{F}{2}$ for each unpaired coordinate z_i. Notice also, that this puts a restriction on F: $F \leq 2n-2$, and since obviously $F \leq N$, we obtain

$$F \leq \min(2n-2, N) \tag{A4}$$

as written above in the main text. The maximum degree of Ω occurs when $F=0$ and is given by $n-1$.

To see how many linearly independent wave functions we have for given $\{N,n,F\}$ we proceed as follows. We choose $N-F$ (necessarily even here) of the N coordinates and group them together in pairs $\{(z_{a_i},z_{b_i})\}$ for $i=1,\ldots,(N-F)/2$ with $a_i,b_i \in [1,\ldots,N]$ and $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ and $a_i \neq b_j$ for all i,j. We then bring together the position of the paired paired particles to coordinates $\tilde{z}_i$. In other words we set $z_{a_i} = z_{b_i} = \tilde{z}_i$ for $i=1,\ldots,(N-F)/2$. Taking this limit selects out a particular group of terms from the original full symmetrization that do not vanish. In particular the nonvanishing terms are the terms in which a factor of $\Phi(z_{a_i},z_{b_i};\ldots)(z_{a_i}-z_{b_i})^{-1}$ appeared for each pair (z_{a_i},z_{b_i}). The other terms will have a factor of $(z_{a_i}-z_{b_i})$ in the numerator, and will vanish in these limits. After taking these limits we are left with

$$\begin{aligned}\tilde{\psi} = S'\Bigg[&\underline{\prod_{i<j}(\tilde{z}_i-\tilde{z}_j)^6\prod_k\Phi(\tilde{z}_k;w_1,\ldots,w_{2n})}\prod_{1\leq a<b\leq F/2}(z_a-z_b)^2\prod_{N/2+1\leq c<d\leq(N+F)/2}(z_c-z_d)^2\\ &\times\underline{\underline{\prod_{l=1}^{(N-F)/2}\prod_{e=1}^{F/2}(z_e-\tilde{z}_l)^3\prod_{m=1}^{(N-F)/2}\prod_{f=N/2+1}^{(N+F)/2}(\tilde{z}_m-z_f)^3}}\prod_{g=1}^{F/2}\prod_{h=N/2+1}^{(N+F)/2}(z_g-z_h)^2\Omega(z_1,\ldots,z_{F/2};z_{N/2+1},\ldots,z_{(N+F)/2})\Bigg],\end{aligned} \tag{A5}$$

where S' symmetrizes over $\{\tilde{z}_i\}$ and $\{z_1,\ldots,z_{F/2},z_{(N+1)/2},\ldots,z_{(N+F)/2}\}$ separately. In other words, S' is what remains of the original full symmetrization over the N particles. The underlined factors contain the dependence of $\tilde{\psi}$ on $\tilde{z}$, and are symmetric in $\{\tilde{z}_i\}$, while the doubly underlined factor is symmetric in $\{z_1,\ldots,z_{F/2},z_{(N+1)/2},\ldots,z_{(N+F)/2}\}$ as well. Thus, the symmetrization S' reduces to S'' which symmetrizes over unpaired particles only (because the expression is already symmetric in $\tilde{z}_i$), and we can rewrite the wave function as

$$\begin{aligned}\tilde{\psi} = (\tilde{\psi}_{\mathrm{LJ}})^2 S''\Bigg\{\Bigg[&\Omega(\ldots)\prod_{l=1}^{(N-F)/2}\prod_{e=1}^{F/2}(z_e-\tilde{z}_l)^3\prod_{m=1}^{(N-F)/2}\prod_{f=N/2+1}^{(N+F)/2}\\ &\times(\tilde{z}_m-z_f)^3\Bigg]\Bigg[\prod_{i<j}(\tilde{z}_i-\tilde{z}_j)^6\prod_k\Phi(\tilde{z}_k)\Bigg]\Bigg\},\end{aligned} \tag{A6}$$

