# Thermodynamical derivation of a hydrodynamical model of inhomogeneous superfluid turbulence

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In this paper, we build up a thermodynamical model of inhomogeneous superfluid turbulence to describe vortex diffusion in inhomogeneous turbulent tangles, and a coupling between second sound and vortex-density waves. The theory chooses as fundamental fields the density, the velocity, the energy density, the heat flux, and the averaged vortex line length per unit volume. The restrictions on the constitutive quantities are deduced from the entropy principle, using the Liu method of Lagrange multipliers. Field equations are written and the wave propagation is studied with the aim to describe the mutual interactions between the second sound and the vortex tangle.

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## I. INTRODUCTION

Turbulence is almost the rule in the flow of classical fluids. It is a complex nonlinear phenomenon for which the development of a satisfactory theoretical framework is still incomplete. Turbulence is often found in the flow of quantum fluids, especially superfluid helium 4, known as liquid helium II.[1–5]

The behavior of superfluid helium II is very different from that of ordinary fluids. One example of nonclassical behavior is the possibility to propagate the second sound, a wave motion in which temperature and entropy oscillate and density and pressure remain essentially constant. Second sound is important in turbulence because it is used to measure the vortex line density $L$. A second example of nonclassical behavior is heat transfer in counterflow experiments. Consider a channel with a heater at a closed end and open to the helium bath at the other end. A heat flux $\mathbf{q}$ is present in the channel. Using an ordinary fluid (such as helium I), a temperature gradient can be measured along the channel, which indicates the existence of a finite thermal conductivity. If helium II is used, and the heat flux inside the channel is not too high, the temperature gradient is so small that it cannot be measured, so indicating that the liquid has an extremely high thermal conductivity (three million times larger than that of helium I). This is confirmed by the fact that helium II is unable to boil.

The most well known hydrodynamical model of superfluid helium is the two-fluid model of Tisza[6] and Landau,[7] which regards helium II as a mixture of two fluid components, the normal fluid and the superfluid, with densities $\rho_n$ and $\rho_s$, respectively, and velocities $\mathbf{v}_n$ and $\mathbf{v}_s$, respectively, with total mass density $\rho$ and velocity $\mathbf{v}$ defined by $\rho=\rho_s+\rho_n$ and $\rho\mathbf{v}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n$. The first component consists of thermally excited states that form a viscous fluid which carries the entire entropy content of the liquid. The second component is related to the quantum ground state and is an ideal fluid, which does not experience dissipation nor carry entropy.[1,8] The two-fluid model explains the experimental situation described above in the following way: In the absence of mass flux ($\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s=0$), the heat is carried toward the bath by the normal fluid only, and $\mathbf{q}=\rho sT\mathbf{v}_n$ where $s$ is the entropy per unit mass and $T$ is the temperature. With the net mass flux being zero, there is superfluid motion toward the heater ($\mathbf{v}_s=-\rho_n\mathbf{v}_n/\rho_s$); hence there is a net internal counterflow $\mathbf{V}_{ns}=\mathbf{v}_n-\mathbf{v}_s=\mathbf{q}/(\rho_s sT)$ which is proportional to the applied heat flux $\mathbf{q}$.

In recent years there has been growing interest in superfluid turbulence, because a better understanding of it can throw new light on problems in classical turbulence, but it is also crucial to explain much observed superfluid behavior, including that relevant for the application of superfluid helium as a coolant for superconducting devices.

In counterflow experiments (the experimental situation described above, characterized by no matter flow but only heat transport, exceeding a critical heat flux $q_c$) one observes an extra attenuation of second sound, which grows with the square of the heat flux. This damping force, known as *mutual friction*, finds its origin in the interaction between the flow of excitations and an array of quantized vortex filaments in helium II.[1,8] A first thermodynamic study of these interesting phenomena was made in Ref. 9, where the presence of vortices was modeled through a pressure tensor $\mathbf{P}_\omega$ for which a constitutive relation was written.

Quantum turbulence is described as a chaotic tangle of quantized vortices of equal circulation:

$$\vec{\kappa}=\oint \mathbf{v}_s\cdot dl. \tag{1.1}$$

$\kappa=|\kappa|$ is called quantum of vorticity and results in $\kappa=h/m_4$, with $h$ the Planck constant and $m_4$ the mass of $^4$He atom: $\kappa\simeq 9.97\times10^{-4}$ cm$^2$/s. The vortex tangle is assumed to be isotropic and may be described by introducing a scalar quantity $L$, the average vortex line length per unit volume (briefly called vortex line density).

The evolution equation for $L$ in counterflow superfluid turbulence has been formulated by Vinen. Neglecting the influence of the walls, such an equation is[10]

$$\frac{dL}{dt}=\alpha_v V_{ns}L^{3/2}-\beta_v\kappa L^2, \tag{1.2}$$

 

with $V_{ns}$ the averaged magnitude of the counterflow velocity $\mathbf{V}_{ns}$ and $\alpha_v$ and $\beta_v$ dimensionless parameters. This equation assumes homogeneous turbulence, i.e., that the value of $L$ is the same everywhere in the system. In fact, homogeneity may be expected if the average distance between the vortex filaments, of the order of $L^{-1/2}$, is much smaller than the size of the system, but it will not be so for dilute vortex tangles.

Recent experiments[3,5] show the formation of another type of superfluid turbulence, which has some analogies with a classical one, as for instance using towed or oscillating grids, or stirring liquid helium by means of propellers. In this situation, which has been called *coflow*, both components, normal and superfluid, flow along the same direction. To describe these experiments it is necessary to build up a hydrodynamic model of quantum turbulence, in which the interactions between both fields can be studied and the role of inhomogeneities is explicitly taken into account.

In a hydrodynamic model of superfluid turbulence the line density $L$ acquires field properties: It depends on the coordinates, it has a drift velocity $\mathbf{v}_L$, and it has associated a diffusion flux. These features are becoming increasingly relevant, as the local vortex density may be measured with higher precision, and the relative motion of vortices is observed and simulated. Thus it is important to describe situations going beyond the usual description of the vortex line density averaged over the volume. Our aim is to formulate a hydrodynamical framework sufficiently general to encompass vortex diffusion and to describe the interactions between the usual waves and the vortices, instead of considering the latter as a rigid framework where such waves are simply dissipated. This is important because second sound provides the standard methods of measuring the vortex line density $L$, and the mentioned dynamical mutual interplay between second sound and vortex lines may modify the standard results.

In this paper, using extended thermodynamics (ET),[11,12] we want to build up a hydrodynamical model for turbulent superfluids. We will choose as fundamental fields the density $\rho$, the velocity $\mathbf{v}$, the internal energy density $E$, the heat flux $\mathbf{q}$, and the averaged vortex line density $L$. The relations which constrain the constitutive quantities are deduced from the entropy principle, using the Liu method of Lagrange multipliers.[12,13] In Sec. II, a brief recall of the model formulated in Ref. 9 is made. In Sec. III, the Liu procedure is applied to elaborate the constitutive theory. In the following sections, the constitutive relations are analyzed and field equations are written, with special emphasis on vortex diffusion and second-sound propagation and possible experiments to measure the coefficients appearing in the formalism are suggested. In the final section a comparison with previous hydrodynamical models is presented.

## II. BRIEF RECALL OF A PREVIOUS MODEL OF SUPERFLUID HELIUM DEDUCED FROM EXTENDED THERMODYNAMICS

A thermodynamic formalism, known as extended thermodynamics (ET),[11,12] was developed, in order to describe rapid phenomena or materials in which the relaxation times of some fluxes are long. This theory, in fact, uses the dissipative fluxes, besides the traditional variables, as independent fields.

ET offers a natural framework for the macroscopic description of liquid helium II. In analogy with heat transport problem, using ET, the relative motion of the excitations is well described by the dynamics of the heat flux. For this reason, in the one-fluid model of liquid helium II, it is rather natural to select as fundamental fields the density $\rho$, the velocity $\mathbf{v}$, the absolute temperature $T$, and the heat flux $\mathbf{q}$. In a previous paper,[14] ET was applied to formulate a nonstandard one-fluid model of liquid helium II, for laminar flows. The proposed model describes the propagation in bulk helium II of two waves, namely a sound and a temperature wave, and their attenuation, in agreement with experimental data. The model explains also some thermomechanical peculiarities, as the fountain effect, and the propagation of the fourth sound, a wave which propagates in helium II, when it flows in very thin capillaries or in porous media.[15–18]

In Ref. 9 a first thermodynamic study of turbulent flows was made. In that work, we restricted our consideration to stationary situations, in which the vortex filaments were supposed fixed, and we focused our attention on their action on the second sound propagation, through a vorticity tensor $\mathbf{P}_\omega$ which describes the vortex contribution to the internal friction in the system. Thus we did not assume that $\mathbf{P}_\omega$ itself is governed by an evolution equation, but that it was given by a constitutive relation.

The linearized set of field equations, neglecting viscous phenomena, written in an inertial frame, is[9,14]

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0, \tag{2.1a}$$

$$\rho \dot{\mathbf{v}} + \nabla p^* = 0, \tag{2.1b}$$

$$\rho \dot{\epsilon} + \nabla \cdot \mathbf{q} + p^* \nabla \cdot \mathbf{v} = 0, \tag{2.1c}$$

$$\dot{\mathbf{q}} + \zeta^* \nabla T = \sigma_{\mathbf{q}}. \tag{2.1d}$$

In these equations, the quantity $\epsilon$ is the specific internal energy per unit mass, $p^*$ the thermostatic pressure, and $\zeta^* = \lambda_1/\tau$ a coefficient linked to the second sound velocity[14] ($\tau$ being the relaxation time of the heat flux and $\lambda_1$ the thermal conductivity). In the equation for the heat flux all the nonlinear terms were neglected, with the exception of the production term $\sigma_{\mathbf{q}}$, for which the following constitutive relation was chosen:

$$\sigma_{\mathbf{q}} = -\mathbf{P}_\omega \cdot \mathbf{q}, \quad \text{where} \tag{2.2a}$$

$$\mathbf{P}_\omega = \frac{1}{2} \kappa L [B_{\mathrm{HV}} \langle \mathbf{U} - \mathbf{s}'\mathbf{s}' \rangle + B'_{\mathrm{HV}} \langle \mathbf{W} \cdot \mathbf{s}' \rangle], \tag{2.2b}$$

which takes into account the interaction between vortex lines and heat flux. In the tangle, the vortex line is described by a vectorial function $\mathbf{s}(\xi,t)$, $\xi$ being the arc length measured along the curve of vortex filament oriented in the direction of the vorticity vector; a prime denotes differentiation with respect to the arc length, so that $\mathbf{s}'$ is the unit vector tangent to the vortex segment.

In Eq. (2.2) $\mathbf{U}$ is the second-order unit tensor, $\mathbf{s}'\mathbf{s}'$ is the diadic product, and $\mathbf{W}$ is a completely antisymmetric third-order tensor which makes the third matrix of Eq. (2.2) an antisymmetric matrix, in such a way that, for example, it is $\mathbf{W}\cdot\mathbf{s}'\cdot\mathbf{V}=-\mathbf{s}'\times\mathbf{V}$; $B_{\mathrm{HV}}$ and $B'_{\mathrm{HV}}$ are the Hall-Vinen coefficients.[1] Tensor $\mathbf{P}_\omega$ is linked to the tensor $\mathbf{\Pi}$ introduced in our work[19] by the relation $(1/3)\kappa L B_{\mathrm{HV}}\mathbf{\Pi}$.

