Comment on “London model for the levitation force between a horizontally oriented point magnetic dipole and superconducting sphere”

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In a recent paper the magnetostatic boundary-value problem for a magnetic dipole with transverse direction in the presence of a superconducting sphere was solved in both cases when the London penetration depth is zero and finite. It was concluded that the levitation force on the transverse magnetic dipole is exactly half that for a magnetic dipole with radial direction. We show that this conclusion is incorrect in either case. In the former case it is due to an incorrect boundary condition. In the latter case it is caused by calculational errors. Corrected results are presented. The distribution of supercurrent and the associated magnetic moment are also calculated.

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In a recent paper Coffey solved the magnetostatic boundary-value problem for a magnetic dipole with transverse direction in the presence of a superconducting sphere in both cases when the London penetration depth is zero and finite.[1] The result for the former case was also published in a separate paper.[2] The latter case involves some mathematical difficulty and is interesting. From these studies it was concluded that the levitation force on the transverse magnetic dipole is exactly half that for a magnetic dipole with radial direction. Unfortunately, this conclusion appears to be incorrect in either case. In the former case it is due to an incorrect boundary condition employed. In the latter case it is caused by calculational errors. Because the conclusion is impressive it deserves some clarification. In addition to discussions of the errors, the corrected results are presented here. We also calculate the distribution of supercurrent and the associated magnetic moment.

For the convenience of comparison, we will use similar notations as in Ref. 1. We use both the rectangular coordinates (x,y,z) and the spherical ones (r,θ,ϕ). The unit vectors of these coordinate systems are denoted by $(\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z)$ and $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_\phi)$, respectively. The position vector is denoted by $\boldsymbol{r}$. As in Ref. 1 we use MKS units.

Consider a superconducting sphere with radius b, whose center is located at the origin of the coordinate system. There is a point magnetic dipole located at the position $\boldsymbol{d}=d\boldsymbol{e}_z$ where $d>b$ (d is denoted by a in Ref. 1), the magnetic dipole moment being $\boldsymbol{m}_0=\boldsymbol{e}_x m_0\cos\phi_0+\boldsymbol{e}_y m_0\sin\phi_0$. Here we are considering a somewhat more general case; when $\phi_0=0$ it reduces to the case in Ref. 1. (m_0 is denoted by m in Ref. 1. We change the notation to avoid confusion with the angular eigenvalue in the spherical harmonics.) The problem is to find the magnetic induction in the whole space.

The magnetic induction outside the sphere is $\boldsymbol{B}=\boldsymbol{B}_1+\boldsymbol{B}_2$, where $\boldsymbol{B}_1$ is the field of $\boldsymbol{m}_0$ in free space, and $\boldsymbol{B}_2$ is the induced field produced by the supercurrent in the sphere. $\boldsymbol{B}_1$ can be described by a scalar potential $\boldsymbol{B}_1=-\mu_0\nabla\Phi_1$, where

$$\Phi_1(\boldsymbol{r})=\frac{\boldsymbol{m}_0\cdot(\boldsymbol{r}-\boldsymbol{d})}{4\pi|\boldsymbol{r}-\boldsymbol{d}|^3}=\frac{m_0 r\sin\theta\cos(\phi-\phi_0)}{4\pi(r^2+d^2-2rd\cos\theta)^{3/2}}. \quad (1)$$

The Maxwell equation for $\boldsymbol{B}_2$ is obviously

$$\nabla\cdot\boldsymbol{B}_2=0,\quad \nabla\times\boldsymbol{B}_2=0,\quad r>b. \quad (2)$$

Therefore, $\boldsymbol{B}_2$ can also be described by a scalar potential $\boldsymbol{B}_2=-\mu_0\nabla\Phi_2$, where Φ_2 satisfies the Laplace equation $\nabla^2\Phi_2=0$, and thus can be expanded as

$$\Phi_2(\boldsymbol{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{A_{lm}^{>}}{r^{l+1}}Y_{lm}(\theta,\phi),\quad r>b, \quad (3)$$

where $A_{lm}^{>}$ are constants to be determined by the boundary condition. The magnetic induction outside the sphere is then $\boldsymbol{B}=-\mu_0\nabla\Phi$ where $\Phi=\Phi_1+\Phi_2$.

