Nonequilibrium entanglement and noise in coupled qubits

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We study the charge entanglement and noise spectrum of two Coulomb-coupled double quantum dots under stationary nonequilibrium transport conditions. In the transport regime, the entanglement exhibits a clear switching threshold and various limits due to suppression of tunneling by quantum Zeno localization or by an interaction-induced energy gap. We also discuss the interdot current correlation as a possible indicator of the entanglement in transport experiments.

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I. INTRODUCTION

Precise engineering and preparation of entangled states forms the backbone of many quantum information schemes.[1] The complete control of interactions between two or more parties is a sangraal that is not without cost. For example, in superconducting nanocircuits[2] there has been much success in devising schemes for *tunable* capacitative couplings,[3] but thermal fluctuations, background noise, and limited control over *natural* interactions must be dealt with and overcome in increasingly imaginative ways.

Here we take a slightly different point of view and ask for the degree of entanglement between two parallel, interacting electronic conductors under the "unfavorable" condition of stationary currents passing through both of them. As this is clearly a mixed-state situation, we specifically consider a nonequilibrium version of the concurrence as entanglement measure for an electron charge double qubit (DQ), realized in Coulomb-coupled double quantum dots[4,5] that are strongly coupled to external electron reservoirs at high voltage bias. We compare this to the same closed device in equilibrium with a heat bath.We find that effects such as suppression of nonresonant tunneling and the quantum Zeno effect (QZE) have a direct impact on the entanglement, to which we also establish a further link by calculating the nonequilibrium quantum shot-noise tensor whose off-diagonal elements, as a function of the system parameters, show a behavior very similar to the concurrence. In Appendix A we present a derivation of the MacDonald formula, and additional results for the noise spectra of a single quantum dot with two internal levels. In certain limits this system exhibits a divergence of the zero frequency noise.

II. MODEL: TWO COUPLED DOUBLE QUANTUM DOTS

For the sake of clarity, we define the double qubit by "left" and "right" orbital charge states $|\alpha_i\rangle$, $\alpha=L,R$ of one additional electron on top of the many-body ground state $|0_i\rangle$ (limit of *intradot* Coulomb interaction $U_{\rm in}\to\infty$) of two double quantum dots $i=1,2$ which are coupled by a single matrix element U for *interdot* "same site" interactions (left-left and right-right), cf. Fig. 1. Tunneling of electrons occurs only within but not between the qubits due to coupling T_i in each double dot. Using projectors onto these orbital states, $(\hat{n}_L^{(i)}=|L\rangle\langle L|_i, \hat{n}_{LR}^{(i)}=|L\rangle\langle R|_i,\ldots)$, the total Hamiltonian is

$$\mathcal{H}_0=\sum_{i=1,2}\left[\varepsilon_i(\hat{n}_L^{(i)}-\hat{n}_R^{(i)})+T_i(\hat{n}_{LR}^{(i)}+\hat{n}_{RL}^{(i)})\right]+U(\hat{n}_L^{(1)}\hat{n}_L^{(2)}+\hat{n}_R^{(1)}\hat{n}_R^{(2)}). \tag{1}$$

The electron spin label is suppressed here and in the following, as only charge states (acting as pseudospin) play a role. This description has turned out to be useful for modeling charge-related properties such as decoherence and noise in individual double quantum dots.[5,6]

We "open" the DQ by coupling it to four external electron reservoirs, $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_T+\mathcal{H}_{\rm res}$, with $\mathcal{H}_{\rm res}=\sum_{i=1,2}\sum_{\alpha\in L,R}\sum_{ki\alpha}\epsilon_{ki\alpha}c_{ki\alpha}^{\dagger}c_{ki\alpha}$ ($\alpha=L/R$ refers to left and right reservoirs for qubit number i, $i=1,2$) and $\mathcal{H}_T=\sum_{i=1,2}\sum_{\alpha\in L,R}\sum_k(V_k^{\alpha i}c_{ki\alpha}^{\dagger}\hat{s}_\alpha^i+\text{H.c.})$, with Hubbard operators $\hat{s}_\alpha^i=|0_i\rangle\langle\alpha_i|$ that couple the qubits to the continuum.

A. Equilibrium entanglement

If the DQ is disconnected from the reservoirs ($\mathcal{H}_T=0$) but in contact with a heat bath at temperature $T=1/\beta$, the equi-

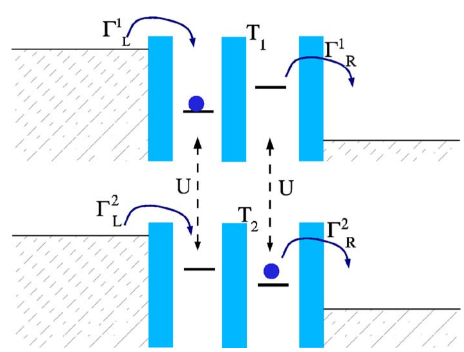

FIG. 1. (Color online) Transport double qubit model: left-right charge states in two Coulomb-coupled double quantum dots with one additional electron each and "on-site" (LL,RR) interaction U, coherent tunnel couplings T_1 and T_2, and electron reservoir tunnel rates $\Gamma_{L/R}^{1/2}$.