where $\tilde{\psi}_{\mathrm{LJ}}$ is a Laughlin-Jastrow factor in the unpaired particle coordinates. We thus discover that $\Omega(\cdots)$ can always be taken to be fully symmetric in its arguments (any nonsymmetric parts vanish when symmetrized). We can thus think of this as a bosonic wave function for the zero modes. However, we have already determined the maximal degree of $\Omega(\cdots)$ to be $n-1-\frac{F}{2}$. The minimal degree is obviously 0, so we have a total of $n-\frac{F}{2}$ orbitals in which to put F bosons. There are $\binom{(n-\frac{F}{2})+F-1}{F}=\binom{n+\frac{F}{2}-1}{F}$ such linear independent wave functions. This is equivalent to placing F fermions in $n+F/2-1$ orbitals. Since the number of orbitals changes half as fast as the number of particles we put in them, these particles have semionic exclusion statistics.[20]

2. Zero energy

We will continue on to demonstrate that the linearly independent set of wave functions we have just written down is in fact a complete set of zero energy states of the Hamiltonian P_3^3. First, however, we show that these wave functions are indeed zero energy states. The wave function for any zero energy state must vanish as three or more powers when three particle positions come to the same point. On the sphere,[12,14] this is equivalent to restricting the total angular momentum of the cluster of three particles to be no greater than $3N_\phi/2-3$.

First we will show that the proposed wave functions ψ_G are zero energy eigenstates of this Hamiltonian. For the ground state, i.e., no additional flux $(n=0)$ we have $N_\phi=3(\frac{N}{2}-1)$. Let us look at the $(ijk)\equiv(z_i,z_j,z_k)$ triplet. We want to know what the highest total angular momentum is for this triplet in our wave function ψ_G. The wave function can be rewritten (in the manner of Haldane[12]) as a sum of terms proportional to $f_{\rm rel}(z_i,z_j,z_k)f_{\rm tot}(z_i,z_j,z_k)$, where $f_{\rm rel}(\,)$ is an eigenstate of l_{ijk}, the three particle relative angular momentum operator, and $f_{\rm tot}(\,)$ is an eigenstate of L_{ijk}, the three particle total angular momentum operator. Note that, as above, we will always focus on $q=0$ for simplicity. (The $q\neq 0$ case is a relatively minor generalization.) To find the total angular momentum, we look at the maximal degree of $z_i^\alpha z_j^\beta z_k^\gamma$ in $f_{\rm tot}(\,)$, and find the total angular momentum $L=\max[\frac{1}{2}(\alpha+\beta+\gamma)]$. To find the maximum total angular momentum we must consider all possible ways to have chosen the triplet (z_i,z_j,z_k) from the many terms in the wave function. In particular, we must look at all cases of which coordinate is one of the paired variables, and which is unpaired, as well as looking at which variable is an A coordinate, and which is a B coordinate. Here we are looking at the relative angular momentum of a given triplet in each of the many terms of the symmetrization sum. All possibilities are enumerated next.

Case 1: $i,\ j,\ k\in A,\ i<j<k$. Here we have

$$\alpha=\deg_{z_i}\psi_G=2\left[\left(\frac{N}{2}-1\right)-2\right]+\frac{N}{2}-1,$$

$$\beta=\deg_{z_j}\psi_G=2\left[\left(\frac{N}{2}-2\right)-1\right]+\frac{N}{2}-1,$$

$$\gamma=\deg_{z_k}\psi_G=2\left[\frac{N}{2}-3\right]+\frac{N}{2}-1. \tag{A7}$$

Using $L=\frac{1}{2}(\alpha+\beta+\gamma)$ and with $N_\phi=3(N/2-1)$ we obtain in this case $L=\frac{3}{2}N_\phi-6$.

