# Thermal propagation of fluxons in two-dimensional Josephson junction arrays

G. Filatrella,[1] S. Girotti,[2] and G. Rotoli[3]
[1]*Laboratorio Regionale SuperMat CNR-INFM Salerno and Dipartimento di Scienze Biologiche ed Ambientali, Università del Sannio, Via Port'Arsa, 11, I-82100 Benevento, Italy*
[2]*Dipartimento di Fisica, Università di Camerino, Via Madonna delle Carceri, 9, I-62032 Camerino (MC), Italy*
[3]*CNISM and DIMEG, Università di L'Aquila, Localitá Monteluco, I-67040 L'Aquila, Italy*
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We have modeled flux quanta propagation in two-dimensional square arrays of small Josephson junctions when the motion is hampered by the presence of an effective barrier due to the superconducting loops. The energy barriers have been estimated simulating the effect of thermal fluctuations and evaluating the barrier via the Arrhenius factor. The results have been compared with a much simpler semianalytic method, showing that the method is able to give an acceptable estimate. The strength of the fluxon-(anti)fluxon interaction as a function of the loop inductance and the distance between the excitations has been also evaluated. It is reported that the presence of a finite inductance substantially affects the interaction potential, and the contributions due to mutual inductances are found to further change the behavior of the interaction.

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## I. INTRODUCTION

Dissipation in superconductors can be ascribed to the motion of fluxons: the voltage due to the flux change causes the quasiparticles to move and to absorb energy.[1,2] This mechanism would always appear in superconductors above the maximum reversible magnetic field or the minimum field for the entrance of fluxons unless some pinning mechanism occured in the superconductor to hamper the motion, keeping the fluxons pinned for a sufficiently long time. In granular superconductors, the stopping mechanism is believed to be the cell of nonsuperconducting material surrounded by a superconducting path broken with weak links.[3] On a general ground, the pinning mechanism is explained by the breaking of spatial translational invariance in discrete systems that gives rise to the Peierls-Nabarro barrier.[4] The existence of a barrier to flux motion in artificial Josephson junctions arrays[5] (JJA) has indeed been proven before the discovery of high $T_c$ superconductors.[6] Lobb and co-workers have found that the intrinsic inhomogeneity causes the system to show an energy barrier to the motion of fluxons for all values of the physical parameters. The same authors also found the minimum value of the barrier, which is topology dependent. Thus, the minimum energy reads $\Delta U_{min} \simeq 0.2\hbar I_0/2e$ for square arrays and $\Delta U_{min} \simeq 0.04\hbar I_0/2e$ for triangular arrays ($I_0$ is the mean critical current of the individual Josephson junctions of the array and $\hbar I_0/2e$ is the Josephson energy, or the energy necessary to increase the current to the maximum Josephson current). For a given geometry and critical current, the main parameter that governs the system and determines the energy barriers is the self-inductance $L$ of the smallest loops, combined in the so-called superconducting quantum interference device (SQUID) parameter $\beta_L = 2eLI_0/\hbar$. The minimum-energy barrier is retrieved for $\beta_L \ll 1$, in which limit the model of JJA can be approximated by the $XY$ model.[7] The analysis of the large $\beta_L$ limit leads to the Bean model of a critical state pictures.[3,8]

A more detailed comparison has been proposed for a one-dimensional array case: a critical state model has been analyzed by Parodi and Vaccarone[9] and by Fehrenbacher *et al.*[10] A description of the critical state behavior in two-dimensional JJA has also been proposed by Pace *et al.*[11] Majhofer *et al.*[12] and Wolf and Majhofer[13] have proposed a model that for $\beta_L \gg 1$ retrieves the results of the Bean model.

A systematic study of the dependence of the energy barriers on the external current and fields, varying the SQUID parameter $\beta_L$, was proposed in Ref. 14. The basic idea is to evaluate the energy barrier with a semianalitic approach to avoid brute force simulation of the Langevin equation, as proposed by Lobb *et al.*[6] In fact, the energy barrier $\Delta U$ for a single degree of freedom is found by evaluating the difference between the minimum and maximum of the energy $U$: $\Delta U = U^{max} - U^{min}$, without solving the corresponding Langevin equation (we have adopted here the obvious notation that $U^{max}$ is the saddle point corresponding to the maximum potential, and correspondingly $U^{min}$ is the stable minimum). For a JJA with many degrees of freedom, the approach cannot be used because the noise acts on all of them.[15]

To circumvent these difficulties, it has been proposed[14] that one can estimate the escape times by taking advantage of a variation of a technique called the pseudopotential.[16] This technique is based on the minimum available noise energy, which deals with systems where there exist many trajectories to leave a potential minimum, each with a different activation energy. The principle states that the global escape time is governed, for sufficiently weak noise, by the Arrhenius factor $\exp(\Delta U/k_B T)$, where $\Delta U$ is the energy barrier corresponding to the minimum energy to be supplied by the noise. It has been found[14] that the energy barriers $\Delta U(\beta_L)$ exhibit at zero external field and applied current a monotone growth, S shaped, which can be summarized by a heuristic interpolating formula between the minimum-energy barrier for $\beta_L \ll 1$ and the energy barrier of decoupled arrays ($\beta_L \gg 1$). Introducing the normalized energy $\varepsilon = U/(\hbar I_0/2e)$ for square arrays, the two energies read $\Delta\varepsilon_0 = 0.203$ and $\Delta\varepsilon_{max} = 2$, respectively (in excellent agreement with the findings of Ref. 6).

