Superconducting ground states of the extended *t*-*J* model

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In this work, we study an extended version of the *t*-*J* model. For special values of the hopping parameters we obtain exactly the ground state of the model. We found that depending on the values of the system parameters, the ground state may be superconducting, a charge ordered insulator, or a phase separated state. As holes are added to the system, it may experience a transition from an insulator to a superconductor. The doping dependence of the superfluid density and the complex symmetry of the superconducting parameter are in agreement with the phenomenology of the cuprates.

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I. INTRODUCTION

In recent years, there has been great interest in theoretical models for superconductivity in strongly correlated electron systems (SCESs).[1,2] It is widely accepted that in many SCESs such as heavy fermion systems or high-temperature superconductors (HTSCs), the electron pairing is caused by other than the usual BCS mechanism. This picture is supported by the fact that the superconducting states possess properties that cannot seem to be explained with the BCS theory. These include the high anisotropy of the superconducting gap or the dependence with doping in the HTSC. To describe the HTSC, the common starting point are the different versions of the Hubbard model and the *t*-*J* model.[2,3] The *t*-*J* model is the strong coupling limit of the Hubbard model, and it is believed that it could contain the basic physics of the cuprates. It was proposed by Anderson that the ground state of this Hamiltonian at half filling is a linear superposition of singlet states without long-range magnetic order[3] (called resonating valence bond, or RVB, state), which becomes superconducting in the presence of doping. Since then, much work was devoted to study the *t*-*J* model using both analytical and numerical methods.[2] A rigorous confirmation of the above claim is still lacking, although it is known that the RVB state is realized in some one- and two-dimensional systems.[4]

It is now generally accepted that the electronic interactions play a fundamental role in the formation of the electron pairs in HTSC and heavy fermion systems, but the origin of the pairing mechanism is still controversial. In part, this loss of consensus is caused by the fact that approximate schemes cannot give conclusive results when the interactions are strong. As a consequence, rigorous results for SCES are very valuable. Exact demonstrations of superconductivity driven only by electronic interactions in SCES have been performed for some few particular systems. These demonstrations are usually based on the concept of off-diagonal long-range order (ODLRO), which is a property that implies the Meissner effect and flux quantization.[5,6] However, in many cases, the theory used contains some unjustified assumptions. For instance, in Refs. 7–9 exact results were obtained for extensions of the Hubbard model including the so-called bond charge interaction X. In all the cases, the superconducting ground states were found, imposing the condition $X=t$ (where t is the hopping matrix element) and neglecting the Coulomb repulsion between neighbor sites V. While the assumption $V=0$ is, in principle, valid due to screening effects, this implies that X/t must be very small,[12] in contradiction with the condition $X=t$. In other series of studies, ground states with ODLRO of Hamiltonians that contain an infinite range pair hopping term were obtained.[13,14] Although these results are rigorous, an infinite range hopping is, as a starting point, very unrealistic. It is worth mentioning that Yang[15] obtained exact eigenstates of a modified version of the Hubbard model having ODLRO, even though these eigenfunctions are not ground states.

In the present work, we study a modified version of the *t*-*J* model in dimensions $d=1$ and $d=2$. For a special relation between the hopping parameters, we found exactly the ground state of this Hamiltonian and show that it is superconducting if the coupling parameters satisfy some inequalities. To obtain the ground states, we start by following the method proposed by Brandt and Giesekus[16–19] to derive exact solutions for the Anderson and Hubbard models. We shall show how superconducting solutions can be constructed with this method.

II. HAMILTONIAN MODEL

We begin by considering a four-leg *t*-*J* ladder system, whose Hamiltonian is

$$H=\mathcal{P}\left[t\sum_{\substack{iab\sigma\\a>b}}(c^\dagger_{ia\sigma}c_{ib\sigma}+\text{H.c.})+t'\sum_{\substack{iab\sigma\\\langle ij\rangle}}(c^\dagger_{ia\sigma}c_{jb\sigma}+\text{H.c.})\right]\mathcal{P}+J\sum_{\substack{iab\\a>b}}\left(\mathbf{S}_{ia}\cdot\mathbf{S}_{ib}-\frac{1}{4}n_{ia}n_{ib}\right)+J'\sum_{\substack{iab\\\langle ij\rangle}}\left(\mathbf{S}_{ia}\cdot\mathbf{S}_{jb}-\frac{1}{4}n_{ia}n_{jb}\right)+V\sum_{\substack{iab\\a>b}}n_{ia}n_{ib}+V'\sum_{\substack{iab\\\langle ij\rangle}}n_{ia}n_{jb},\qquad(1)$$

