Ground state, quasihole, a pair of quasihole wave functions, and instability in bilayer quantum Hall systems

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The bilayer quantum Hall system (BLQH) differs from its single layer counterparts (SLQH) by its symmetry breaking ground state and associated neutral gapless mode in the pseudospin sector. Due to the gapless mode, qualitatively good ground-state and low energy excited-state wave functions at any finite distance are still unknown. We investigate this important open problem by the composite boson (CB) theory developed by one of the authors to study BLQH systematically. We derive the ground state, quasihole, and a pair of quasihole wave functions from the CB theory and its dual action. We find that the ground-state wave function is the product of two parts, one in the charge sector which is the well known Halperin (111) wave function and the other in the spin sector which is nontrivial at any finite d due to the gapless mode. So the total ground-state wave function differs from the well known (111) wave function at any finite d. In addition to commonly known multiplicative factors, the quasihole and a pair of quasihole wave functions also contain nontrivial normalization factors multiplying the correct ground state wave function. We expect that the quasihole and pair wave function not only has logarithmically divergent energy and well localized charge distribution, but also correct interlayer correlations. All the distance dependencies in all the wave functions are encoded in the spin part of the ground-state wave function. The instability encoded in the spin part of the ground-state wave function leads to the pseudo-spin-density wave proposed by one of the authors previously. Some subtleties related to the Lowest Landau Level (LLL) projection and shortcomings of the CB theory are also noted.

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I. INTRODUCTION

The wave function approach has been very successfully applied to study single layer quantum Hall (SLQH) systems at Laughlin series $\nu=\frac{1}{2k+1}$[1] and Jain series at $\nu=\frac{p}{2sp\pm1}$.[2] One of main reasons for the success of the wave function approach in SLQH is that there is a gap in the bulk, suitable wave functions[1,2] can describe both the ground-state and low energy excitations quite accurately. Its accuracy can be checked easily by exact diagonalization in a finite size system whose size need only go beyond a few magnetic lengths. Spherical geometry can be used to get rid of edge state effects quite efficiently. In general, trial wave function approach is very robust to study SLQH and multi component systems as long as there is a gap in the bulk. The gap protects the many properties of the system such as charge density distributions and energies from being sensitive to some subtle details of wave functions.

However, the situation could be completely different in the spin-polarized bilayer quantum Hall system (BLQH) at total filling factor $\nu_T=1$. This system has been under enormous experimental and theoretical investigations over the last decade.[3] When the interlayer separation d is sufficiently large, the bilayer system decouples into two separate compressible $\nu=1/2$ layers.[4] However, when d is smaller than a critical distance d_{c1}, even in the absence of interlayer tunneling, the system undergoes a quantum phase transition into a novel spontaneous interlayer coherent incompressible phase which is an excitonic superfluid state (ESF) in the pseudospin channel.[5–7] In Ref. 10, Halperin proposed the (111) wave function to describe the ground state of the ESF state. Starting from the (111) wave function, using various methods, several authors[11] discovered a neutral gapless mode (NGM) with linear dispersion relation $\omega\sim vk$ and that there is a finite temperature Kosterlitz-Thouless (KT) phase transition associated with this NGM. By treating the two layer indices as two pseudo-spin indices, Girvin, Macdonald, and collaborators mapped the bilayer system into an easy plane quantum ferromagnet (EPQFM)[3,12] (which is equivalent to the ESF) and explored many rich and interesting physical phenomena in this system.

As pointed out in Ref. 12, the (111) wave function may not be qualitatively good at finite d, because (111) is a broken symmetry state in a direction in XY plane of isotropic ferromagnet at $d=0$ instead of a easy-plane ferromagnet at finite d. The NGM is a hallmark of the interlayer coherent quantum Hall state. Its existence is expected to dramatically alter the properties of the wave functions of the ground state, quasihole and quasiparticle. In Ref. 13, Jeon and one of the authors studied properties of essentially all the known trial wave functions of ground state and excitations in bilayer quantum Hall systems at the total filling factor $\nu_T=1$. The results indicated that qualitatively good trial wave functions for the ground state and the excitations of the interlayer coherent bilayer quantum Hall system at finite d are still not available and searching for them remains an important open problem. Specifically, they investigated the properties of the quasihole wave function, meron wave function and a pair of meron wave function built on the (111) state which have superscripts "prime" in this paper:

$$\Psi'_{qh}=\left(\prod_{i}^{N_1} z_i\right)\Psi_{111},$$

$$\Psi'_{\text{meron}} = \left(\prod_{i}^{N_1} \frac{z_i}{|z_i|} \right) \Psi_{111},$$

$$\Psi'_{\text{pair}} = \prod_{i}^{N_1} (z_i - z_0) \prod_{i}^{N_2} (w_i - w_0) \Psi_{111}, \tag{1}$$

where Ψ_{111} is the Halperin's (111) wave function:

$$\Psi_{111}(z,w) = \prod_{i<j}^{N_1} (z_i - z_j) \prod_{i<j}^{N_2} (w_i - w_j) \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} (z_i - w_j) \times \exp\left(- \frac{1}{4l_0^2} \sum_i |z_i|^2 + |w_i|^2 \right), \tag{2}$$

where $N_1 = N_2 = N$ in the balanced case and z and w are the coordinates in layer 1 and layer 2, respectively. In the following, we suppress the exponential factor.