A principal question about these results is whether or not the assumptions can be checked against brute force numeri-

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cal simulations, at least in some selected points, to control if the pseudopotential approach is able to estimate the energy barrier with good approximation. This will be the first objective of the present work. We will prove that the pseudopotential method is indeed acceptable, and we will therefore take advantage of the simplicity of this approach to tackle two problems that otherwise would be very challenging.

(a) The role of mutual inductances. An accurate description of a two-dimensional array requires not only inclusion of the elementary geometrical self-inductances of the smallest loops but also the mutual interactions of each loop upon another loop.[17] The resulting model is a global interaction of each Josephson junction with another. We will determine the corrections to the pinning barriers when such mutual inductance terms are included.

(b) The estimate of the vortex-vortex interaction potential. In the continuous limit, the dipole interaction between vortices gives rise to a repulsive (attractive) potential for homopolar (heteropolar) vortices. In the limit of negligible inductances, the same potential applies to the Josephson fluxons interactions. We will extend this calculation to the case of finite self- and mutual inductances.

This paper is organized as follows. in Sec. II, we will write the model equations for the JJA with inductances and the method to compute the pinning energy both in pseudopotential approach and via the Langevin simulations; the latter will be employed in some selected points to check that the simplified method is appropriate for the barrier estimate. In Sec. III, we will discuss the effective vortex-vortex potential both in presence of self-inductance alone and with mutual inductance terms. Finally, in Sec. IV we will summarize the results.

## II. MODEL AND NUMERICAL RESULTS

### A. Models of JJA

A network of Josephson junctions can be modeled with different degrees of accuracy. To include the magnetic field produced by the screening currents in the model, one starts writing the fluxoid quantization in a loop containing Josephson junctions. In vector form for an $n_m = N \times M$ loops array, this reads (we denote a vector with an arrow and a matrix with a boldface character)

$$2\pi \frac{\vec{\Phi}_{ext}}{\Phi_0} + 2\pi \frac{\mathbf{L}\vec{I}^s}{\Phi_0} + \mathbf{M}\vec{\varphi} = 2\pi \vec{n}. \tag{1}$$

Here, $\vec{\Phi}_{ext}$ is the vector of the external magnetic flux in all the $n_m$ loops of the array, $\Phi_0 = \hbar/2e$ is the magnetic quantum flux, and $\vec{\varphi}$ is the phase vector of the Josephson junctions, which is formed with the gauge invariant phase differences, hereafter simply the junction phase, of all horizontal and vertical junctions in the array branches. For an array consisting of $N$ horizontal rows and $M$ vertical columns, we have $n_b = N \times (M+1) + M \times (N+1)$ branches. $\mathbf{M}$ is the $n_m \times n_b$ matrix summing up the junction phases in each array loop and finally, $\mathbf{L}$ is the full mutual inductance matrix between the array loops and the screening current vector $\vec{I}^s$.[17] In doing so, one includes mutual inductance terms that link the current circulating in any loop to all other loops in Eq. (1); this coupling leads to a model of considerable numerical complexity.[17] To explicitly write the dynamic equations, it is convenient to normalize the time with respect to the inverse characteristic frequency $\omega_c^{-1} = (2eRI_0/\hbar)$ and the current with respect to the critical current $I_0$, so the following equations for the phase can be deduced from Eq. (1) and current conservation at array nodes (see Fig. 1 for notation):

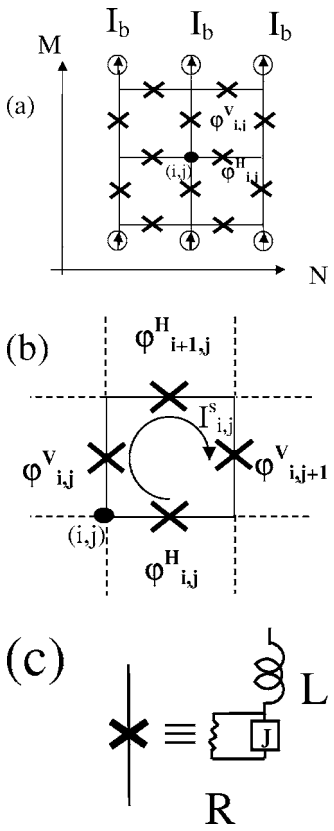


FIG. 1. Schematic circuit model for a two-dimensional $N \times M$ array (a). The screening currents appearing in Eq. (1) are the currents circulating in each loop, as shown in (b). Finally, each branch is modeled as a lumped inductance ($L$) and a Josephson junction consisting of an ideal element and a shunt resistor (c).

$$\dot{\vec{\varphi}} + \sin \vec{\varphi} = \frac{1}{\beta_L} \mathbf{K}_L \cdot \vec{\varphi} + \vec{\gamma}_b + \vec{\vartheta}, \tag{2}$$

with

$$\mathbf{K}_L = \frac{1}{L} \mathbf{M}^T (\mathbf{L})^{-1} \mathbf{M},$$

where $\vec{\gamma}_b$ is a bias current vector, which is the external bias, $I_b$ as modified by the mutual inductances and which depends on the bias current distribution over the array boundary. $\sin \vec{\varphi}$ stands for $\sin \varphi_j \mathbf{1}$. One usually assumes that a uniform current is fed to the array in the vertical direction (see Fig. 1), and the appropriate boundary conditions determined by the magnetic field at the edges. The current term $\vec{\vartheta}$ is the

Johnson noise current due to the resistors, which for the generic $j$th junction is assumed to have zero average and to be Gaussian correlated with intensity $\theta_0$,

$$\langle \vartheta_j(\tau) \rangle = 0, \tag{3}$$

$$\langle \vartheta_j(\tau) \vartheta_j(\tau') \rangle = 2\theta_0 \delta(\tau - \tau'), \tag{4}$$

where $\theta_0 = (K_B T / R I_0^2)\omega_c$ is the normalized temperature. This full model, which includes all mutual inductances *and* random fluctuations, is numerically very heavy and has not been actually employed in this work. Instead, we have made some simplifications of the full mutual inductance model [Eq. (2)], described in Sec. II B.

