Effect of long-range connections on an infinite randomness fixed point associated with the quantum phase transitions in a transverse Ising model

Amit Dutta* and R. Loganayagam

Department of Physics, Indian Institute of Technology Kanpur - 208016, India
(Received 16 August 2006; revised manuscript received 19 October 2006; published 9 February 2007)

We study the effect of long-range connections on the infinite-randomness fixed point associated with the quantum phase transitions in a transverse Ising model (TIM). The TIM resides on a long-range connected lattice where any two sites at a distance r are connected with a nonrandom ferromagnetic bond with a probability that falls algebraically with the distance between the sites as $1/r^{d+\sigma}$. The interplay of the fluctuations due to dilutions together with the quantum fluctuations due to the transverse field leads to an interesting critical behavior. The exponents at the critical fixed point (which is an infinite randomness fixed point) are related to the classical "long-range" percolation exponents. The most interesting observation is that the gap exponent ψ is exactly obtained for all values of σ and d. Exponents depend on the range parameter σ and show a crossover to short-range values when $\sigma \geq 2 - \eta_{SR}$ where η_{SR} is the anomalous dimension for the conventional percolation problem. Long-range connections are also found to tune the strength of the Griffiths phase.

The presence of quenched randomness drastically modifies the nature of the zero-temperature (quantum) phase transitions.[1–3] In contrary to the pure systems, many low-energy and low frequency properties of systems with quenched randomness are dictated by locally ordered "rare regions."[4] These active rare regions produce stronger effects on zero-temperature transitions than on finite-temperature classical transitions.[1,5]

Some of the features associated with low-dimensional random quantum transitions include activated (quantum) dynamical scaling at the quantum critical point and the existence of Griffiths-McCoy (GM) singular regions[6,7] where the response function diverges even away from the critical point. The existence of the abovementioned features are well established in the case of one dimensional random quantum Ising models by using a renormalization group (RG) technique[8] which is exact in the asymptotic limit.[1] For $d=1$, the RG flow on the critical manifold for strong randomness is towards an infinite randomness fixed point where the quenched randomness effectively grows stronger and stronger as the system is coarse grained.[1] Numerical studies using an extension of the above RG scheme[9] as well as quantum Monte Carlo studies[10] predict a similar scenario of IRFP even for random quantum Ising transitions for spatial dimension $d=2$. Also at an IRFP, stronger couplings always dominate and hence frustration turns out to be irrelevant. Therefore, the critical behavior of quantum Ising spin-glass and random ferromagnetic models are expected to be governed by the same fixed point.[5] Long-range spatial correlations of disorder are found to enhance the off-critical singularities.[11]

Situation becomes very interesting when the quenched randomness arises due to dilution or vacancies in a quantum magnetic model.[12–16] The problem is immediately connected to the well-studied problem of the percolation transition of the underlying lattice.[17] The study of quantum phase transitions in a TIM at the percolation threshold of a dilute lattice shows the existence of an infinite randomness critical point scenario in dimensions greater than unity.[13]

The experimentally studied systems LiHoF$_4$ which are suitably modeled by transverse Ising spin glass models however are complicated because of the presence of long-range dipolar interactions.[18] Moreover in a metallic system, the presence of long-range Rudermann-Kasuna-Kittel-Yosida (RKKY) interaction is expected to modify the critical behavior.[19] Question therefore remains whether a nontrivial long-range interaction modifies the exponents associated with an IRFP. To address this particular issue, a quantum Ising spin-glass model with long-range random interactions, where interaction between any two spins separated by a distance r_{ij} is J_{ij}/r_{ij}^α with random J_{ij}'s, was introduced.[20] The perturbative RG calculations[21] of the above spin-glass model around the upper critical dimension failed to locate any stable weak coupling fixed point in the non-mean-field region and a RG flow towards the strong coupling was observed. This immediately points to the existence of an IRFP for the above spin glass.[20] However, no quantitative information can be derived regarding the nature IRFP in the above spin-glass model.