These quasihole and meron wave functions differ only by "normalization factors." As shown in Ref. 13, the normalization factor $|z_i|$ is accurate only at long distance limit $|z_i| \to \infty$ limit. Near the origin, the "meron" and the "quasihole" have similar behaviors. Normalization factors have been shown not to be important in single layer quantum Hall systems. However, as shown in Ref. 13, they make a dramatic difference in the BLQH. Although the smallest meron has a localized charge 1/2 and logarithmically divergent energy, the charge of the quasihole excitation extends over the whole system and its energy also diverges linearly as the area of the system size. This indicates that the quasihole wave function is not a good trial wave function for any low energy excitations. The meron wave function is not a good trial wave function either, because it ignores the strong interlayer correlations.[13] It was found the energy of the possible wave function of a pair of merons in Eq. (1) increases quadratically $\sim |z_0 - w_0|^2$ instead of logarithmically as the separation of the pair increases. All the results achieved in Ref. 13 indicate that qualitatively good trial wave functions in the interlayer coherent bilayer quantum Hall system at finite d, both for ground state and excitations, are still unknown and searching for them remains an important open problem. Therefore the wave function approach to BLQH is much more difficult and far less powerful in BLQH than in SLQH. Fortunately, effective theory approaches such as the EPQFM approach[12,3] and composite boson theory approach[11,12,14–16] circumvent this difficulty associated with the unknown wave function at any finite d and are very effective to bring out most of the interesting phenomena in the pseudospin sector in this system. In fact, all these effective theories start from the insights gained from Halperin's (111) wave function which is exact at $d=0$.

In a series of papers,[14–16] one of the authors developed a systematic composite boson approach to study balanced and imbalanced bilayer quantum Hall systems in rather details. The theory puts spin and charge degree freedoms in the same footing, explicitly bring out the spin-charge connection and classify all the possible excitations in a systematic way. Then He pushed the theory further to understand novel phases and quantum phase transitions as the distance between the two layers is changed. He found that starting from the well studied excitonic superfluid (ESF) state, as distance increases, the instability driven by magnetoroton minimum collapsing at a finite wavevector in the pseudospin channel leads to the formation of a pseudo-spin-density wave (PSDW) at some intermediate distances. He constructed a quantum Ginsburg-Landau theory to study the transition from the excitonic superfluid (ESF) to the PSDW and analyze in detail the properties of the PSDW. He showed that a square lattice is the favorite lattice and the correlated hopping of vacancies in the active and passive layers in the PSDW state leads to very large temperature-dependent drag observed in the experiment. In the presence of disorders, the properties of the PSDW are consistent with all the experimental observations[6,8] in the intermediate distances. Further experimental implications of the PSDW are given. Then he extended the composite boson theory to study slightly imbalanced BLQH. In the global U(1) symmetry breaking excitonic superfluid side, as the imbalance increases, the system supports continuously changing fractional charges. In the translational symmetry breaking excitonic solid side, there are two quantum phase transitions from the commensurate excitonic solid to an incommensurate excitonic solid and then to the excitonic superfluid state. These results explained the experimental observations in Ref. 9 very nicely. The author found that the theory can be easily extended to study some additional interesting phenomena in trilayer quantum Hall systems.[17] It was concluded in Ref. 16 that field theory approaches are much more powerful in BLQH than in SLQH.

Obviously, the CB theory circumvent this difficulty associated with the unknown wave function at any finite d and is used to achieve all these interesting and important results at two different distance regimes without knowing the precise wave functions for the ground state and excitations. It would be interesting to use the CB theory to address the important and outstanding problem avoided in Ref. 13 and in all the other pervious work that finding the good ground state and low energy excited wave function for BLQH at any finite d. In SLQH, the CB theory developed in Ref. 18 was used to rederive the already well known Laughlin wave functions for ground state and quasihole at $\nu = \frac{1}{2k+1}$. As said previously, the gap in the bulk protects the properties of the system such as charge density distributions and energies from being sensitive to some subtle details of wave functions. Here we are facing a more difficult and interesting task: to derive these unknown wave functions at finite d.

The rest of the paper is organized as follows. In Sec. II, in order to be self-contained, we review briefly the CB approach and its dual action developed in Ref. 16 which are needed to derive the wave functions in the following sections. In Sec. III, using the formalism presented in Sec. II, we derive the ground state wave function which is different from the (111) wave function at any finite d. In Sec. IV, using the dual action presented in Sec. II, we derive the quasihole wave function and compare it with the quasihole and meron wave functions built on the (111) wave function listed in Eq. (1). In Sec. V, we derive a pair of meron wave functions with charge 1 and compare it with the pair meron wave function built on (111) listed in Eq. (1). In Sec. VI, we look at the instability in the ground state wave function as distance approaches d_{c1}. Finally, we reach conclusions in Sec. VII. Some caveats related to the lowest Landau level (LLL) projection of the wave functions are also pointed out. We note that there is also an alternative approach in Ref. 19.

II. COMPOSITE BOSON APPROACH AND ITS DUAL ACTION IN BLQH

In this section, we briefly review the formalism developed in Ref. 16 which is needed to derive the wave functions in the following sections. Consider a bi layer system with N_1 (N_2) electrons in left (right) layer and with interlayer distance d in the presence of magnetic field $\vec{B}=\nabla\times\vec{A}$ (Fig. 1):

$$H = H_0 + H_{\rm int},$$

$$H_0 = \int d^2x c_\alpha^\dagger(\vec{x}) \frac{\left[-i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}(\vec{x})\right]^2}{2m} c_\alpha(\vec{x}),$$

$$H_{\rm int} = \frac{1}{2}\int d^2x d^2x'\, \delta\rho_\alpha(\vec{x}) V_{\alpha\beta}(\vec{x}-\vec{x}')\,\delta\rho_\beta(\vec{x}'), \qquad (3)$$

where electrons have *bare* mass m and carry charge $-e$, $c_\alpha,\alpha=1,2$ are electron operators in top and bottom layers, $\delta\rho_\alpha(\vec{x})=c_\alpha^\dagger(\vec{x})c_\alpha(\vec{x})-n_\alpha,\alpha=1,2$ are normal ordered electron densities on each layer. The intralayer interactions are $V_{11}=V_{22}=e^2/\epsilon r,$ while the interlayer interaction is $V_{12}=V_{21}=e^2/\epsilon\sqrt{r^2+d^2}$, where ϵ is the dielectric constant.