### B. Approximated inductances models

Different models correspond to different approximations of the matrix $\mathbf{L}$. The simplest level is to assume negligible inductance, a model which would lead to the $XY$ model. The physical approximation is that the screening currents $I^s$ would not produce any magnetic field trapped in the loops.[18] So the $XY$ approximation assumes that the magnetic flux is only due to the external flux and that $\mathbf{L}\vec{I}^s \simeq 0$. Therefore, it is believed to be a good approximation of arrays where $\mathbf{L} \simeq 0$.[7] Here, we want to investigate the properties of the array as a function of the inductance matrix itself, so the $XY$ model is not appropriate. We have, therefore, assumed in Eq. (1) that the matrix $\mathbf{L}$ is diagonal; i.e., only the loop self-inductance is retained. Such degree of approximation leads to the Nakajima-Sawada (NS) model.[19]

In the NS case,[19] the matrix $\mathbf{L} = L\mathbf{1}$, so $\mathbf{K}_L = \mathbf{M}^T\mathbf{M}$. This matrix is local; i.e., each phase $\varphi_j$ is coupled to the nearest six phases of the neighbor junctions for any array size. Thus, the algorithm complexity for the numerical solution is $O(n_b)$ rather than $O(n_b^2)$, and the associated Langevin equations require a moderate effort. This is the model utilized in the simulations of the Langevin equations.

The NS approximation[19] can be improved to include the magnetic field produced by the current $\vec{I}^s$ also in the other cells of the array.[17] Here, we use an approximated model which is able to include the relevant part of mutual inductance coupling in the array[20] with minimal memory requirements. Practically, we have developed a method to include all the mutual coupling in a radius equivalent to the distance of the fluxon from the array boundary, so the method works well for relatively large arrays with fluxons nearly at the array center [we successfully tested it for $\sim 100 \times 100$ arrays,[20] notwithstanding, the method is nevertheless $O(n_b^2)$ (Ref. 21)]. The mutual coupling model (MC) leads to a modification of the fluxon shape, but the pseudopotential method described in Sec. II C can be implemented including such mutual inductance effects without modification. On the other hand, the direct use of Langevin equations for the MC model is numerically very heavy being $O(n_b^2)$, so we do not use this approach in the MC model in connection with Langevin simulations but only in connection with the pseudopotential approach described in Sec. II C.

### C. Pseudopotential approach

In the pseudopotential approach, one tries to reconstruct the energy barrier for a multidimensional system to retrieve an effective activation energy.[16] The pseudopotential assumption states that the effective energy that will dominate the escape rate for the transition between two states will be the energy barrier corresponding to the lowest barrier over all possible trajectories. Put it another way, one can say that the escape time depends only marginally on trajectories which correspond to energy variations larger than the smallest variation. This is justified by the fact that if one has the competition between many different escape times $\tau_i$, the average escape rate $\tau^{-1}$ is given by

$$\frac{1}{\tau} = \sum_i \frac{1}{\tau_i} = \sum_i \frac{1}{\tau_0^i} e^{\Delta_i / k_B T}. \tag{5}$$

The exponential dependence of the escape rates on the energy barrier guarantees that even small changes in the energy will result in significant changes of the escape rates. Therefore, just the lower energy barrier is a good estimate of $\tau$ in Eq. (5).

To find the minimal trajectory without searching among all possible trajectories, we have made some further assumptions.

(1) To retrieve the minimal energy barrier, one can just move a single junction along a predetermined trajectory and let the other junctions relax in the equilibrium position. This would physically correspond to activating the noise term of one Josephson junction, while the other junctions just follow the deterministic trajectory. The idea behind this is that, at least in the limit of long escape times (the exponential part of the Kramer theory), a transition happens only when fluctuations will occur in an unusually strong intensity in a site, and it is unlikely that the other fluctuations will achieve in the same time a significant size. However, this leaves the question of which junction will cause the trajectory.

(2) Thus, we further assume that the noise term is effective only on the central junction which undergoes the maximum phase change. In fact, in Ref. 22 the energy barrier contributions due to junctions which do not undergo the maximum phase change have been computed, and it has been proven that the lowest barrier always corresponds to the central junction, such barrier being at least some 30% lower than the next barrier. The actual trajectory will be the one corresponding to the minimal energy variations, as stated by the pseudopotential approach.

The simplicity of the pseudopotential or adiabatic approach is particularly useful in exploring large arrays (the size scale of the escape phenomenon is dictated by the parameter $\beta_L$, whose inverse gives the number of cells over which the fluxon is spread) or the multidimensional parameter space (we recall that the relevant parameters are the driving current, the SQUID parameter, and the externally applied magnetic field).

The results for pseudopotential approach with self-inductances (NS model) and mutual inductances (MC model)

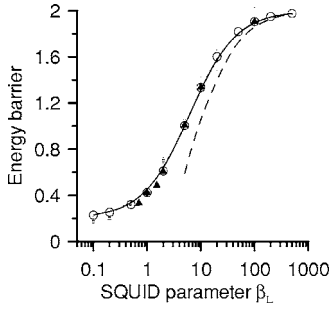


FIG. 2. Normalized pinning barriers as function of $\beta_L$ for an unbiased two-dimensional array. Circles represent the results of the adiabatic approach with the NS approximation, while the data computed including mutual inductance terms are shown as triangles. The solid line is the interpolating approximate equation [Eq. (7)] and the dashed curve is the estimate via the Frenkel-Kontorova approach (Ref. 23) for discrete one-dimensional array of JJ. Square symbols represent the energy barrier evaluated via simulations of the Langevin equations Eq. (2), reported together with the standard deviation. The barriers have been evaluated at the center of the array.