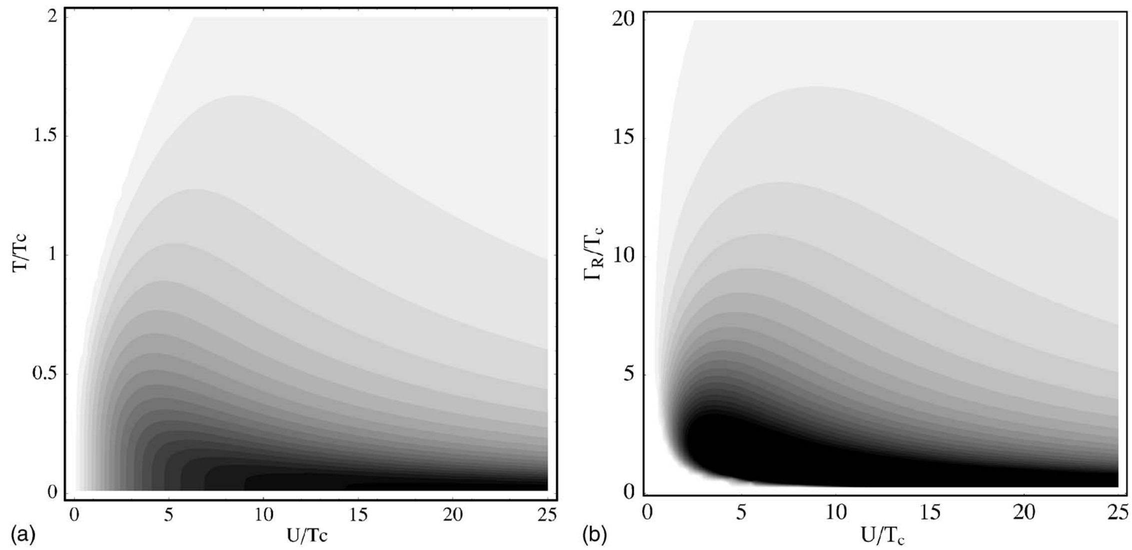

FIG. 2. Left: Grey-scale plot of double qubit equilibrium concurrence as a function of interaction U/T_c and temperature T/T_c (white is zero, black is maximum $C=1$). In all of the following results both devices have identical parameters $\Gamma_L^{(i)} \equiv \Gamma_L$, $\Gamma_R^{(i)} \equiv \Gamma_R$, $T_i \equiv T_c$, for $i=1,2$. Right: Concurrence of nonequilibrium double qubit as a function of interaction U/T_c and reservoir tunnel rate Γ_R for large Γ_L/T_c $=50$. Zero entanglement occurs below a threshold $\propto 1/\Gamma_R$ in the weak tunneling regime, and for very strong tunneling $\Gamma_R \gg T_c$ due to Zeno localization, cf. text (black is the maximum $C=0.3$).

librium entanglement between qubit 1 and 2 is trivially obtained from the concurrence[7] (a well-known entanglement measure of mixed states of two qubits) $C(\beta)$ for the canonical ensemble state $\rho(\beta)=e^{-\beta\mathcal{H}_0}/Z$, $Z=\mathrm{Tr}\, e^{-\beta\mathcal{H}_0}$. For simplicity we restrict ourselves to the unbiased, symmetric case $\varepsilon_i=0$, $T_i=T_c(i=1,2)$. The eigenvectors of $\mathcal{H}_0$ correspond to eigenvalues $E_0=0$, $E_1=U$, and $E_\pm=(U\pm\sqrt{16T_c^2+U^2})/2$ and are expressed in the basis of singlet and triplet states, $S_0=1/\sqrt{2}(|L_1R_2\rangle-|R_1L_2\rangle)$, $T_+=|L_1L_2\rangle$, $T_-=|R_1R_2\rangle$, $T_0=1/\sqrt{2}(|L_1R_2\rangle+|R_1L_2\rangle)$. It turns out that the equilibrium case already exhibits some interesting features, cf. Fig. 2. At any finite temperature T, the entanglement is zero below a certain threshold value of the interaction U where the state $\rho(\beta)$ is too mixed in order to be entangled which is, e.g., in analogy with the corresponding transition in the (abstract) example of the Werner state.[8] Furthermore, the concurrence shows a nonmonotonic behavior as a function of U at fixed T, with an entanglement maximum at an optimal U value.

B. Stationary transport

The limit $t\to\infty$ in the dynamical evolution of the reduced DQ density operator ρ defines a stationary nonequilibrium state ρ_∞ which usually is much more difficult to determine than in the equilibrium case. Transport properties of models such as Eq. (1) can be analyzed by using various nonequilibrium techniques. Here, we consider a specific limit of infinite source-drain bias in order to obtain quasianalytic results from a generalized master equation, $\dot{\rho}=L[\rho]$. The superoperator L is parametrized by the Markovian DQ-lead tunnel rates $\Gamma_\alpha^i \equiv 2\pi\Sigma_k|V_k^{\alpha i}|\delta(\varepsilon-\epsilon_{ki\alpha})$ (of which the energy dependence is neglected), and the DQ parameters $\varepsilon_i=0$, $T_i=T_c(i=1,2)$. Analytical expressions for the stationary solution of the 25 coupled equations of motion (EOM) can then be found in an approximation where the broadening due to tunneling of the DQ levels is neglected, which for $\varepsilon_i=0$, however, is only a very crude approximation.

One obtains better results for the stationary currents $\langle I_i\rangle_\infty$ by second-order perturbation theory in the intradot tunnel couplings T_i, which clearly show a tunnel-broadened resonance

$$\langle I_i\rangle_\infty = -e\frac{\Gamma_R^i T_i^2}{(\Gamma_R^i/2)^2+U^2} \tag{2}$$

($-e$ is the electron charge). In this limit, the resonance is determined by the energy gap U between the localized eigenstates of the DQ: at large U, the triplet $T_+=|L_1L_2\rangle$ becomes populated (note that this state is always available because of the infinite voltage approximation). The energy gap to any other state involving delocalized electrons (e.g, the triplet T_0 or the singlet S_0) then suppresses the elastic current. In analogy to single charge qubits, where the energy gap is given by the internal bias ε, we expect this suppression to be lifted in the presence of inelastic processes.[9]

Furthermore, as a function of the coupling Γ_R^i to the drain, the current first increases and then becomes smaller again. With the drains acting as broadband measuring devices (electron on right side or not), strong couplings $\Gamma_R^i\to\infty$ completely freeze the charges on the left sides which is a "transport version" example[10] of the QZE. Alternatively, this localization can be interpreted as an infinite level broadening and the corresponding suppression of the local spectral density due to the decay to the drain. Finally, the behavior of the current, cf. Eq. (2), follows the occupation of the entangled singlet state S_0 as illustrated in Fig. 3. The main current contribution therefore stems from two-particle tunneling