where $c^\dagger_{ia\sigma}$ ($c_{ia\sigma}$) are the creation (destruction) operators for electrons at the rung i and the leg a (a goes from 1 to 4) with spin σ, $n_{ia}=\Sigma_\sigma c^\dagger_{ia\sigma}c_{ia\sigma}$ are the number operators, t and t' are the hopping integrals between orbitals at the same rung and at nearest-neighbor rungs, respectively, J and J' are the ex-

TABLE I. Lowest eigenvalues of H_i.

n_i	E_i
0	0
1	0
2	$-J+V$
3	$-3J/2+3V$
4	$-3J+6V$

change interactions, and V and V' are the Coulomb repulsion between nearest-neighbor sites. Here, the operators $\mathcal{P}$ exclude states with double occupation at any site.

We now introduce the operators defined as

$$\alpha_{i,\sigma} = \beta_{i,\sigma} + \beta_{i+1,\sigma}, \tag{2}$$

with

$$\beta_{i,\sigma} = \sum_{a=1}^{4} c_{ia\sigma}.$$

Using these operators, the first term of the Hamiltonian (1) can be expressed as

$$H_t = \mathcal{P}\left(t' \sum_{i\sigma} \alpha^\dagger_{i\sigma}\alpha_{i\sigma} - 2t'\hat{N}\right)\mathcal{P}, \tag{3}$$

if the condition $t' = \frac{1}{2}t$ is satisfied, where $\hat{N}$ is the total number of particle operator. In the following, we shall assume that this relation between t and t' holds, with $t>0$. As is well known, the t-J model can be obtained from the Hubbard model in the limit of strong interactions, in which case $J = 4t^2/U$, where U is the strength of the on-site Coulomb repulsion of the Hubbard model. Since the value of t' is half the value of t, it is consistent to assume that J' is in the order of a quarter of the value of J. Thus, in a prime analysis we shall omit the interactions J' and V' and will consider their effects later. In order to obtain a ground state for the Hamiltonian (1), we determine a lower bound for the ground-state energy. We first note that the term $t\Sigma_{i\sigma}\alpha^\dagger_{i\sigma}\alpha_{i\sigma}$ is semidefinite positive, so that a lower bound for its expectation value is zero. To obtain a lower bound for the other part of the Hamiltonian, we diagonalize the local Hamiltonian

$$H_i = V \sum_{ab,a>b} n_{ia}n_{ib} + J \sum_{ab,a>b}\left(\mathbf{S}_{ia}\cdot\mathbf{S}_{ib} - \frac{1}{4}n_{ia}n_{ib}\right) \tag{4}$$

for different values of $n_i = \Sigma_a n_{ia}$. As asserted by the variational principle of quantum mechanics, the value of $\langle H_i\rangle$ is minimal in the ground state.[10] This allows us to obtain a lower bound for $\langle\Sigma_i H_i\rangle = \Sigma_i\langle H_i\rangle$ diagonalizing H_i.[11] The lowest eigenvalues for each value of n_i are shown in Table I. In particular, for $n_i=2$, the state that minimizes the energy are the singlet states.

The values of n_i that minimize the energy *per particle* are $n_i=1$ if $V>J$, $n_i=2$ if $J>V>J/4$, and $n_i=4$ if $J/4>V$. Thus, for the case $J>V>J/4$, a lower bound for $\Sigma_i\langle H_i\rangle$ is $\frac{1}{2}N\epsilon_2$, where N is the total number of particles. For simplicity sake, we will assume here that N is even (in the thermodynamic limit, the consideration of N or $N+1$ particles is irrelevant for our purpose). Then, a lower bound for the ground-state energy is given by

$$\lambda_l = -\left[2t' + \frac{1}{2}(J-V)\right]N. \tag{5}$$

Now we attempt to construct a wave function that satisfies this lower bound. At this point, we define the operators