In relation (2.2) the tangent vector $\mathbf{s}'$ appears, but not the curvature vector $\mathbf{s}''$, although it is known that $\mathbf{s}''$ plays an important role in the microscopic dynamics of the vortices forming the tangle, as shown by numerical simulations of Schwarz.[20] Thus relation (2.2) is suitable when the vortices are either rectilinear lines, as in cylinders under pure rotation, or when the vortex tangle is assumed to be formed by many short rectilinear segments, randomly oriented. However, the influence of $\mathbf{s}''$ should also be taken into account for a more realistic description. In this work, we will modify relation (2.2) to incorporate these effects [see relation (3.6) and its microscopic derivation in the Appendix].

In counterflow superfluid turbulence, the vorticity tensor $\mathbf{P}_\omega$ is frequently supposed isotropic and it is assumed equal to $\mathbf{P}_\omega=KL\mathbf{U}$, with $K=(1/3)\kappa B_{HV}$. In this hypothesis, Eq. (2.2) gets simply

$$\sigma_{\mathbf{q}} = -KL\mathbf{q} \quad \text{with } K=\frac{1}{3}\kappa B_{\mathrm{HV}}. \tag{2.3}$$

Equations (2.1) describe the propagation of two longitudinal waves. Their respective phase speeds, neglecting the thermal expansion, are[9,14]

$$u_1^2=\left(\frac{\partial p}{\partial\rho}\right)_T, \quad u_2^2=\frac{\zeta^*}{\rho c_V}, \tag{2.4}$$

where $c_V$ is the specific heat at constant volume. The first one is the normal sound wave—a pressure or density wave—and is referred to as the first sound; the second is a temperature wave and is called the second sound.

In Refs. [9 and 14] a comparison with the two-fluid model was performed. It was shown that, both in the absence and in the presence of a vortex tangle, denoting with $s$ the specific entropy, putting

$$\zeta^*=\rho\frac{\rho_s}{\rho_n}Ts^2 \tag{2.5}$$

and making the change of variables

$$\mathbf{v}=\frac{\rho_s}{\rho}\mathbf{v}_s+\frac{\rho_n}{\rho}\mathbf{v}_n, \quad \mathbf{q}=\rho_s T s\mathbf{V}_{ns}, \tag{2.6}$$

system (2.1) can be identified with the equations of the two-fluid model.

The conceptual advantage of the one-fluid model is that, in fact, from the purely macroscopic point of view one sees only a single fluid, rather than two physically different fluids. The internal degree of freedom arising from the relative motion of the two fluids is here taken into account by the heat flux, whose relaxation time is very long. However, the two-fluid model provides a very appealing image of the *microscopic* helium behavior, and therefore is the most widely known.

## III. CONSTITUTIVE THEORY AND FIELD EQUATIONS

In the model just described there is no field equation for the evolution of $L$, including the influence of inhomogeneities or the coupling with the heat flux. This will be the aim of the present section. Indeed, one cannot simply add an evolution equation for $L$ to the evolution equations (2.1), but the presence of $L$ will also modify such equations, through several coupling terms which must be studied in a coherent framework.

### A. Balance equations

We consider, as a starting point to build up a nonlinear extended theory for a turbulent superfluid, the formulation of ET which uses the method of Lagrange multipliers accounting for the restrictions set by the balance equations[13] (see also Refs. [12 and 18]). Since our aim is to obtain an evolution equation for $L$ taken as a field variable, we consider for the fields $\Gamma=[\rho,\rho\mathbf{v},\rho\epsilon+(1/2)\rho v^2,\mathbf{q},L]$ general balance equations of the type

$$\frac{\partial\Gamma}{\partial t}+\nabla\cdot\mathbf{J}^\Gamma=\sigma^\Gamma, \tag{3.1}$$

with $\mathbf{J}^\Gamma$ and $\sigma^\Gamma$ the respective flux and production of the quantity $\Gamma$. The balance equations for the fields (3.1) can be written in terms of nonconvective quantities in the following way:

$$\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0, \tag{3.2a}$$

$$\rho\frac{d\mathbf{v}}{dt}+\nabla\cdot\mathbf{J}^{\mathbf{v}}=0, \tag{3.2b}$$

$$\frac{dE}{dt}+E\nabla\cdot\mathbf{v}+\nabla\cdot\mathbf{q}+\mathbf{J}^{\mathbf{v}}\cdot\nabla\mathbf{v}=0, \tag{3.2c}$$

$$\frac{d\mathbf{q}}{dt}+\mathbf{q}\nabla\cdot\mathbf{v}+\nabla\cdot\mathbf{J}^{\mathbf{q}}=\sigma^{\mathbf{q}}, \tag{3.2d}$$

$$\frac{dL}{dt}+L\nabla\cdot\mathbf{v}+\nabla\cdot\mathbf{J}^L=\sigma^L, \tag{3.2e}$$

where $E=\rho\epsilon$ is the specific energy per unit volume, $\mathbf{J}^{\mathbf{v}}$ the stress tensor, $\mathbf{J}^{\mathbf{q}}$ the intrinsic part of the flux of the heat flux, a quantity often used in the study of nonlocal effects in heat transport,[11,21] and $\mathbf{J}^L$ the flux of vortex lines; $\sigma^{\mathbf{q}}$ and $\sigma^L$ are terms describing the net production of heat flux and vortices. In this system $d/dt$ denotes the material time derivative $[d/dt=(\partial/\partial t)+\mathbf{v}\cdot\nabla]$. Observe that $E$ is the internal energy density of the whole system, which, if we neglect the interaction energy, can be considered as the sum of the energy $E_0$ of the helium background (normal and superfluid components) and of the energy of the vortex tangle $E_V=\epsilon_V L$, where $\epsilon_V$ is the energy per unit length of the vortex, which is given by[1,22]

$$\epsilon_V = \rho_s \kappa \tilde{\beta} \quad \text{with } \tilde{\beta} = \frac{\kappa}{4\pi} \ln\left(\frac{c}{a_0 L^{1/2}}\right), \tag{3.3}$$

where $a_0$ is the radius of the vortex core, which is of the order of the atomic radius, and $c$ is a constant of the order of unity.

### B. Constitutive theory

Constitutive equations for the fluxes $\mathbf{J}^{\mathbf{v}}$, $\mathbf{J}^{\mathbf{q}}$, and $\mathbf{J}^L$ and the productions $\sigma^{\mathbf{q}}$ and $\sigma^L$ are necessary to close the set of equations (3.2). As a consequence of the material objectivity principle these constitutive relations, to the first order in $\mathbf{q}$, can be expressed in the form

$$\mathbf{J}^{\mathbf{v}} = p_0(\rho,E,L)\mathbf{U}, \tag{3.4a}$$

$$\mathbf{J}^{\mathbf{q}} = \beta_0(\rho,E,L)\mathbf{U}, \tag{3.4b}$$

$$\mathbf{J}^L = \nu_0(\rho,E,L)\mathbf{q}. \tag{3.4c}$$

For the production term in the equation for the line density $L$ we will choose Vinen's production and destruction terms (1.2), which here we rewrite, using the second part of Eq. (2.6) to write the relative velocity $\mathbf{V}_{ns}$ in terms of the heat flux:

$$\sigma^L = -BL^2 + AqL^{3/2}, \tag{3.5}$$

with $A=\alpha_v/\rho_s Ts$ and $B=\beta_v \kappa$. For the production term in the equation for the heat flux we will choose the expression

$$\sigma^{\mathbf{q}} = -KL\mathbf{q} \pm HL^{3/2}\hat{\mathbf{q}}, \tag{3.6}$$

which includes, compared with Eq. (2.3), an additional term in the force acting on a superfluid component. This term, which has been found independently by various authors,[23–26] does not depend on counterflow velocity and is called "dry friction term" (see also the review by Nemirovskii and Fiszdon,[27] and Ref. 28). In the steady state the fully turbulent regime $L$ is roughly proportional to $q^2$, and therefore both terms in Eq. (3.6) yield in this case a contribution proportional to $q^3$, the well-known Gorter-Mellink expression; but Eq. (3.6) is more general because it is also valid in the unsteady regime.

This term has different signs in papers by Nemirovskii *et al.*[23] and by Yamada *et al.*[24] In papers by Guerst[25,26] this term is also present and its sign depends on the drift velocity of the vortex tangle. In our work[29] and in a recent paper of Lipniacki,[30] this term is also present, and it is linked to the expression of mutual friction force, when the anisotropy of the tangle is taken in consideration. In the Appendix, expression (3.6) for the production term $\sigma^{\mathbf{q}}$ will be determined on a microscopic basis, using the vortex filament model.

Further restrictions on these constitutive relations are obtained imposing the validity of the entropy principle, applying the Liu method of Lagrange multipliers.[13] This method requires the existence of a scalar function $S$ and a vector function $\mathbf{J}^S$ of the fundamental fields, namely the entropy density and the entropy flux density respectively, such that the following inequality,

$$\begin{aligned}\dot{S} + S\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^S - \Lambda^\rho[\dot{\rho} + \rho\nabla\cdot\mathbf{v}] - \Lambda^{\mathbf{v}}\cdot[\rho\dot{\mathbf{v}} + \nabla\cdot\mathbf{J}^{\mathbf{v}}] \\ - \Lambda^E[\dot{E} + E\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{q} + \mathbf{J}^{\mathbf{v}}\cdot\nabla\mathbf{v}] - \Lambda^{\mathbf{q}}\cdot[\dot{\mathbf{q}} + \mathbf{q}\nabla\cdot\mathbf{v} \\ + \nabla\cdot\mathbf{J}^{\mathbf{q}} - \sigma^{\mathbf{q}}] - \Lambda^L[\dot{L} + L\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{J}^L - \sigma^L] \geq 0,\end{aligned} \tag{3.7}$$

is satisfied for arbitrary fields $\rho$, $\mathbf{v}$, $E$, $\mathbf{q}$, and $L$. This inequality expresses the restrictions coming from the second law of thermodynamics.

In this inequality, $S=S(\rho,E,q^2,L)$ and $\mathbf{J}^S=\phi(\rho,E,q^2,L)\mathbf{q}$ are objective functions of the fundamental fields. In order to make the theory internally consistent, we must consider for $S$ and $\mathbf{J}^S$ approximate constitutive relations to second order in $\mathbf{q}$:

$$S = s_0(\rho,E,L) + s_1(\rho,E,L)q^2, \quad \mathbf{J}^s = \phi_0(\rho,E,L)\mathbf{q}. \tag{3.8}$$

The quantities $\Lambda^\rho$, $\Lambda^{\mathbf{v}}$, $\Lambda^E$, $\Lambda^{\mathbf{q}}$, and $\Lambda^L$ are Lagrange multipliers, which are also objective functions of $\rho, E, \mathbf{q}$, and $L$. To the first order in $\mathbf{q}$, it results in $\Lambda^\rho=\Lambda_0^\rho(\rho,E,L)$, $\Lambda^{\mathbf{v}}=\Lambda_0^v(\rho,E,L)\mathbf{q}$, $\Lambda^E=\Lambda_0^E(\rho,E,L)$, $\Lambda^{\mathbf{q}}=\lambda_0^q(\rho,E,L)\mathbf{q}$ and $\Lambda^L=\Lambda_0^L(\rho,E,L)$.