The magnetic induction $\boldsymbol{B}_3$ inside the sphere and the boundary condition at the surface $r=b$ depend on the model, so the two cases when the London penetration depth is zero and finite should be treated separately.

First consider the case where the London penetration depth is zero. This is a fairly good approximation for macroscopic problems. This case is studied in Sec. II of Ref. 1. In this case the superconducting sphere behaves as a perfect diamagnet and the magnetic induction inside the sphere is

$$\boldsymbol{B}_3(\boldsymbol{r})=0,\quad r<b. \quad (4)$$

Since the normal component of the magnetic induction is always continuous, the boundary condition in this case is

$$B_r|_{r=b}=0. \quad (5)$$

To determine the coefficients $A_{lm}^{>}$, we expand Φ_1 in terms of the spherical harmonics. As in Sec. III of Ref. 1, one can actually consider a more general magnetic source which can be described in the region outside the sphere but near its surface by a scalar potential of the form

$$\Phi_1(\boldsymbol{r})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}b_{lm}r^lY_{lm}(\theta,\phi),\quad r\gtrsim b, \quad (6)$$

where b_{lm} are known coefficients. The Φ_1 in Eq. (1) can indeed be expanded in such a form with

$$b_{l,\pm 1} = \mp \frac{m_0}{8\pi d^{l+2}}\sqrt{\frac{4\pi l(l+1)}{2l+1}}e^{\mp i\phi_0}, \quad l \geq 1, \tag{7}$$

and all other coefficients vanishing. (In Sec. II of Ref. 1 only terms with $m=1$ were included and the real part of the field was taken at the end. This is equivalent.) With the boundary condition (5) it is easy to find that

$$A^>_{lm} = \frac{l}{l+1}b^{2l+1}b_{lm}. \tag{8}$$

This is what was obtained in Sec. III of Ref. 1 [see Eq. (30) in Ref. 1 where $A^>_{lm}$ is denoted by a_{lm}]. However, the results for the case of a magnetic dipole in Sec. II of Ref. 1 was not obtained through these coefficients. (See the discussions below.) Substituting these results into Eq. (3) we obtain

$$\Phi_2(\boldsymbol{r}) = \frac{m_0}{4\pi}\sum_{l=1}^{\infty}\frac{lb^{2l+1}}{(l+1)d^{l+2}}\frac{1}{r^{l+1}}P_l^1(\cos\theta)\cos(\phi-\phi_0) \tag{9}$$

for the case of the magnetic dipole described above. We use the Ferrer definition of the associated Legendre functions $P_l^m(x)=(1-x^2)^{m/2}P_l^{(m)}(x)$, while in Ref. 1 the Hobson definition is used, which has an additional factor $(-)^m$, but the spherical harmonics are the same.[3] Note that Φ_1 is written in the form of Eq. (6) for the convenience of more general discussions. If it is written as $\Phi_1(\boldsymbol{r}) = (m_0/4\pi)\sum_{l=1}^{\infty}(r^l/d^{l+2})P_l^1(\cos\theta)\cos(\phi-\phi_0)$, then the validity of the result (9) seems more obvious.