Performing a singular gauge transformation[14,16]

$$\phi_a(\vec{x}) = e^{i\int d^2x'\,\phi(\vec{x}-\vec{x}')\rho(\vec{x}')} c_a(\vec{x}), \qquad (4)$$

where $\phi(\vec{x}-\vec{x}')=\arg(\vec{x}-\vec{x}')$ is the angle between the vector $\vec{x}-\vec{x}'$ and the horizontal axis. $\rho(\vec{x})=c_1^\dagger(\vec{x})c_1(\vec{x})+c_2^\dagger(\vec{x})c_2(\vec{x})$ is the total density of the bilayer system. Note that this transformation treats both c_1 and c_2 on the same footing. This is reasonable only when the distance between the two layers is sufficiently small. It can be shown that $\phi_a(\vec{x})$ satisfies all the boson commutation relations. We can transform the Hamiltonian (3) into the Lagrangian in Coulomb gauge:

$$\mathcal{L} = \phi_a^\dagger(\partial_\tau - i a_0)\phi_a + \phi_a^\dagger(\vec{x})\frac{\left[-i\hbar\vec{\nabla}+\frac{e}{c}\vec{A}(\vec{x})-\hbar\vec{a}(\vec{x})\right]^2}{2m}\phi_a(\vec{x}) + \frac{1}{2}\int d^2x'\,\delta\rho(\vec{x})V_+(\vec{x}-\vec{x}')\,\delta\rho(\vec{x}') + \frac{1}{2}\int d^2x'\,\delta\rho_-(\vec{x}) \times V_-(\vec{x}-\vec{x}')\,\delta\rho_-(\vec{x}') - \frac{i}{2\pi}a_0(\nabla\times\vec{a}), \qquad (5)$$

where $V_\pm=\frac{V_{11}\pm V_{12}}{2}$ and $V_{11}=V_{22}=\frac{2\pi e^2}{\epsilon q}, V_{12}=\frac{2\pi e^2}{\epsilon q}e^{-qd}$. The Chern-Simon gauge field is $\vec{a}=\int d^2\vec{r}'\,\nabla\phi(\vec{x}-\vec{x}')\rho(\vec{x}') =\int d^2\vec{r}'\frac{\hat{z}\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^2}\rho(\vec{x}')$.

In the Coulomb gauge, integrating out a_0 leads to the constraint $\nabla\times\vec{a}=2\pi\phi_a^\dagger\phi_a$. Note that if setting $V_-=0$, then the above equation is identical to a single layer with spin in the absence of Zeeman term, so the Lagrangian has a SU(2) pseudospin symmetry. The V_- term breaks the SU(2) symmetry into U(1) symmetry. In the BLQH at finite d, $V_->0$, so the system is in the easy-plane limit.

We can write the two bosons in terms of magnitude and phase

$$\phi_a = \sqrt{\bar{\rho}_a+\delta\rho_a}\,e^{i\theta_a}. \qquad (6)$$

The boson commutation relations imply that $[\delta\rho_a(\vec{x}),\theta_b(\vec{x})]=i\hbar\,\delta_{ab}\delta(\vec{x}-\vec{x}')$. After absorbing the external gauge potential $\vec{A}$ into $\vec{a}$, we get the Lagrangian in the Coulomb gauge

$$\mathcal{L} = i\delta\rho^+\left(\frac{1}{2}\partial_\tau\theta^+ - a_0\right) + \frac{\bar{\rho}}{2m}\left[\frac{1}{2}\nabla\theta_+ + \frac{1}{2}(\nu_1-\nu_2)\nabla\theta_- - \vec{a}\right]^2 + \frac{1}{2}\delta\rho^+V_+(\vec{q})\,\delta\rho^+ - \frac{i}{2\pi}a_0(\nabla\times\vec{a}) + \frac{i}{2}\delta\rho^-\partial_\tau\theta^- + \frac{\bar{\rho}f}{2m}\left(\frac{1}{2}\nabla\theta_-\right)^2 + \frac{1}{2}\delta\rho^-V_-(\vec{q})\,\delta\rho^- - h_z\delta\rho^-, \qquad (7)$$

where $f=4\nu_1\nu_2$ which is equal to 1 at the balanced case and $h_z=V_-\bar{\rho}_-=V_-(\bar{\rho}_1-\bar{\rho}_2)$ plays a similar role as the Zeeman field.

