# Comment on "London model for the levitation force between a horizontally oriented point magnetic dipole and superconducting sphere"

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The levitation force due to a horizontally oriented point magnetic dipole placed in front of a superconducting sphere is calculated and is shown to be different from the result given recently by Coffey [Phys. Rev. B **65**, 214524 (2002)]. The present calculation for the levitation force is based on the formulas developed long ago for an equivalent Neumann boundary value problem in hydrodynamics. The result derived here demonstrates that the value of the levitation force for a horizontally oriented dipole lies between one-fourth and one-half the value of the configuration with a radially oriented point dipole, thus providing upper and lower bounds for the levitation force due to a tangentially oriented magnetic dipole.

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In Refs. 1 and 2 Coffey investigated the magnetic interaction between a horizontally oriented dipole and a superconducting sphere in the Meissner state within London theory. By the use of spherical harmonics, Coffey solved the Neumann problem for a perfectly diamagnetic sphere in the presence of a point dipole of horizontally oriented moment and extended the analysis to the case of nonvanishing penetration depth governed by the vector London equation. The levitation force is then extracted from the series solutions for the magnetostatic scalar potentials in both cases. It is elucidated in this comment that for a perfectly conducting diamagnetic sphere the problem can be solved in closed form using the formulas developed for an equivalent mathematical problem in hydrodynamics. More specifically, the relationship between magnetostatics and hydrodynamics is utilized here to derive the solution for an arbitrarily oriented magnetic dipole located in front of a superconducting sphere. The levitation force acting on the superconducting sphere is computed using a rather simple formula used earlier in hydrodynamics. The calculation yields an expression for the vertical component of the levitation or lifting force that is different from the one given by Coffey. The difference is due to the choice of the boundary condition used by Coffey, which is not appropriate for a superconducting surface with zero penetration depth.

We begin by mentioning the relationship between the problem for a superconducting surface in magnetostatics and the corresponding problem for the flow field around a bounding surface in inviscid hydrodynamics. A perfectly *superconducting* surface is described by the boundary condition that the magnetic field has no component that is normal to the surface. The formulation in terms of a magnetostatic potential yields a Neumann boundary value problem for the superconducting surface placed in an inhomogeneous magnetic field. An equivalent boundary value problem exists in inviscid hydrodynamics where consequences of the flow field of an ideal fluid past an impermeable body are of interest.[3,4] This simple[5] analogy may be utilized to translate results from hydrodynamics to magnetostatics and vice versa. It is worthwhile to mention that many of the practically important cases have already been solved in hydrodynamics and by analogy these results can be understood in magnetostatics, avoiding unnecessary duplications.

Recently, much interest has been shown in the calculation of levitation force between a point magnetic dipole and superconducting surfaces.[1,2,6,7] This requires the construction of solutions to the magnetostatic potential due to a magnetic dipole located in front of a superconducting plane or sphere. For a superconducting sphere, a closed form analytic solution for the scalar potential was obtained in the case of a dipole oriented vertically,[6] while an infinite-series solution in spherical harmonics was derived for the magnetic dipole oriented horizontally.[1,2] The results for the magnetic scalar potential for the same problems were obtained earlier in Refs. 8 and 9 in the context of superconducting imaging. But the corresponding results in the context of hydrodynamics have been known for a long time.[10–14] In fact, Weiss[11] treated a more general problem in magnetostatics and derived solutions for a point dipole problem as a special case. It appears that Neumann was the first to derive the solution for a magnetic source (monopole) located in front of a sphere, and documented the results in an appendix of his book.[15] Below, we utilize the results in Ref. 11 and show that the expression for levitation force due to a horizontally or tangentially oriented point dipole calculated by Coffey[1,2] is quite different from ours. For the sake of completeness, we give here a detailed account of the general solution for an arbitrarily oriented magnetic point dipole located in front of the superconducting sphere and then discuss the special cases.