Case 2a: $i,\ j\in A,\ i<j;\ k\in B$ *with pairing of the form* $(ia)(jb)(ck)$, i.e., *terms of the form*

$$\frac{\Phi(z_i,z_a)}{(z_i-z_a)}\frac{\Phi(z_j,z_b)}{(z_j-z_b)}\frac{\Phi(z_c,z_k)}{(z_c-z_k)}. \tag{A8}$$

Here we have

$$\alpha=\deg_{z_i}\psi_G=2\left[\left(\frac{N}{2}-1\right)-1\right]+\left(\frac{N}{2}-1\right)-1, \tag{A9}$$

$$\beta=\deg_{z_j}\psi_G=2\left[\frac{N}{2}-2\right]+\left(\frac{N}{2}-1\right)-1,$$

$$\gamma=\deg_{z_k}\psi_G=2\left[\frac{N}{2}-1\right]+\left(\frac{N}{2}-2\right)-1.$$

Similarly, adding up these powers we obtain an angular momentum $L=\frac{3}{2}N_\phi-4$.

Case 2b: $i,\ j\in A,\ i<j;\ k\in B$ *with pairing of the form* $(ia)(jk)$, i.e., *terms of the form*

$$\frac{\Phi(z_i,z_a)}{(z_i-z_a)}\frac{\Phi(z_j,z_k)}{(z_j-z_k)}. \tag{A10}$$

Here we have

$$\alpha=\deg_{z_i}\psi_G=2\left[\left(\frac{N}{2}-1\right)-1\right]+\left(\frac{N}{2}-1\right)-1,$$

$$\beta=\deg_{z_j}\psi_G=2\left[\frac{N}{2}-2\right]+\left(\frac{N}{2}-1\right),$$

$$\gamma=\deg_{z_k}\psi_G=2\left[\frac{N}{2}-1\right]+\left(\frac{N}{2}-2\right) \tag{A11}$$

which results in an angular momentum $L=\frac{3}{2}N_\phi-3$.

These cases are the only possibilities. Thus the highest total angular momentum for any triplet is $\frac{3}{2}N_\phi-3$ and so the proposed wave function ψ_G is a zero energy eigenstate of the Gaffnian Hamiltonian P_3^3 as claimed.

3. Completeness

Now we show that the proposed wave functions span the complete set of zero energy states of the Gaffnian Hamiltonian. To do this we will construct the most general zero energy eigenstate and show that it takes the form of our proposed wave function. Take the following zero energy wave function $\psi_G=\psi_{\rm LJ}^2\phi_G$, where here $\psi_{\rm LJ}$ is the Laughlin-Jastrow factor for all of the particles. Consider the behavior of ψ_G as particles in an arbitrary triplet (ijk) approach each other, while the other particles remain far away from the three:

$$\psi_G\propto\underbrace{(z_i-z_j)^2(z_j-z_k)^2(z_k-z_i)^2}_{\alpha+\beta+\gamma=6,\ \text{part of}\ \psi^2_{LJ}}\underbrace{(z_i-z_j)^{q_{ij}}(z_j-z_k)^{q_{jk}}(z_k-z_i)^{q_{ki}}}_{\alpha+\beta+\gamma=q_{ij}+q_{jk}+q_{ki}=Q,\ \text{part of}\ \phi_G}\quad. \tag{A12}$$

The wave function vanishes as $6+Q$ powers as these three particles come together. This is equivalent[12,14] to saying that the total angular momentum of three particles is $L=\frac{3}{2}N_\phi-(6+Q)$. (Since we are on the sphere, the maximum angular momentum of each particle is $\frac{N_\phi}{2}$. Any relative angular momentum reduces the total by a corresponding amount.[12,14]) Furthermore, by analyticity of ψ_G we must have $q_{mn}\geqslant -2$.

Now, in order for ψ_G to be a zero energy state of the Gaffnian Hamiltonian, we must have $Q\geqslant -3$ (so that the relative angular momentum of the cluster is greater than or equal to $3=6+Q$). From here on we will concentrate on the ϕ_G factor of the eigenstates, restoring the ubiquitous ψ^2_{LJ} at the end. Allowed forms in the Laurent expansion of ϕ_G as (ijk) approach each other are

$$\frac{1}{(z_i-z_j)^2(z_j-z_k)}, \tag{A13}$$

$$\frac{z_k-z_i}{(z_i-z_j)^2(z_j-z_k)^2}, \tag{A14}$$

$$\frac{1}{(z_i-z_j)(z_j-z_k)(z_k-z_i)} \tag{A15}$$

as well as the same terms with (ijk) permuted. However, it is easy to see that the second two forms reduce to the first since expression (A14) is equivalent to