Performing the duality transformation on Eq. (7) leads to the dual action in terms of the vortex degree of freedoms $J_\mu^{v\pm}=\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\mu\partial_\nu\theta_\pm=J_\mu^{v1}\pm J_\mu^{v2}$ and the corresponding dual gauge fields $b_\mu^\pm$:

$$\mathcal{L}_d = -i\pi b_\mu^+\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda^+ - iA_{s\mu}^+\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda^+ + i\pi b_\mu^+J_\mu^{v+} + \frac{m}{2\bar{\rho}f}(\partial_\alpha b_0^+ - \partial_0 b_\alpha^+)^2 + \frac{1}{2}(\nabla\times\vec{b}^+)V_+(\vec{q})(\nabla\times\vec{b}^+) - iA_{s\mu}^-\epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda^- + i\pi b_\mu^-J_\mu^{v-} - h_z(\nabla\times\vec{b}^-) + \frac{m}{2\bar{\rho}f}(\partial_\alpha b_0^- - \partial_0 b_\alpha^-)^2 + \frac{1}{2}(\nabla\times\vec{b}^-)V_-(\vec{q})(\nabla\times\vec{b}^-) - \frac{m}{\bar{\rho}f}(\nu_1-\nu_2)(\partial_\beta b_0^- - \partial_0 b_\beta^-)(\partial_\beta b_0^+ - \partial_0 b_\beta^+), \qquad (8)$$

where $A_{s\mu}^\pm=A_{s\mu}^1\pm A_{s\mu}^2$ are the two source fields. It is useful to stress that the dual CS term only appears in the charge sector.

For simplicity, we only consider the balanced case. Putting $\nu_1=\nu_2=1/2$ and $h_z=0$ into Eq. (7), we get the Lagrangian in the balanced case where the symmetry is enlarged to $U(1)_L\times U(1)_G\times Z_2$.

III. GROUND STATE WAVE FUNCTION

In this section, we derive the ground state wave function from the formalism reviewed in the last section. From Eq. (7), we can find the corresponding Hamiltonian in the charge sector

$$\mathcal{H}_c = \frac{1}{2}\sum_q \left[4\Pi_+(-\vec{q})\left(V_+(q) + \frac{4\pi^2\bar{\rho}}{m}\frac{1}{q^2}\right)\Pi_+(\vec{q}) + \frac{\bar{\rho}q^2}{4m}\theta_+(-\vec{q})\theta_+(\vec{q})\right], \quad (9)$$

where $\Pi_+(\vec{q})=\delta\rho^+/2$ and $[\theta_+(\vec{q}),\Pi_+(\vec{q}\,')]=i\hbar\,\delta(\vec{q}+\vec{q}\,')$.

Representing $\theta_+(\vec{q})$ by $-i\frac{\partial}{\partial\Pi_+(-\vec{q})}$, in the long wavelength limit, neglecting the Coulomb interaction $V_+(q)\sim 1/q$ which is less singular than $1/q^2$, we find the ground-state wave function in the charge sector:

$$\Psi^b_{c0}[\Pi_+(\vec{q})] = \exp\left[-\frac{1}{2}\sum_q \frac{2\pi}{q^2}\delta\rho_+(-\vec{q})\delta\rho_+(\vec{q})\right]. \quad (10)$$

In the following, we use $\vec{x}$ to stand for the complex coordinate $x+iy$. Using $\delta\rho_+(x)=\Sigma\delta(x-z_i)+\Sigma\delta(x-w_i)-\bar{\rho}$ and transforming to the position space, it is simply the modulus of the (111) wave function in the balanced case $\Psi^b_{c0}=|\Psi_{111}|$. This is the wave function in the bosonic picture. In order to get the wave function in the original fermionic picture, we need to perform the inverse of the SGT in Eq. (4) on the bosonic wave function. In the first quantization form, the inverse is

$$U_0 = \prod_{i<j}^{N_1}\frac{(z_i-z_j)}{|z_i-z_j|}\prod_{i<j}^{N_2}\frac{(w_i-w_j)}{|w_i-w_j|}\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{(z_i-w_j)}{|z_i-w_j|}. \quad (11)$$

Performing the inverse transformation on the modulus leads to the (111) wave function in the fermionic coordinates

$$\Psi_{c0} = U_0\Psi^b_{c0} = \Psi_{111}(z,w). \quad (12)$$

In contrast to the SLQH, there is also an additional pseudospin sector in the BLQH which contains the most interesting physics. From Eq. (7), we can find the corresponding Hamiltonian in the spin sector is

$$\mathcal{H}_s = \frac{1}{2}\sum_q \left[4V_-\Pi_-(-\vec{q})\Pi_-(\vec{q}) + \frac{\rho_E q^2}{4}\theta_-(-\vec{q})\theta_-(\vec{q})\right], \quad (13)$$

where $\Pi_-(\vec{q})=\delta\rho^-/2$ and $[\theta_-(\vec{q}),\Pi_-(\vec{q}\,')]=i\hbar\,\delta(\vec{q}+\vec{q}\,')$; $\rho_E=\bar{\rho}/m$ is the spin stiffness.[21] At small q, $V_-(q)=a-bq+cq^2$,[21] where $a\sim d^2, b\sim d^2$ and c remains a constant at small distances.[12,16] It is important to stress that this form of $V_-(q)$ has the shape displayed in Fig. 1, it not only has a phonon part near $q=0$, also has a roton part near $q=q_0\sim 1/l_B$, where l_B is the magnetic length.

Representing $\theta_-(\vec{q})$ by $-i\frac{\partial}{\partial\Pi_-(-\vec{q})}$, we find the ground-state wave function in the spin sector

$$\Psi_{s0}[\Pi_-(\vec{q})] = \exp\left[-\frac{1}{2}\sum_q \frac{\sqrt{V_-(q)/\rho_E}}{q}\delta\rho_-(-\vec{q})\delta\rho_-(\vec{q})\right]. \quad (14)$$

It is easy to see that the above equation make senses only when $V_-(q)$ is positive for all q.