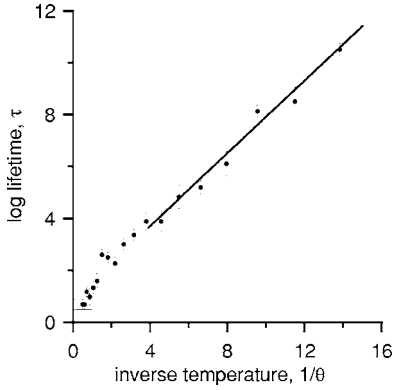


FIG. 3. Typical average escape time (on a logarithmic scale) as a function of the inverse normalized temperature. Symbols refer to the numerically evaluated average escape times, and error bars denote the standard deviation over realizations (10 in this example). The solid line is the interpolating linear fit from which the energy barrier is estimated using the last six points. The parameters of the simulations are $\beta_L=0.5$, $\gamma=0$, and $N=8$.

are reported in Fig. 2 circles and triangles, respectively). The findings for the MC model are consistent with previous calculations of barrier by Phillips *et al.*[17] Thus, this is not a new result, but it is a useful test of the mutual inductance algorithm, which is very involved even for the simple pseudopotential approach. In Sec. II D, we will further comment on these results in comparison with escape rates calculated by direct numerical simulations of Eq. (2) for the NS model.

### D. Numerically computed escape rates

To compute the activation energy, we have chosen as initial conditions a fluxon trapped in the central cell of the array. The array size is varied through simulations to ensure that the results are independent of the size (i.e., to avoid edge effects). For the lowest values of the inductance parameter $\beta_L$, we have simulated arrays of size $O(100^2)$. It is worth underscoring that the effort to compute numerically a single value of the barrier height is considerable, especially for low values of the screen parameter where the characteristic size requires simulations of large arrays.

The system evolves in time because of the presence of the random current term $\vec{\theta}$ in Eq. (2). The question we ask is how long will the fluxon be trapped in the same cell? This corresponds to numerically computing the escape time $\tau$. Assuming that some energy barrier prevents the fluxon form jumping, we expect the escape time to have the functional form

$$\tau = \tau_0 e^{\Delta U/\Delta U_0 \theta_0} = \tau_0 e^{\Delta \epsilon/\theta_0} \tag{6}$$

(here, $\Delta U_0=\hbar I_0/2e$ is the normalizing energy). This is indeed what can be observed in Fig. 3, where we have plotted the average escape time $\langle\tau\rangle$ (on a logarithmic scale) versus the inverse normalized temperature. It is evident that the functional behavior is of the type of behavior in Eq. (6). From the slope of the linear part (small escape rates), one can retrieve the normalized energy barrier to be compared with the estimate of the approximate method.[14] The normalized barrier $\Delta\varepsilon=\Delta U/\Delta U_0$ will be a function of the parameters $\beta_L$ and $\gamma$. In Fig. 2, we show the dependence on the SQUID parameter $\beta_L$ (symbols) together with the results from the pseudopotential method (circles for NS model) and the theoretical estimate for a discrete one-dimensional array of JJ via the Frenkel-Kontorova approach.[23] It is clear that the energy for two-dimensional arrays tends to be constant for very low $\beta_L$. The agreement between the Langevin simulations and the adiabatic method seems quite good, as can be checked by assuming the pseudopotential estimate to be "exact," and then using the $\chi^2$ test to estimate if the energy barriers (and the associated standard deviations) evaluated with the Arrhenius formula are compatible with the pseudopotential method. The result for the data in Fig. 2 is that the likelihood that the barriers retrieved with the two methods are drawn from the same underlying model is above 10%, which is acceptable for a comparison without free parameters.

The data presented in Fig. 2 can also be summarized by the heuristic interpolating formula,

$$\Delta\varepsilon \equiv \Delta\varepsilon(\gamma=0,\beta_L) \simeq \Delta\varepsilon_{max} - (\Delta\varepsilon_{max} - \Delta\varepsilon_0)\frac{\beta_L^0}{\beta_L^0+\beta_L} \tag{7}$$

(we recall that for a square array in these normalized units, $\Delta\varepsilon_{max}=2$ and $\Delta\varepsilon_0=0.203$). The value $\beta_L^0$ can be found by the least-squares method that yields $\beta_L^0=5.7\pm0.3$ for the data estimated with Langevin simulations, and $\beta_L^0=5.43\pm0.01$ for the data obtained via the adiabatic approach. The physical interpretation of the parameter $\beta_L^0$ is that it corresponds to the inductance at which the energy barrier is half of the maximum value.