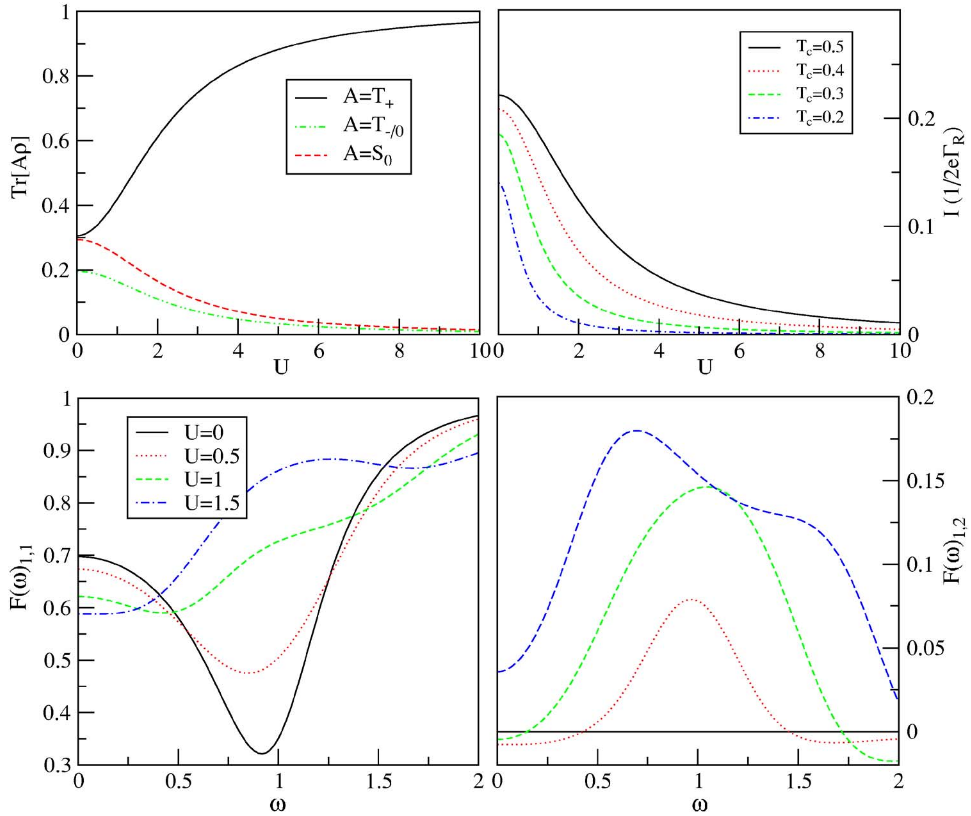

FIG. 3. (Color online) For $\Gamma_L=50$, $\Gamma_R=0.5$, $T_c=0.5$. Bottom left: Diagonal noise spectrum $F(\omega)_{1,1}=S(\omega)_{1,1}/2eI$. The resonance at $\omega=2T_c$ splits into new resonance points at the Bohr frequencies $\lambda_\pm=E_1-E_\pm=1/2(U\mp\sqrt{16T_c^2+U^2})$. Top right: Stationary current I_{stat}. Top left: Occupation of several singlet and triplet states. Bottom right: The cross-correlation frequency spectrum $F(\omega)_{1,2}=S(\omega)_{1,2}/2eI$. Again resonance points manifest, however the correlation is always zero for $U=0$ and can assume negative values for $U\neq 0$.

events, which in turn motivates our later comparison of the concurrence with the current fluctuations.

C. Nonequilibrium entanglement

We now define the nonequilibrium entanglement via the concurrence C of the stationary state $\hat{P}\rho_\infty$, where $\hat{P}$ is the projection onto doubly occupied states including proper normalization; i.e., we calculate the concurrence when both double dots have a single electron in them and there are thus two two-state systems to be entangled. The projection $\hat{P}$ corresponds to taking the limit $\Gamma_L^i\to\infty$ where both qubits are always occupied with one single electron. For example, for $U=0$ and $\Gamma_L\to\infty$, the stationary state of a single charge qubit is described by the (Bloch) vector of pseudospin Pauli matrices ($\varepsilon\equiv\varepsilon_L-\varepsilon_R$)

$$\langle\vec{\sigma}\rangle=\left(\frac{2T_c\varepsilon}{\mathcal{N}},\frac{\Gamma_R T_c}{\mathcal{N}},\frac{\Gamma_R^2/4+\varepsilon^2}{\mathcal{N}}\right),\tag{3}$$

with $N\equiv\Gamma_R^2/4+\varepsilon^2+2T_c^2$ and in the $L-R$ basis where $\sigma_z\equiv|L\rangle\langle L|-|R\rangle\langle R|$, etc. For $U\neq 0$, we numerically checked that $\hat{P}\rho_\infty=\lim_{\Gamma_L^i\to\infty}\rho_\infty$ which means that the two-qubit concurrence C defined in this way does no longer depend on the left tunnel rates. This is a good description of nonequilibrium entanglement in a real system as long as $\Gamma_L^i \gg \max(U,\Gamma_R^i,T_c,\varepsilon)$.

As C is zero to second order in T_c, we use numerical results, cf. Fig. 2, which shows an intriguing behavior of the concurrence as a function of U and the tunnel rate $\Gamma_R^i\equiv\Gamma_R$. We find a switching threshold in that below an interaction strength $U\sim 2T_c^2/\Gamma_R$ the entanglement is zero: for small Γ_R, the stationary currents become very small, cf. Eq. (2), and thus strong interactions are required in order to entangle the dots. The DQ state becomes strongly mixed for $\Gamma_R\to 0$ (note that we have not taken into account any additional, internal relaxation processes here); its zero entanglement along the axis $\Gamma_R=0$ is in fact a continuation of the point $U=0$ where the states of both qubits are located at the origins of their Bloch spheres, cf. Eq. (3).