At $d=0$, $a=b=0$, $V_-(q)=cq^2$, then Eq. (14) becomes

$$\Psi_{s0}[\Pi_-(\vec{q});d=0] = \exp\left[-\frac{1}{2}\sum_q \sqrt{c/\rho_E}\,\delta\rho_-(-\vec{q})\delta\rho_-(\vec{q})\right] = \text{const}. \quad (15)$$

At any finite distance, as long as $d<d_{c1}\sim l_B$ in Fig. 1, so the roton has a large gap, we can neglect the contributions from the roton part in Fig. 1 and only focus on the phonon contributions. In the long wavelength limit $q\ll q_0\sim 1/l_B$, $V_-(q)\to a\sim d^2$. Using $\delta\rho_-(x)=\Sigma\delta(x-z_i)-\Sigma\delta(x-w_i)$ and transforming to the coordinate space, it is

$$\Psi_{s0}(z,w) = \exp\left[-2\sqrt{1/\rho_E}\left(\sum_{i<j}\frac{d}{|z_i-z_j|} - \sum_{i,j}\frac{d}{|z_i-w_j|} + \sum_{i<j}\frac{d}{|w_i-w_j|}\right)\right]. \quad (16)$$

Obviously, the above equation only holds at small distance $d<d_{c1}$ and in the long distance limit $|z_i-w_j|\gg l_B$.[22] The total wave function is

$$\Psi_0(z,w) = \Psi_{111}(z,w)\Psi_{s0}(z,w). \quad (17)$$

It is easy to see that the total wave function coincides with the (111) wave function only in $d\to 0$ limit. At any finite d, it has an extra factor from the gapless spin sector $\Psi_{s0}(z,w)$. Note that this extra spin factor Eq. (16) is not in the LLL, this should not be too worrisome, because similar to the meron wave function listed in Eq. (1), modulus of the coordinates could appear in the long distance limit where Eq. (16) hold.

IV. QUASIHOLE WAVE FUNCTIONS

By inserting one static vortex at the origin in layer 1 by setting $J_0^{+v}=J_0^{-v}=\delta(x)$ or layer 2 by setting $J_0^{+v}=-J_0^{-v}=\delta(x)$, from the dual action Eq. (8), we will first try to derive the quasihole wave function and compare it with the known quasihole wave function and meron wave function written down in Ref. 13

In order to derive the quasihole wave function, we have to resort to the dual action Eq. (8) in the balanced case where the last term vanishes. Setting the two sources $A^{\pm}_{s\mu}=0$, in the Coulomb gauge $\nabla b^{\pm}_\alpha=0$, Eq. (8) becomes

$$\begin{aligned}\mathcal{L}_d = &-i2\pi b_0^+\epsilon_{\alpha\beta}\partial_\alpha b_\beta^+ + i\pi b_0^+J_0^{v+} + i\pi b_\alpha^+J_\alpha^{v+} + \frac{m}{2\bar{\rho}}(\partial_\alpha b_0^+)^2 \\ &+ \frac{m}{2\bar{\rho}}(\partial_0 b_\alpha^+)^2 + \frac{1}{2}(\nabla\times\vec{b}^+)V_+(\vec{q})(\nabla\times\vec{b}^+) \\ &+ i\pi b_0^-J_0^{v-} + i\pi b_\alpha^-J_\alpha^{v-} + \frac{m}{2\bar{\rho}}(\partial_\alpha b_0^-)^2 + \frac{m}{2\bar{\rho}}(\partial_0 b_\alpha^-)^2 \\ &+ \frac{1}{2}(\nabla\times\vec{b}^-)V_-(\vec{q})(\nabla\times\vec{b}^-). \end{aligned} \quad (18)$$

Note the absence of CS term in the spin sector.

We only consider static vortices, so $J_\alpha^{v+}=J_\alpha^{v-}=0$. Integrating out b_0^+ and b_0^- and transforming into the coordinate space lead to

$$\mathcal{L}_d = -\frac{1}{4\pi}\frac{\bar{\rho}}{2m}(\pi J_0^{v+} - 2\pi\epsilon_{\alpha,\beta}\partial_\alpha b_\beta^+)\ln|x-y|(\pi J_0^{v+} - 2\pi\epsilon_{\alpha,\beta}\partial_\alpha b_\beta^+) + \frac{m}{2\bar{\rho}}(\partial_0 b_\alpha^+)^2 + \frac{1}{2}(\nabla\times\vec{b}^+)V_+(\vec{q})(\nabla\times\vec{b}^+) + \frac{m}{2\bar{\rho}}(\partial_0 b_\alpha^-)^2 - \frac{1}{4\pi}\frac{\bar{\rho}}{m}(\pi J_0^{v-})\ln|x-y|(\pi J_0^{v-}) + \frac{1}{2}(\nabla\times\vec{b}^-)V_-(\vec{q})(\nabla\times\vec{b}^-) \quad (19)$$

The corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\frac{\bar{\rho}}{m}q^2\theta_+(-\vec{q})\theta_+(\vec{q}) + \frac{1}{2}\frac{\bar{\rho}}{m}[\pi J_0^{v+}(-\vec{q}) - 4\pi\Pi_+(-\vec{q})]\frac{1}{q^2}[\pi J_0^{v+}(\vec{q}) - 4\pi\Pi_+(\vec{q})] + 2\Pi_+(-\vec{q}) V_+(\vec{q})\Pi_+(\vec{q}) + \frac{1}{2}\frac{\bar{\rho}}{m}q^2\theta_-(-\vec{q})\theta_-(\vec{q}) + \frac{1}{2}\frac{\bar{\rho}}{m}[\pi J_0^{v-}(-\vec{q})] \times\frac{1}{q^2}[\pi J_0^{v-}(\vec{q})] + \frac{1}{2}\Pi_-(-\vec{q})V_-(\vec{q})\Pi_-(\vec{q}), \quad (20)$$

where $[\theta_\pm(\vec{q}),\Pi_\pm(\vec{q}\,')] = i\hbar\,\delta(\vec{q}+\vec{q}\,')$.