$$b^\dagger_{iab} = \frac{1}{\sqrt{2}}(c^\dagger_{ia\uparrow}c^\dagger_{ib\downarrow} - c^\dagger_{ia\downarrow}c^\dagger_{ib\uparrow}), \tag{6}$$

which create a singlet pair at the rung i with electrons at the sites a and b (usually called *bond* operators). Now let us consider the wave function defined as

$$|\psi\rangle = \sum_{\mathcal{C}}\prod_{i\in\mathcal{C}}(b^\dagger_{i12} + b^\dagger_{i34} - b^\dagger_{i14} - b^\dagger_{i23})|0\rangle, \tag{7}$$

where i takes the values of a set $\mathcal{C}$ of $\frac{1}{2}N$ different natural numbers p_m, with $p_m < M+1$, M being the number of rungs of the ladder, and $|0\rangle$ is the vacuum state. The summation extends over all the possible sets. We note that $|\psi\rangle$ is formed by singlet local states, which minimize the energy of the local Hamiltonian H_i in the case $J>V>J/4$. In addition, it is easy to check that the following property holds:

$$\beta_{j,-\sigma}b^\dagger_{iab}|0\rangle = \frac{\delta_{ij}}{\sqrt{2}}[c^\dagger_{ia\sigma}|0\rangle + c^\dagger_{ib\sigma}|0\rangle]. \tag{8}$$

In virtue of this property, it follows that $\alpha_{i\sigma}|\psi\rangle=0$. Thus, $\langle\psi|H|\psi\rangle = -\left[2t' + \frac{1}{2}(J-V)\right]N$. This value equals the lower bound λ_l previously obtained. Then, no other wave function can make $\langle H\rangle$ lower than $|\psi\rangle$. As a consequence, $|\psi\rangle$ is a ground state of the Hamiltonian (1) (for $J>V>J/4$, $J' = V' = 0$). Now we show that this state possesses ODLRO and, consequently, is superconducting. A system exhibits ODLRO if the off-diagonal matrix element

$$\langle lm|\rho_2|pq\rangle = \frac{\langle\psi|c^\dagger_{l\uparrow}c^\dagger_{m\downarrow}c_{p\uparrow}c_{q\downarrow}|\psi\rangle}{\langle\psi|\psi\rangle} \tag{9}$$

approaches a nonzero value as the distance between the sites l and p tends to infinity (here, l and p are at the neighborhood of m and q, respectively, i.e., $\mathbf{r}_l \approx \mathbf{r}_m, \mathbf{r}_p \approx \mathbf{r}_q$ with $|\mathbf{r}_l - \mathbf{r}_p| \to \infty$).[5]

For the sake of simplicity, we will calculate the value of

$$\rho_{abcd} = \frac{\langle\psi|b^\dagger_{iab}b_{jcd}|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{10}$$

If the value of ρ_{abcd} is nonvanishing in the limit $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$, $\langle lm|\rho_2|pq\rangle$ will not vanish in this limit. The value of ρ_{abcd} is obtained from the expression of $|\psi\rangle$ with the result

$$\rho_{abcd} = f\frac{(2M-N)N}{16M(M-1)}, \tag{11}$$

where f is given by

$$f=-\frac{1}{4}(-1)^{(a+b+c+d)/2}[1-(-1)^{a+b}][1-(-1)^{c+d}], \quad (12)$$

so that the value of ρ_{abcd} may be positive, negative, or zero, depending on the indices [the relation (12) is not general but it is valid for the wave function (7)]. This reveals the complex symmetry of the order parameter. For instance, if the correlation (11) is calculated with $a=c$, $b=d$ (i.e., involving only two legs), its value is positive. However, it can be negative or zero if the correlation involves more than two legs. At this point, we note that in Bose-Einstein condensates, the value of the off-diagonal elements of the reduced matrix density of order 1, ρ_1, represents the condensate fraction.[5] Since the superfluid density is proportional to the density of superconducting electrons (i.e., the fraction of superconducting pairs),[20] it is natural to interpret here that the average of the *module* of ρ_{abcd} is a measure of the superfluid density ($\rho_s \propto \overline{|\rho_{abcd}|}$). On the other hand, there is a phenomenological relation between the superfluid density and the critical temperature, known as the Uemura relation, which establishes that these two properties are proportional. For some systems, this relation is not accurate,[20] but we can use it to roughly estimate the dependence of the critical temperature with doping. From Eq. (11), the superfluid density can be expressed in the thermodynamic limit as $\rho_s=\text{const}\times(1-x)(x-\frac{1}{2})$, where $x=1-n$ and $n=N/N_s$, with N_s being the number of sites ($N_s=4M$). Thus, the system exhibits a parabolic dependence of ρ_s and T_c, as is typically observed in the HTCS.