Now let us examine the treatment on the problem in Sec. II of Ref. 1. Instead of Eq. (5), the boundary condition $\boldsymbol{B}|_{r=b}=\boldsymbol{0}$ is used. [See Eq. (11) in Ref. 1. It is not a printing error since the subsequent calculations are based on it.] This is incorrect physically. If the tangential components of $\boldsymbol{B}$ vanish at $r=b$, then there should be no surface current on the surface of the sphere. Because $\boldsymbol{B}_3=\boldsymbol{0}$, there is no volume current inside the sphere either. Then what would be responsible for the cancellation of $\boldsymbol{B}_1$ inside the sphere? From the above results it is easy to see that once B_r vanishes at $r=b$, then the other components B_θ and B_ϕ do not. Then how can the boundary condition $\boldsymbol{B}|_{r=b}=\boldsymbol{0}$ be satisfied? We see from their Eq. (10) that the three components of $\boldsymbol{B}_2$ are given separately with independent coefficients. With the subsequent results for the coefficients given in their Eq. (12), the boundary condition $\boldsymbol{B}|_{r=b}=\boldsymbol{0}$ is indeed satisfied, but the field $\boldsymbol{B}_2$ so obtained is not a solution of the Maxwell equation (2). There is still a mathematical error in the process. In the expression for $\boldsymbol{B}_2$ given in their Eq. (10), there are four unknown coefficients (actually four sequences: A_l, C_l, D_l, and E_l). To solve for four coefficients from three equations, the two sets $\{Y_{l1}(\theta,\phi)\}_{l=1}^{\infty}$ and $\{\cos\theta Y_{l1}(\theta,\phi)\}_{l=1}^{\infty}$ were treated as linearly independent, but they are actually not.

In Sec. III of Ref. 1, the more general case with Φ_1 given by Eq. (6) was considered. In this case the result (8) was obtained. This is correct. However, it was stated that this is obtained by using the boundary condition $\boldsymbol{B}|_{r=b}=\boldsymbol{0}$. If so, then how can the conditions $B_\theta|_{r=b}=0$ and $B_\phi|_{r=b}=0$ be satisfied? This was not considered. The problem in Sec. III is essentially the same as that in Sec. II (just more complicated), but the treatment in the two sections appears to be inconsistent.

Now let us recalculate the levitation force on the magnetic dipole. We first calculate the spherical components of $\boldsymbol{B}_2$ and then combine them to give the rectangular ones. It turns out that

$$\boldsymbol{B}_2(\boldsymbol{d}) = -\frac{\mu_0\boldsymbol{m}_0}{8\pi}\frac{b}{d^4}\sum_{l=1}^{\infty}l^2\left(\frac{b^2}{d^2}\right)^l. \tag{10}$$

The self-interaction energy is then

$$U = -\frac{1}{2}\boldsymbol{m}_0\cdot\boldsymbol{B}_2(\boldsymbol{d}) = \frac{\mu_0 m_0^2}{16\pi}\frac{b}{d^4}\sum_{l=1}^{\infty}l^2\left(\frac{b^2}{d^2}\right)^l. \tag{11}$$

Compared with the corresponding result in Ref. 1, the factor l^2 in the summation above is $l(l+1)$ in their Eq. (16). Working out the summation, we have

$$U = \frac{\mu_0 m_0^2}{16\pi}\frac{b^3(d^2+b^2)}{d^2(d^2-b^2)^3}, \tag{12}$$

and the levitation force on the magnetic dipole is

$$F_z = -\frac{\partial U}{\partial d} = \frac{3\mu_0 m_0^2}{4\pi}\left(\frac{b^3 d}{(d^2-b^2)^4} - \frac{b^3(3d^2-b^2)}{6d^3(d^2-b^2)^3}\right). \tag{13}$$

In a recent paper[4] we calculated the levitation force by the standard formula in classical electrodynamics[5] that $\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{m}_0\cdot\boldsymbol{B}_2)|_{\boldsymbol{r}=\boldsymbol{d}}$ (note that $\boldsymbol{r}$ is replaced by $\boldsymbol{d}$ only after the differentiation is carried out) and obtained $\boldsymbol{F}=F_z\boldsymbol{e}_z$ where F_z is given above. This is a more general result since it also gives $F_x=F_y=0$. It confirms the above result. The first term in the above equation is half the value of the levitation force on a magnetic dipole with radial direction, but we still have the second term. The conclusion in Ref. 1 that the levitation force on the transverse magnetic dipole is half that for one with radial direction is thus incorrect. It holds approximately only when $d-b \ll b$ such that the second term in the above result is negligible compared with the first one. In this limit the surface of the sphere can be approximately regarded as an infinite plane for which the conclusion holds exactly.