As in Refs. 1, 2, and 6 let the origin of coordinates be at the center of a sphere of radius $b$. The field solution is sought for the magnetostatic potential $\Phi(\mathbf{x})$ such that the magnetic field is given by $\mathbf{H}=-\nabla\Phi$ and induction $\mathbf{B}=\mu_0\mathbf{H}$. The Neumann problem for the magnetic scalar potential is then

$$\nabla^2\Phi=0, \tag{1}$$

$$\frac{\partial\Phi}{\partial r}=0 \quad \text{on } r=b. \tag{2}$$

Notice that condition (2) implies vanishing of the normal component of the magnetic field at the surface of the sphere $r=b$. Coffey[1,2] used the condition that the magnetic field vector vanishes (that is, both tangential and normal components of the magnetic field vanish) at the spherical surface $r=b$ in the derivation of his solutions for point-magnetic-dipole–sphere interaction problems. Below, we present a general solution for the boundary value problem described by (1) and (2) for an arbitrary magnetic potential applied externally.

Let $\phi_0$ be the scalar potential of the unperturbed magnetic field in the absence of the sphere. If the superconducting sphere of radius $b$ is introduced into the field of $\phi_0$, then the modified potential $\Phi$ is[10,11]

$$\Phi = \phi_0 + \phi_1, \tag{3}$$

where

$$\phi_1 = \frac{b}{r}\phi_0\left(\frac{b^2}{r}\right) - \frac{b}{r}\int_0^1 \phi_0\left(\frac{sb^2}{r}\right)ds. \tag{4}$$

The expressions given in (3) and (4) can be used to find solutions of (1) and (2) for various externally imposed potentials in the presence of a superconducting sphere. For a general magnetic dipole of moment $\mathbf{M}=\mathbf{M}_{\parallel}+\mathbf{M}_{\perp}$ positioned at $\mathbf{a}$ outside the sphere, the perturbed magnetostatic potential is[11]

$$\Phi = \frac{\mathbf{M}\cdot(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} + \left(\frac{b}{a}\right)^3\left(\frac{\mathbf{M}\cdot[\mathbf{r}-(r^2/b^2)\mathbf{a}]}{|\mathbf{r}-(b^2/a^2)\mathbf{a}|^3} - \int_0^1 \frac{\mathbf{M}\cdot[s\mathbf{r}-(r^2/b^2)\mathbf{a}]}{|\mathbf{r}-(sb^2/a^2)\mathbf{a}|^3}ds\right), \tag{5}$$

where $\mathbf{r}$ is the position vector with $|\mathbf{r}|=r$. The integral in the above expression can be evaluated and the resulting magnetostatic potential now reads

$$\Phi = \frac{(\mathbf{M}_{\perp}+\mathbf{M}_{\parallel})\cdot(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3} + \left(\frac{b}{a}\right)^3\left(\frac{(\mathbf{M}_{\perp}-\mathbf{M}_{\parallel})\cdot[\mathbf{r}-(b^2/a^2)\mathbf{a}]}{|\mathbf{r}-(b^2/a^2)\mathbf{a}|^3}\right) - \frac{\mathbf{M}_{\perp}\cdot\mathbf{r}}{ba[r^2-(\mathbf{a}\cdot\mathbf{r}/a)^2]}\left(r-\frac{r^2-(b^2/a^2)(\mathbf{a}\cdot\mathbf{r})}{|\mathbf{r}-(b^2/a^2)\mathbf{a}|}\right). \tag{6}$$

Note that the expression for the modified potential given in (6) represents the general solution for an arbitrarily oriented magnetic point dipole in the presence of a superconducting sphere. The terms in the closed form expression (6) may be interpreted as images inside the sphere. The image system can be best understood by analyzing the radial (the vertical case in Ref. 6) and tangential (the horizontal case in Refs. 1 and 2) initial dipole orientations separately.

(1) For the initial radial dipole ($\mathbf{M}_{\parallel}$) the image system consists of an image dipole at the Kelvin image point $(b^2/a^2)\mathbf{a}$.

(2) For the tangentially oriented initial dipole ($\mathbf{M}_{\perp}$) the image system consists of an image dipole at the Kelvin inverse point together with a distribution of magnetic dipoles from the origin to the Kelvin's image point.[9,11]

Thus, there is a fundamental difference in the image systems for radial and tangential dipole-sphere configurations. The present image representation for point-dipole–sphere configurations agree with those already given in Refs. 8 and 9. The image system for the radial case is also given in Ref. 6.