$$\frac{-1}{(z_i-z_j)^2(z_j-z_k)}+\frac{-1}{(z_i-z_j)(z_j-z_k)^2} \tag{A16}$$

and expression (A15) is equivalent to

$$\frac{-1}{(z_k-z_i)^2(z_j-z_k)}+\frac{-1}{(z_i-z_j)(z_k-z_i)^2}. \tag{A17}$$

It follows then, that it is enough to consider forms of the type of Eq. (A13) for triplets (ijk) [as well as the same form with permutations of (ijk)]. When $(ijk)\to\tilde{z}$, the most general zero energy eigenstate should have the form

$$\phi_G \propto \frac{F(z_i,z_j,z_k)}{(z_i-z_j)^2(z_j-z_k)} \tag{A18}$$

[or a form similar to this with any permutation of (ijk)], where $F(\cdots)$ must be analytic (i.e., with no poles).

Now arbitrarily pair up and relabel the particles, e.g., $(z_{A_1},z_{B_1}),(z_{A_2},z_{B_2}),\ldots,(z_{A_{N/2}},z_{B_{N/2}})$. Look at the most singular part of ϕ_G as particles within these pairs approach each other, while pairs are kept separated:

$$\phi_G \propto \frac{1}{(z_{A_1}-z_{B_1})^2}\frac{1}{(z_{A_2}-z_{B_2})^2}\cdots\frac{1}{(z_{A_{N/2}}-z_{B_{N/2}})^2}\phi_{N/2}. \tag{A19}$$

Since we have isolated the most singular part of ϕ_G, it is clear that $\phi_{N/2}$ cannot contain factors $(z_{A_i}-z_{B_i})^{-1}$. If we now consider triplets $(A_1,B_1,k)\,\forall\, k\notin\{A_1,B_1\}$, and bring particle k close to (A_1,B_1) it is clear that $\phi_{N/2}$ must contain a factor of $(z_{A_1}-z_k)^{-1}$ or $(z_{B_1}-z_k)^{-1}$, but not both, in order to satisfy the requirements on the pole structure deduced above. We might be led to naively define

$$\phi_{N/2}=\sum_j\left(\prod_{m,n}\frac{\phi_{1,j}}{(z_{A_1}-z_{\mathcal{D}^j_m})(z_{\mathcal{C}^j_n}-z_{B_1})}\right), \tag{A20}$$

where $\mathcal{C}^j\cup\mathcal{D}^j=\{z_i\}$; $\mathcal{C}^j\cap\mathcal{D}^j=\varnothing$ (i.e., a partition of the set of particle coordinates), and j indexes all possible partitions. However, the pole structure places further restrictions on the sets $\mathcal{C}$ and $\mathcal{D}$. In particular, A_i and B_i cannot both be in $\mathcal{C}$ or in $\mathcal{D}$. Otherwise, supposing $A_i,B_i\in\mathcal{C}$, we have (schematically) the following:

$$\begin{aligned}\phi_G \propto{}& \frac{1}{(z_{A_1}-z_{B_1})^2}\cdots\frac{1}{(z_{A_i}-z_{B_i})^2}\\ &\cdots\left[\cdots\frac{1}{(z_{A_1}-z_{A_i})(z_{A_1}-z_{B_i})}\cdots\right]\end{aligned} \tag{A21}$$

and we immediately recognize, that the triplet (A_1,A_i,B_i) has too many poles ($Q<-3$). We conclude, that for ith pair only one factor is allowed in $\phi_{N/2}$, either $(z_{A_1}-z_{A_i})^{-1}$ or $(z_{A_1}-z_{B_i})^{-1}$. That is, the partitions are such that $\mathcal{C}$ includes only one representative of any pair, and $\mathcal{D}$ includes the complementary member of this pair, i.e., necessarily $A_i\in\mathcal{C}$, $B_i\in\mathcal{D}$ or $B_i\in\mathcal{C}$, $A_i\in\mathcal{D}$. At this point we can recognize that our notation of $\mathcal{C}$ and $\mathcal{D}$ is redundant, and that we can rewrite