The constitutive theory is obtained substituting Eq. (3.4) in Eq. (3.7) and imposing that the coefficients of all derivatives must vanish.[12,13] After some lengthy calculations, we obtain

$$\Lambda_v^0 = 0, \tag{3.9a}$$

$$dS_0 = \Lambda_0^\rho d\rho + \Lambda_0^E dE + \Lambda_0^L dL, \tag{3.9b}$$

$$S_0 - \rho\Lambda_0^\rho - \Lambda_0^E(E + p_0) - \Lambda_0^L L = 0, \tag{3.9c}$$

$$\phi_0 = \Lambda_0^E + \Lambda_0^L \nu_0, \tag{3.9d}$$

$$d\phi_0 = \lambda_0^q d\beta_0 + \Lambda_0^L d\nu_0, \tag{3.9e}$$

$$S_1 = \frac{1}{2}\lambda_0^q. \tag{3.9f}$$

The following residual inequality for the entropy production remains:

$$\sigma^S = \mathbf{\Lambda}^{\mathbf{q}}\cdot\sigma^{\mathbf{q}} + \Lambda^L\sigma^L \geq 0. \tag{3.10}$$

In the following section the coefficients appearing in Eq. (3.4) will be examined in depth, and related to specific physical situations specially suitable to stress their physical meaning.

## IV. PHYSICAL INTERPRETATION OF THE CONSTITUTIVE RELATIONS

In order to single out the physical meaning and relevance of the constitutive quantities and of the Lagrange multipliers, we analyze now in detail the relations obtained in the previous section. The reader interested in the concrete physical applications may skip this section and go directly to Secs. V and VI.

### A. Generalized temperature and chemical potentials

We first introduce a "generalized temperature" as the reciprocal of the first-order part of the Lagrange multiplier of the energy:

$$\Lambda_0^E = \left[\frac{\partial S_0}{\partial E}\right]_{\rho,L} = \frac{1}{T}. \tag{4.1}$$

Observe that, in the laminar regime (when $L=0$), $\Lambda_0^E$ reduces to the absolute temperature of thermostatics. In the presence of a vortex tangle the quantity (4.1) depends also on the line density $L$ [see relations (3.8) and (3.9)].

If we now write Eqs. (3.9b) and (3.9d) in the following way,

$$dE = TdS_0 - T\Lambda_0^\rho d\rho - T\Lambda_0^L dL, \tag{4.2}$$

$$-T\Lambda_0^\rho = \frac{E}{\rho} - T\frac{S_0}{\rho} + \frac{p_0 + LT\Lambda_0^L}{\rho}, \tag{4.3}$$

we can define the quantity $-\Lambda_0^\rho/\Lambda_0^E = -T\Lambda_0^\rho$ as the "mass chemical potential" in the turbulent superfluid

$$-T\Lambda_0^\rho = -T\left[\frac{\partial S_0}{\partial \rho}\right]_{E,L} = \mu_0^\rho, \tag{4.4}$$

while the quantity $-\Lambda_0^L/\Lambda_0^E = -T\Lambda_0^L$ can be identified with the "chemical potential of vortex lines" $\mu_0^L$:

$$-T\Lambda_0^L = -T\left[\frac{\partial S_0}{\partial L}\right]_{\rho,L} = \mu_0^L. \tag{4.5}$$

Indeed, in the absence of vortices ($L=0$) Eq. (4.2) is just the Gibbs equation of thermostatics and the quantity (4.3) is the equilibrium chemical potential. The presence of vortices modifies the energy density $E$ and the chemical potentials. For the chemical potential of vortex lines we will take the expression

$$\mu_0^L = \epsilon_V \ln\left(\frac{L}{L^*}\right), \tag{4.6}$$

where $\epsilon_V$ is the energy per unit length of the vortex lines, given by Eq. (3.3). Usually one takes $\epsilon_V$ as practically constant, as its dependence on $L$ is very mild.

In Eq. (4.6) $L^*$ is a reference vortex line density, defined as the average length $\langle l\rangle$ of the vortex loops composing the tangle, divided by the volume of the system, namely $L^* \equiv \langle l\rangle/V$. Then $\ln(L/L^*)$ vanishes when there is only one vortex loop in the whole volume. Note that Eq. (4.6) will be positive for $\langle l\rangle/V \le L \le 1/a_0^2$. This will always be the case, as $LV$, the total vortex length, will always be higher than the average length of one vortex loop. On the other side, the average distance between vortices is $L^{-1/2}$; thus the last inequality will always be fulfilled because the average vortex separation must certainly be higher than the radius of the vortex core. Equation (4.6) has a similarity with the corresponding expression for ideal gases, where $\mu = k_B T \ln N + \mu_0(T)$, $k_B T$ being related to the energy per particle. In the following, we will consider $\epsilon_V$ as depending only on $T$—through $\rho_s(T)$.

Our consideration of $\epsilon_V$ as practically constant—independent on $L$—is for the sake of simplicity. We will see that, from a quantitative point of view, the main thermodynamic consequences will follow from the sign of $\partial\mu_0^L/\partial L$ [see Eqs. (4.18a) and (4.18b) and the comments below Eq. (5.10)]. If it is assumed that the only dependence of $\mu_0^L$ on $L$ in Eq. (4.6) is through $\ln(L/L^*)$, one has

$$\frac{\partial \mu_0^L}{\partial L} = \frac{\epsilon_V}{L} > 0. \tag{4.7}$$

If one takes the full dependence in $\ln[1/a_0 L^{1/2}]\ln[L/L^*]$, it is found that

$$\frac{\partial \mu_0^L}{\partial L} = \frac{\epsilon_V}{L} - \frac{\rho_s \kappa^2}{8\pi L}\ln\left(\frac{L}{L^*}\right) = \frac{\rho_s\kappa^2}{4\pi L}\ln\left(\frac{(L^*)^{1/2}}{a_0 L}\right), \tag{4.8}$$

and the second factor is positive for $L \le (L^*/a_0^2)^{1/2} = (\langle l\rangle/V a_0^2)^{1/2}$. In practical terms, most of the experimental situations satisfy this inequality. Indeed, $a \simeq 10^{-8}$ cm and a typical length of vortex loops is of the order of $10^{-3}$ or $10^{-4}$ cm.[19] When these data are considered, we have that the mentioned inequality for $L$ will be satisfied for $L$ up to $10^6$ cm$^{-2}$, which is a value higher than experimentally known observations. Thus here, for simplicity, we will take $\epsilon_V \simeq \rho_s \kappa^2/4\pi$.

In Eq. (4.6) $\rho_s$ is the density of the superfluid component in helium II, which depends essentially on its temperature. Therefore, the chemical potential of the vortex line depends on $T$ and on $L$, and it follows that

$$\frac{\partial \mu_0^L}{\partial \rho} = 0, \quad \frac{\partial \mu_0^L}{\partial T} = \frac{\epsilon_V}{\rho_s}\frac{\partial \rho_s}{\partial T}\ln\left(\frac{L}{L^*}\right) = \frac{\mu_0^L}{\rho_s}\frac{\partial \rho_s}{\partial T} < 0, \quad \frac{\partial \mu_0^L}{\partial L} \simeq \frac{\epsilon_V}{L}. \tag{4.9}$$

### B. Generalized Gibbs equation

We consider again Eqs. (4.3) and (4.2), which we rewrite as

$$\rho\mu_0^\rho + L\mu_0^L = E - TS_0 + p_0, \tag{4.10}$$

$$dS_0 = \frac{1}{T}dE - \frac{\mu_0^\rho}{T}d\rho - \frac{\mu_0^L}{T}dL. \tag{4.11}$$

In the absence of vortices ($L=0$) the latter equation is just the Gibbs equation of thermostatics. Note that this equation is analogous to the Gibbs equation for a mixture: This suggests that the turbulent superfluid may be considered as a mixture of laminar superfluid, described by $\rho$, and of vortex lines, described by $L$.

Note that if we work with the specific quantities (per unit mass) $\epsilon = E/\rho$ and $s_0 = S_0/\rho$, from Eqs. (4.3)–(4.6) and (4.11), we obtain the equation

$$ds_0 = \frac{1}{T}\left[d\epsilon - \frac{p_0 - L\mu_0^L}{\rho^2}d\rho - \frac{\mu_0^L}{\rho}dL\right]. \tag{4.12}$$

In Eq. (4.12) $s_0$ and $\epsilon$ are specific quantities per unit mass, but $L$ is the line density per unit volume. We introduce then the quantity $\mathcal{L}=Lv$ (where $v=1/\rho$ is the specific volume), and express (4.12) using these quantities; we obtain

$$ds_0 = \frac{1}{T}[d\epsilon + p_0 dv - \mu_0^L d\mathcal{L}]. \quad (4.13)$$

From these two latter equations and from Eq. (4.10) we also obtain the relation which will be useful in the following:

$$d\mu_0^\rho + \mathcal{L} d\mu_0^L = dp_0 - s_0 dT. \quad (4.14)$$

In the following we will stick to the use of $L$ as a variable, because it is often used in the literature and because in the usual analysis of counterflow $\rho$ is constant, in such a way that $d\mathcal{L}=dL/\rho$ and it is indifferent to take $\mathcal{L}$ or $L$ as a variable.

Obviously, in this model $p_0$ cannot be identified with the equilibrium pressure $p^*$, as $p_0$ depends also on the line density $L$, with $p_0=p_0(\rho,T,L)$. To determine the dependence of the pressure $p_0$ on $L$, we use the integrability condition:

$$\frac{\partial p_0}{\partial L} = L\frac{\partial \mu_0^L}{\partial L} \simeq \epsilon_V. \quad (4.15)$$

As a consequence, we can write $p_0$ in the following way:

$$p_0(\rho,T,L) \simeq p^*(\rho,T) + \epsilon_V L, \quad (4.16)$$

with $p^*$ being the thermostatic pressure. As we see, the theory allows us to determine the dependence of the pressure on the vortex line density. In particular, we observe that, in the presence of vortex tangle, the total pressure is the pressure of the liquid helium $p^*$ plus pressure of the vortex tangle, given by $\epsilon_V L$. If the full dependence (4.8) of $\mu_0^L$ on $L$ is taken into account one would find $\partial p_0/\partial L=(\rho_s\kappa^2/4\pi)[\epsilon_V-(1/2)\ln(L/L^*)]$ and the expression for the pressure would be $p_0=p^*(\rho,T)+\epsilon_V L-(1/2)[\ln(L/L^*)-1]L$.