Now the levitation force exerted by the dipole $\mathbf{M}$ on the superconducting sphere is given by (see the end of the discussion for force calculations)

$$\mathbf{F} = 4\pi\mu_0\frac{b^3}{(b^2-a^2)^4}\left[\left(6-4\frac{b^2}{a^2}+\frac{b^4}{a^4}\right)\left(M^2+\frac{(\mathbf{M}\cdot\mathbf{a})^2}{a^2}\right)\mathbf{a} - \frac{1}{2}\left(3-\frac{b^2}{a^2}\right)\left(1-\frac{b^2}{a^2}\right)[(\mathbf{M}\cdot\mathbf{a})\mathbf{M}+3M^2\mathbf{a}]\right]. \tag{7}$$

The force due to a radially and/or a tangentially oriented dipole can be extracted from the above expression (7). For the radial dipole with strength $|\mathbf{M}_{\parallel}|=m/4\pi$ located at $\mathbf{a}=(0,0,a)$, the levitation force (7) becomes

$$F_{z\parallel} = \frac{3\mu_0 m^2}{2\pi}\frac{b^3 a}{(a^2-b^2)^4}, \tag{8}$$

which is in agreement with the force derived in Ref. 6. In the limit $a^2 \gg b^2$, we have

$$F_{z\parallel} \to \frac{3\mu_0 m^2}{2\pi}\frac{b^3}{a^7}, \tag{9}$$

while for close dipole-sphere separation, $a-b=h\ll b$, Eq. (8) yields

$$F_{z\parallel} = \frac{3\mu_0 m^2}{32\pi h^4}. \tag{10}$$

The expressions (9) and (10) are simply those given in Ref. 6.

For the tangential dipole of strength $|\mathbf{M}_{\perp}|=m/4\pi$ positioned on the $z$ axis at $\mathbf{a}=(0,0,a)$, the levitation force extracted from (7) is

$$F_{z\perp} = \frac{\mu_0 m^2}{4\pi}\frac{b^3 a}{(a^2-b^2)^4}\left(\frac{3}{2}+2\frac{b^2}{a^2}-\frac{1}{2}\frac{b^4}{a^4}\right). \tag{11}$$

We note that the functional form in (11) is much different from that derived recently[1,2] [see Eq. (20a) in the cited references]. It is seen from Eqs. (8) and (11) that the levitation force in the tangential case is *not* one-half of the levitation force in the radial case (except in the limiting case for a half space). The plots of the normalized levitation force components for the dipole oriented along the radial and tangential directions, respectively, are depicted in Fig. 1. The plots show that the levitation force for the tangentially oriented dipole is less than the levitation force due to the radially oriented dipole for all dipole-sphere separations. In the limit $a^2 \gg b^2$, we have

$$F_{z\perp} \to \frac{3\mu_0 m^2}{8\pi}\frac{b^3}{a^7}. \tag{12}$$

Again, the above result is in disagreement with the corresponding result in Ref. 1. For close dipole-sphere separation, $a-b=h\ll b$, Eq. (11) yields

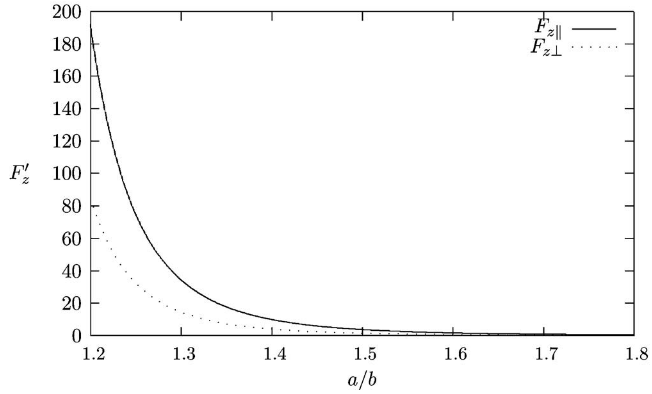


FIG. 1. The levitation force due to radial and tangential dipoles located in front of a superconducting sphere. Here $F'_z$ is the normalized force component.

$$F_{z\perp} = \frac{3\mu_0 m^2}{64\pi h^4}. \tag{13}$$

This result is the same as in Refs. 1, 2, and 7 and in this limiting case (for a semi-infinite superconductor) the levitation force is one-half the value of the configuration with a radially (vertically) oriented point dipole.