III. NONLOCAL INTERACTIONS

Now we shall consider the effect of V' and J' on the properties of the system. For this purpose, we shall obtain a lower bound for the ground-state energy of the Hamiltonian (1). For convenience sake, we rewrite the Hamiltonian (1) as $H=H_t+\Sigma_i H_{ij}$ with

$$H_{ij}=\frac{J}{2}\sum_{\substack{ab\\a>b}}(\mathbf{S}_{ia}\cdot\mathbf{S}_{ib}+\mathbf{S}_{ja}\cdot\mathbf{S}_{jb})+J'\mathbf{S}_i\cdot\mathbf{S}_j + \frac{1}{2}\left(V-\frac{J}{4}\right)\sum_{\substack{ab\\a>b}}(n_{ia}n_{ib}+n_{ja}n_{jb})+\left(V'-\frac{J'}{4}\right)n_in_j, \quad (13)$$

where $\mathbf{S}_i=\Sigma_a\mathbf{S}_{ia}$ (notice the $\frac{1}{2}$ factor in J and V). If J is the dominant interaction (as we assume that is the case with $J'\sim\frac{1}{4}J$), the energy is minimized when the electrons form local singlets, which, in turn, makes the magnetic interaction between neighbor rungs null. We verify this by diagonalizing the local Hamiltonian H_{ij}. We shall denote as E_{nm} and ϵ_{nm} the energies and the corresponding energies per particle for $n_i=n$, $n_j=m$. The lowest ground-state energies in each case are shown in Table II. Consideration of other values of n_i,n_j than those showed in Table II gives upper values of E_{nm}. Whether or not $\{n_i,n_j\}=\{2,0\}$ is the configuration that minimizes the energy per particle depends on the values of V' and J'. We shall now consider the case $V'>\frac{1}{4}J'$. In this instance, the lower energy per particle is ϵ_{20} if $J/4<V<J+2(V'-J')$ (with $J>J'$). So, a lower bound for $\langle\Sigma_i H_{ij}\rangle=\Sigma_i\langle H_{ij}\rangle$ is equal to $\lambda=(N/2)(V-J)$. As previously obtained, a minimal value for $\langle H_t\rangle$ is $-2t'N$. Then, a lower bound for $\langle H\rangle$ is $\lambda_l=-2t'N-\frac{1}{2}(J-V)N$, for the conditions

TABLE II. Lowest eigenvalues of H_{ij}.

nm	E_{nm}
10	0
11	$V'-J'$
20	$(V-J)/2$
22	$V+4V'-J-J'$
40	$3V-3J/2$

$$\frac{J}{4}<V<J+2(V'+J'),V'>\frac{J'}{4}. \quad (14)$$

Now let us show that Eq. (1) has ground states with ODLRO.

Consider the following wave function:

$$|\psi'\rangle=\sum_{\mathcal{C}'}\prod_{i\in\mathcal{C}'}(b^{\dagger}_{i12}+b^{\dagger}_{i34}-b^{\dagger}_{i14}-b^{\dagger}_{i23})|0\rangle, \quad (15)$$

where now $\mathcal{C}'$ is a set of $N/2$ odd natural numbers p_m, such that $p_m<M+1$. In this wave function, the singlet pairs are located only at rungs with even indices. Since $|\psi'\rangle$ does not have electrons at neighbor sites, it is made up of local singlets and $\Sigma\alpha^{\dagger}_{i\sigma}\alpha_{i\sigma}|\psi'\rangle=0$, the expectation value $\langle\psi'|H|\psi'\rangle$ equals the lower bound λ_l. Then, $|\psi'\rangle$ is a ground state of Eq. (13). In similar form to that done for $|\psi\rangle$, it is easy to show that $|\psi'\rangle$ is superconducting, with

$$\rho_{abcd}=f\frac{(M-N)N}{4M(M-2)},$$

where f is that defined in Eq. (12).