### C. Consequences for the fluxes

Consider now the consequences of Eqs. (3.9d) and (3.9e) which concern the expressions of the fluxes. From Eqs. (3.9d) and (3.9e), using definitions (4.1) and (4.5), we get

$$\lambda_0^q d\beta_0 = d\left(\frac{1}{T}\right) - \nu_0 d\left(\frac{\mu_0^L}{T}\right). \quad (4.17)$$

From this equation, recalling that $\mu_0^L$ depends only on $T$ and $L$, we obtain $\partial\beta_0/\partial\rho=0$ and we put

$$\frac{\partial\beta_0}{\partial T} = -\frac{1}{T^2\lambda_0^q}\left[1 + \nu_0 T^2\frac{\partial}{\partial T}\left(\frac{\mu_0^L}{T}\right)\right] = \zeta_0, \quad (4.18a)$$

$$\frac{\partial\beta_0}{\partial L} = -\frac{\nu_0}{T\lambda_0^q}\frac{\partial\mu_0^L}{\partial L} = \chi_0. \quad (4.18b)$$

From Eq. (4.17), we also obtain as integrability conditions

$$\frac{\partial\lambda_0^q}{\partial\rho} = 0, \qquad \frac{\partial\nu_0}{\partial\rho} = 0, \quad (4.19)$$

$$\left[1 + T^2\nu_0\frac{\partial(\mu_0^L/T)}{\partial T}\right]\frac{\partial\lambda_0^q}{\partial L} - T^2\nu_0\frac{\partial(\mu_0^L/T)}{\partial L}\frac{\partial\lambda_0^q}{\partial T} = \lambda_0^q T^2\left[\frac{\partial\nu_0}{\partial L}\frac{\partial(\mu_0^L/T)}{\partial T} - \frac{\partial\nu_0}{\partial T}\frac{\partial(\mu_0^L/T)}{\partial L}\right]. \quad (4.20)$$

Since $\lambda_0^q<0$ [as will be seen in Eq. (4.29)], it follows that $\nu_0$ and $\chi_0$ must have the same sign, which will be important in Eq. (5.10). Furthermore, with $\nu_0\leq 0$ [see discussion below Eq. (5.5)], and $\partial\rho_s/\partial T<0$ and $\lambda_0^q<0$, we deduce that $\zeta_0\geq 0$ and $\chi_0\leq 0$.

Using Eq. (4.18b) we can eliminate $\nu_0$ from Eq. (4.18a), obtaining

$$\zeta_0 = -\frac{1}{T^2\lambda_0^q} - \frac{\chi_0}{T}\left(1 - \frac{T}{\rho_s}\frac{\partial\rho_s}{\partial T}\right)\frac{\mu_0^L}{\epsilon_V}L, \quad (4.21)$$

an expression which will be of interest for the expression of the speed of second sound, as seen below in Eq. (6.11). In the absence of vortex tangle, we get

$$\zeta^* = -\frac{1}{T_*^2\lambda_*^q}, \quad (4.22)$$

with $\zeta^*$, $T_*$, and $\lambda_*^q$ being the values of $\zeta_0$, $T$, and $\lambda_0^q$ for $L=0$. This equation is identical to that found in [18] in the study of laminar flows of nonviscous fluids in the presence of heat flux.

### D. Approximate expressions for the entropy density and the entropy flux

We consider finally approximate expressions of the entropy density and entropy flux. Concerning the entropy density $S$, from Eqs. (4.6) and (4.1) we can immediately write

$$S \simeq S_0(\rho,T,L) + \frac{1}{2}\lambda_0^q(\rho,T,L)q^2, \quad (4.23)$$

where $\lambda_0^q$ must satisfy Eq. (4.20).

Using Eq. (3.8), we obtain the following expression, to the first order in $\mathbf{q}$, for the entropy flux:

$$\mathbf{J}^S = \phi_0\mathbf{q} = \frac{1}{T}(1 - \nu_0\mu_0^L)\mathbf{q}. \quad (4.24)$$

This equation shows that the entropy flux is different from the product of the reciprocal temperature and the heat flux, but it contains, also to the lowest order, an additional term depending on the energy per unit length of the vortex line. This result recalls the usual expression $\mathbf{J}^s=T^{-1}\mathbf{q}-\mu T^{-1}\mathbf{J}$ for the entropy flux in the presence of a mass flux, because, according to Eq. (3.4c), the flux of vortices $\mathbf{J}^L$ is equal to $\nu_0\mathbf{q}$; when this relation is used to write the flux of vortices, the second term in Eq. (4.24) may be interpreted as the vortices contribution to the entropy flux.

### E. Consequences of the entropy inequality

We now study the consequences of the entropy inequality (3.10). Using Eqs. (3.5) and (3.6), we get for the entropy production

$$\sigma^S = -\lambda_0^q(KLq^2 \mp HL^{3/2}q) + \frac{1}{T}\mu_0^L(BL^2 - AqL^{3/2}) \geq 0. \tag{4.25}$$

In a counterflow stationary situation, in which $L^{1/2}=(A/B)q$ and $\sigma^L=0$, it is

$$\sigma^S = -\lambda_0^q(KLq^2 \mp HL^{3/2}q) = -\lambda_0^q\frac{A^2}{B^2}\left(K \mp H\frac{A}{B}\right)q^4 \geq 0. \tag{4.26}$$

We study now the sign of the coefficient $K \mp H(A/B)$. We can write

$$K \mp H\frac{A}{B} = \frac{1}{3}\kappa B_{\text{HV}} \mp \frac{|\mathbf{V}_{ns}|}{|\mathbf{q}|}\frac{\alpha_v}{\beta_v\kappa} \tag{4.27}$$

recalling that (see the Appendix) it is $\alpha_v=\alpha c_1 I_0$ and $\beta_v=\alpha\tilde{\beta}c_2$, and substituting in Eq. (4.27), we obtain for the sign of the quantity (4.27) the following results:

$$K \mp H\frac{A}{B} = \frac{1}{3}\kappa B_{\text{HV}} \mp \frac{|\mathbf{V}_{ns}|}{|\mathbf{q}|}\frac{2c_1I_0}{\tilde{\beta}c_2} = \frac{1}{3}\kappa B_{\text{HV}} \mp \frac{1}{\rho_s Ts}\frac{2c_1I_0}{\tilde{\beta}c_2}. \tag{4.28}$$

As we see, if $H\geq 0$ $(I_0<0)$ this coefficient is always positive, while if $H\leq 0$ $(I_0>0)$ this coefficient is positive if $I_0 \leq (1/6c_1)\kappa\rho_s Ts\tilde{\beta}c_2 B_{\text{HV}}$, from which we deduce, if the results are $K\mp HA/B\geq 0$,

$$\lambda_0^q \leq 0, \tag{4.29}$$

and, as a consequence of this latter equation and of Eq. (4.17b), we conclude that the two coefficients $\nu_0$ and $\chi_0$ have the same sign. In the following we will make this assumption.

In a nonstationary situation, under the hypothesis (4.29), to satisfy relation (4.23), the coefficients $K$, $B$, and $A$ must satisfy, for all values of $T$, $L$, and $\mathbf{q}$, the following relation:

$$\sigma^S = \frac{1}{T}B\mu_0^L L^2 - \left(\frac{1}{T}\mu_0^L A \pm |\lambda_0^q|H\right)qL^{3/2} + |\lambda_0^q|KLq^2 \geq 0. \tag{4.30}$$

This concludes the analysis of the thermodynamic restrictions on the coefficients of the constitutive relations. In the next two sections, we will explore two simple but physically relevant situations where the terms introduced here—namely, those related to vortex density inhomogeneities—play an especially explicit role.

## V. FIELD EQUATIONS. I: VORTEX DIFFUSION

Now we will apply the general set of equations derived up to here to the analysis of two specific situations: here, we will use it to describe vortex diffusion. A diffusion equation for the vortex line density was proposed by van Beelen *et al.*,[31] in an analysis of vorticity in capillary flow of superfluid helium, in situations with a step change in $L$ arising when the tube is divided in a region with laminar flow and another one with turbulent flow.

First of all, we note that, substituting in (3.2) the constitutive expressions obtained in Sec. IV, the following system is obtained:

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0, \tag{5.1a}$$

$$\rho\dot{\mathbf{v}} + \nabla p_0 = 0, \tag{5.1b}$$

$$\rho\dot{\epsilon} + \nabla\cdot\mathbf{q} = 0, \tag{5.1c}$$

$$\dot{\mathbf{q}} + \zeta_0\nabla T + \chi_0\nabla L = \sigma^{\mathbf{q}}, \tag{5.1d}$$

$$\dot{L} + L\nabla\cdot\mathbf{v} + \nabla\cdot(\nu_0\mathbf{q}) = \sigma^L, \tag{5.1e}$$

with $\zeta_0$ and $\chi_0$ defined by Eq. (4.18) and satisfying

$$\frac{\nu_0}{\chi_0} = -\frac{T\lambda_0^q L}{\epsilon_V}. \tag{5.2}$$

For the production terms $\sigma^{\mathbf{q}}$ and $\sigma^L$, we will simply take Eqs. (3.5) and (3.6), respectively.

In this approximation, the unknown coefficients, which must be determined from experimental data, are the specific energy $\epsilon$, the pressure $p_0$, and the three coefficients $\zeta_0$, $\chi_0$, and $\nu_0$, which are functions only of $T$ and $L$. Here, we will focus special attention on the coefficients $\chi_0$ and $\nu_0$, which are the ones appearing in the present formulation, as compared with the formulation presented in Eq. (2.1).

### A. The drift velocity of the tangle

As observed in the Introduction, in a hydrodynamical model of turbulent superfluids, the line density $L$ acquires field properties and its rate of change must obey a balance equation of the general form

$$\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v}^L) = \sigma^L, \tag{5.3}$$

with $\mathbf{v}^L$ the drift velocity of the tangle. If we now observe that Eq. (5.1e) can be written

$$\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v} + \nu_0\mathbf{q}) = \sigma^L, \tag{5.4}$$

we conclude that the drift velocity of the tangle, with respect to the container, is given by

$$\mathbf{v}^L = \mathbf{v} + \frac{\nu_0}{L}\mathbf{q}. \tag{5.5}$$

Note that the velocity $\mathbf{v}^L$ does not coincide with the microscopic velocity of the vortex line element, but represents an averaged macroscopic velocity of this quantity. It is to bring attention to the fact that often in the literature the microscopic velocity $\dot{\mathbf{s}}$ is denoted with $\mathbf{v}_L$.

Observing that in counterflow experiments $(\mathbf{v}=0)$ results in $\mathbf{v}^L=\nu_0\mathbf{q}/L$, and recalling that measurements[1] (in developed superfluid turbulence) show that the vortex tangle drifts as a whole toward the heater, we conclude that $\nu_0 \leq 0$. The measurement of the drift velocity $\mathbf{v}^L$ of the vortex tangle, together with the measurement of $\mathbf{q}$ and $L$, would allow one to obtain quantitative values for the coefficient $\nu_0$. In the following section we will propose also a way to measure the coefficient $\chi_0$.

Another possibility is to interpret $\nu_0\mathbf{q}=\mathbf{J}^L$ as the diffusion flux of vortices, which since $\nu_0 \leq 0$, would be opposite to the direction of $\mathbf{q}$. Note that, in this model, if $\mathbf{q}=0$, $\mathbf{J}^L$ also is zero.

### B. Vortex diffusion

An interesting physical consequence from the generalized equations (5.1d) and (5.1e) is the description of vortex diffusion. Assume, for the sake of simplicity, that $T$=const and that $\mathbf{q}$ varies very slowly, in such a way that $\dot{\mathbf{q}}$ may be neglected.

We find from Eqs. (5.1d) and (2.3) that

$$\chi_0 \nabla L = \sigma^{\mathbf{q}} = - KL\mathbf{q} \pm HL^{3/2}\hat{\mathbf{q}}. \tag{5.6}$$

To separate the dynamics of $T$ and of $L$ is not easy in general. However, one may devise some situations in which this may be done, at least as an illustrative approximation to understand the approximation which has lead to Eq. (5.6). Indeed, the heat perturbations propagate in the superfluid with the characteristic velocity $V_2$ of the second sound; instead, vortex density propagation is a diffusion process, characterized by a diffusion coefficient $\kappa$ (the quantum of vorticity). Then, we could consider a thin cylinder of length $l$, and to heat suddenly and for a short time one of its ends; the temperature propagation will arrive at the other end in a time $t_1 = l/V_2$; instead, the vortex density perturbation will arrive in a characteristic time $t_2 = l^2/\kappa$. Thus, if the cylinder is long enough, the inhomogeneity in the temperature will disappear faster than that in $L$, and one may make the approximation of homogeneous $T$ but inhomogeneous $L$. In general, however, both the inhomogeneities in $T$ and $L$ should be taken into consideration.