$$\phi_{N/2}\propto\sum_{k\in\text{partitions}}\left(\prod_{i\neq j}\frac{\tilde{\phi}_k}{(z_{A^k_i}-z_{B^k_j})}\right), \tag{A22}$$

where $\tilde{\phi}_k$ cannot contain any more poles, and the conventions are that for all k the same coordinates are paired up, i.e., $\{A^k_i,B^k_i\}=\{A^{k'}_i,B^{k'}_i\}$ are equal as sets. The difference between partition k and partition k' is the order of coordinates within a pair, i.e., we could have $z_{A^k_i}=z_{B^{k'}_i}$, $z_{B^k_i}=z_{A^{k'}_i}$ or $z_{A^k_i}=z_{A^{k'}_i}$, $z_{B^k_i}=z_{B^{k'}_i}$. Clearly the sum over k is a subset of symmetrization over all particles.

Finally, we should also include the exchange of pairs $(z_{A^k_i},z_{B^k_i})\leftrightarrow(z_{A^k_j},z_{B^k_j})$, since the most singular part is symmetric under this exchange, and arrive at

$$\phi_{N/2}=\sum_{\text{pairings}}\sum_{k\in\text{partitions}}\left(\prod_{i\neq j}\frac{\tilde{\phi}_k}{(z_{A^k_i}-z_{B^k_j})}\right). \tag{A23}$$

Then for the particular choice of pairs we will have

$$\phi_G\propto\prod_i\frac{1}{(z_{A_i}-z_{B_i})^2}\left[\sum_{\text{pairings}}\sum_{k\in\text{partitions}}\left(\prod_{i\neq j}\frac{\tilde{\phi}_k}{(z_{A^k_i}-z_{B^k_j})}\right)\right] \tag{A24}$$

and recover the whole eigenfunction by symmetrization over all particles and multiplication by the Jastrow factor squared:

$$\psi_G = \psi_{\mathrm{LJ}}^2 S\left\{\prod_i \frac{1}{(z_{A_i}-z_{B_i})^2} \times\left[\sum_{\text{pairings}}\sum_{k\in\text{partitions}}\left(\prod_{i\neq j}\frac{\tilde{\phi}_k}{(z_{A_i^k}-z_{B_j^k})}\right)\right]\right\}. \tag{A25}$$

The wave function of the densest state has as few zeros as possible, and to find it we may choose $\tilde{\phi}_k \equiv 1$, then

$$\psi_G = \psi_{\mathrm{LJ}}^2 S\left\{\sum_{\text{pairings}}\sum_{k\in\text{partitions}}\left(\prod_i \frac{1}{(z_{A_i^k}-z_{B_i^k})^2}\prod_{i\neq j}\frac{1}{(z_{A_i^k}-z_{B_j^k})}\right)\right\} \tag{A26}$$

$$\equiv \psi_{\mathrm{LJ}}^2 S\left[\prod_i \frac{1}{(z_{A_i}-z_{B_i})^2}\prod_{i\neq j}\frac{1}{(z_{A_i}-z_{B_j})}\right]. \tag{A27}$$

This can be recognized as the proposed Gaffnian wave function with no broken pairs and no added flux. To obtain the states of lower density we need to consider the case of nonconstant $\tilde{\phi}_k(z_{A_1},z_{B_1};\ldots;z_{A_{N/2}},z_{B_{N/2}})$. By analyzing the symmetry of denominators we find that $\tilde{\phi}_k$ must be symmetric under the exchange of pairs $(z_{A_i},z_{B_i})\leftrightarrow(z_{A_j},z_{B_j})$. We now claim that a complete basis for functions that satisfy this symmetry condition is given by functions of the form

$$\sum_{\tau\in S_{N/2}}\prod_{i=1}^{N/2} f_i(z_{A_{\tau(i)}},z_{B_{\tau(i)}}), \tag{A28}$$

where the f_i's are chosen from a basis for arbitrary polynomials of their two arguments. While this may seem to be a strange way to write a basis for the polynomial $\tilde{\phi}_k(z_{A_1},z_{B_1};\ldots;z_{A_{N/2}},z_{B_{N/2}})$ this is actually a form well known to physicists. To see this, imagine a system of $N/2$ bosons where the "position" of each boson is specified by two coordinates (z_1,z_2). The functions f_i are basis functions for the single "particle" positions. All multiparticle states can be written as symmetrized (bosonic) linear combinations of the occupancies of these basis states.