Now we calculate the supercurrent and the associated magnetic moment. There is no volume current inside the sphere. The surface current density is

$$\boldsymbol{K}_s = \mu_0^{-1}\boldsymbol{e}_r\times\boldsymbol{B}|_{r=b} = -\boldsymbol{e}_r\times\boldsymbol{\nabla}\Phi|_{r=b}. \tag{14}$$

The associated magnetic moment is

$$\boldsymbol{m} = \frac{1}{2}\int_{r=b}\boldsymbol{r}\times\boldsymbol{K}_s r^2 d\Omega = \frac{1}{2}b^3\int_{r=b}\boldsymbol{e}_r\times\boldsymbol{K}_s d\Omega$$

$$= \frac{1}{2}b^3\int_{r=b}\boldsymbol{\nabla}\Phi\, d\Omega, \tag{15}$$

where $d\Omega$ is the element of solid angle, and the boundary condition (5) has been used. After some algebra, we obtain

$$m=\frac{1}{8}m_0\sum_{l=1}^{\infty}\frac{2l+1}{l+1}\frac{b^{l+2}}{d^{l+2}}\int_0^{\pi}\sin\theta\left(1+\cos^2\theta+\sin\theta\cos\theta\frac{d}{d\theta}\right)P_l'(\cos\theta)d\theta, \tag{16}$$

where $P_l'(\cos\theta)$ is the derivative of $P_l(\cos\theta)$ with respect to the argument $\cos\theta$. The integral can be shown to be $(8/3)\delta_{l1}$, so that

$$m=\frac{1}{2}\left(\frac{b}{d}\right)^3 m_0. \tag{17}$$

According to the image method,[4] the image of the currently considered magnetic dipole contains two parts: a point magnetic dipole with dipole moment $(b/d)^3 m_0$ located at the image point $(b^2/d)e_z$, and a continuous distribution of magnetic dipoles on the straight line from the origin to the image point, the dipole moment from ue_z to $(u+du)e_z$ being $-(m_0/bd)u\,du$. The total dipole moment of the latter part is $-(m_0/bd)\int_0^{b^2/d}u\,du=-\frac{1}{2}(b/d)^3 m_0$, so the total magnetic dipole moment of all images is $\frac{1}{2}(b/d)^3 m_0$. This is the same as obtained above.

Next we consider the case where the London penetration depth is finite. In this case the magnetic induction B_3 and the supercurrent density J_s inside the sphere satisfy the Maxwell-London equation

$$\nabla\cdot B_3=0,\quad \nabla\times B_3=\mu_0 J_s, \tag{18a}$$

$$\nabla\cdot J_s=0,\quad \nabla\times J_s=-\frac{k^2}{\mu_0}B_3, \tag{18b}$$

where k is a phenomenological parameter with $1/k$ (denoted by λ in Ref. 1) being the London penetration depth. These are similar to the Maxwell equations for monochromatic fields in free space. The approach[5] to the solutions of those Maxwell equations can be slightly modified to derive the following solution of the above equations:

$$B_3(r)=\mu_0\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\{-iA_{lm}^{<}\nabla\times[z_l(r)LY_{lm}(\theta,\phi)]+B_{lm}^{<}z_l(r)LY_{lm}(\theta,\phi)\}, \tag{19a}$$

$$J_s(r)=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\{ik^2A_{lm}^{<}z_l(r)LY_{lm}(\theta,\phi)+B_{lm}^{<}\nabla\times[z_l(r)LY_{lm}(\theta,\phi)]\}, \tag{19b}$$

where $A_{lm}^{<}$ and $B_{lm}^{<}$ are constants, $L=-ir\times\nabla$ is the angular momentum operator, and $z_l(r)=I_{l+1/2}(kr)/\sqrt{kr}$ where $I_{l+1/2}(kr)$ are Bessel functions of imaginary argument.[3,6] [$z_l(r)$ is denoted as $z_l(r/\lambda)$ in Ref. 1.] The boundary condition in this case is

$$B|_{r=b}=B_3|_{r=b}. \tag{20}$$

The field outside the sphere is still given by Eqs. (3) and (6).