The wave function (15) is not the most general ground state of Eq. (1). It is contained in the ground state $|\psi''\rangle=\mathcal{P}_C|\psi\rangle$ [where $|\psi\rangle$ was defined in Eq. (7)]. Here, $\mathcal{P}_C$ sets out all the states that contain at least two electrons at neighbor rungs. It is not difficult to show that $|\psi''\rangle$ possesses ODLRO and, thus, it is superconducting.

For $N=M$ $(x=\frac{3}{4})$, the ground state $|\psi''\rangle$ is not superconducting but it is a charge ordered insulator, with a two subchain configuration. In each site of one subchain, there are two electrons and no electrons in the other subchain. For this number of electrons, $|\psi''\rangle$ and $|\psi'\rangle$ coincide. If holes are added to the system, in order to make $0<N<M$, the system becomes superconducting.

To complement the above discussion, valid for the case $J/4<V<J+2(V'-J'),V'>J'/4$, we now consider that $J/4<V<J+2(V'-J')$ but $V'<J'/4$. In this case, the energy is lowered if the electron pairs are located at neighbor sites; i.e., $\{n_i,n_j\}=\{2,2\}$ is the configuration that reduces the energy per particle, and the ground-state exhibits phase separation instead of superconductivity. A corresponding lower

bound for the ground-state energy is $\lambda_l' = -2t'N - \frac{1}{2}[J - V + 4(J'/4 - V')]N$. The exact ground state that satisfies this lower bound is

$$|\psi_{ps}\rangle = \sum_{k=1}^{M-N/2} \prod_{i=k}^{k+N/2-1} (b_{i12}^\dagger + b_{i34}^\dagger - b_{i14}^\dagger - b_{i23}^\dagger)|0\rangle. \quad (16)$$

At this point, it is interesting to compare our results with those obtained in previous works. The four-leg t-J ladder was investigated by White and Scalapino[21] using density-matrix renormalization-group techniques. They found similar phases that were obtained here, namely, phase separation, superconductivity (singlet pairs), and charge ordered phases.

Analogous results were also obtained by Ledermann *et al.* for the four-leg Hubbard model using bosonization techniques.[22] The latter authors showed that at low doping and small values of the coupling U, the ground state of the four-leg Hubbard ladder is C1S0 type (a phase with m gapless charge modes and n charge modes is denoted as CmSn). These authors also found, in accordance with the results of Ref. 21, that for small values of the coupling J/t, the singlets form with more probability in the top and bottom legs.

In order to compare with these results, we will examine the presence of gaps in the charge and spin modes. Introducing the creation operators in k space, $c_{k\sigma}^\dagger$, defined as

$$c_{k\sigma}^\dagger = \sum_j c_{j\sigma}^\dagger e^{ikr_j},$$

the noninteracting Hamiltonian H_t may be expressed as

$$H_t = \sum_{k\sigma} -8t(B_{1k\sigma}^\dagger B_{1k\sigma} + B_{2k\sigma}^\dagger B_{2k\sigma} + B_{3k\sigma}^\dagger B_{3k\sigma}) + 8(3t + \epsilon_k)B_{4k\sigma}^\dagger B_{4k\sigma}, \quad (17)$$

where $\epsilon_k = 2t\cos k$ and the $B_{ak\sigma}^\dagger$ are fermionic operators defined as

$$B_{1k\sigma}^\dagger = \frac{1}{2}(c_{1k\sigma}^\dagger + c_{2k\sigma}^\dagger - c_{3k\sigma}^\dagger - c_{4k\sigma}^\dagger),$$

$$B_{2k\sigma}^\dagger = \frac{1}{2}(c_{1k\sigma}^\dagger - c_{2k\sigma}^\dagger + c_{3k\sigma}^\dagger - c_{4k\sigma}^\dagger),$$

$$B_{3k\sigma}^\dagger = \frac{1}{2}(c_{1k\sigma}^\dagger - c_{2k\sigma}^\dagger - c_{3k\sigma}^\dagger + c_{4k\sigma}^\dagger),$$

$$B_{4k\sigma}^\dagger = \frac{1}{2}(c_{1k\sigma}^\dagger + c_{2k\sigma}^\dagger + c_{3k\sigma}^\dagger + c_{4k\sigma}^\dagger). \quad (18)$$