On the other hand, for very large Γ_R one runs again into the QZE with electrons becoming trapped on the left ($\langle\sigma_z\rangle\to 1$, cf. Eq. (3)), and $\hat{P}\rho_\infty$ approaching the (pure) localized state $|L_1L_2\rangle$ which has zero entanglement. Finally, an increase from small to larger Γ_R at fixed U yields the reentrance behavior visible in the "teardrop"-shaped region of large entanglement in Fig. 2.

D. Nonequilibrium noise: Formalism

Turning now to our description of nonequilibrium shot noise and its relation to entanglement, the stationary state ρ_∞ on its own is not sufficient in order to describe intrinsic properties of the DQ: for example, only limited information on the spectrum can be obtained from stationary quantities such as the current. In contrast, the shot-noise spectrum exhibits resonances at the transition frequencies of the system and contains furthermore useful information on its relaxation and dephasing properties.[6,11–13] We will now also show an emergent resemblance in the behavior of the current *cross noise* and the nonequilibrium concurrence as a function of the system parameters.

In general, the finite-frequency noise has contributions from particle currents as well as contributions from displacement currents.[6,14] In our case $(\Gamma_L^i \gg \Gamma_R^i)$, however, it is a good approximation to consider only particle currents. Our starting point is the generating function

$$\hat{G}(s_1, \ldots, s_m, t) = \sum_{n_1, \ldots n_m=0}^{\infty} s_1^{n_1} \cdot \cdot \cdot s_m^{n_m} \rho^{(n_1),\ldots,(n_m)}(t) \quad (4)$$

which, for an arbitrary number of m qubits, contains the complete information on the tunneling process as a function of time via the counting variables $s \equiv \{s_i\}$ and the conditional density matrices $\rho^{(n_1),(n_2),\cdots}(t)$ for n_i tunneling events ("jumps") to the drain i after time t.

The conditional density matrices $\rho^{(n_1),(n_2),\cdots}(t)$ arise[16] from the equation of motion of the reduced density operator $\dot{\rho}(t) = L\rho(t)$ by splitting the superoperator $L = L_0 + L_1 + L_2 + \cdots + L_m$ such that L_i describes the "previous" electron leaving system "i." In analogy with the quantum jump approach, one introduces an interaction picture for $\dot{\rho}(t)$ with respect to L_0,

$$\bar{\rho}(t) \equiv e^{-L_0 t}\rho(t), \quad (5)$$

$$\overline{L_1}(t) \equiv e^{-L_0 t} L_1 e^{L_0 t}, \quad (6)$$

so that the time evolution is governed by

$$\begin{aligned}\frac{d}{dt}\bar{\rho}(t) &= -L_0\bar{\rho}(t) + e^{-L_0 t}(L_0 + L_1 + L_2 + \ldots)e^{L_0 t}\bar{\rho}(t) \\ &= [\overline{L_1}(t) + \overline{L_2}(t) + \cdots]\bar{\rho}(t). \quad (7)\end{aligned}$$

We can then integrate this equation, and substitute the result back into the equation of motion iteratively,

$$\begin{aligned}\bar{\rho}(t) &= \rho(0) + \int_0^t dt_1 \overline{L_1}(t_1)\bar{\rho}(t_1) + \cdots = \rho(0) \\ &+ \int_0^t dt_1 \overline{L_1}(t_1)\bar{\rho}(0) + \int_0^t dt_1 \int_0^{t_1} dt_2 \overline{L_1}(t_1)\overline{L_1}(t_2)\bar{\rho}(t_2) \\ &+ \cdots . \quad (8)\end{aligned}$$

We leave this effective interaction picture to give

$$\begin{aligned}\rho(t) &= U_0(t,0)\rho(0) + \int_0^t dt_1 U_0(t,t_1) L_1 U_0(t_1,0)\rho(0) \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 U_0(t,t_1) L_1 U_0(t_1,t_2) L_1 U_0(t_2,0)\rho_0 + \cdots . \\ & \quad (9)\end{aligned}$$

$U_0(t,0)$ is the time evolution operator due to L_0, $U_0(t,0) = \exp\int_0^t dt L_0$. We can see that the term in L_i is identified with the rate of electrons leaving the system i into the right reservoir. Thus, the first term in the expansion describes time evolution where no electrons have left any system. The second term describes when one electron has left system i at time t_1, and so on. This transmission is an incoherent process, and we can write the density matrix as

$$\rho(t) = \sum_{(n_1),(n_2),\ldots,(n_3)} \rho^{(n_1),(n_2),\ldots,(n_m)}(t), \quad (10)$$

where $\rho^{(n_1),(n_2),\ldots,(n_m)}(t)$ contains a sum of the possible orderings of products of $(n_1+1)+(n_2+1)+\cdots+(n_m+1)$ free time evolution operators and the jump operators $L_1, L_2, \ldots.$ at $(n_1),(n_2),\ldots$ intermediate times $t_1, \ldots, t_{(n_1+n_2+n_3+..)}$ which are integrated over. This series is defined with $\rho^{(-1),(-1),\cdots} = 0$. Time evolution of this series gives an equation of motion for each term as

$$\begin{aligned}\dot{\rho}^{(n_1),(n_2),\ldots,(n_m)} &= L_0\rho^{(n_1),(n_2),\cdots} + L_1\rho^{(n_1-1),(n_2),\cdots} + \cdots \\ &+ L_m\rho^{(n_1),(n_2),\ldots,(n_m-1)}. \quad (11)\end{aligned}$$