In this Brief Report, we study the quantum transitions in a TIM with ferromagnetic interactions defined on a long-range bond dilute lattice. It should be emphasized at the outset that in this model there is no long-range random interaction, nevertheless, there are long-range connection probabilities which effectively simulate a long-range interaction that nontrivially affects the exponents at IRFP. The Hamiltonian of the model is

$$H = -\sum_{(i,j)} J_{ij} S_i^z S_j^z - \Gamma \sum_i S_i^x, \quad (1)$$

where S_i's are Pauli spin matrices and Γ is a non-commuting transverse field. The bond J_{ij} is chosen from the binary distribution

$$P(J_{ij}) = \frac{p}{r_{ij}^{d+\sigma}} \delta(J_{ij} - J) + \left(1 - \frac{p}{r_{ij}^{d+\sigma}}\right) \delta(J_{ij}), \quad (2)$$

where p $(0 < p \leq 1)$ denotes the connection probability between the nearest neighbor sites, d is the spatial dimension
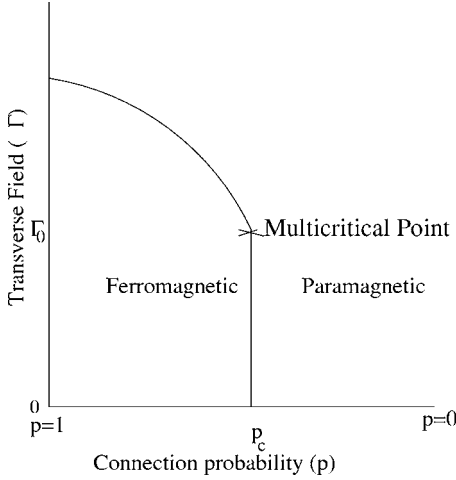

FIG. 1. The phase diagram of a dilute TIM in the Γ-p plane. Senthil and Sachdev[13] studied the quantum transition along the vertical line which terminates at the multicritical point (p_c, Γ_o). In the long-range connected case, the phase diagram is qualitatively similar. However, p_c depends on σ.

and the parameter σ determines the range of connection. The parameter σ is restricted to nonzero positive values to ensure extensivity. The Hamiltonian refers to a pure TIM which resides on a lattice where any two sites separated by a distance r is connected with a ferromagnetic bond of strength J with a probability falling algebraically with the distance between the sites as $p/r_{ij}^{d+\sigma}$. Clearly, for smaller values of σ long-range connections are dominant.

An infinite randomness critical fixed point is characterised by three exponents[1]: At the critical point, the smallest energy gap ΔE is related to the linear dimension L of the rare region as $-\ln \Delta E \sim L^{\psi}$ which is a signature of activated quantum dynamical scaling. Analogously, the magnetic moment of the cluster grows as $L^{\phi\psi}$. The typical size of such a cluster (the correlation length) diverges as $\xi \sim \delta^{-\nu}$ where δ measures the deviation from the quantum critical point. All the bulk exponents are expressed in terms of the exponents ϕ, ψ and ν.[1,5] Senthil and Sachdev[13] explored the phase transition in the same TIM described by the Hamiltonian (1) on a conventional dilute lattice where only the nearest neighbor sites are connected with a ferromagnetic bond of strength J with probability p so that

$$P(J_{ij}) = p \, \delta(J_{ij} - J) + (1-p) \, \delta(J_{ij}), \qquad (3)$$

where (i,j)'s are nearest neighbor sites. The zero-temperature phase diagram of the model in the Γ-p plane is shown in the Fig. 1.[12] For $\Gamma=0$, the system undergoes a geometrical phase transition at the percolation threshold p_c. For $p > p_c$, at a sufficiently large $\Gamma = \Gamma_c(p)$, the quantum fluctuations destroy the long-range order. However, even at p_c there exists an infinitely connected cluster with fractal dimensions $d_f(<d)$. Even for a one-dimensional TIM, the critical transverse field is finite and thus a finite amount of transverse field is required to destroy the long-range order at $p = p_c$. This leads to the existence of a multicritical point with $\Gamma = \Gamma_o$ and $p = p_c$ as pointed out by Harris.[12] Senthil and Sa-

chdev studied the transition along the vertical line ($p = p_c$ and $\Gamma < \Gamma_o$) where the quantum transition is dictated by the percolation fixed point p_c. The quantum critical exponents are solely determined by the classical percolation exponents.