For the determination of the electrical properties of a superconductor, the behavior of dissipation in the presence of a bias current is of crucial importance. The resulting Lorentz

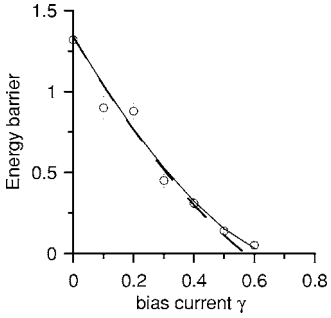


FIG. 4. Normalized pinning barriers as a function of bias current. The thin solid line is the curve obtained by interpolating the results of the adiabatic approach and the symbols represent the energy barrier evaluated via simulations of the Langevin equations [Eq. (2)] reported together with the standard deviation. The dashed line represents the interpolating approximation equations [Eq. (8)].

force between the bias and the flux quanta results in a tendency of the vortices to move, thus determining the effective critical current.[24] We have, therefore, estimated the energy barrier behavior as a function of the bias current by direct numerical simulations of Eq. (2) (see Fig. 4). Using a formula to extrapolate from $\gamma=0$ to finite values of the current (without further parameters), the energy barriers read[14,22]

$$\Delta\varepsilon(\gamma,\beta_L) \simeq [\Delta\varepsilon(0,\beta_L)\sqrt{1-\gamma^2} - \gamma(\pi - 2\sin^{-1}(\gamma))]. \quad (8)$$

Also in this case the $\chi^2$ test has been applied to verify the compatibility of the pseudopotential method with the Langevin equation estimates, obtaining a significant agreement without additional free parameters (the likelihood of the $\chi^2$ test being above 5%).

To summarize this section, the parameter $\beta_L$ affects the pinning energy $\Delta\varepsilon$ of an isolated vortex, lowering the value of an isolated junction $\Delta\varepsilon=2$ (corresponding to $\beta_L=\infty$) to the limit of the $XY$ model $\Delta\varepsilon=0.203$ (corresponding to $\beta_L=0$) with an approximated behavior given by Eq. (7). For each value of the screening parameter, the bias current lowers the energy barrier, with a functional form closely reproduced by Eq. (8). The results obtained for the NS model with the Langevin simulations and the adiabatic method are consistent.

## III. FLUXON INTERACTIONS

Fluxons interact; the effective potential of a fluxon in an array is not only due to the interaction with the network but also with other fluxons. In the simplest case, let us assume that there are just two fluxons at a distance of $d$ cells. In this case, the energy to move a fluxon will depend on the superposition of the interactions of the fluxon with the grid and the fluxon-fluxon (F-F) potential. With the quasistatic method employed for the single fluxon, we have evaluated the energy to move two excitations apart from a cell, the results for NS model being reported in Figs. 5(a) and 5(b). The fluxon-fluxon $\Delta\varepsilon^{int}_{\text{F-F}}(d)$ and the fluxon-antifluxon (F-AF) $\Delta\varepsilon^{int}_{\text{F-AF}}(d)$ interaction potentials, which are a function of the distance $d$, can be estimated from the data. In general, the barrier should be increased by the attractive potential between two vortices

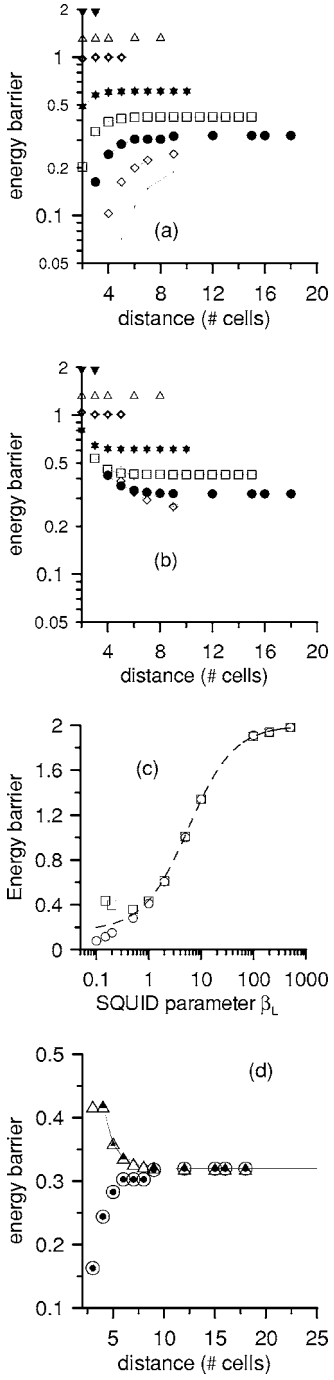


FIG. 5. Normalized pinning barriers for two excitations: (a) fluxon-fluxon and (b) fluxon-antifluxon pairs as a function of the distance between the excitations for various values of $\beta_L$ ($\beta_L$ takes the values, of 100, 10, 5, 2, 1, 0.5, 0.2, and 0.1, from the top to the bottom). (c) describes the behavior of the interaction energy as a function of $\beta_L$ at a distance of five cells ($d=5$), and the dashed line is the interpolation formula [Eq. (7)]. (d) represents the comparison of the NS simulations (filled symbols) with mutual inductances (open symbols) for $\beta_L=0.5$ in the case of fluxon-fluxon (circles) and fluxon-antifluxon (triangles) interactions.

with different polarities if the vortices are moved apart, and should be decreased by the same potential when vortices, again with opposite polarities, are moved closer. The con-

verse is true for vortices of the same polarity. The example reported in Fig. 5(a) refers to two vortices moved apart. For instance, the result for $\beta_L=100$ at $d=2$ reports the energy barrier to move a fluxon from the distance of two cells to a distance of three cells. For F-AF pairs [Fig. 5(b)], the barriers are at a maximum when the vortices are closest, and tend towards the energy barrier of isolated vortices at an infinite distance. Thus, we expect that the energy barrier,

$$\Delta\varepsilon_{\text{F-AF}}(d) = \Delta\varepsilon(\beta_L) + \Delta\varepsilon^{int}_{\text{F-AF}}(d), \tag{9}$$

has the obvious limit $\Delta\varepsilon_{\text{F-AF}}(\infty)=\Delta\varepsilon(\beta_L)$, as in Eq. (7). Another limit is that the attractive or repulsive potentials should always be smaller than the pinning potential; otherwise, the two vortices would either attract and annihilate each other, or move to the next cell even without noise. Since the pinning potential decreases when $\beta_L$ decreases, the minimum distance for the existence of static F-AF pairs increases when $\beta_L$ decreases. For instance, the minimum distance for a stable pair reads three cells for $\beta_L=0.5$, and increases up to six cells for $\beta_L=0.1$. The minimum distance could therefore be interpreted as the unbinding length, or the minimal distance to have isolated vortex-antivortex pairs.