In matrix form, the EOM of the generating function follows from the Liouville equation for the conditional density matrices and reads $\dot{\mathbf{G}}(s,t) = M(s)\mathbf{G}(s,t)$ with formal solution $\mathbf{G}(s,t) = \exp[tM(s)]\mathbf{G}(s,0)$. General expectation values can be extracted from derivatives of $\mathrm{Tr}[\mathbf{G}(s,\tau)]$ with respect to the counting variables. In particular, the symmetrized noise correlation function $S(\omega)_{i,j} \equiv \int_{-\infty}^{\infty} e^{i\omega\tau}\langle\{\delta I_i(t+\tau), \delta I_j(t)\}\rangle$ between qubit i and j can then be written as a MacDonald formula[15,17]

$$\frac{S(\omega)_{i,j}}{2e^2\omega} = \int_0^{\infty} d\tau \sin(\omega\tau)\frac{\partial}{\partial\tau}\left\langle n_i n_j - \frac{\tau^2 \bar{I}_i \bar{I}_j}{e^2}\right\rangle, \quad (12)$$

where $\langle n_i n_j\rangle = \hat{D}_{ij}\mathrm{Tr}\,[\mathbf{G}(s,\tau)]|_{s=1}$ with the differential operator $\hat{D}_{ij} \equiv \partial_{s_i,s_j} + \delta_{ij}\partial_{s_i}$. A derivation of this formula is given in Appendix A. We simplify this expression following Flindt *et al.*[18] by introducing jump operators L_i for qubit sources i and writing $\partial_\tau\langle n_i n_j\rangle = \mathrm{Tr}[L_i\Sigma_{n_1,\ldots,n_m} n_j \rho^{(n_1),\ldots,(n_m)}(\tau)] + (i \leftrightarrow j)$. This can be further evaluated by Laplace transforming the EOM $\partial_t\hat{G}(s,t) = (L_0 + \Sigma_i s_i L_i)\hat{G}(s,t)$ and taking derivatives in counting variables, giving $\partial_{s_i}\tilde{G}(s,-i\omega)|_{s=1} = F_\omega L_i F_\omega \rho_0$, where $F_\omega = (-i\omega - L)^{-1}$ and ρ_0 is the steady-state initial condition. Using the projections $F_\omega = -P/i\omega - R_\omega$, $R_\omega = Q(i\omega + L)^{-1}Q$, $(P = \rho_0 \otimes 1,\ Q = 1 - P)$ with $P\rho_0 = \rho_0$ and $Q\rho_0 = 0$ leads to

$$\frac{S(\omega)_{i,j}}{-2e^2} = \mathrm{Re}\,\mathrm{Tr}\left[\left(L_i R_\omega + \frac{\delta_{ij}}{2}\right)L_j\rho_0\right] + (i \leftrightarrow j). \quad (13)$$

In the zero frequency limit, we verify that the noise is determined as usual[12,19] by the lowest eigenvalue $\lambda_0(s)$ of the matrix $M(s)$, namely by the long-time behavior $G(s,t\to\infty) \propto \exp[t\lambda_0(s)]$ and therefore $S(0)_{i,j} = 2e^2\hat{D}_{ij}\lambda_0(s=1)$. As mentioned, at finite frequencies the noise has contributions from current fluctuations in the left and right reservoir as well as noise contributions from displacement currents[6,14] $S_Q(\omega)$. In the single qubit case, according to the Ramo-Shockley

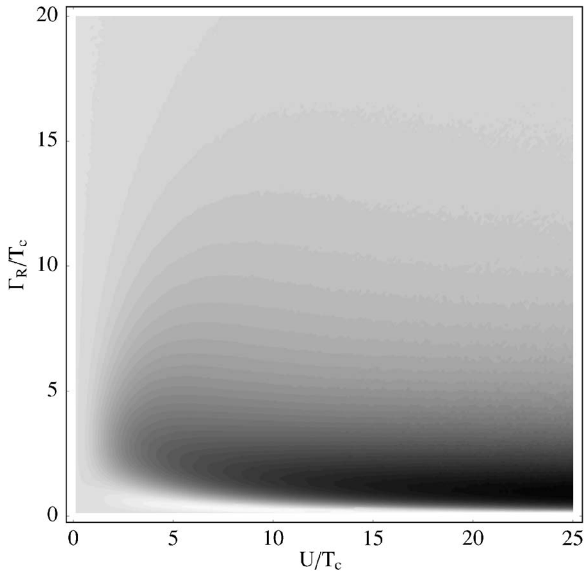

FIG. 4. The cross-correlation zero-frequency spectrum Fano factor $F(0)_{1,2}=S(0)_{1,2}/2eI$. The resemblence to the concurrence, Fig. 2, is qualitative. (White is minimum, $F_{1,2}=-0.12$, black is maximum, $F_{1,2}=0.84$).

theorem,[6,14] $S(\omega)=\alpha S_L(\omega)+\beta S_R(\omega)-\alpha\beta S_Q(\omega)$ with capacitance coefficients $\alpha\equiv c_L/(c_L+c_R)$ and $\beta\equiv c_R/(c_L+c_R)$. We can assume here extremely asymmetric junctions such that the left and right effective capacitances c_R, c_L are extremely asymmetric so that $c_L c_R \ll 1$ and here this contribution is small. However in general one can use the multivariable approach in order to calculate cross terms in the decomposition of the total current fluctuations rather trivially,

$$\delta I(t+\tau)\delta I(t)=\alpha^2\delta I_L(t+\tau)\delta I_L(t+\tau)+\beta^2\delta I_R(t+\tau)\delta I_R(t) + \alpha\beta[\delta I_L(t+\tau)\delta I_R(t)+\delta I_R(t+\tau)\delta I_L(t)]. \tag{14}$$

E. Nonequilibrium noise: Results

The currents through our two parallel charge qubits give rise to two diagonal and one off-diagonal component in the tensor $S(\omega)_{i,j}$ of the noise spectrum. In Fig. 3, we present results for the diagonal noise, i.e., the noise spectrum $S(\omega)_{1,1}=S(\omega)_{2,2}$ of the individual, interacting qubits. This spectrum clearly displays resonances at the Bohr frequencies as given by the excitation energies of the closed system. At $U=0$, there is one single resonance at $\omega=2T_c$ that splits up when U is increased. Similar to light emission spectra in real molecules, frequency-dependent shot-noise spectra thus provide direct information about the correlated energy levels in artificial molecules.