We suppose here, in accord with experiments, that the first term in the right-hand side of Eq. (5.6) is prevalent with respect to the second one, so that $\nabla L$ is always collinear but opposite to the heat flux. Then, we may write $\hat{\mathbf{q}} = -\nabla L/|\nabla L|$ and

$$\mathbf{q} = -\left(\frac{\chi_0}{KL} \mp \frac{H}{K}\frac{L^{1/2}}{|\nabla L|}\right)\nabla L. \tag{5.7}$$

Introducing this expression in Eq. (5.1e), we find

$$\frac{dL}{dt} + L\nabla\cdot\mathbf{v} - \frac{\nu_0\chi_0}{K}\nabla\cdot\left(\frac{\nabla L}{L}\right) \pm \frac{\nu_0 H}{K}\nabla\cdot\left(L^{1/2}\frac{\nabla L}{|\nabla L|}\right) = \sigma^L = -BL^2 + AqL^{3/2}, \tag{5.8}$$

where $q$ denotes the modulus of $\mathbf{q}$. Equation (5.8) can be written (if $\nabla L \neq 0$)

$$\frac{dL}{dt} + L\nabla\cdot\mathbf{v} - \frac{\nu_0\chi_0}{KL}\Delta L + \left(\frac{\nu_0\chi_0}{KL^2} \pm \frac{\nu_0 HL^{-1/2}}{2K|\nabla L|}\right)(\nabla L)^2 = \sigma^L. \tag{5.9}$$

Then, we have for $L$ a reaction-diffusion equation, which generalizes the usual Vinen's equation (1.2) to inhomogeneous situations. The diffusivity coefficient is found to be

$$D = \frac{\nu_0\chi_0}{KL}. \tag{5.10}$$

Since $\lambda_0^q < 0$ and $K > 0$, it turns out that $D > 0$, as is expected. Thus the vortices will diffuse from regions of higher $L$ to those of lower $L$. Note that $D$ must have dimensions $(\text{length})^2/\text{time}$, the same dimensions as $\kappa$. Then, a dimensional ansatz could be $D \propto \kappa$. Indeed, Tsubota *et al.*[32] have studied numerically the spatial vortex diffusion in a localized initial tangle allowed to diffuse freely, and they found for $D$ at very low temperatures (when there is practically no normal fluid) a value $D \approx (0.1 \pm 0.05)\kappa$. Note that $KL$ in Eq. (5.7) plays the role of a friction coefficient. Then it is natural that it appears in the denominator of $D$. This has some analogy with Einstein relation $D = k_B T/\zeta$, $\zeta$ being the friction coefficient.

If $\mathbf{v}$ vanishes, or if its divergence vanishes, Eq. (5.8), neglecting also the term in $(\nabla L)^2$, yields

$$\dot{L} = -BL^2 + AqL^{3/2} + D\Delta L. \tag{5.11}$$

Equation (5.11) indicates two temporal scales for the evolution of $L$: One of them is due to the production-destruction term ($\tau_{\text{decay}}$) and another one to the diffusion:

$$\tau_{\text{decay}} \approx [BL - AqL^{1/2}]^{-1}, \quad \tau_{\text{diff}} \approx \frac{X^2}{D}, \tag{5.12}$$

where $X$ is the size of the system. For large values of $L$, $\tau_{\text{decay}}$ will be much shorter and the production-destruction dynamics will dominate over diffusion; for small $L$, instead, diffusion processes may be dominant. This may also be understood from a microscopic perspective because the mean free path of vortex motion is of the order of intervortex spacing, of the order of $L^{-1/2}$, and therefore it increases for low values of $L$.

A more general situation for the vortex diffusion flux is to keep the temperature gradient in Eq. (5.1d). In this more general case, $\mathbf{q}$ is not more parallel to $\nabla L$ but results in

$$\left(q \mp \frac{H}{K}L^{1/2}\right)\hat{\mathbf{q}} = -\frac{\chi_0}{KL}\nabla L - \frac{\zeta_0}{KL}\nabla T, \tag{5.13}$$

in which case, it would become

$$\mathbf{J}^L = \nu_0\mathbf{q} = -D\nabla L - D\frac{\zeta_0}{\chi_0}\nabla T \mp \frac{\nu_0 H}{K}L^{1/2}\frac{\chi_0\nabla L + \zeta_0\nabla T}{|\chi_0\nabla L + \zeta_0\nabla T|}. \tag{5.14}$$

Thus, if $\nabla L = 0$, Eq. (5.14) will yield

$$\mathbf{q} = -\lambda_{\text{eff}}\nabla T, \tag{5.15}$$

with an effective thermal conductivity $\lambda_{\text{eff}} = D\zeta_0/\chi_0\nu_0 \pm (H/K)(L^{1/2}/|\Delta T|) > 0$. As in the case of the diffusion coefficient $D$, $\lambda_{\text{eff}}$ is expected to be positive, in the usual circumstances; a negative sign would imply a thermal instability in the system, which cannot be dismissed *a priori* but that we will not study here because we do not have enough precise information.

The second term in Eq. (5.14) plays a role analogous to thermal diffusion—or Soret effect—in the usual diffusion of particles. In this case, Eq. (5.8) modifies as

$$\frac{dL}{dt} + L\nabla\cdot\mathbf{v} - \frac{\nu_0\chi_0}{KL}\Delta L - \frac{\nu_0\zeta_0}{KL}\Delta T \mp \nabla\cdot\left(\frac{\nu_0 H}{K}L^{1/2}\frac{\chi_0\nabla L + \zeta_0\nabla T}{|\chi_0\nabla L + \zeta_0\nabla T|}\right) = \sigma^L. \tag{5.16}$$

These kinds of situations have not been studied enough in the context of vortex tangles, but they would arise in a natural way when trying to understand the behavior of quantum turbulence in the presence of a temperature gradient.

Expression (5.14) yields a coupling between the heat flux and an inhomogeneity in $L$; in other terms, it means that a heat flux may influence the vortex line density. It follows, in contrast with the standard assumption that the vortex line density is longitudinally homogeneous in counterflow experiments, that the vortex tangle would be slightly inhomogeneous. In view of Eqs. (5.14) and (5.10), this longitudinal inhomogeneity along $\mathbf{q}$ would be given by

$$-\chi_0\nabla L = KL\mathbf{q} + \zeta_0\nabla T \mp \frac{H}{\nu_0}L^{3/2}\hat{\mathbf{q}}. \tag{5.17}$$

In the linear approximation $\zeta_0$ is related to the second sound velocity in the absence of vortex lines [see Eq. (6.11)]. We have seen that $\nu_0<0$ and that $\chi_0$ must have the same sign as $\nu_0$, according to the comments below Eq. (5.5). From here, it follows that there should be a slight inhomogeneity in $L$ in such a way that $\nabla L$ points in the same direction as $KL\mathbf{q}+\zeta_0\nabla T\mp(H/\nu_0)L^{3/2}\hat{\mathbf{q}}$. Thus an experiment suggested by our formalism would be to carefully measure the longitudinal profile of $L$ along the heat flux, to check whether there is a slight increase in $L$. Furthermore, Eq. (5.17) would allow one to measure the coefficient $\chi_0$, in the linear approximation.

Since below Eq. (5.5) we have mentioned a way to measure $\nu_0$, it turns out that the coefficients $\nu_0$ and $\chi_0$ could be measured independently of each other. A further quantitative check of our formalism would be to check the relation (5.10) between $\nu_0$, $\chi_0$, and the diffusion coefficient $D$, which could be measured independently by studying the evolution of an inhomogeneity in the vortex line density under constant temperature.

## VI. FIELD EQUATIONS. II: WAVE PROPAGATION IN COUNTERFLOW VORTEX TANGLES

Here, we will study wave propagation in counterflow vortex tangles. Experiments show that in this case the velocity $\mathbf{v}$ is zero, and only the fields $T$, $\mathbf{q}$, and $L$ are involved. The equations for these fields, under these hypotheses, expressing the energy in terms of $T$ and $L$, are simply

$$\rho c_V\frac{dT}{dt} + \rho\epsilon_L\frac{dL}{dt} + \nabla\cdot\mathbf{q} = 0, \tag{6.1a}$$

$$\frac{d\mathbf{q}}{dt} + \zeta_0\nabla T + \chi_0\nabla L = -KL\mathbf{q} \pm HL^{3/2}\hat{\mathbf{q}}, \tag{6.1b}$$

$$\frac{dL}{dt} + \nabla\cdot(\nu_0\mathbf{q}) = -BL^2 + AqL^{3/2}, \tag{6.1c}$$

where $c_V=\partial\epsilon/\partial T$ is the specific heat at constant volume, $\epsilon_L=\partial\epsilon/\partial L\simeq\epsilon_V$.

These equations are enough for the discussion of the physical effects of the coupling of second sound and the distorsion of the vortex tangle (represented by the inhomogeneities in $L$), which must be taken into account in an analysis of the vortex tangle by means of second sound. In fact, some of the previous hydrodynamical analyses of turbulent superfluids had this problem as one of their main motivations.[32]

As we can easily see, a stationary solution of system (6.1) is

$$\mathbf{q} = \mathbf{q}_0 = (q_{10},0,0),$$
$$L = L_0 = \frac{A^2}{B^2}[q_{10}]^2, \quad T = T_0(\mathbf{x}) = T^* - \frac{KL_0q_{10} - HL_0^{3/2}}{\zeta_0}x_1, \tag{6.2}$$

with $q_{10}>0$. We have chosen here the sign in front of $H$ opposite to the sign in front of $K$, because in stationary situations the quantity $I_0$ introduced in the Appendix has positive sign.

To study the wave propagation in a neighborhood of this solution, we substitute $\sigma^{\mathbf{q}}$ and $\sigma^L$ with

$$\sigma^{\mathbf{q}} \simeq -K[L_0\mathbf{q} + \mathbf{q}_0(L-L_0)] + H\left[L_0^{3/2}\hat{\mathbf{q}}_0 + \frac{3}{2}L_0^{1/2}(L-L_0)\hat{\mathbf{q}}_0 + \frac{L_0^{3/2}}{|\mathbf{q}_0|}(\mathbf{q}-\mathbf{q}_0)\right], \tag{6.3}$$

$$\sigma^L \simeq -\left[2BL_0 - \frac{3}{2}Aq_{10}L^{1/2}\right](L-L_0) + AL_0^{3/2}\hat{\mathbf{q}}_0\cdot(\mathbf{q}-\mathbf{q}_0), \tag{6.4}$$

obtaining

$$\rho c_V\partial_t T + \rho\epsilon_L\partial_t L + \nabla\cdot\mathbf{q} = 0, \tag{6.5a}$$

$$\partial_t\mathbf{q} + \zeta_0\nabla T + \chi_0\nabla L = -K[L_0\mathbf{q} + \mathbf{q}_0(L-L_0)] + H\left[\frac{3}{2}L_0^{1/2}(L-L_0)\hat{\mathbf{q}}_0 + \frac{L_0^{3/2}}{|\mathbf{q}_0|}\mathbf{q}\right], \tag{6.5b}$$

$$\partial_t L + \nu_0\nabla\cdot\mathbf{q} = -\left[2BL_0 - \frac{3}{2}Aq_{10}L_0^{1/2}\right](L-L_0) + Aq_{10}L_0^{3/2}(q_1-q_{10}), \tag{6.5c}$$

where $\partial_t$ stands for $\partial/\partial t$.