These are similar to the band operators obtained in Ref. 18. The real space representation of the operators $B_{ak\sigma}^\dagger$ are $B_{1i\sigma}^\dagger = \frac{1}{2}(c_{1i\sigma}^\dagger + c_{2i\sigma}^\dagger - c_{3i\sigma}^\dagger - c_{4i\sigma}^\dagger)$, $B_{2i\sigma}^\dagger = \frac{1}{2}(c_{1i\sigma}^\dagger - c_{2i\sigma}^\dagger + c_{3i\sigma}^\dagger - c_{4i\sigma}^\dagger)$, $B_{3i\sigma}^\dagger = \frac{1}{2}(c_{1i\sigma}^\dagger - c_{2i\sigma}^\dagger - c_{3i\sigma}^\dagger + c_{4i\sigma}^\dagger)$, and $B_{4i\sigma}^\dagger = \frac{1}{2}(c_{1i\sigma}^\dagger + c_{2i\sigma}^\dagger + c_{3i\sigma}^\dagger + c_{4i\sigma}^\dagger)$. Thus, $B_{4i\sigma}^\dagger = \beta^\dagger$, where β was already defined. We note that many different ground states similar to those in Eq. (15) can be constructed. Consider the wave function

$$|\psi_3\rangle = \sum_{\mathcal{C}'} \prod_{i\in\mathcal{C}'} [A(b_{i12}^\dagger + b_{i34}^\dagger - b_{i14}^\dagger - b_{i23}^\dagger) + B(b_{i14}^\dagger + b_{i23}^\dagger - b_{i13}^\dagger - b_{i24}^\dagger) + C(b_{i13}^\dagger + b_{i24}^\dagger - b_{i12}^\dagger - b_{i34}^\dagger)]|0\rangle, \quad (19)$$

where $\mathcal{C}'$ was already defined and A,B,C are arbitrary coefficients. This wave function is also a ground state of Eq. (1) for the conditions (14). We note that the following relations hold:

$$B_{1i\uparrow}^\dagger B_{1i\downarrow}^\dagger = \frac{\sqrt{2}}{4}(b_{i12}^\dagger + b_{i34}^\dagger - b_{i13}^\dagger - b_{i14}^\dagger - b_{i23}^\dagger - b_{i24}^\dagger),$$

$$B_{2i\uparrow}^\dagger B_{2i\downarrow}^\dagger = \frac{\sqrt{2}}{4}(b_{i13}^\dagger + b_{i24}^\dagger - b_{i12}^\dagger - b_{i14}^\dagger - b_{i23}^\dagger - b_{i34}^\dagger),$$

$$B_{3i\uparrow}^\dagger B_{3i\downarrow}^\dagger = \frac{\sqrt{2}}{4}(b_{i14}^\dagger + b_{i23}^\dagger - b_{i13}^\dagger - b_{i12}^\dagger - b_{i34}^\dagger - b_{i24}^\dagger). \quad (20)$$

Using the above relations and Eq. (8), we can immediately evaluate the correlations

$$\langle\psi_3|B_{1j\downarrow}^\dagger B_{1j\uparrow}^\dagger B_{1i\uparrow} B_{1i\downarrow}|\psi_3\rangle = \frac{|A-C|^2}{2(|A-C|^2 + |B-A|^2 + |C-B|^2)} \frac{(M-N)N}{4M(M-2)},$$

$$\langle\psi_3|B_{2j\downarrow}^\dagger B_{2j\uparrow}^\dagger B_{2i\uparrow} B_{2i\downarrow}|\psi_3\rangle = \frac{|C-B|^2}{2(|A-C|^2 + |B-A|^2 + |C-B|^2)} \frac{(M-N)N}{4M(M-2)},$$

$$\langle\psi_3|B_{3j\downarrow}^\dagger B_{3j\uparrow}^\dagger B_{3i\uparrow} B_{3i\downarrow}|\psi_3\rangle = \frac{|A-B|^2}{2(|A-C|^2 + |B-A|^2 + |C-B|^2)} \frac{(M-N)N}{4M(M-2)},$$

$$\langle\psi_3|B_{4j\downarrow}^\dagger B_{4j\uparrow}^\dagger B_{4i\uparrow} B_{4i\downarrow}|\psi_3\rangle = 0. \quad (21)$$