The *cross-noise* spectrum exhibits a somewhat more complicated resonance structure (Fig. 3). More interesting is however the behavior of the cross-correlation Fano factor at zero frequency, $F(0)_{1,2}\equiv S(0)_{1,2}/2eI$, which becomes positive as U increases. This positive cross correlation is an indication of correlated emission of electron pairs into different exit right leads.[20] As a function of U, cf. in Fig. 4, there is a strong analogy between the cross noise $F(0)_{1,2}$, its first derivative $F'(0)_{1,2}$, and the nonequilibrium concurrence C, at least on a qualitative level.

In particular, the nonanalytic switching of C with increasing U from unentangled to entangled states translates into a strongly delayed (though smooth) onset of the increase in

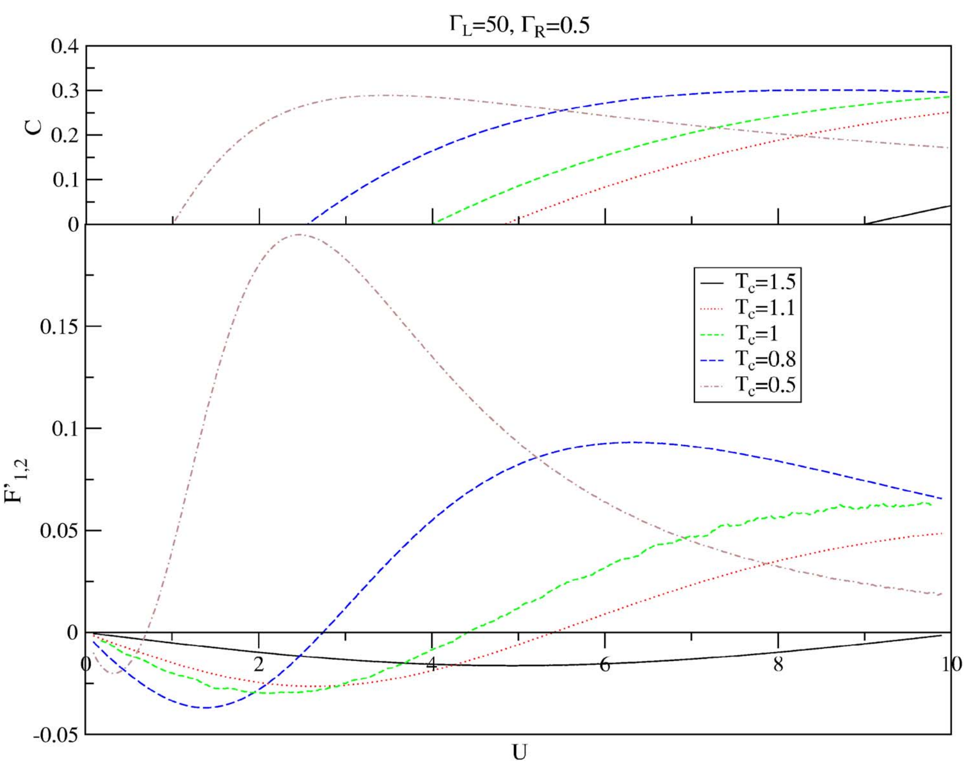

FIG. 5. (Color online) The switching phenomenon in the concurrence is more clearly seen, as well as a negative to positive reemergence in the first derivative of the noise around the same point.

$F(0)_{1,2}$, and the transition of $F'(0)_{1,2}$ from negative to positive. For example, in Fig. 5 we see $F'(0)_{1,2}$ for $T_c=0.5$ becomes positive around $U=1$ in agreement with the switching of C at $U\sim 2T_c^2/\Gamma_R=1$. This analogy between noise and entanglement so far holds on a qualitative level only.

We have not included the effect of dissipation in our calculations for two interacting dots so far. Weak decoherence processes can in principle be easily incorporated through additional terms within the master equation. In Ref. 6 it was shown how to use the resulting changes in the noise spectrum in order to extract, e.g., relaxation and decoherence times T_1 and T_2. This can also be done for the interacting qubits discussed here.

III. CONCLUSIONS

We have shown how the entanglement of a nonequilibrium double qubit differs from its thermal-equilibrium relative by exhibiting a $1/\Gamma$ switching threshold for weak tunnelling rates Γ. The cross-correlation noise reflects this threshold at $\omega=0$ and shows resonances at the Bohr frequencies of the double qubit for finite ω. Future theoretical work may include clarifying this relationship and checking the influence of decoherence on the correlated noise power spectrum.

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APPENDIX A: MACDONALD FORMULA

As it is often omitted in the literature we give a derivation of the MacDonald formula.[15] Using the definition,

$$S(\omega)=\int_{-\infty}^{\infty} dt e^{i\omega\tau}\langle[\delta I(t+\tau),\delta I(t)]_+\rangle, \tag{A1}$$

where $\delta I(t)=I(t)-\langle I\rangle$. The correlation function in this expression is a statistically stationary variable (it measures fluctuations around a stationary average), is only a function of τ, and is symmetric in τ and ω.[15]

The inverse Fourier transform of the noise power gives

$$\langle[\delta I(t+\tau),\delta I(t)]_+\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}d\omega. \tag{A2}$$

If $en(\tau)=\int_t^{t+\tau}I(t')dt'$ (where e is the electron charge),

$$\int_t^{t+\tau}\delta I(t')dt'=en(\tau+t)-en(t)-\tau\langle I\rangle. \tag{A3}$$