From Eq. (19b) we have $J_{sr}(r)=i\Sigma_{l=1}^{\infty}\Sigma_{m=-l}^{l}l(l+1)B_{lm}^{<}[z_l(r)/r]Y_{lm}(\theta,\phi)$. We assume that there is no electric current when $r\gtrsim b$; then in that region we have $\nabla\times B=0$. Using the second equation in Eq. (18a) (Ampere's law) and the boundary condition (20) we have $J_{sr}|_{r=b}=0$. This leads to $B_{lm}^{<}=0$ [so that $J_{sr}(r)=0$]. Therefore the second part in Eq. (19) vanishes, and the spherical components of B_3 read

$$B_{3r}(r)=\mu_0\sum_{l=1}^{\infty}\sum_{m=-l}^{l}l(l+1)A_{lm}^{<}\frac{z_l(r)}{r}Y_{lm}(\theta,\phi), \tag{21a}$$

$$B_{3\theta}(r)=\mu_0\sum_{l=1}^{\infty}\sum_{m=-l}^{l}A_{lm}^{<}\frac{[rz_l(r)]'}{r}\partial_\theta Y_{lm}(\theta,\phi), \tag{21b}$$

$$B_{3\phi}(r)=\mu_0\sum_{l=1}^{\infty}\sum_{m=-l}^{l}imA_{lm}^{<}\frac{[rz_l(r)]'}{r}\frac{1}{\sin\theta}Y_{lm}(\theta,\phi), \tag{21c}$$

where the prime indicates the derivative with respect to r. These solutions were written down directly in Ref. 1. [See their Eqs. (42) and (60). There is a superfluous factor $1/r$ in Eq. (60c).] It is inconvenient to verify that these are solutions of Eq. (18). On the other hand, it is much easier to do that for the form in Eq. (19). Since $B_{lm}^{<}=0$, Eq. (19b) also gives the components of J_s immediately. With the boundary condition (20), the coefficients are found to be

$$A_{lm}^{<}=-\frac{2l+1}{l+1}\frac{b^l}{\sqrt{kb}I_{l-1/2}(kb)}b_{lm},\quad l=1,2,\ldots,\quad m=0,\pm1,\ldots,\pm l, \tag{22a}$$

$$A_{lm}^{>}=\frac{l}{l+1}\frac{I_{l+3/2}(kb)}{I_{l-1/2}(kb)}b^{2l+1}b_{lm},\quad l=0,1,\ldots,\quad m=0,\pm1,\ldots,\pm l. \tag{22b}$$

This is equivalent to the result (61) and (62) in Sec. V of Ref. 1, where $A_{lm}^{>}$ is denoted by a_{lm}. For the special case of the magnetic dipole (studied separately in Sec. IV of Ref. 1), this together with Eq. (7) is equivalent to the result given in their Eq. (45). (Note that B_2 was written in different forms in their Sec. IV and Sec. V; the equivalence is not obvious.) Unfortunately, when calculating the self-interaction energy, an error occurred and the result in their Eq. (50) is incorrect, where $l(l+1)$ should be replaced by l^2. This renders the subsequent results and conclusion incorrect.

Let us recalculate the levitation force. For the special case of the magnetic dipole, we have

$$\Phi_2(r)=\frac{m_0}{4\pi}\sum_{l=1}^{\infty}\frac{lb^{2l+1}}{(l+1)d^{l+2}}\frac{I_{l+3/2}(kb)}{I_{l-1/2}(kb)}\frac{1}{r^{l+1}}P_l^1(\cos\theta)\cos(\phi-\phi_0). \tag{23}$$