To comprehend the nature of IRFP associated with the quantum transition of the TIM defined through Eqs. (1) and (2), it is necessary to explore the associated percolation transition with long-range connection probabilities.[22] To study the critical behavior of the above percolation model, the mapping of the same to the corresponding q-state Potts model (in the limit $q \rightarrow 1$)[23] turns out to be useful. The resulting Hamiltonian is

$$H = -\sum_{ij} \frac{K}{r_{ij}^{d+\sigma}} \left(\prod_{\alpha=1}^{q} \delta_{S_i^{\alpha}, S_j^{\alpha}} - 1 \right), \qquad (4)$$

where S_i^{α}'s are the q-state Potts model variables and the interaction K is a function of nearest neighbor connection probability p. Here, the connection probability between two sites gets translated into the interaction between the spins as we have only retained most dominant (or relevant) long-range interacting term in Eq. (4).

Long-range interacting ferromagnetic Potts Model described by Hamiltonian (4), exhibits a nontrivial order-disorder transition at a finite temperature $T_c(\sigma)$ [even when the spatial dimensionality $d=1$ if the range parameter $\sigma < 1$ (Refs. 22 and 24)]. The marginal case $d = \sigma = 1$ shows a topological transition as in the inverse-square Ising model.[16,25] In the corresponding percolation model (with connection probability $p/r_{ij}^{d+\sigma}$) therefore, an infinite percolation cluster always exists if the nearest neighbor connection probability p is greater than $p_c(\sigma)$ and there is a percolation transition at $p = p_c(\sigma)$. For $p < p_c(\sigma)$, infinitely connected cluster disappears. The percolation threshold $p_c(\sigma)$ shifts to the lower value of p as the range parameter σ decreases, i.e., the connection probability between two distant sites gets lower. At $p_c(\sigma)$, however, there exists an infinite cluster with fractal dimension $d_f(d > d_f > 1)$. Using the same argument as given above, we conclude that the phase diagram of the TIM on a long-range connected lattice is qualitatively similar to that of the conventional dilute magnet[12] (Fig. 1). A similar phase diagram is also expected for the long-range connected TIM even for $d=1$ if $\sigma < 1$.[16]

In the same spirit as in Ref. 13, we shall now examine the nature of transition at the percolation threshold $p_c(\sigma)$ with $\Gamma < \Gamma_o$ (Fig. 1). Along this vertical line, the disorder averaged static correlation function between two spins separated by a distance x (which is actually the probability that two sites separated by a distance x belongs to the same cluster) is given as[17] $\overline{C}(x) \sim 1/(x^{[d-2+\eta_{LR}(\sigma)]})$, where $\eta_{LR}(\sigma)$ is the anomalous dimension exponent of the percolation transition with long-range connections. This algebraic fall shows that the vertical line is critical with transitions dictated by the percolation fixed point $p_c(\sigma)$.

To derive the percolation exponents, let us use the Potts model Hamiltonian (4) in the $q \rightarrow 1$ limit (where the order disorder transition in the Potts model is always continuous). In the continuum limit, the transition in the above Potts model is described in terms of a cubic model with a long-

range interaction term. The corresponding Landau-Ginzburg-Wilson action is[26]

$$\mathcal{H} = -\frac{1}{2} \int (r + ak^2 + bk^\sigma) \sum_{i=1}^{q} Q_{ii}(k) Q_{ii}(-k)$$

$$+ w \int \int \int \sum_i Q_{ii}(k_1) Q_{ii}(k_2) Q_{ii}(k_3) \delta(k_1 + k_2 + k_3), \tag{5}$$