In Fig. 5(c), the behavior of the barrier at a fixed distance ($d=5$) is reported as a function of the coupling for the attractive (F-AF) and repulsive (F-F) cases. A nontrivial behavior of the energy barrier as a function of the coupling parameter $\beta_L$ is observed at relatively short distances for the opposite polarity case: the barrier first decreases when the SQUID parameter is lowered, and then reaches a minimum at $\beta_L \simeq 0.5$ to increase again for $\beta_L<0.5$. This behavior can be understood noting that $\beta_L$ affects the interaction in two opposite ways: it increases the pinning energy $\Delta\varepsilon$ [see Eq. (7)], but it decreases the attractive potential $\Delta\varepsilon^{int}_{\text{F-AF}}$. The latter effect can be understood, at least qualitatively, following the argument (borrowed from Tinkham[24]) that the interaction potential is due to field cancellation: since the higher the screening current, the lower the cancellation, one expects that the effect of $\beta_L$ on the interaction is to decrease the interaction energy, as also demonstrated in Ref. 25.

The same argument also gives the dependence of the energy on the distance between the excitations. Consider a vortex in a square array when the elementary cell size is $a$. Let us assume that in Eq. (1), the screening currents $I^s$ are so negligible ($\beta_L \to 0$) that there is no external applied field and $n=1$ (this corresponds just to a single trapped vortex),

$$\sum_{j\epsilon \text{ loop}} \varphi_j = N_L \Delta\varphi^* = 2\pi. \tag{10}$$

Here, $N_L$ is the number of junctions in a loop of radius $r$ and $\Delta\varphi^*$ is the phase difference assumed constant for each junction in the loop. One can estimate $N_L \simeq 2\pi r/a$ and consequently, $\Delta\varphi^* \simeq a/r = 1/d$ (we recall that $d$ is the number of cells). The estimate can be inserted into the energy of an isolated junction $\varepsilon = 1-\cos\varphi$. Moreover, one can assume for large $r$ that $\Delta\varphi^*$ is small enough to approximate the cosine with $1-(\Delta\varphi^*)^2/2$, so finally, the total energy over a circle of radius $R$ from the vortex center is

$$\varepsilon(R) \simeq \frac{1}{2}\sum_{r\leqslant R}\sum_{j\epsilon \text{ loop}} \varphi_j^2 \simeq \frac{1}{2}\sum_{r\leqslant R}\left(\frac{a}{r}\right)^2 N_L \simeq \frac{1}{2}\sum_{r\leqslant R}\left(\frac{a}{r}\right)^2 \frac{2\pi r}{a}. \tag{11}$$

Approximating the last sum with the integral, one gets the logarithmic divergence,

$$\varepsilon(R) \simeq \frac{1}{2}\int_{r\leqslant R}\left(\frac{2\pi a}{r}\right)\frac{dr}{a} \equiv E_V \log(d). \tag{12}$$

The constant $E_V$ summarizes the numerical factors, including the corrections at small distances where approximations such as Eq. (10) are poor. If just a vortex is present, $d$ can be as large as the array; however, if within a distance $d$, another excitation of opposite polarity is present and the sum of the phases including a fluxon *and* an antifluxon is zero, so $\Delta\varphi^*$ vanishes for distances larger than the F-AF distance. Put another way, the energy of a pair F-AF is given by Eq. (12), where $d$ is the distance with the nearest excitation of the opposite polarity, if the distance is large enough to ensure that the approximation [Eq. (10)] is acceptable. With this interpretation, one retrieves that two excitations interact with a logarithmic potential, or with a force that decreases with the inverse of the distance. One can argue that the same argument implies that $\beta_L$ decreases the interaction energy. In fact, increasing $\beta_L$, i.e., in the presence of a finite inductance, the approximation [Eq. (10)] should be replaced by the more complex Eq. (1), solving consistently for the phases and the screening currents. However, by simple inspection of the equations, one obtains the result that the phases $\Delta\varphi^*$ should be smaller in the case of finite inductance because of the additional screening due to the currents on the right-hand side, and therefore the energy [Eq. (12)] is lowered when $\beta_L$ increases. Another way to see this is to notice that if the junctions are interacting more weakly (i.e., when $\beta_L$ gets higher) and if the vortices are kept at a fixed distance, the phases tend to zero more quickly, and therefore the vortex interaction also is weaker.

In the case of two excitations of the same polarity, assuming that the profile of the second fluxon is the same as the reverse sign, the interaction potential reverses: so it keeps the same behavior, but becomes *repulsive*. Therefore, for the F-F interaction if $\beta_L$ decreases, both contributions (the array barrier and the interaction energy) decrease, and no minimum is observed as a function of the distance. Moreover, in Fig. 5(c) a slight asymmetry between the F-F and the F-AF interactions is evident for very low $\beta_L$.