Using the expectation value of the square of this expression, the inverse Fourier transform and setting $\tau=t'-t''$,

$$\begin{aligned}
&2e^2\langle[n(t+\tau)-n(t)-\tau\langle I\rangle/e]^2\rangle\\
&\quad=\left\langle\int_t^{t+\tau}[\delta I(t')\delta I(t'')+\delta I(t'')\delta I(t')]dt'dt''\right\rangle\\
&\quad=\int_t^{t+\tau}dt'dt''\int_{-\infty}^{\infty}\frac{1}{2\pi}S(\omega)e^{-i\omega(t'-t'')}d\omega.
\end{aligned} \tag{A4}$$

Rearranging and performing the time integrals we obtain

$$=\int_{-\infty}^{\infty}\frac{1}{2\pi}S(\omega)\frac{1}{\omega^2}[e^{-i\omega\tau}-1][e^{i\omega\tau}-1]d\omega, \tag{A5}$$

$$=\int_{-\infty}^{\infty}\frac{1}{\pi}S(\omega)\frac{1}{\omega^2}[1-\cos(\omega\tau)]d\omega, \tag{A6}$$

Differentiating both sides with respect to τ gives

$$2e^2\frac{\partial}{\partial\tau}\langle[n(t+\tau)-n(t)-\tau\langle I\rangle/e]^2\rangle=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{S(\omega)}{\omega}\sin(\omega\tau)d\omega, \tag{A7}$$

and performing the Fourier transform with $\int_{-\infty}^{\infty}e^{i\omega'\tau}d\tau$,

$$\begin{aligned}
&2e^2\int_{-\infty}^{\infty}d\tau e^{i\omega'\tau}\frac{\partial}{\partial\tau}\langle[n(t+\tau)-n(t)-\tau\langle I\rangle/e]^2\rangle\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{S(\omega)}{\omega}\frac{1}{2i}(e^{i\omega\tau}-e^{-i\omega\tau})e^{i\omega'\tau}d\omega d\tau,\\
&\quad=\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\frac{S(\omega)}{\omega}\frac{1}{2i}[2\pi\delta(\omega'+\omega)-2\pi\delta(\omega'-\omega)],\\
&\quad=\frac{i}{\omega'}[S(\omega')+S(-\omega')].
\end{aligned} \tag{A8}$$

At this point we can use the even nature of the noise power (and setting $\omega'=\omega$),

$$2e^2\int_{-\infty}^{\infty}d\tau e^{i\omega\tau}\frac{\partial}{\partial\tau}\langle[n(t+\tau)-n(t)-\tau\langle I\rangle/e]^2\rangle=2i\frac{S(\omega)}{\omega}. \tag{A9}$$

We can match the odd (and imaginary) parts of this equation to give

$$e^2\int_{-\infty}^{\infty}d\tau\frac{\partial}{\partial\tau}\langle[n(t+\tau)-n(t)-\tau\langle I\rangle]^2\rangle\sin(\omega\tau)=\frac{S(\omega)}{\omega}. \tag{A10}$$

Again using the fact that $S(\omega)=S(-\omega)$, and that the original correlator is symmetric in τ, implies the integral over τ can be written

$$2e^2\int_0^\infty \frac{\partial}{\partial \tau} d\tau \langle [n(t+\tau) - n(t) - \tau\langle I\rangle]^2\rangle \sin(\omega\tau) = \frac{S(\omega)}{\omega}. \tag{A11}$$

This expression allows us to calculate the full frequency spectrum of the noise from just the fluctuations of the charge $\langle n^2\rangle$ and the average current $\langle I\rangle$. We apply the initial condition that $n(t=0)=0$, and since the ergodic theorem still holds (the result is independent of t) the term in the integrand can also be written $\langle [en(\tau)-\tau\langle I\rangle]^2\rangle = \langle [\int_t^{t+\tau} dt' I(t') - \tau\langle I\rangle]^2\rangle$, and

$$e\langle n(\tau)\rangle = \left\langle \int_t^{t+\tau} dt' I(t') \right\rangle = \langle I\rangle \int_t^{t+\tau} dt' = \langle I\rangle \tau, \tag{A12}$$

giving the expansion $\langle Q^2(\tau)\rangle = e^2\langle n^2(\tau)\rangle - 2e\tau\langle n(\tau)\rangle\langle I\rangle + e\tau\langle I\rangle\langle n(\tau)\rangle$.

Macdonald's final expression for the noise power is

$$\frac{S(\omega)}{2e^2\omega} = \int_0^\infty d\tau \sin(\omega\tau) \frac{\partial}{\partial\tau}[\langle n^2(\tau)\rangle - \tau^2\langle I\rangle^2/e^2]. \tag{A13}$$

There is also a $\omega \to 0$ limit to this equation where the derivative is taken at $\tau\to\infty$, so that the integrand is τ independent and

$$\begin{aligned} S(\omega\to 0) &= 2e^2\omega\left[\int_0^\infty d\tau \sin(\omega\tau)\right] \\ &\quad \times \frac{\partial}{\partial\tau}_{\tau\to\infty}[\langle n^2(\tau)\rangle - \tau^2\langle I\rangle^2/e^2], \\ &= 2e^2[\langle n^2(\tau)\rangle - 2\tau\langle I\rangle^2/e^2]. \end{aligned} \tag{A14}$$

The integral of $\omega\int_0^\infty d\tau \sin(\omega\tau)$ follows from introducing a parameter δ,

$$\omega\, \mathrm{Im}\left[\int_0^\infty d\tau e^{(i\omega-\delta)\tau}\right]_{\delta\to 0} = \omega\, \mathrm{Im}\left[\frac{1}{\delta - i\omega}\right]_{\delta\to 0} = 1. \tag{A15}$$