where Q_{ii} is a diagonal traceless $q \times q$ tensor and $\int$ denotes the integration over $\vec{k}$ in the first Brillioun zone, r and w are the mass and the coupling terms, respectively. The term bk^σ arises due to the long-range interaction in the Hamiltonian (4). At the Gaussian level this term dominates over the short-range term ak^2 as long as $\sigma < 2$. For higher values of σ (≥ 2), long-range interactions are irrelevant and therefore exponents of the Potts model are of short-range nature. In higher-orders when the effect of the coupling term w is perturbatively included, the short-range term ak^2 picks up the anomalous dimension η_{SR}. We therefore conclude that the long-range interaction is relevant as long as $\sigma < 2 - \eta_{SR}$. The most interesting feature of any long-range interacting spin model, described by action (5) is that the term bk^σ does not get renormalised at any order of perturbation and hence the anomalous dimension for the long-range interacting system sticks to the value $\eta_{LR} = 2 - \sigma$ for all values of $\sigma < 2 - \eta_{SR}$.[27] It can also be checked by simple dimensional counting of different parameters of action (5) that the upper-critical dimension and the range follow the relation $d_u = 3\sigma_u$, i.e., for a given d if the range parameter $\sigma < \sigma_u (=d/3)$, the interactions are long-ranged enough to render mean field theory to be exact. For $\sigma = 2$ (short-range case), the upper critical dimension is as usual 6.

Let us now restate the above results for the equivalent long-range percolation problem. As expected, the exponents depend on σ. In particular, the anomalous dimension exponent $\eta_p(\sigma)$ of a percolation transition is related to the fractal dimension of the lattice at p_c through the scaling relation[17] $d_f(\sigma) = [d + 2 - \eta_p(\sigma)]$. In the long-range model, $\eta_p(\sigma) = \eta_{LR}(\sigma) = 2 - \sigma$ which immediately leads to the result $d_f(\sigma) = (d + \sigma)/2$. The fractal dimension at the percolation threshold is therefore exactly known for long-range percolation problem for all values of d and σ. The other exponents of the percolation transition can be obtained by perturbative calculation around the upper critical dimension,[26] e.g., to the first order in $\epsilon (=3\sigma - d)$, the exponent $\nu_p = 1/\sigma + \epsilon/4\sigma^2$. The exponents not only depend on the range parameter σ but also cross over to the short-range value as σ approaches $2 - \eta_{SR}$ when the long-range interaction term bk^σ becomes irrelevant. In the following, we shall use the standard percolation scaling relation for the probability that a site falls in a finite cluster of size s given as[17]

$$P(s,p) = s^{1-\tau} f\{s[p - p_c(\sigma)]^{1/\nu_p d_f}\}, \tag{6}$$

where $\tau = (d + d_f)/d_f$.

We are now in a position to obtain the critical exponents related to the quantum transitions at $p = p_c$ and $\Gamma < \Gamma_0$.

Let us first look at the exponent ψ. The energy gap ΔE at the percolation threshold of a cluster of size L scales as $\Delta E \sim \exp(-cL^{d_f})$ where c is a nonuniversal constant. The logarithmic scaling of the correlation length with the frequency can be easily verified also by calculating the dynamical response function using the percolation scaling relation (6).[13] The exponent ψ is therefore given as $\psi = d_f = (d + \sigma)/2$. The important features which constitute the essential theme of this communication need to be highlighted here: (i) the exponent $\psi(\sigma)$ is exactly known for all values of σ and d while in the conventional case the exponent is exactly known only in the spatial dimension $d = 2$,[17] (ii) ψ depends on the range of interaction σ and thus gets modified as the range parameter σ is tuned, and (iii) $\psi(\sigma)$ undergoes a crossover to the short-range value when $\sigma \geq 2 - \eta_{SR}$. To evaluate the exponent ϕ, let us now use the scaling relation of magnetization in a weak magnetic field h at the quantum critical point given as[1]

$$m(h) \sim \left[\ln\left(\frac{h_o}{h}\right) \right]^{\phi - d/\psi}, \tag{7}$$

where h_o is a nonuniversal constant. In the present TIM, the magnetization at $p = p_c$ scales as

$$m(h) \sim \left[\ln\left(\frac{h_o}{h}\right) \right]^{2 - (d + d_f)/d_f}. \tag{8}$$