These results show that there is pair charge transport in the modes $m=1,2,3$. However, there is no type of charge transport in the mode $m=4$. Then, the ground state has three gapless charge modes. On the other hand, the absence of spin triplets in the ground states indicates that there is a gap between the singlet and triplet states. As a consequence, all the spin modes are gapped. Then, $|\psi_3\rangle$ is of C3S0 type. As already mentioned, the ground state of the four-leg Hubbard ladder is C2S0. The difference between the ground states of the models is probably due to the following reason. In Ref. 22, it was shown by renormalization-group method that the four-leg Hubbard ladder may be represented as two two-leg Hubbard ladders. At low energies, the two inner legs become decoupled from the two other legs. However, in the present model, all the legs are equivalent (a permutation of the leg indices makes no modification in the Hamiltonian), which probably avoids this decoupling.

In addition, for the model considered here, the superconducting phase arises when a minimal repulsive interaction is present, in order to satisfy the condition $J > V > J/4$. In fact, it is the value of J/V that determines the phase, independent of the value of J/t. A consequence of this is that the superconducting states may exist for arbitrarily small values of J/t. This is in contrast to what occurs in the conventional t-J and Hubbard models studied in Refs. 21 and 22, where, at fixed doping, it is the value of J/t that determines the phase. This is probably caused by the fact that the exchange interactions are distinct in each case. In the model considered here, the exchange interactions between two electrons at the same rung are the same for any two given legs. This enhances the effects of the exchange interactions, which could increase the tendency of phase separation. This trend is avoided if a sufficiently strong Coulomb repulsion is present. The condition for superconductivity $J/4<V<J$ means that V must be sufficiently large to avoid phase separation, but not too large to break the singlet pairs. This result is similar to those obtained in Refs. 23 and 24 for the one-dimensional t-J model including intersite Coulomb repulsion. In these studies, it was found that the interaction V suppresses phase separation and enhances the superconducting correlations, in agreement with our results.

The tendency to form singlets with more probability in some legs is also not observed in the ground state $|\psi'\rangle$ [Eq. (15)]. This difference is explained by the fact that, as already mentioned, in the present model all legs are equivalent.

Now we consider the possible relevance of the present solution to the superconducting cuprates. We notice that the t-J model must be regarded as an effective model for the cuprates, where the values of t and J must be calculated from the parameters of the three-band Hubbard model.[25] The same also holds for the interaction V. As noted before, the solution obtained here is valid for any value of J/t. However, to stabilize the superconducting state, the condition $J/4<V<J$ must be satisfied (if we consider nonlocal interactions V' and J', with $V'\sim V/4$ and $J'\sim J/4$, the condition is the same). Recently, Neudert *et al.*[26] obtained that the value of V may be estimated as $V=0.125V_{pd}$, where V_{pd} is the Coulomb repulsion between O and Cu neighbor sites (this estimation agrees with the results of Feiner *et al.*[27] for the two-dimensional model). Using the value of $V_{pd}=2$ eV for Sr_2CuO_3,[26] we obtain $V=0.25$ eV. On the other hand, Neudert *et al.* also calculated the value of J in Sr_2CuO_3 using different methods, with an average result of $J=300$ meV (Table II of Ref. 26). These values satisfy the conditions for the validity of the superconducting solution, suggesting that these impositions may be attained in physically realizable systems. On the other hand, the present model cannot be viewed as an accurate description of the cuprates due to the type of hoppings that we assumed.