In Fig. 5(d), the energy barrier is shown for the two models (NS and MC). It is clear that the mutual inductance terms affect little the pinning barriers as a whole, because the pinning energy due to the loops is essentially unchanged, while the interaction energy is strongly affected by mutual inductance terms. An analytic insight is possible for the asymptotic behavior at large distance,[25] and if just self-inductances are retained, it has been predicted that the potential should decay exponentially,

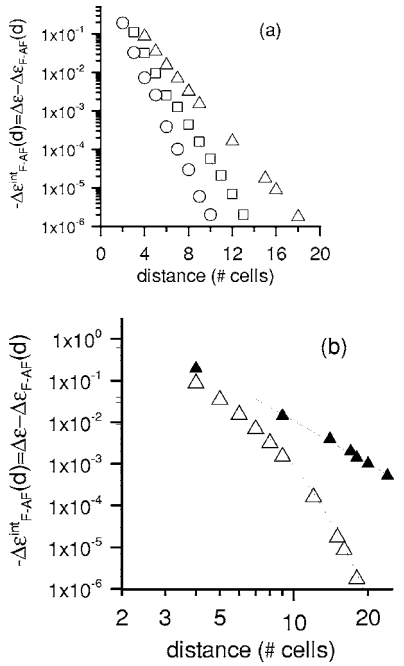


FIG. 6. Normalized pinning barriers for a fluxon-antifluxon pair as a function of the distance for (a) $\beta_L=2$ (circles), 1 (squares), and 0.5 (triangles) with the NS model (the solid lines represent the fitted exponential behavior [Eq. (15)] and (b) $\beta_L=0.5$ for both the NS approach (open triangles) and the MC approach (filled triangles). The solid line through the mutual inductance values has been obtained with a power-law function [see Eq. (16)]. Scales on the horizontal axis in (a) and (b) are linear and logarithmic, respectively.

$$U(d \gg 1) \propto e^{-d\sqrt{\beta_L}}. \tag{13}$$

This makes good physical sense because the screening currents should have an effect over a distance of $\simeq 1/\sqrt{\beta_L}$,[17,19,25] since $\beta_L$ plays the role of the $\Delta x^2$ in the discrete approximation of the second spatial derivative in Eqs. (2). Thus, the screening would change the potential with respect to the *XY* limit [Eq. (12)]. However, an algebraic dependence should be recovered if the effects of mutual inductance terms are considered to all orders as in Eq. (2), and should read[25]

$$U(d \gg 1) \propto \frac{1}{\beta_L d}. \tag{14}$$

These analytic predictions are only valid for the asymptotic behavior and, to our knowledge, have never been checked against numerical simulations. To estimate how the interaction is modified by the presence of inductances, we have simulated the energy barrier for a fluxon and an antifluxon for the case $\beta_L=0.5$, 1 and 2, as shown in Fig. 6. On the vertical axis, the interaction potential $-\Delta\varepsilon^{int}_{\text{F-AF}}(d)=\Delta\varepsilon-\Delta\varepsilon_{\text{F-AF}}(d)$ [see Eq. (9)] is plotted as a function of the distance $d$. The fit gives an exponential behavior,

$$-\Delta\varepsilon^{int}_{\text{F-AF}}(d) = Ae^{-d/d_0}, \tag{15}$$

with $d_0=1.35$ for $\beta_L=0.5$, $d_0=0.95$ for $\beta_L=1$, and $d_0=0.70$ for $\beta_L=2$, in agreement with Eq. (13). To obtain the data shown in Fig. 6(a), we have made sure that the energy barrier has been computed with an accuracy of $10^{-6}$. The main difficulty is to simulate arrays whose size is large enough to ensure that the energy barrier does not depend on border effects. We have found that the results stabilize with the desired accuracy only for arrays of about $100\times 100$. For such arrays, we have not been able to validate the energy barriers with Langevin simulations as accurately as required by the exponential behavior.

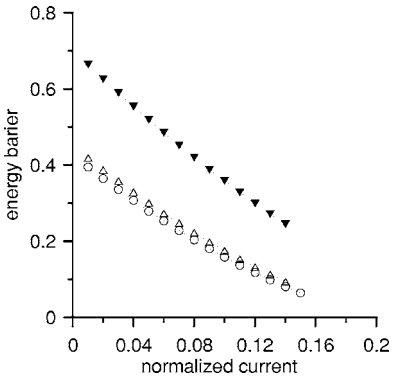


FIG. 7. Normalized pinning barriers for two fluxons as a function of the bias current for various distances ($d=3$, filled triangles; $d=5$, open triangles; and $d=7$, circles) between the fluxons and for an array with $\beta_L=1$.

When mutual inductance effects are added, the interaction potential appears to approach zero more slowly, as in Fig. 6(b), where we display the interaction potential for $\beta_L=0.5$ [notice that because of the slower dependence, we have used in Fig. 6(b) logarithmic horizontal scale, rather than the linear scale, as in Fig. 6(a)]. In particular, using our approximate mutual inductance matrix, we are able to show that the behavior tends to be algebraic rather than exponential, as can be made evident in Fig. 6(b), comparing the logarithmic plot of numerical data with exponential and power-law behaviors. It is clear from the figure that the power law offers a better approximation of the numerical data. The fit of power law gives

$$-\Delta\varepsilon^{int}_{\text{F-AF}}(d) = Ad^{-\alpha}, \tag{16}$$

with $\alpha\simeq 3.1$ and $A\simeq 3$. The accuracy of this result is not critically dependent on array dimension because energies are larger, and the above fits have been obtained using smaller arrays of about $75\times 25$. On the other hand, the use of smaller arrays will introduce finite size effects which imply that the exponent is not near to 1, as in Eq. (14), but is higher. Also in this case the calculation of barriers uses the pseudopotential method because Langevin simulations would be much heavier when mutual inductance terms are included.