Consider now the case, when we have added n quanta of flux to the Gaffnian ground state. The highest degree of $f_i(\,)$ is n, and we could choose basis polynomials $f_i(z_1,z_2)$ of the form $z_1^{n_1}z_2^{n_2}$ with $0\leq n_1,n_2\leq n$. The dimension of this space is $(n+1)^2$. However, a different basis set turns out to be more useful. Specifically, it is useful to separate functions f_i that vanish in the limit $z_1\to z_2$, from ones that do not.

We choose a basis for our space of f_i which decomposes into two disjoint sub-bases: the symmetric $z_1^{n_1}z_2^{n_2}+z_1^{n_2}z_2^{n_1}$ with $0\leq n_1\leq n_2\leq n$ and the antisymmetric $z_1^{n_1}z_2^{n_2}-z_1^{n_2}z_2^{n_1}$ with $0\leq n_1\leq n_2\leq n-1$. The dimensions of subspaces spanned by them are $\frac{1}{2}(n+2)(n+1)$ and $\frac{1}{2}(n+1)n$, respectively. Clearly the span of the antisymmetric sub-basis vanishes as $z_1\to z_2$. The quotient of the full space by the span of the antisymmetric sub-basis is just the span of the symmetric sub-basis ($\mathcal{S}$), i.e., symmetric polynomials. Of these, polynomials which vanish as $z_1\to z_2$ are spanned by $(z_1-z_2)^2(z_1^{n_1}z_2^{n_2}+z_1^{n_2}z_2^{n_1})$, with $0\leq n_1\leq n_2\leq n-2$. The dimension of this subspace is $\frac{1}{2}n(n-1)$.

The quotient of $\mathcal{S}$ by the subspace of the vanishing symmetric polynomials has dimension $2n+1$ and contains symmetric polynomials in two variables that do not vanish in the limit $z_1\to z_2$, we will call this quotient $\mathcal{Q}$. However, by considering the Taylor expansion[16] of Read-Rezayi pairing form $\Phi(z_1,z_2)$ given in Eq. (A2) we have already found a set of $2n+1$ linearly independent symmetric polynomials in two variables, thus we may choose them as the basis of this quotient space.

Now given a choice of $\tilde{\phi}_k$ we obtain a zero energy state of the Hamiltonian. Further, all possible zero energy states can be written in this way. We can now decompose any $\tilde{\phi}_k$ into basis polynomials f_i of the above described form. Let our choice be such that $f_i(\,)$ for $1\leq i\leq\frac{F}{2}$ belong to the subspace of polynomials that vanish as $z_1\to z_2$ and $f_i(\,)$ for $\frac{F}{2}+1\leq i\leq\frac{N}{2}$ belong to the complementary subspace, i.e., $\mathcal{Q}$. Then each vanishing $f_i(\,)$ simplifies with the appropriate factor in the denominator of ϕ_G producing a "broken pair," and the remaining factors form what we above called $\Omega(\,)$, whereas the product of nonvanishing $f_i(\,)$ can be reexpressed as a linear combination of Read-Rezayi pairing forms $\Phi(\,)$. Thus we conclude that the most general zero energy eigenstate of H_G is of the conjectured form, and therefore we counted the complete degeneracy of eigenstates for a given value of additional flux n.