Comparing Eqs. (7) and (8) and using $\psi = d_f$ and the percolation exponents[17] $\beta_p = (\tau - 2)/\sigma$, $\gamma_p = (3 - \tau)/\sigma$, we have $\phi = (d - \beta_p/\nu_p)/d_f = 1$.

As in the conventional dilute magnet,[13] in the present long-range case also both the sides of the transition point p_c are found to be flanked with Griffiths phase with continually varying exponents. For example, in the disordered phase ($p < p_c(\sigma)$), the disorder averaged imaginary part of local dynamical susceptibility goes as

$$\overline{\chi_L''(\omega)} \sim (\omega)^{d/z} - 1, \tag{9}$$

where in arriving at Eq. (9), we have used the same functional form of the percolation scaling function $f(x)$ [Eq. (6)] as the conventional case[17] with exponents depending on σ. Equation (9) shows that the paramagnetic phase is gapless with a power-law density of states at low energy. This power-law singularity leads to off-critical GM singularities, e.g., the average local susceptibility diverges even in the paramagnetic phase as $T^{d/z-1}$ when the temperature $T \to 0$.[4] The dynamical exponent z is related to the percolation correlation length ξ_p as $z = \xi_p^{d_f} = (p_c - p)^{-\nu d_f}$. The dynamical exponent varies continuously with $(p_c - p)$ and diverges at the percolation threshold. We immediately conclude that the dynamical exponent z increases with increasing σ leading to a stronger divergence of z at p_c, indicating an enhancement in Griffiths-McCoy singularities. Therefore, the strength of the Griffiths phase also varies as σ is tuned.

In conclusion, the transitions in a TIM at the percolation threshold of a dilute lattice with long-range connection probabilities provide an ideal situation where the effect of long-range connections on the IRFP can be extensively studied. The exponents ψ and $\nu(=\nu_p)$ of IRFP are found to depend

nontrivially on the range parameter σ and cross over to the short-range values when the range parameter $\sigma \gtrsim 2 - \eta_{SR}$. However, for lower values of $\sigma(<\sigma_u)$, when the long-range connections are prominent, the exponents are described in terms of mean field percolation exponents.[17] Most importantly, the exponent ψ is exactly known for all values of d and σ and increases as σ increases. The strength of GM singularities also get enhanced for higher values of σ. This is expected because for smaller values of $\sigma(<\sigma_u)$ the model becomes more mean-field-like and GM singularities should be weak. A recent numerical study (based on the finite size scaling argument) of a percolation transition in a power-law dilute chain[28] in $d=1$ suggests that the scaling relation $d_f = (d+\sigma)/2$, obtained from renormalisation group arguments, may not be valid when long-range connection probabilities are dominant, i.e., for $0 < \sigma \leq 1/2$. However, for higher values of σ the numerical results corroborate the renormalization group results. No such numerical study is available for the percolation transition with power-law dilution in dimen-

sionality greater than unity. Question therefore remains whether one should use the value of d_f as obtained from the Potts model analogy in smaller σ region for $d=1$. Also, Vojta and Schmalian[14] showed that for dilute quantum rotors the quantum dynamics is of conventional (power-law) nature (for transitions below the multicritcal point) with the dynamical exponent $z=d_f$ and the correlation length exponent $\nu = \nu_p$. The scaling relation employed in Ref. 14 should in principle also work in the case of the present dilute lattice with long-range connections. We should also mention that the present work may also be relevant for the studies of quantum phase transitions on some interesting network models.[29]

*Electronic address: dutta@iitk.ac.in

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