Consider the propagation of harmonic plane waves, seeking solutions of Eqs. (6.1) of the form

$$T = T_0(\mathbf{x}) + \tilde{T}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \tag{6.6a}$$

$$\mathbf{q} = \mathbf{q}_0 + \tilde{\mathbf{q}}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \tag{6.6b}$$

$$L = L_0 + \tilde{L}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)}, \tag{6.6c}$$

where $k=k_r+ik_s$ is the complex wave number, $\omega$ the real frequency, and $\mathbf{n}$ the unit vector in the direction of the wave propagation. Furthermore, we suppose that the oversigned quantities denote small amplitudes, whose products can be neglected. Inserting Eqs. (6.6) in the linearized field equations (6.1), and making the positions

$$N_1 = KL_0 - H\frac{L_0^{3/2}}{q_{10}}, \quad N_2 = 2BL_0 - \frac{3}{2}AL_0^{1/2}q_{10}, \tag{6.7a}$$

$$N_3 = Kq_{10} - \frac{3}{2}HL_0^{1/2}, \quad N_4 = Aq_{10}L_0^{3/2}, \tag{6.7b}$$

we obtain the following algebraic set of equations for the amplitudes:

$$-[\rho c_V]_0\omega\tilde{T} - [\rho\epsilon_L]_0\omega\tilde{L} + k\tilde{\mathbf{q}}\cdot\mathbf{n} = 0, \tag{6.8a}$$

$$(-\omega - iN_1)\tilde{\mathbf{q}} + k[\zeta_0]_0\tilde{T}\mathbf{n} + \left(k[\chi_0]_0\mathbf{n} - i\frac{N_3}{q_{10}}\mathbf{q}_0\right)\tilde{L} = 0, \tag{6.8b}$$

$$(-\omega - iN_2)\tilde{L} + k[\nu]_0\tilde{\mathbf{q}}\cdot\mathbf{n} + iN_4\tilde{q}_1 = 0, \tag{6.8c}$$

where subscript 0 denotes quantities referring to the unperturbed state; in what follows, this subscript will be neglected to simplify the notation.

This system possesses nontrivial solutions if and only if its determinant vanishes. Imposing this condition, in the case $\mathbf{n}=(1,0,0)$, i.e., when the wave is collinear with the heat flux $\mathbf{q}$, one obtains

$$\omega^2 = k^2[V_2^2(1-\rho\epsilon_L\nu_0) + \nu_0\chi_0] + N_1N_2 - i\omega(N_1+N_2) + i\frac{k^2}{\omega}V_2^2N_2 - ik[(\chi_0 + V_2^2\rho\epsilon_L)N_4 - \nu_0N_3], \tag{6.9}$$

while, in the case $\mathbf{n}=(0,0,1)$, i.e., when the wave is orthogonal with the heat flux $\mathbf{q}$, one obtains

$$\omega^2 = k^2[V_2^2(1-\rho\epsilon_L\nu_0) + \nu_0\chi_0] + N_1N_2 - i\omega(N_1+N_2) + i\frac{k^2}{\omega}V_2^2\left(N_2 - \frac{N_3N_4}{\omega + iN_1}\right). \tag{6.10}$$

In both Eqs. (6.9) and (6.10), we have denoted with $V_2$ the quantity

$$V_2^2 = \frac{\zeta_0}{\rho c_V}, \tag{6.11}$$

which, in the absence of vortices ($\zeta_0=\zeta^*$), coincides with the usual velocity of the second sound,[8,18] mentioned in Eq. (2.4) and, in the presence of vortices, includes a positive contribution proportional to $L$, according to Eqs. (4.21), which shows that the speed of the waves increases when $L$ increases. In the first term on the right-hand side of Eq. (6.8)—the term related with the speed of waves—this effect is further enhanced when the vortex tangle is deformed by the second sound.

We compare the result (6.10) with the result obtained in Ref. 9, where we supposed $L$ a fixed quantity, and the term $\nu_0$ was assumed to vanish, eliminating in this way the effects of the oscillations of $\mathbf{q}$ on the vortex line density $L$ of the tangle. In that work, the dispersion relation for the second sound was

$$\omega^2 = V_2^2k^2 - i\omega KL_0. \tag{6.12}$$

Comparison of Eqs. (6.9) and (6.10) with Eq. (6.12) shows that the distortion of the vortex tangle under the action of the heat wave, and its corresponding back reaction on the latter, implies remarkable changes in the velocity and the attenuation of the second sound, the latter effect depending on the relative direction between $\mathbf{q_0}$ and $\mathbf{n}$. Thus, if one uses Eq. (6.12) instead of Eq. (6.8) one obtains erroneous values for the average vortex line density $L_0$ and the friction coefficient, leading to an incorrect interpretation of the physical results. Introduction of nonlinear effects in our equations would yield further corrections.

## VII. COMPARISON WITH TWO-FLUID MODELS

### A. Field equations in the variables of the two-fluid model

Equations (5.1a), (5.1b), (5.1c), (5.1d), and (5.1e) can be written using the most familiar variables of the two fluid model, performing in them the change of variables (2.6) and substituting Eq. (2.5) with

$$\zeta_0 = \rho\frac{\rho_s}{\rho_n}Ts_0^2. \tag{7.1}$$

From Eqs. (2.6) and (7.1), we can easily obtain the following expressions of $\rho_n/\rho$, $\mathbf{v}_s$ and $\mathbf{v}_n$ as functions of $\zeta_0$, $\mathbf{V}_{ns}$, and $\mathbf{q}$:

$$\frac{\rho_n}{\rho} = \frac{\rho Ts_0^2}{\zeta_0 + \rho Ts_0^2}, \quad \mathbf{v}_s = \mathbf{v} - \frac{s_0}{\zeta_0}\mathbf{q}, \quad \mathbf{v}_n = \mathbf{v} + \frac{1}{\rho Ts_0}\mathbf{q}. \tag{7.2}$$

If we perform in the field equations (5.1a), (5.1b), (5.1c), (5.1d), and (5.1e) this change of variables, we check immediately that the first three equations are identical to the ones of the two-fluid model for helium II. We concentrate therefore on the field equation (5.1d) for the heat flux. We obtain

$$\frac{d\mathbf{V}_{ns}}{dt} + \frac{\rho}{\rho_n}s_0\nabla T + \frac{\rho}{\rho_n}\frac{s_0}{\zeta_0}\chi_0\nabla L = \frac{\rho}{\rho_n}\frac{s_0}{\zeta_0}\sigma^{\mathbf{q}} - \frac{\mathbf{V}_{ns}}{\rho Ts_0}\frac{d(\rho Ts_0)}{dt}. \tag{7.3}$$

The equation for the velocity of the superfluid component can be obtained multiplying Eq. (7.3) by $-\rho_n/\rho$ and adding it to the balance equation for the velocity (5.1b). Recalling that one can write $\mathbf{v}_s=\mathbf{v}-(\rho_n/\rho)\mathbf{V}_{ns}$, we find

$$\frac{d\mathbf{v}_s}{dt} - s_0\nabla T + \frac{1}{\rho}\nabla p_0 - \frac{s_0}{\zeta_0}\chi_0\nabla L = -\frac{s_0}{\zeta_0}\sigma^{\mathbf{q}} + \frac{\mathbf{V}_{ns}}{\rho T s_0}\frac{d(\zeta_0/s_0)}{dt}. \tag{7.4}$$

Finally, using Eq. (4.14), which relates the chemical potentials $\mu_0^\rho$ and $\mu_0^L$ to the equilibrium variables, the field equation for the superfluid velocity takes the form

$$\frac{d\mathbf{v}_s}{dt} + \nabla\mu_0^\rho + \frac{L}{\rho}\nabla\mu_0^L - \frac{s_0}{\zeta_0}\chi_0\nabla L = -\frac{s_0}{\zeta_0}\sigma^{\mathbf{q}} + \frac{\mathbf{V}_{ns}}{\rho T s_0}\frac{d(\zeta_0/s_0)}{dt}. \tag{7.5}$$

We conclude that field equations (5.1a), (5.1b), (5.1c), (5.1d), and (5.1e), in the variables of the two-fluid model, can be written

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0, \tag{7.6a}$$

$$\rho\frac{d\mathbf{v}}{dt} + \nabla p_0 = 0, \tag{7.6b}$$

$$\rho\frac{d\epsilon}{dt} + \nabla\cdot(\rho_s T s_0 \mathbf{V}_{ns}) = 0, \tag{7.6c}$$

$$\frac{d\mathbf{v}_s}{dt} + \nabla\mu_0^\rho + \frac{L}{\rho}\nabla\mu_0^L - \frac{s_0}{\zeta_0}\chi_0\nabla L = -\frac{s_0}{\zeta_0}\sigma^{\mathbf{q}} + \frac{\mathbf{V}}{\rho T s_0}\frac{d(\zeta_0/s_0)}{dt}, \tag{7.6d}$$

$$\frac{dL}{dt} + L\nabla\cdot\mathbf{v} + \nabla\cdot(\nu_0\rho_s T s_0 \mathbf{V}_{ns}) = \sigma^L. \tag{7.6e}$$

In the following subsections, we will compare these equations with those obtained using directly the two-fluid model by other authors.

### B. Comparison with Nemirowskii-Lebedev and Geurst models

Hydrodynamical descriptions of superfluid turbulence incorporating the inhomogeneities of the vortex-line density and an evolution equation for $L$ have been proposed previously using different methods by Nemirovskii and Lebedev[23] (phenomenological), Yamada *et al.*[24] (stochastic), and Geurst[25,26] (variational). We will briefly compare some of the differences of our work and theirs.

Nemirovskii and Lebedev[23] provide a thermodynamical analysis of the superfluid turbulence in the framework of Bekarewich-Khalatnikov method.[8] They start from evolution equations for $\rho$ (density), $S$ (entropy), $\mathbf{J}$ (momentum), $\mathbf{v}_s$ (superfluid velocity), energy $E$ of the system (supposing it dependent on the other variables), and $L$ (vortex line density). The evolution equations for $\rho$, $\mathbf{J}$, and $E$ are the usual conservation equations for mass, momentum, and energy, whereas the evolution equations for $\mathbf{v}_s$ and $L$ are found to be

$$\frac{\partial\mathbf{v}_s}{\partial t} + \nabla\mu + \mathbf{v}_s\cdot\nabla\mathbf{v}_s = \chi\mathbf{V}_{ns}L - \chi\rho\kappa\beta\mathbf{V}_{ns}L^{3/2} \tag{7.7}$$

and

$$\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v}^L) = \alpha_v V_{ns}L^{3/2} - \beta_v\kappa L^2, \tag{7.8}$$

analogous to our Eq. (5.3). They assume from the start that the drift velocity is proportional to $V_{ns}$, namely $\mathbf{v}^L = b(T)\mathbf{V}_{ns}$, but do not study the sign of $b(T)$, which has been proved to be negative in our analysis. Their Gibbs equation (in the superfluid reference frame) has the form

$$dE = \mu\, d\rho + T\, dS + \mathbf{V}_{ns}\cdot d\mathbf{J} + \epsilon_V dL. \tag{7.9}$$

Comparing Eq. (7.7) with our corresponding equation (7.6d) we note that in the equation deduced from our model the terms $\mathcal{L}\nabla\mu_0^L$ and $-(s_0/\zeta_0)\chi_0\nabla L$ are present, which are not considered in Ref. 23. Comparing Eq. (7.9) with our Eq. (4.11), we see that Nemirovskii and Lebedev identify the energy $\epsilon_V$ per unit length with the chemical potential $\mu_0^L$ of the vortex tangle. A term of the type $\lambda_q\mathbf{q}\cdot d\mathbf{q}$ (corresponding to the term $\mathbf{V}_{ns}\cdot d\mathbf{J}$) does appear in our model, owing to our choice (3.4). A more general theory, in which nonlinear terms in $\mathbf{q}$ will be considered, will be the object of a subsequent paper.