APPENDIX B: A SINGLE DOT WITH TWO DISSIPATIVE LEVELS

Finally we consider the slightly simpler case of a two-level quantum dot with internal dissipation at rate γ. In this case, the matrix $M(s)$ depends on only one counting variable,

$$M(s) = \begin{pmatrix} -\Gamma_L^{(1)} - \Gamma_L^{(2)} & s\Gamma_R^{(1)} & s\Gamma_R^{(2)} \\ \Gamma_L^{(1)} & -\Gamma_R^{(1)} & \gamma \\ \Gamma_L^{(2)} & 0 & -\Gamma_R^{(2)} - \gamma \end{pmatrix}. \tag{B1}$$

We begin by looking at the nondecaying regime $\gamma=0$. We only consider symmetric rates $\Gamma_L^{(1)} = \Gamma_R^{(1)} = \Gamma^{(1)}$ and $\Gamma_L^{(2)} = \Gamma_R^{(2)} = \Gamma^{(2)}$ in the following. For identical rates $\Gamma^{(1)} = \Gamma^{(2)} = \Gamma$ the addition of the extra level only effects the rate of tunneling "in," so that the results are those found for the single level with $\Gamma_L^{(1)} \to 2\Gamma_L^{(1)}$.[14,19]

For asymmetric rates $\Gamma^{(1)} \neq \Gamma^{(2)}$, the eigenvalue $\lambda_0(s,\Gamma^{(1)})$ of $M(s)$ belonging to the stationary solution at fixed $\Gamma^{(2)}$ is nonanalytic in the vicinity of $s=1$ and for $\Gamma^{(1)}\to 0$. In particular, the double degeneracy at $s=1$ in the eigenvalues $\lambda_0 = -\Gamma^{(2)} + \Gamma^{(2)}\sqrt{s}, \lambda_1 = 0, \lambda_2 = -\Gamma^{(2)} - \Gamma^{(2)}\sqrt{s}$ for $\Gamma=0$ is lifted for $\Gamma > 0$. We omit the complex analytical form here.

As the second but not the first derivative of $\lambda_0(s)$ diverges for $\Gamma^{(1)}=0$, the stationary current is $\langle I\rangle = \frac{\Gamma^{(1)}+\Gamma^{(2)}}{3}$, which stays finite (as must be) when one of the transport channels becomes closed, whereas the Fano factor $F \equiv S(0)/2e\langle I\rangle$ diverges as

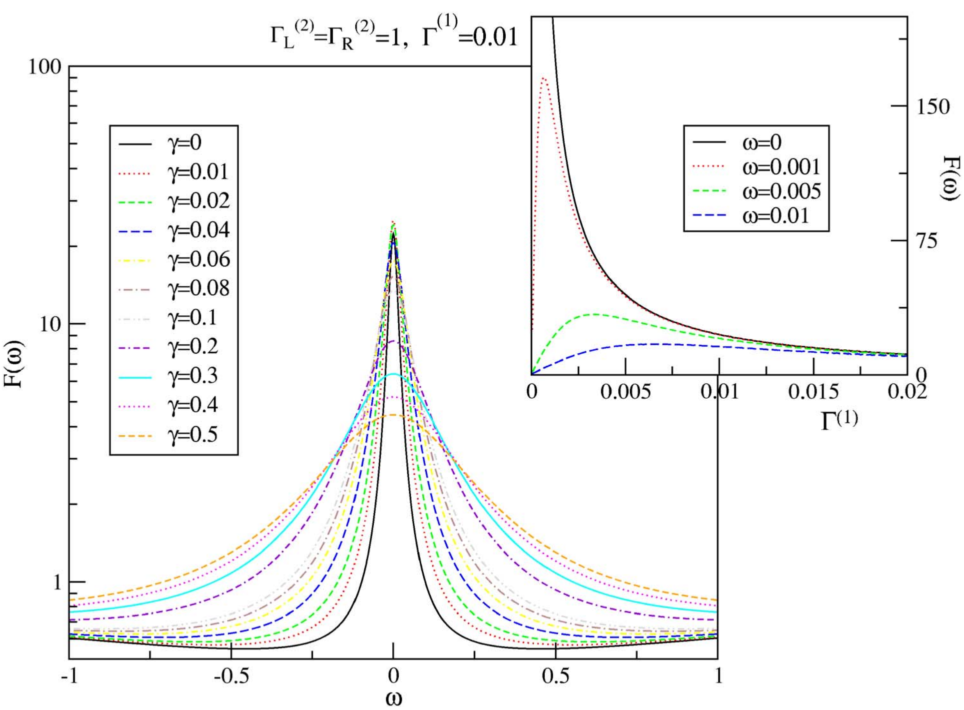

FIG. 6. (Color online) Main: Frequency-dependent Fano factor $F(\omega)=S(\omega)/2eI$ for a symmetric two-level single dot with zero and nonzero dissipation γ. The linewidth becomes proportional to γ. Inset: For zero dissipation $\gamma=0$, $F(0)$ diverges as $1/\Gamma$ for small Γ, cf. Eq. (B2).

$$F = \frac{4\beta^2}{9\Gamma^{(1)}(\Gamma^{(1)} + \Gamma^{(2)})}, \quad \Gamma^{(1)} \to 0 \tag{B2}$$

for $\Gamma^{(1)} \to 0$. The divergence for $\omega=0$ translates into a peak in the ω dependent noise $S(\omega)$ which is shown in Fig. 6 as a function of $\Gamma^{(1)}$. A similar "huge" Fano factor for two-channel transport has been identified in the coexistence regime of quantum shuttles[18] and in electrostatically coupled single dots.[13]

In addition, in Fig. 6, we show $F(\omega)$ as a function of ω for fixed $\Gamma^{(2)}$ and for a range of dissipation rates (γ). We see that the noise is reduced as γ is increased and the width of the lineshapes is increased. For $\gamma=0$ the width is proportional to $\Gamma^{(1)}/\Gamma^{(2)}$, but for larger dissipation the width is proportional to γ.

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