APPENDIX B: AN EXAMPLE OF ANGULAR MOMENTUM ADDITION

We would like to determine the full angular momentum spectrum of the zero energy states of the Hamiltonian P_3^3 using Eq. (5). Here we will consider the example of $N=4$ particles and $n=3$ (six quasiholes). Equation (5) tells us that we should have a total number of zero energy states given by the sum of three terms corresponding to $F=0,2,4$. For $F=4$ we have (6 bosons in 1 orbitals) $\otimes$ (4 fermions in 4 orbitals). Both six bosons in one orbital on four fermions in four orbitals have $L=0$, so overall this is an $L=0$ state. The $F=2$ case is more tricky. Here we have (6 bosons in 2 orbitals) $\otimes$ (2 fermions in 3 orbitals). First we take six bosons in two orbitals. When there are two orbitals on a sphere, we clearly have $L=1/2$. So the two orbitals have $L_z=\pm 1/2$. There are seven ways to fill these orbitals with six bosons, which we can write as (6,0), (5,1),…,(0,6). Counting the total L_z of each of these states, we get $3,2,1,0,-1,-2,-3$ which we recognize as being $L=3$. Thus, six bosons in two orbitals is $L=3$. Similarly, we look at two fermions in three orbitals. The three orbitals must be $L=1$ with $L_z=1,0,-1$. We can fill the three orbitals with two fermions in three ways, which have $L_z=1,0,-1$ so we recognize this as $L=1$. Now we must add together the angular momentum of (6 bosons in 2 orbitals) $\otimes$ (2 fermions in 3 orbitals). This means we need to add $L=3$ with $L=1$. By the usual angular momentum addition rules we obtain $L=2,3,4$. Finally, we turn to the $F=0$ case. Here we have (6 bosons in 3 orbitals)

$\otimes$ (0 fermions in 2 orbitals). The 0 fermions in two orbitals clearly has $L=0$. It is a simple exercise to count up the possibilities for six bosons in three orbitals. We discover that this has $L=0,2,4,6$. Putting together all of the results we find that the zero energy states of the Hamiltonian P_3^3 for N particles with $n=3$ occur at angular momentum $L=0,0,2,2,3,4,4,6$ which agrees with the results of exact diagonalizations.

APPENDIX C: FURTHER GENERALIZED WAVE FUNCTIONS

Although there may be many possible ways to generalize Gaffnian wave function,[19] the form written in Eq. (1) suggests a generalization from paired to clustered wave functions where instead of dividing the particles into two groups, we divide the particles into g groups. Let us assume the number of particles N in the system is divisible by g and write $N=gn$. We then write the wave function

$$\Psi = \tilde{S}\left[\left\{\prod_{a=1}^{g}\left[\prod_{(a-1)n<i<j\le an}(z_i-z_j)\right]\right\} \times\left\{\prod_{1\le a<b\le g}\left[\prod_{i=1}^{n}\frac{1}{z_{(a-1)n+i}-z_{(b-1)n+i}}\right]\right\} \times\prod_{1\le i<j\le N}(z_i-z_j)^m\right] \tag{C1}$$

with $m\geq 1$ where again $\tilde{S}$ symmetrizes or antisymmetrizes for bosons (odd m) or fermions (even m), respectively. Counting powers of z we discover that this wave function occurs at flux

$$N_\phi = (N/g-1)-(g-1)+m(N-1) \tag{C2}$$

$$=(1/g+m)N-(g+m) \tag{C3}$$

corresponding to a filling fraction $\nu=g/(gm+1)$ which is just the Jain sequence. Furthermore the precise value of the flux (the shift) is also in agreement with the Jain series. This construction clearly reproduces the Gaffnian for $g=2$. For the bosonic case $(m=1)$ for general g this construction produces a wave function that does not vanish when g particles come to the same point, but vanishes as $g+1$ powers as the $(g+1)$ st particle arrives at that point. However, for $g>2$ this trial wave function is not the densest possible wave function with this particular property.[14] Nonetheless, we believe that this, and other related wave functions can generally be constructed with simple projection rules. For example, for the $g=3$, $m=1$ case of Eq. (C1) this wave function is the unique densest wave function that does not vanish as three particles come together, that always vanishes as at least four powers when four particles come together, and vanishes faster than four powers if particles are brought together in groups of two and then two groups of two are brought together.

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