IV. TWO-DIMENSIONAL MODEL

Now we consider the two-dimensional version of the model, whose Hamiltonian is

$$H=H_t+J\sum_{\substack{iab\\a>b}}\left(\mathbf{S}_{ia}\cdot\mathbf{S}_{ib}-\frac{1}{4}n_{ia}n_{ib}\right)+V\sum_{\substack{iab\\a>b}}n_{ia}n_{ib}, \quad (22)$$

where

$$H_t=\mathcal{P}\Bigg[t\sum_{\substack{iab\sigma\\a>b}}(c^\dagger_{ia\sigma}c_{ib\sigma}+\text{H.c.})+t'\sum_{\substack{iab\sigma\\\langle ij\rangle}}(c^\dagger_{ia\sigma}c_{jb\sigma}+\text{H.c.})+t''\sum_{\substack{iab\sigma\\\langle\langle ij\rangle\rangle}}(c^\dagger_{ia\sigma}c_{jb\sigma}+\text{H.c.})\Bigg]\mathcal{P}. \quad (23)$$

Here, we assume that around any site of a square lattice, there are four basis sites. The operator $c^\dagger_{ia\sigma}$ creates an electron on a basis site a of the lattice site i and t, t', and t'' are the hoppings between basis sites at the same, nearest-neighbor, and next-nearest-neighbor lattice sites. For simplicity, we shall not consider here nonlocal exchange interactions.

A ground state for this Hamiltonian can be constructed in similar form to those done for the one-dimensional model. Using the α operators, the Hamiltonian (23) can be expressed as

$$H_t=\mathcal{P}\left(t'\sum_{i\sigma}\alpha^\dagger_{i\sigma}\alpha_{i\sigma}-4t'\hat{N}\right)\mathcal{P}, \quad (24)$$

if the conditions $t'=\frac{1}{2}t$ and $t''=\frac{1}{4}t$ are satisfied. Then, since the first term of H_t is semidefinite positive, a lower bound for $\langle H_t\rangle$ is $\lambda_t=-4t'N$. We note that the other part of the Hamiltonian (22), containing the local interactions J and V, is a sum of the local Hamiltonians H_i defined in Sec. II. Thus, a lower bound for the expectation value of this term is equal to the one obtained for the one-dimensional case. Therefore, a lower bound for $\langle H\rangle$ is $\lambda_2=-4t'N-\frac{1}{2}(J-V)N$, if $J>V>J/4$. Consider now the wave function

$$|\psi_2\rangle=\sum_{\mathcal{C}}\prod_{i\in\mathcal{C}}(b^\dagger_{i12}+b^\dagger_{i34}-b^\dagger_{i14}-b^\dagger_{i23})|0\rangle. \quad (25)$$

where i takes the values of a set $\mathcal{C}$ of $N/2$ numbers that label the sites of the bidimensional lattice. The expectation value $\langle\psi_2|H|\psi_2\rangle=-4t'N-\frac{1}{2}(J-V)N$. Thus, $|\psi_2\rangle$ is a ground state of the Hamiltonian (22), if $J>V>J/4$. Since $|\psi_2\rangle$ has the same formal structure as $|\psi\rangle$, it follows that it is also superconducting, with the same ρ_{abcd} given in Eq. (11). Similar superconducting ground states can be easily obtained for $d=3$ or higher dimensions. Clearly, as the dimension increases, the conditions involving the hopping integrals also increase.

V. CONCLUSIONS

In summary, the aim of the present work was twofold. In first place, we presented a procedure to construct superconducting ground states for SCES based on the method of Brandt and Giesekus. In second place, using this method, we derived exactly the ground state of an extended t-J model and showed that it is superconducting. In order to do that, we had to assume a particular relationship between the hopping integrals. Special values of the interactions J and V are not required but some inequalities must be satisfied in order to stabilize the different phases. The superconducting ground states may exist for arbitrarily small values of J/t. These are spin singlets without magnetic long-range order, with similar structure to that of the RVB state proposed by Anderson.[3]

The model cannot be viewed as an accurate description of the cuprates due to the restrictions involving the hoppings that were assumed. Nevertheless, the doping dependence of the superfluid density and the complex symmetry of the superconducting order parameter are in qualitative agreement with the phenomenology of the HTSC.

We would like to remark that the ground state was constructed taking advantage of the fact that, for the special hoppings integrals that were assumed, the ground states are a tensorial product of local wave packets. The possibility of that this can be a property of the ground states of the HTSC cannot be excluded. In particular, the fact that the size of the wave packets is of same order as the lattice constant explains why the HTSCs are superconductors with short coherence length. Thus, the present solution confirms that an extended *t*-*J* model can possess superconducting ground states, and that much of the physics of the cuprates is contained in this model. In addition, it suggests to us a hypothesis about the ground states of the HTSCs.

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