The magnetic induction at the position d and the self-interaction energy are found to be

$$B_2(d)=-\frac{\mu_0 m_0}{8\pi}\frac{b}{d^4}\sum_{l=1}^{\infty}l^2\left(\frac{b^2}{d^2}\right)^l\frac{I_{l+3/2}(kb)}{I_{l-1/2}(kb)} \tag{24}$$

and

$$U = -\frac{1}{2}\boldsymbol{m}_0 \cdot \boldsymbol{B}_2(\boldsymbol{d}) = \frac{\mu_0 m_0^2}{16\pi}\frac{b}{d^4}\sum_{l=1}^{\infty} l^2\left(\frac{b^2}{d^2}\right)^l \frac{I_{l+3/2}(kb)}{I_{l-1/2}(kb)}. \quad (25)$$

It is obvious that when $k\to\infty$ all results reduce to the above ones for the case of zero penetration depth as expected. For comparison we write down the self-interaction energy for a magnetic dipole with radial direction[7]

$$\tilde{U} = \frac{\mu_0 m_0^2}{8\pi}\frac{b}{d^4}\sum_{l=1}^{\infty} l(l+1)\left(\frac{b^2}{d^2}\right)^l \frac{I_{l+3/2}(kb)}{I_{l-1/2}(kb)}. \quad (26)$$

It is rather clear that there is no simple relation between U and $\tilde{U}$, or between the corresponding levitation forces. Therefore the conclusion in Ref. 1 that the levitation force on the transverse magnetic dipole is half that for the radial one is again incorrect when the London penetration depth is finite. This is just as expected since it is already not true in the limit $k\to\infty$.

Using the functional relation[3,6] $I_{l+3/2}(x)/I_{l-1/2}(x)=1-(2l+1)I_{l+1/2}(x)/xI_{l-1/2}(x)$, we can recast U in the form

$$U = \frac{\mu_0 m_0^2}{16\pi}\frac{b^3(d^2+b^2)}{d^2(d^2-b^2)^3} - \frac{\mu_0 m_0^2}{16\pi}\frac{b}{d^4}\sum_{l=1}^{\infty} l^2(2l+1) \times\left(\frac{b^2}{d^2}\right)^l \frac{I_{l+1/2}(kb)}{kbI_{l-1/2}(kb)}, \quad (27)$$

where the first term is the result (12), and the second term is a correction due to the finite penetration depth, which vanishes in the limit $k\to\infty$. Similarly, the levitation force can be put in the following form which shows that the correction is negative:

$$F_z = \frac{3\mu_0 m_0^2}{4\pi}\left(\frac{b^3 d}{(d^2-b^2)^4} - \frac{b^3(3d^2-b^2)}{6d^3(d^2-b^2)^3}\right) - \frac{\mu_0 m_0^2}{8\pi}\frac{b}{d^5}\sum_{l=1}^{\infty} l^2(l+2)(2l+1)\left(\frac{b^2}{d^2}\right)^l \frac{I_{l+1/2}(kb)}{kbI_{l-1/2}(kb)}. \quad (28)$$

As before we can also calculate the levitation force by the formula $\boldsymbol{F}=\boldsymbol{\nabla}(\boldsymbol{m}_0\cdot\boldsymbol{B}_2)|_{\boldsymbol{r}=\boldsymbol{d}}$. After much algebra, we obtain $\boldsymbol{F}=F_z\boldsymbol{e}_z$ where F_z is given above. This confirms the above result and also gives $F_x=F_y=0$.

Now we calculate the supercurrent and the associated magnetic moment. In this case there is no surface current. The volume current density inside the sphere can be calculated according to Eq. (19b) and the subsequent result for the coefficients. The nonvanishing components are

$$J_{s\theta}(\boldsymbol{r}) = -\frac{m_0 k^2}{4\pi}\sum_{l=1}^{\infty}\frac{2l+1}{l+1}\frac{b^l}{d^{l+2}}\frac{z_l(r)}{\sqrt{kb}I_{l-1/2}(kb)} \times P_l'(\cos\theta)\sin(\phi-\phi_0), \quad (29a)$$

$$J_{s\phi}(\boldsymbol{r}) = -\frac{m_0 k^2}{4\pi}\sum_{l=1}^{\infty}\frac{2l+1}{l+1}\frac{b^l}{d^{l+2}}\frac{z_l(r)}{\sqrt{kb}I_{l-1/2}(kb)} \times\frac{d}{d\theta}[\sin\theta P_l'(\cos\theta)]\cos(\phi-\phi_0). \quad (29b)$$