From the above Hamiltonian in the bosonic representation, we can see the charge sector and spin sector remain decoupled.[20] Due to the absence of the CS term in the spin sector, the inserted vortex only shifts the total density variable in the charge sector, but does not couple to the relative density in the spin sector, so the Hamiltonian in the spin sector remains the same as the ground state one Eq.(13), the corresponding wave function remains the same as the ground state one in the spin sector Eq. (14). All the effects of the inserted vortex are encoded in the charge sector. Again, neglecting the Coulomb interaction $V_+(q)$ in the long wavelength limit, we find that the wave function in the charge sector is

$$\Psi_{cqh}^b = \exp\left[\frac{1}{2}\sum_q\left(\frac{1}{2}J_0^{v+}(-\vec{q}) - \delta\rho_+(-\vec{q})\right)\left(-\frac{2\pi}{q^2}\right)\times\left(\frac{1}{2}J_0^{v+}(\vec{q}) - \delta\rho_+(\vec{q})\right)\right]. \quad (21)$$

Transforming to the coordinate space and setting $J_0^{v+}(x) = J_0^{v-}(x) = -\delta(x)$[20] lead to

$$\Psi_{cqh}^b = \exp\left\{\frac{1}{2}\int dxdy\left[\frac{1}{2}\delta(\vec{x}) + \left(\sum_i\delta(\vec{x}-z_i) + \delta(\vec{x}-w_i) - \bar{\rho}\right)\right]\ln|x-y|\left[\frac{1}{2}\delta(\vec{y}) + \left(\sum_i\delta(\vec{y}-z_i) + \delta(\vec{y}-w_i) - \bar{\rho}\right)\right]\right\}$$

$$= \prod_i^{N_1}|z_i|^{\frac{1}{2}}\prod_i^{N_2}|w_i|^{\frac{1}{2}}\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}|z_i-w_j|\prod_{i<j}^{N_1}|z_i-z_j|\prod_{i<j}^{N_2}|w_i-w_j|. \quad (22)$$

The SGT for the quasihole could be different from that for the ground state. If one inserts a vortex at the origin at the layer 1 in the boson Lagrangian (5), in order to recover the original electronic Hamiltonian (3), U_0 in Eq. (11) is needed to remove the CS term, an additional SGT U_{v1} is needed to remove the effects of the inserted vortex. In the first quantization, it is easy to show that[23]

$$U_{v1} = e^{i\sum_i \arg(z_i/|z_i|)} = e^{\sum_i \ln(z_i/|z_i|)} = \prod_i \frac{z_i}{|z_i|}. \quad (23)$$

The total SGT for the quasihole at the layer 1 is $U_{\mathrm{qh}} = U_0 U_{v1}$.

Performing the SGT on Eq. (22), we get the quasihole wave function

$$\Psi_{\mathrm{qh}}(z,w) = \left(\prod_i^{N_1} z_i\right)\prod_i\left|\frac{w_i}{z_i}\right|^{1/2}\Psi_0(z,w), \quad (24)$$

where $\Psi_0(z,w)$ is the ground-state wave function (17) and $N_1 = N_2 = N$ in the balanced case. Note that there is no singularity at the origin.

Note that even in the $d\to 0$ limit, Eq. (24) differs from both the quasihole and the meron wave function listed in Eq. (1). This should not cause any problem. It is known that (111) wave function is the exact wave function in the $d\to 0$ limit. But both the quasihole and the meron wave function listed in Eq. (1) make sense only at finite distances. It is known that the lowest energy excitation at $d=0$ is a skymion carrying charge 1, while the quasihole and the meron carry total charge 1/2, so they are not valid excitations anymore at $d=0$. We conclude the quasihole Eq. (24) make sense only at finite d. It is not interesting to take $d\to 0$ limit to this equation.[24]

Compared to the quasihole and the meron wave function listed in Eq. (1), we can see that there are two modifications in the Eq. (24): (1) The ground-state wave function is the correct one Eq. (17) instead of the (111) wave function. (2) The prefactor is different from both the quasihole and the meron. We expect this prefactor takes care of the strong interlayer correlations. All the wave functions in Eq. (1) are built upon the (111) wave function. As suggested in Ref. 13 and explicitly shown in this paper, the (111) wave function is not even qualitatively correct at any finite d. As shown in Ref. 13, although prefactors are not important in SLQH due to the gap in the bulk, they maybe crucial in BLQH due to the gapless mode in the interlayer correlations. The two factors maybe responsible for the quasihole's charge distribution spreading over the whole system and its energy diverges linearly with the area of the system. Although, the meron wave function's energy is only logarithmically divergent, it ignores the strong interlayer correlations, so it is not a good trial wave function either. We propose that the quasihole wave function Eq. (24) not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations.