They arrive at a rich set of equations, which is applied to study the propagation of linear and nonlinear second sound through the tangle, incorporating the distortion of the tangle produced by the second sound and the corresponding back reaction on the latter. In contrast with the present analysis, they do not pay attention to vortex diffusion, which at the time when their paper was written was not receiving special attention, in contrast with second-sound propagation, which has always been a powerful experimental tool. In Ref. 24, Yamada *et al.* have discussed the model by Nemirowski and Lebedev by starting from a stochastic theory of vortex tangle in superfluid turbulence, whose averaged form leads to the three equations in Ref. 23 plus the possible contribution of an eddy viscosity.

The Geurst proposal[25,26] is based on six variables: mass density $\rho$, entropy density $S$, vortex line density $L$, mass velocity $\mathbf{v}$, normal flow velocity $\mathbf{v}_n$, and tangle velocity $\mathbf{v}^L$, for which he obtains six corresponding equations, expressing the balance of the first three quantities, and the equations of motion of the latter three velocities. Thus he is using one more variable than those appearing in our analysis of Sec. III, namely, the vortex velocity $\mathbf{v}^L$. In our formalism, we have been led to the relation (5.5) between $\mathbf{v}_L$ and $\mathbf{v}$, namely $\mathbf{v}_L = \nu_0\mathbf{q}/L$, and therefore a supplementary equation for $\mathbf{v}_L$ is not strictly needed in our formalism, but $\mathbf{v}_L$ could indeed be taken as an independent variable in more general situations than those considered here. Geurst derivation is strictly valid for homogeneous turbulence whereas we have focused our interest on inhomogeneous effects. Therefore, he is especially interested in the motion of an homogeneous tangle, whereas we are especially interested in the interplay between a heat flux and the inhomogeneities in the vortex tangle, which is not described in Geurst formalism.

To obtain the evolution equation for these variables, Geurst applies a generalized form of Hamilton's least action principle for the nondissipative part and a nonequilibrium thermodynamic formalism for the dissipative part. The five unknown coefficients appearing in the theory are obtained by comparing the hydrodynamic equations with similar equations obtained by Schwarz on vortex microscopic dynamics.[20] The Gibbs equation he uses has the form

$$dE = \mu\, d\rho + T\, dS + \mu_L dL - P_n \cdot d(v_n - v) - P_L d(v_L - v), \tag{7.10}$$

where $P_n$ and $P_L$ are the impulse densities of the normal fluid and the vortex tangle respectively. Instead of our expression (4.6) for the chemical potential of vortex lines, Geurst takes

$$\mu_L = \epsilon_V - \frac{1}{2}\rho_s \kappa \beta L^{1/2} |v_L - v| \tag{7.11}$$

with $\epsilon_V$ as given by Eq. (4.7). The corresponding expression for the entropy production is

$$\sigma_S = - r_L \mu_L - (v_L - v) F_{sL} - (v_L - v_n) \cdot F_{nL}, \tag{7.12}$$

where $r_L$ is the production term in the Vinen's equation, and $F_{sL}$ and $F_{nL}$ are the forces exerted by the superfluid and the normal fluid on the vortex tangle, respectively. The terms $r_L$, $F_{sL}$, and $F_{nL}$ are expressed in terms of $m_l$, $v_L - v$, and $v_L - v_n$ by using the usual formalism of nonequilibrium thermodynamics. The expressions for $F_{sL}$ and $F_{nL}$ have the form

$$F_{nL} = \rho_s \kappa L \left[ a(v_n - v) - bL^{1/2} \ln \frac{c}{a_0 L^{1/2}} \operatorname{sgn}(v_L - v) \right] \tag{7.13}$$

and

$$F_{sL} = \rho_s \kappa L \left[ a'(v_n - v) - bL^{1/2} \ln \frac{c}{a_0 L^{1/2}} \operatorname{sgn}(v_L - v) \right], \tag{7.14}$$

where $a$ and $b$ can be written in terms of the phenomenological coefficients appearing in the equations for these forces, and sgn $v_L - v$ is $+1$ or $-1$ according to $v_L - v > 0$ or $v_L - v < 0$, respectively (this version is valid for one-dimensional situations, and $\operatorname{sgn}(v_L - v)$ is considered as an independent parameter in the formalism by Geurst. Since we have one variable less than those by Geurst, our Gibbs equation for the entropy (3.8) and our entropy production (4.25) also have one less term; in fact, in counterflow situations, $v = 0$, $v_n = q/(\rho T s)$, and $v_L = \nu_0 q/L$. Thus the contributions from $v_n$ and $v_L$ could be written in terms of the heat flux, and could be related—though not in a straightforward way—to the second term on the right-hand side of our first equation in (3.8).

It is interesting to point out that Geurst has been able to describe Vinen's equation in variational terms. He achieves this purpose, in a one-dimensional setting, by showing that Eq. (7.8) may be written in the form

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(L v^L) = - \frac{1}{4\pi \beta_v} \frac{L^2}{\rho_s \kappa} \frac{\partial}{\partial L} [U_l - (v^L - v) P_l], \tag{7.15}$$

with $U_l$ the internal energy density associated with the motion of the superfluid around the core of the quantized vortices, namely $U_l = \epsilon_V L$, and $P_l = m_l (v_l - v)$ being the impulse density of the vortex tangle and $m_l$ its virtual mass density (for detailed expressions see Ref. [25]). In the presence of high gradients he introduces $\partial L/\partial x$ as an additional independent variable in the energy $U_l$ (and in $m_l$) and he obtains a generalization of Vinen's equation which, written in our notation, has the form

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}\left[ L v^L - 4\pi \beta_v \kappa \gamma_l \frac{\partial L}{\partial x} \right] = \alpha_v V_{ns} L^{3/2} - \beta_v \kappa L^2 - \frac{8\pi \beta_v \kappa \gamma_l}{L} \left( \frac{\partial L}{\partial x} \right)^2 + (4\pi \beta_v)^2 \frac{\kappa \gamma_l}{\rho_s} \frac{\partial}{\partial x} \left( \frac{\rho_s}{4\pi \beta_v} \right) \frac{\partial L}{\partial x}, \tag{7.16}$$

with $\gamma_l$ a dimensionless coefficient. This equation describes diffusion when $v^L$ and the last two (nonlinear) terms are neglected, yielding

$$\frac{\partial L}{\partial t} = \alpha_v V_{ns} L^{3/2} - \beta_v \kappa L^2 + D \frac{\partial^2 L}{\partial x^2}, \tag{7.17}$$

with the diffusion coefficient $D$ identified as $D = 4\pi \beta_v \kappa \gamma_l$. Since $\beta_v$ and $\gamma_l$ are dimensionless coefficients, it turns out that, as in our previous analysis, the diffusion coefficient $D$ is proportional to the quantum of vorticity $\kappa$. In contrast to our paper and to Ref. [26], $v_L$ is taken here as an independent variable, rather than being expressed in terms of $q$ or $V_{ns}$.

In summary, in this brief comparison we have pretended to show that the hydrodynamics of superfluid turbulence is still an open topic, especially in the nonlinear regime.

## VIII. CONCLUSIONS

The study of quantum turbulence in superfluids often assumes homogeneity of the vortex tangle line density $L$. In several situations this homogeneity will not hold, and the vortex lines will diffuse from the most concentrated to the less concentrated zones. For instance, vortex lines could be produced near the walls and migrate by diffusion to the bulk of the container until a homogeneous situation is reached. Furthermore, if vortex lines are flexible, they will be bent and their density will be compressed and rarefied by second-sound waves and this will produce an inhomogeneity in $L$, which, in its turn, will influence the propagation of second sound. This may be relevant in the interpretation of the experimental results on the speed and the attenuation of the second sound in terms of the average vortex line density of the system.

To incorporate these effects has been the main motivation of this paper. We have not limited ourselves to adding a more general evolution equation for $L$, namely Eq. (5.3), but we have tried to insure thermodynamical consistency of the mutual coupling of this equation and the equations considered previously for the other fields. We have worked in a macroscopic thermodynamic framework, which yields several consequences of incorporating the additional terms to the evolution equations for the heat flux and the vortex line density through the term with the coefficient $\nu_0$ in Eqs. (5.1) and (6.1). The thermodynamic consequences are shown as restrictions on the coefficients of the new terms, as for instance, Eqs. (4.18)–(4.22). We have also commented how the choice for the chemical potential of vortex lines (4.6) influences such restrictions. We have obtained, in this way, field equations for the relevant quantities, and we have studied two simple but directly significant situations, as an illustration of their usefulness. In particular, we have compared the dispersion relation (6.10), where the mentioned coupling is included, with the dispersion relation (6.12), where the tangle was taken as rigid, presenting only a frictional resistance to the second sound. Furthermore, we have paid special attention to Eq. (5.16) describing vortex diffusion.

We have presented a brief comparison of our formalism and the previous hydrodynamic formalisms by Nemirovskii and Lebedev and by Geurst—and we have also mentioned the stochastic approach by Yamada *et al.* Each of the mentioned formalisms uses a different set of basic variables; mass density, energy, and vortex length are common to all, but the choices for the velocities are different. Nemirovskii and Lebedev take the momentum density $\mathbf{J}$ and the superfluid velocity $\mathbf{v}_s$; Geurst uses three velocities: Mass velocity $\mathbf{v}$, normal velocity $\mathbf{v}_n$, and vortex tangle velocity $\mathbf{v}_L$; we take mass velocity $\mathbf{v}$ and the heat flux $\mathbf{q}$ (related to $\mathbf{v}_n-\mathbf{v}_s$). The different choices are due the different aims in several works. The characteristic aim of our work is to describe the coupling between the heat flux and the inhomogeneities in the vortex line density. It is seen that, in contrast with laminar superfluid hydrodynamics, turbulent superfluid hydrodynamics is an open topic where several different approaches should be compared in deeper detail, as they seem to yield results which are not entirely equivalent to each other because of the different hypotheses, the different methods used, and the different situations considered.

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## APPENDIX: Microscopic determination of the production terms $\sigma^{q}$ and $\sigma^{L}$

In the vortex filaments model, a quantized vortex line is thought of as a classical vortex line in the superfluid with a hollow core of radius $a_0$ of about 1 Å, and quantized circulation $\kappa$. The vortex line is described by a vectorial function $\mathbf{s}(\xi,t)$, $\xi$ being the arc length measured along the curve of the vortex filament. The first two derivatives of $\mathbf{s}$ with respect to $\xi$, which we will denote with a prime, play an essential role in this description: $\mathbf{s}'$ is the unit vector tangent along the vortex line at a given point and $\mathbf{s}''$ is the curvature vector. Another relevant vector is the binormal, defined by $\mathbf{s}'\times\mathbf{s}''$. All these three vectors and their relative orientations with respect to the counterflow velocity $\mathbf{V}_{ns}$ are important in the microscopic vortex dynamics.