The magnetic moment associated with this current distribution is $\boldsymbol{m}=\frac{1}{2}\int_{r\le b}\boldsymbol{r}\times\boldsymbol{J}_s d\boldsymbol{r}$. After some algebra, we obtain

$$\boldsymbol{m} = \frac{1}{8}k^2\boldsymbol{m}_0\sum_{l=1}^{\infty}\frac{2l+1}{l+1}\frac{b^l}{d^{l+2}}\frac{\int_0^b r^3 z_l(r)dr}{\sqrt{kb}I_{l-1/2}(kb)}\int_0^{\pi}\sin\theta\left(1+\cos^2\theta + \sin\theta\cos\theta\frac{d}{d\theta}\right)P_l'(\cos\theta)d\theta. \quad (30)$$

The integral over θ is $(8/3)\delta_{l1}$ as before, so we are left with only one term. Working out the integral over r, we arrive at

$$\boldsymbol{m} = \frac{1}{2}\left(\frac{b}{d}\right)^3\left[1+\frac{3}{(kb)^2}-\frac{3\coth(kb)}{kb}\right]\boldsymbol{m}_0. \quad (31)$$

When $k\to\infty$, this reduces to the result (17) as expected. So the function of k in the square brackets is a correction factor due to the finite penetration depth. It is less than 1.

By the way we briefly discuss the corresponding result for a radial magnetic dipole, since there exists a mistake in the literature.[7] The current density in this case is

$$\boldsymbol{J}_s(\boldsymbol{r}) = \boldsymbol{e}_\phi J(r,\theta) = -\boldsymbol{e}_\phi\frac{m_0 k^2}{4\pi}\sum_{l=1}^{\infty}(2l+1)\frac{b^l}{d^{l+2}}\frac{z_l(r)}{\sqrt{kb}I_{l-1/2}(kb)}P_l^1(\cos\theta). \quad (32)$$

Therefore, $\boldsymbol{r}\times\boldsymbol{J}_s=-rJ(r,\theta)\boldsymbol{e}_\theta$. It was concluded in Ref. 7 that $\boldsymbol{m}=\boldsymbol{0}$, and it was pointed out that "mathematically, this result follows from the integral $\int_{-1}^{1}P_l^1(x)dx=0$." [See Eq. (30) and the following discussions in that paper. There seems to be a sign error in that equation.] There exist two errors here. First, the result $\int_{-1}^{1}P_l^1(x)dx=0$ is not true. For example, $P_1^1(x)=\sqrt{1-x^2}$ and its integral is not zero. Second, even if the integral were zero, the conclusion would still be incorrect because $\boldsymbol{e}_\theta$ is a function of $\boldsymbol{r}$. Actually, by calculations similar to the above ones we obtain

$$\boldsymbol{m} = -\frac{1}{4}k^2\boldsymbol{m}_0\sum_{l=1}^{\infty}(2l+1)\frac{b^l}{d^{l+2}}\frac{\int_0^b r^3 z_l(r)dr}{\sqrt{kb}I_{l-1/2}(kb)} \times\int_0^{\pi}\sin^3\theta P_l'(\cos\theta)d\theta. \quad (33)$$

The integral over θ is found to be $(4/3)\delta_{l1}$, so we are left with only one term. The integral over r is the same as above; thus we obtain

$$\boldsymbol{m} = -\left(\frac{b}{d}\right)^3\left[1 + \frac{3}{(kb)^2} - \frac{3\coth(kb)}{kb}\right]\boldsymbol{m}_0. \tag{34}$$

When $k \to \infty$, this reduces to $\boldsymbol{m} = -(b/d)^3\boldsymbol{m}_0$. This is equal to the dipole moment of the image dipole obtained in the image method.[4,7,8]

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