V. A PAIR OF QUASIHOLE EXCITATIONS WITH CHARGE 1

Now we put two vortices into the BLQH system. One is on the top layer at z_0 and the other is on the bottom layer at w_0. The only change from the quasi-hole calculation in the last section is $J_0^{v\pm}=\delta(\vec{x}-z_0)\pm\delta(\vec{x}-w_0)$.[20] Again, due to the lack of the CS term in the spin sector, the Hamiltonian in the spin sector remains the same as the ground state Eq. (13), so the wave function is not affected at all by the insertion of the two vortices and remains the same as the ground state one in the spin sector Eq. (14). All the effects of the inserted two vortices are encoded in the charge sector. The wave function in the charge sector in the bosonic picture is

$$\Psi_{c1}^b=\exp\left(\frac{1}{2}\int dxdy\left[\frac{1}{2}[\delta(\vec{x}-z_0)+\delta(\vec{x}-w_0)]+\left(\sum_i \delta(\vec{x}-z_i)+\delta(\vec{x}-w_i)-\bar{\rho}\right)\right]\ln|x-y|\left\{\left[\frac{1}{2}[\delta(\vec{y}-z_0)+\delta(\vec{y}-w_0)]+\left(\sum_i \delta(\vec{y}-z_i)+\delta(\vec{y}-w_i)-\bar{\rho}\right)\right]\right\}=\prod_i^{N_1}|z_i-z_0|^{1/2}\prod_i^{N_1}|z_i-w_0|^{1/2}\prod_i^{N_2}|w_i-z_0|^{1/2}\prod_i^{N_2}|w_i-w_0|^{1/2}. \quad (25)$$

It is easy to see that in the bosonic picture, the above wave function in the charge sector is symmetric under $z_i\leftrightarrow w_i$ or $z_0\leftrightarrow w_0$ separately. This is under expectation, because the two layers are completely symmetric in the charge sector.

Just as U_{v1} is derived for the quasihole, we can get U_{v2} for a pair of vortices inserted at z_0 in top layer and w_0 at the bottom layer:

$$U_{v2}=\prod_i\left(\frac{z_i-z_0}{|z_i-z_0|}\right)\left(\frac{w_i-w_0}{|w_i-w_0|}\right). \quad (26)$$

The total SGT for the meron pair is $U_{\text{pair}}=U_0U_{v2}$.

Performing the SGT on Eq. (25), we get the wave function for a pair of quasihole

$$\Psi_{\text{pair}}(z,w;z_0,w_0)=\left(\prod_i (z_i-z_0)(w_i-w_0)\times\prod_i\left|\frac{(z_i-w_0)(w_i-z_0)}{(z_i-z_0)(w_i-z_0)}\right|^{1/2}\Psi_0(z,w)\right), \quad (27)$$

where $\Psi_0(z,w)$ is the ground state wave function (17) and $N_1=N_2=N$ in the balanced case. Note that the pair wave function is not symmetric under $z_i\leftrightarrow w_i$ or $z_0\leftrightarrow w_0$ separately anymore. This is because the SGT U_{v2} Eq. (26) is not. Of course, it is still symmetric under $z_i\leftrightarrow w_i, z_0\leftrightarrow w_0$ simultaneously.

If we insert the two vortices at the same point, namely, putting $z_0=w_0=0$ in the above equation, as expected, we get

$$\Psi_{\text{pair}}(z,w,0)=\left(\prod_i z_iw_i\right)\Psi_0(z,w). \quad (28)$$

This corresponds to insert a single vortex through the two layers. In contrast to the quasihole excitation Eq. (24), Eq. (28) carries charge 1 and remains a valid wave function even at $d=0$. Indeed, in the $d\to 0$ limit, it recovers the meron pair wave function listed in Eq. (1). If one splits the whole vortex, it will evolve into the pair wave function Eq. (27).

The pair meron wave function built on (111) is listed in Ref. 1 [essentially Eq. (110) in Ref. 10)]. It was shown that its energy $E_{\text{pair}}\sim|z_0-w_0|^2$ instead of logarithmically as naively expected, because the charges are extended between z_0 and w_0. Similar to the quasihole wave function Eq. (24), there are two modifications in the Eq. (27). (1) The ground state wave function is the correct one [Eq. (17)]. (2) The prefactor is different. We expect this prefactor takes care of the strong interlayer correlations between the two vortices, the pair wave function not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations.

VI. INSTABILITY IN THE WAVE FUNCTION AS THE DISTANCE INCREASES

When the distance is sufficiently small, the BLQH is in the ESF phase, we expect the ground state, quasihole, and pair wave functions Eqs. (17), (24), (27) only hold in the ESF phase. When the distance becomes sufficiently large, the two layers become two weakly coupled Fermi liquid (FL) layers. All these wave functions completely break down. A new set of wave functions are needed. Although the ESF phase and FL phase at the two extreme distances are well established, the picture of how the ESF phase evolves into the two weakly coupled FL states was not clear, namely, the nature of the intermediate phase at $d_{c1}<d<d_{c2}$ was still under debate. Recently, starting from the well studied excitonic superfluid (ESF) state, as distance increases, one of the authors found[14] that the instability driven by magnetoroton minimum collapsing at a finite wavevector in the pseudospin channel leads to the formation of a excitonic normal solid (PSDW) at some intermediate distances. He constructed a quantum Ginsburg-Landau theory to study the transition from the ESF to the PSDW and analyze in detail the properties of the PSDW. He showed that a square lattice is the favorite lattice.

As shown in Ref. 14, it is the original instability in $V_-(q)=a-bq+cq^2$ which leads to the magneto-roton minimum in Fig. 1(a). By looking at the two conditions $V_-(\vec{q})|_{q=q_0}=0$ and $\frac{dV_-(\vec{q})}{dq}|_{q=q_0}=0$, it is easy to see that $V_-(q)$ indeed has the shape shown in Fig. 1(b). When $b\sim d^2<b_c=2\sqrt{ac}\sim d$, the minimum of $V_-(q)$ at $q=q_0=\sqrt{a/c}\sim d$ has a gap, the system is in the ESF state, this is always the case when the distance d is sufficiently small. However, when $b=b_c$, the minimum collapses and $S(q)$ diverges at $q=q_0$, which signifies the instability of the ESF to an exciton nor-