The driving force which pushes the vortices is the Magnus force $\mathbf{f}_M$, generated by the relative flow of superfluid with respect to the vortex:[1–3]

$$\mathbf{f}_M = \kappa\rho_s\mathbf{s}'\times(\mathbf{v}_L-\mathbf{v}_{sl}), \tag{A1}$$

where $\mathbf{v}_L=d\mathbf{s}/dt$ is the velocity of the line element and $\mathbf{v}_{sl}=\mathbf{v}_s+\mathbf{v_i}$ is the "local superfluid velocity," sum of the superfluid velocity at a large distance from any vortex line and of the "self-induced velocity," a flow due to all the other vortices including other parts of the same vortex, induced by the curvature of all these lines. In the "local induction approximation," the self-induced velocity $\mathbf{v_i}$ is approximated by[1–3]

$$\mathbf{v_i}^{(\mathrm{loc})} = \tilde{\beta}[\mathbf{s}'\times\mathbf{s}'']_{s=s_0}, \quad \text{with} \quad \tilde{\beta}=\frac{\kappa}{4\pi}\ln\left(\frac{c}{a_0L^{1/2}}\right), \tag{A2}$$

with $c$ a constant of the order of unity and $a_0$ the dimension of the vortex core. The intensity of $\mathbf{v_i}$ is $|\mathbf{v_i}|=\tilde{\beta}/R$, with $R$ the curvature radius of the vortex line. The self-induced velocity is zero if the vortices are straight lines. The coefficient $\tilde{\beta}$ is linked to the internal energy per unit length of the vortex line (the tension of the vortex line) by the relation $\epsilon_V=\rho_s\kappa\tilde{\beta}$.[1–3]

The normal component reacts to a moving vortex by producing a frictional force, the "mutual friction force," which can be written as[1–3]

$$\mathbf{f}_{\mathrm{MF}} = -\alpha\rho_s\kappa\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{v}_n-\mathbf{v}_{sl})]-\alpha'\rho_s\kappa\mathbf{s}'\times(\mathbf{v}_n-\mathbf{v}_{sl}), \tag{A3}$$

[where $\alpha$ and $\alpha'$ are linked to the Hall-Vinen coefficients $B_{\mathrm{HV}}$ and $B'_{\mathrm{HV}}$ by the relations $\alpha=(\rho_n/2\rho)B_{\mathrm{HV}}$ and $\alpha'=(\rho_n/2\rho)B'_{\mathrm{HV}}$], i.e., observing that $\mathbf{v}_n-\mathbf{v}_{sl}=\mathbf{V}_{ns}-\mathbf{v_i}$:

$$\mathbf{f}_{\mathrm{MF}} = -\alpha\rho_s\kappa\mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V}_{ns}-\mathbf{v_i})]-\alpha'\rho_s\kappa\mathbf{s}'\times(\mathbf{V}_{ns}-\mathbf{v_i}). \tag{A4}$$

The force $\mathbf{f}_{\mathrm{MF}}$ is the force on a vortex line element exerted by the superfluid; the force $-\mathbf{f}_{\mathrm{MF}}$ is the force on the superfluid exerted by the vortex element. The force per unit volume exerted by the counterflow on the tangle will be denoted by

$$\mathbf{F} = \langle\mathbf{f}_{\mathrm{MF}}\rangle = \frac{1}{\Lambda L}\int \mathbf{f}_{\mathrm{MF}}d\xi, \tag{A5}$$

where $\Lambda$ denotes the tangle's volume, and it is the force $-\mathbf{F}$ which appears in the evolution equation for the superfluid velocity $\mathbf{v}_s$. It is easy to see that

$$\mathbf{F} = \alpha\rho_s\kappa\langle(\mathbf{U}-\mathbf{s}'\mathbf{s}')\cdot(\mathbf{V}_{ns}-\mathbf{v_i})\rangle+\alpha'\rho_s\kappa\langle\mathbf{W}\cdot\mathbf{s}'\cdot(\mathbf{V}_{ns}-\mathbf{v_i})\rangle. \tag{A6}$$

Using the local induction approximation, and setting symmetry $\langle\mathbf{s}''\rangle=0$, Eq. (A6) can be written as

$$\mathbf{F} = \alpha\rho_s\kappa\langle(\mathbf{U}-\mathbf{s}'\mathbf{s}')\rangle\cdot\mathbf{V}_{ns}+\alpha'\rho_s\kappa\langle\mathbf{W}\cdot\mathbf{s}'\rangle\cdot\mathbf{V}_{ns} - \alpha\rho_s\kappa\tilde{\beta}\langle\mathbf{s}'\times\mathbf{s}''\rangle. \tag{A7}$$

Using now the tensor $\mathbf{\Pi}$ studied in Ref. [19] [linked to the tensor $\mathbf{P}_\omega$ by the relation $\mathbf{P}_\omega=(1/3)\kappa L B_{HV}\mathbf{\Pi}$] and the quantities introduced by Schwarz,[1,20]

$$c_1=\frac{1}{\Lambda L^{3/2}}\int|\mathbf{s}''|d\xi,\quad c_2=\frac{1}{\Lambda L^2}\int|\mathbf{s}''|^2d\xi, \tag{A8}$$

$$\mathbf{I}_l=\frac{1}{\Lambda L^{3/2}}\int\mathbf{s}'\times\mathbf{s}''d\xi,\quad \mathbf{I}=\frac{\mathbf{I}_l}{c_1}=\frac{\int\mathbf{s}'\times\mathbf{s}''d\xi}{\int|\mathbf{s}''|d\xi}, \tag{A9}$$

the expression of the force which the tangle exerts on the fluid is obtained and can be written as

$$\begin{aligned}\mathbf{F}&=\alpha\rho_s\kappa\left[\frac{2}{3}L\mathbf{\Pi}\cdot\mathbf{V}_{ns}-\tilde{\beta}c_1L^{3/2}\mathbf{I}\right]\\&=\frac{\rho_s\rho_n}{\rho}\left[\mathbf{P}_\omega\cdot\mathbf{V}_{ns}-\frac{1}{2}B_{HV}\kappa\tilde{\beta}c_1L^{3/2}\mathbf{I}\right].\end{aligned} \tag{A10}$$

Observe that the last term does not appear when $\mathbf{v_i}=\mathbf{0}$, and is usually neglected in the expression of the force which the tangle exerts on the normal fluid.

This leads us to the following expression for the production term:

$$\sigma^{\mathbf{q}}=-\mathbf{P}_\omega\cdot\mathbf{q}-\frac{B_{HV}}{2}\rho_sTs_0\epsilon_Vc_1L^{3/2}\mathbf{I}. \tag{A11}$$

Supposing that the vector $\mathbf{I}$ is collinear with the counterflow velocity $\mathbf{V}_{ns}$ (and it is true in steady situations), putting $\mathbf{I}=I_0\hat{\mathbf{V}}_{ns}$, we can write

$$\mathbf{F}_{MF}=-\frac{\rho_s\rho_n}{2\rho}\kappa B_{HV}\left[\frac{2}{3}L\mathbf{\Pi}\cdot\mathbf{V}_{ns}-\tilde{\beta}c_1L^{3/2}I_0\hat{\mathbf{V}}_{ns}\right] \tag{A12}$$

and

$$\sigma^{\mathbf{q}}=-\mathbf{P}_\omega\cdot\mathbf{q}+\frac{B_{HV}}{2}\rho_sTs_0\epsilon_Vc_1L^{3/2}I_0\hat{\mathbf{q}}. \tag{A13}$$

Observe that in a stationary situation the coefficient $I_0$ has positive sign, but, in some particular situations, its sign may become negative.

If we suppose isotropy in the distribution of $\mathbf{s}'$, results $\mathbf{\Pi}=\mathbf{U}$, and we are lead to equation

$$\sigma_{\mathbf{q}}=-KL\mathbf{q}\pm HL^{3/2}\hat{\mathbf{q}}, \tag{A14}$$

with $K=(1/3)\kappa B_{HV}$ and $H=B_{HV}/2\rho_sTs_0\epsilon_Vc_1L^{3/2}I_0$.

For the production term in the equation for the line density $L$, we have chosen expression (3.5); $\alpha_v$ and $\beta_v$ are linked to the microscopic quantities by the relations $\alpha_v=\alpha c_1I_0$ and $\beta_v=\alpha\tilde{\beta}c_2$ and coefficients $A$ and $B$ are related to the microscopic quantities (A8) and (A9) by the relations

$$A=\frac{\alpha c_1}{\rho_sTs_0}I_0,\quad B=\alpha\tilde{\beta}c_2. \tag{A15}$$


1. R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, UK, 1991).
2. *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, Berlin, 2001).
3. W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).
4. W. F. Vinen, Phys. Rev. B **61**, 1410 (2000).
5. C. F. Barenghi, J. Phys.: Condens. Matter **11**, 7751 (1999).
6. L. Tisza, Nature (London) **141**, 913 (1938).
7. L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).
8. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Addison-Wesley, Redwood City, CA, 1989).
9. D. Jou, G. Lebon, and M. S. Mongiovì, Phys. Rev. B **66**, 224509 (2002).
10. W. F. Vinen, Proc. R. Soc. London, Ser. A **240**, 493 (1957).
11. D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics* (Springer-Verlag, Berlin, Heidelberg, 2001).
12. I. Müller and T. Ruggeri, *Rational Extended Thermodynamics* (Springer-Verlag, New York, 1998).
13. I. Liu, Arch. Ration. Mech. Anal. **46**, 131 (1972).
14. M. S. Mongiovì, Phys. Rev. B **48**, 6276 (1993).
15. M. S. Mongiovì, Physica A **291**, 518 (2001).
16. M. S. Mongiovì, Phys. Rev. B **63**, 012501 (2001).
17. M. S. Mongiovì and R. A. Peruzza, Int. J. Non-Linear Mech. **39**, 1005 (2004).
18. M. S. Mongiovì, J. Non-Equilib. Thermodyn. **25**, 31 (2000).
19. D. Jou and M. S. Mongiovì, Phys. Rev. B **74**, 054509 (2006).
20. K. W. Schwarz, Phys. Rev. B **31**, 5782 (1988).
21. J. Casas-Vázquez and D. Jou, Rep. Prog. Phys. **66**, 1937 (2003).
22. W. F. Vinen, Proc. R. Soc. London, Ser. A **243**, 400 (1958).
23. S. K. Nemirovskii and V. V. Lebedev, Sov. Phys. JETP **57**, 1009 (1983).
24. K. Yamada, S. Kashiwamura, and K. Mikaye, Physica B **154**, 318 (1989).
25. J. A. Geurst, Physica B **154**, 327 (1989).
26. J. A. Geurst, Physica A **183**, 279 (1992).
27. S. K. Nemirovskii and W. Fiszdon, Rev. Mod. Phys. **67**, 37 (1995).
28. S. K. Nemirovskii, G. Stamm, and W. Fiszdon, Phys. Rev. B **48**, 7338 (1993).
29. D. Jou and M. S. Mongiovì, Phys. Rev. B **72**, 144517 (2005).
30. T. Lipniacki, Eur. J. Mech. B/Fluids **25**, 435 (2006).
31. H. van Beelen, W. van Joolingen, and K. Yamada, Physica B **153**, 248 (1988).
32. M. Tsubota, T. Araki, and W. F. Vinen, Physica B **329-333**, 224 (2003).