From the present results for the levitation forces given in Eqs. (8)–(13), it may be conjectured that

$$\frac{F_{z\parallel}}{4} \le F_{z\perp} \le \frac{F_{z\parallel}}{2} \tag{14}$$

with the limiting values attained in the small and large dipole-sphere separations, respectively. Equation (14) tells that the value of the levitation force for a tangentially/horizontally oriented dipole-sphere configuration lies between one fourth and one half the value of the configuration with a radially/vertically oriented point dipole thus, providing the upper and lower bounds for the levitation force for the tangentially oriented dipole-sphere configuration.

The corresponding correction should be made in the expression for the lift force given in Ref. 1 [Eq. (54)] using London model. These corrected results are important in practice and must be taken into account while modeling magnetic levitation and magnetic force microscopy. The present discussion may also be of interest in the context of current loop dipole theory[16] involving a pair of magnetic dipoles located in the vicinity of a superconducting sphere.

The force in the general case given in Eq. (7) was calculated using a simple formula

$$\mathbf{F} = [-(\mathbf{M}\cdot\nabla)\nabla\phi_1]_{\mathbf{r}=\mathbf{a}}, \tag{15}$$

where $\phi_1$ is found using (4). It may be worthwhile to point out that formula (15) can be used to compute the levitation force in a fairly straightforward fashion. To see this, we first record the explicit expressions for $\phi_1$ in the two cases discussed here. For the radial dipole located at $\mathbf{a}=(0,0,a)$, we have

$$\phi_1 = -\frac{b^3}{a^3}\frac{M_\parallel(z-b^2/a)}{[x^2+y^2+(z-b^2/a)^2]^{3/2}}, \tag{16}$$

and for the transverse dipole along the $x$ direction, $\phi_1$ is

$$\phi_1 = \frac{b^3}{a^3}\frac{M_\perp x}{[x^2+y^2+(z-b^2/a)^2]^{3/2}} - \frac{M_\perp x}{ab(x^2+y^2)}\left(r - \frac{[r^2-(b^2/a)z]}{[x^2+y^2+(z-b^2/a)^2]^{1/2}}\right). \tag{17}$$

Now the substitution of the expressions (16) and (17) in (15), after some algebra, yields the respective forces given in (8) and (11) and this independent calculation was performed to check the results for the two levitation force components. It should be mentioned that the levitation force may also be calculated using a more complicated formula given in Ref. 17 in terms of spherical harmonics. But the present approach is much simpler than the standard spherical harmonics technique.

Finally, the discrepancy between the results obtained here [Eqs. (11) and (12)] and given by Coffey [Eq. (20a) in Refs. 1 and 2] for the vertical component of the levitation force is due to the choice of the boundary conditions employed in the two formalisms. We have used the condition that the normal component of the magnetic field vanishes on the superconducting sphere, whereas Coffey assumed that both normal and tangential components of the magnetic field vanish (that is, $\mathbf{B}=\mathbf{0}$) at the superconducting spherical surface. It should be mentioned that Maxwell's equations do not restrict the tangential component of the magnetic field (see Ref. 18, for example). Therefore, the boundary condition employed in this comment is more appropriate for a superconducting surface. Furthermore, the approach utilized by Coffey involved the calculation of self-interaction energy from the infinite-series expansion of the magnetostatic potential followed by differentiation and summing up of the resulting series to obtain the closed form expression for the force component. In contrast, our approach here utilized an exact closed form solution for the magnetostatic potential in the vector-derivative equation (15) and computed the force component. The latter technique involved the operations of function evaluation, differentiation, and the dot product. Since formula (4) yields the magnetostatic potential in a closed form, one may use the resulting potential in (15) to calculate the required force in a systematic and straightforward fashion, instead of using the self-interaction energy approach. Indeed, Eq. (15) is easy to use for the calculation of the force due to an arbitrarily oriented magnetic dipole positioned in front of a superconducting sphere.

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