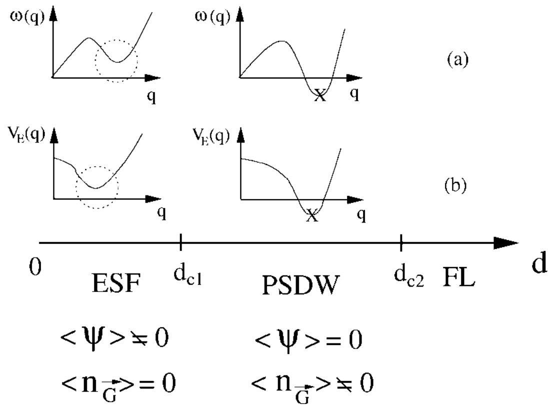

FIG. 1. The zero temperature phase diagram in the balanced case as the distance between the two layers increases. ESF where $\langle\psi\rangle \neq 0$, $\langle n_{\vec{G}}\rangle = 0$ stands for excitonic superfluid PSDW, where $\langle\psi\rangle = 0$, $\langle n_{\vec{G}}\rangle \neq 0$ stands for excitonic normal solid phase, and FL stands for Fermi Liquid. (a) Energy dispersion relation $\omega(q)$ in these phases. (b) $V_{-}(q)$ in these phases. The cross in the PSDW means the negative minimum value of $V_{-}(q)$ is replaced by the PSDW. The two order parameters were defined in Ref. 16. In reality, the instability happens *before* the minimum collapses.

mal solid (PSDW) formation. When $b \sim d^2 > b_c = 2\sqrt{ac} \sim d$, the minimum drops to negative, the system gets to the PSDW state, this is always the case when the distance d is sufficiently large.

We can easily see the instability from the ground-state wave function (17). The only distance dependence in the charge sector is encoded in the $V_{+}(q)$ in the bosonic Hamiltonian in the charge sector Eq. (9), but this dependence is ignored in the (111) wave function in the charge sector. The d dependence in V_{+} is smooth anyway. Essentially all the distance dependence is encoded in the ground-state wave function in the spin sector Eq. (14). As can be seen from Fig. 1, the instability happens at $q = q_0$, where V_{-} becomes negative, but the spin stiffness ρ_E remains noncritical through the ESF to PSDW transition. As $d \to d_{c1}^{-}$, the sum over q in Eq. (14) becomes dominated by the regime $q \sim q_0$. When $d_{c1} < d < d_{c2}$, Eq. (14) breaks down. A new wave function to describe the translational symmetry breaking PSDW state is needed. Some trial wave functions are proposed in Ref. 25. It would be interesting to derive the new wave function of the PSDW state from the CB theory.

VII. CONCLUSIONS

BLQH differs from the SLQH by its symmetry breaking ground state and associated neutral gapless mode in the pseudospin sector. Due to the gapless mode in the bulk, the groundstate wave functions could be considerably different from the well known (111) wave function.[12,13] The low energy excited states could also be sensitive to details such as normalization factors. One important problem is to find good trial wave functions for the ground state and low energy excited states. We investigated this important open problem from the CB theory developed previously in Refs. 14 and 16 to study BLQH systematically. We derived the ground state, quasihole and a pair of quasihole wave functions from CB theory and its dual action by the following procedures: We first performed the singular gauge transformation Eq. (4) to transform a fermionic problem into a bosonic problem, then found that all the wave functions in the bosonic picture are always the product of two parts, one part in the charge sector and the other in the spin sector. All the distance dependence are encoded in the spin part, while all the excitations only happen in the charge sector. After transforming back to the original electron picture by proper inverse SGT's, we get the final wave functions in the electron coordinates. We found that the inverse SGT's are different in the ground state, meron, and a pair of merons. By considering the differences carefully, we derived all these wave functions in the original electronic picture a systematical way.

At any finite d, the ground state wave function in the charge sector is the same as the (111) wave function, while that in the spin sector is highly nontrivial due to the gapless mode. So the total groundstate wave function differs from the well known (111) wave function at any finite d. In the bosonic picture, when inserting vortices in the ground state, the spin part remains the same due to the lack of CS term in this sector, while the charge part changes accordingly. However, due to the insertion of vortices, in order to recover the original electronic problem, the inverse SGT differs from that in the ground state. After transforming back to the original electron problem by the inverse SGT's, we showed that the quasihole and a pair of quasihole wave functions contain nontrivial normalization factors as shown in Refs. 24 and 25. We expect that the quasihole and pair wave function not only has logarithmically divergent energy, well localized charge distribution, but also correct interlayer correlations. It is important to test these trial wave functions by QMC simulations performed in Ref. 13 for the states listed in Eq. (1). We also investigated the instability encoded in the spin sector which leads to the PSDW solid formation proposed in Refs. 14 and 16. Because the CB field theory has been used to describe the trilayer quantum Hall systems very successfully, the analysis in this paper can be easily extended to derive the wave functions in the TLQH.[17]

It is well known that CB approach is not a Lowest Landau Level (LLL) approach,[12,16] it is very difficult to incorporate the LLL projection into the CB approach. This may be partially responsible for the spin part of the ground state wave function (16) not in the LLL level. But as explained below Eq. (17), this should not be too worrisome, because Eq. (16) works only in long distance anyway. As shown in Ref. 13 and listed in Eq. (1), in the long wavelength limit, the meron wave function's normalization factor contains modulus which is not in the LLL either. How to get precise short distance behaviors of these wave functions from the CB theory remains an open problem.

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[21] As shown in detail in Ref. 16, the functional form achieved from the composite boson theory is the same at that achieved from the microscopic LLL+HF approach. But the coefficients should be taken from the LLL+HF approach.
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