We have also investigated the effect of the bias current with the same technique, as reported in Fig. 7. When compared with the results of Fig. 4, the effect of the bias current on a couple seems similar to the effect observed on a single fluxon.

To summarize, when a fluxon-(anti)fluxon pair is present, the interaction energy, defined by Eq. (9), depends on $\beta_L$ and

*decreases* when $\beta_L$ increases. In the F-AF case, the overall behavior of the pinning barriers shows a minimum for $\beta_L \simeq 0.5$ at $d=5$. The minimum tends to move towards lower values of the inductances for larger distances; for instance, it reads for $\beta_L \simeq 0.2$ at $d=9$. The interaction potential [Eq. (9)] is a function of the distance, and for finite inductances (as low as we have been able to simulate, $\beta_L=0.5$) it vanishes exponentially over a distance of $\sim 1/\sqrt{\beta_L}$. This is in contrast with the zero inductance limit ($XY$ model) where a power-law behavior is expected. When mutual inductances are inserted into the model, they give a power-law dependence of $\sim d^{-3}$, instead of the $d^{-1}$ prediction for moderate distances ($d<100$).

## IV. CONCLUSIONS AND DISCUSSIONS

The effect of noise on fluxon propagation in two-dimensional arrays of Josephson junctions has been investigated by simulating the Langevin equation associated with the array. From the Arrhenius-like escape time, we have evaluated the energy barrier and the dependence of the barrier on the inductance and the externally applied current. The results have been compared with a much simpler adiabatic method.[14,16] It has been found that the approximated model represents a fairly accurate estimate of the energy barrier that reproduces the main features of the system. On the basis of such test, one could also expect the approximate model to be accurate at values of the parameters that are difficult to explore numerically via the Langevin equations [Eqs. (2)]. In particular, the role of mutual inductance and the strength of the fluxon interactions in the presence of finite inductances, which would require a major computational effort, have been simulated with the semianalytical method by using an approximate expression for the mutual inductance coupling. The main findings are as follows.

(1) There is an interpolation formula for the pinning energy as a function of the screening parameter $\beta_L = 2\pi L I_0/\Phi_0$, confirmed both by the Langevin simulations and the adiabatic method.

(2) There is validation with Langevin simulations of the interpolation formula for the pinning energy as a function of the bias current.[14]

(3) The pinning barriers for fluxon-(anti)fluxon pairs as a function of the screening parameter $\beta_L$ have been obtained with both Langevin simulations and the adiabatic method. The pinning barrier shows a minimum for $\beta_L \simeq 0.5$ at $d=5$.

(4) The interaction force between two excitations should be less than the pinning force to have a static pair. Since the interaction decreases when the distance increases, this translates into a minimum distance to have a stable pair. Such distance increases when the SQUID parameter $\beta_L$ decreases.

(5) The fluxon-(anti)fluxon interaction energy as a function of the separation between the excitations is obtained with the adiabatic method. This energy shows an exponential behavior over five decades for $\beta_L$ as low as 0.5. This is in striking contrast with the logarithmic behavior expected for very low inductances.[24]

(6) When mutual inductance is considered, the interaction potential vanishes much more slowly and its asymptotic behavior changes from an exponential to an algebraic law that for $d<100$ exhibits an exponent of $-3$ rather than the expected exponent $-1$ of the asymptotic behavior.[25]

We regard the last two results as the most relevant. They suffer the obvious drawback that they have only been found with the adiabatic method. However, the method has proven very reliable in reproducing the correct energy barriers in one fluxon case (see Fig. 2). Thus, we believe that the extent of the results to the fluxon-antifluxon interaction is legitimate. We conclude that to observe the logarithmic behavior of the interaction, one cannot just retain the first-order effect of the inductance matrix, the NS approximation,[19] for it results in a completely different behavior.

The results presented here have been derived under a number of approximations that are worth recalling.

(1) The role of disorder has been neglected. We have imposed both uniformity through the parameters of the array (essentially uniform critical current and dissipation) and through the geometry. Parameter disorder has been extensively investigated in connection with synchronization issues,[26,27] but to our knowledge there is little knowledge of its effect on flux escape phenomena, besides topological disorder that has been investigated by removing one or more junctions from an otherwise regular array.[28]

(2) Quantum fluctuations have been neglected. The role of quantum fluctuations might be neglected as far as $\hbar\omega \ll k_B T$. For higher frequencies, one should replace the quantum equivalent of the correlator [Eq. (4)] in the classical Langevin equation of type of Eq. (2) or replace it by a more refined theory.[29]

(3) Quantum effects have been neglected. When the capacitative energy due to a single electron exceeds the Josephson energy (i.e., $e^2/2C > \hbar I_0/2e$), the system is described by a quantum Hamiltonian. Although many interesting effects have been predicted,[30] also in this context an escape process from potential can occur by means of quantum processes involving fluxons, as tunneling across the above derived barriers[31] and fluxon creation because of quantum fluctuations. While only an indirect evidence that this may be the case for one-dimensional arrays exists,[32] we must mention that quantum effects involving the tunneling of fluxons have been observed both in Josephson systems[33] and for Pearl vortices in (ultra)thin films.[34,35] We remark that Pearl vortices present the same algebraic behavior of fluxons in mutual inductance coupled array.[17]

We remark that the knowledge of energy barriers can help to derive an effective action for the tunneling process. This in turn can be used to predict tunneling rates.

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