

Casimir force and complications in the van Kampen theory for dissipative systems

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In dissipative media it appears as if there are distinct normal modes with complex valued energies. The summation of the zero-point energies of these modes render a complex valued result. Using the contour integration, resulting from the use of the generalized argument principle, gives a real valued and different result. This paper resolves this contradiction and shows that the true normal modes form a continuum with real frequencies. Furthermore, it suggests a way to obtain an approximate result from a summation of the zero-point energy of modes.

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The interaction energy in a system may be expressed in terms of the energy shifts of the electromagnetic normal modes of the system;¹ the longitudinal bulk modes can be used to find, e.g., the polaron energy in a polar semiconductor or the exchange-correlation energy in a metal; the transverse modes to find the Lamb shift;² the surface modes to find the van der Waals interaction between objects; the vacuum modes to find the Casimir³ interaction. In all these cases the energy, or frequency, of a normal mode is found as a solution to an equation of the type

$$f(\omega) = 0, \quad (1)$$

the condition for having a mode. The function $f(\omega)$, which involves the dielectric properties of the system, is often obtained as a determinant of a matrix. The solution to Eq. (1) approaches a pole of the function $f(\omega)$ when the interaction in the system is gradually turned off. When the interaction is turned on the zeros move away from the poles and $\hbar/2$ times this shift in frequency is the contribution to the interaction energy of this particular mode; the interaction energy is the change in total zero-point energy when the interaction in the system is turned on.

It is often straightforward to use this sum-over-modes approach, or mode-summation method, to obtain the interaction energy as the following sum over zeros and poles of the function $f(\omega)$:

$$E = \frac{\hbar}{2} \sum_i (\omega_{0,i} - \omega_{\infty,i}). \quad (2)$$

However, sometimes it is not possible to find an explicit solution to Eq. (1) and sometimes the poles and zeros form a continuum. In those cases the result may be obtained with the so-called generalized argument principle,¹

$$E = \frac{1}{2\pi i} \oint dz \frac{\hbar}{2} z \frac{d}{dz} \ln f(z), \quad (3)$$

where the integration is performed along a contour including the poles and zeros in the right half of the complex frequency plane. The integration should be performed in the positive sense, i.e., in the counterclockwise direction.

The specific problem of the force between two semi-infinite dielectric slabs separated by a dielectric layer has been solved with different approaches; in the Lifshitz⁴ very complicated theory the dielectrics are characterized by randomly fluctuating sources as demanded by the fluctuation-

dissipation theorem; Schwinger *et al.*⁵ derived the force using Schwinger's source theory, where "the vacuum is regarded as truly a state with all physical properties equal to zero;" van Kampen *et al.*⁶ applied Eqs. (2) and (3) to the surface modes of the geometry to find the interaction energy and force.

All these approaches, that appear to be quite different, give one and the same result. Milonni and Shih⁷ made a study, on how these theories are related, based on conventional QED. The result in Eq. (3) is consistent with what one arrives at from many-particle theory where the focus is put on the interacting particles in the system, not the electromagnetic normal modes; there is no explicit reference to zero-point energies. This is demonstrated in detail in Ref. 1 in the case of the exchange-correlation energy of a metal. In Ref. 8 the van der Waals and Casimir forces between two quantum wells were derived both in terms of the zero-point energy of the normal modes and as the result from correlation energy; both approaches produced the same result. Thus, there are different complementary approaches to the interactions in a system. The present work is concerned with the sum-over-modes approach in presence of dissipation in the system.

van Kampen *et al.*⁶ considered nondissipative materials, only, with real-valued dielectric functions. In the case of dissipative materials the sum-over-modes approach runs into yet another problem. The straightforward solution of Eq. (1) produces complex-valued frequencies or energies, signalling that the normal modes are no longer stable—the modes decay. If this were the case they would no longer be true normal modes. The interaction energy in Eq. (2) becomes complex valued. Using instead Eq. (3) produces a real-valued interaction energy. This situation has made many researchers confused and led to believe that there is something fundamentally wrong with the mode-summation method. This work tries to resolve this confusion.

The result of Eq. (3) is the correct result in the case of dissipation; the complex-valued result of Eq. (2) is not correct. Let us choose to study a simple example, a longitudinal bulk mode in a metal represented by a dielectric function of the Drude type,

$$\varepsilon(\omega) = 1 - \omega_p^2 / \omega(\omega + i\eta). \quad (4)$$

We let the parameter η be a positive real-valued constant; η , which is the result of electron scattering against impurities or other defects, is really ω dependent and complex valued; the

approximation we use is good for a metal in the low momentum limit, for frequencies below the plasma frequency, ω_{pl} . To find a longitudinal bulk mode we let $f(\omega)$ be $\varepsilon(\omega)$. Equation (1) then gives us two zeros,

$$\omega = -i\eta/2 \pm \sqrt{\omega_{pl}^2 - \eta^2/4}. \quad (5)$$

We are interested in the zero in the right half-plane,

$$\omega = \sqrt{\omega_{pl}^2 - \eta^2/4} - i\eta/2. \quad (6)$$

Thus, the zero is below the real frequency axis. The problem is that the expression in Eq. (4) for the dielectric function is only valid above the real axis. Below the real axis η has the opposite sign. The zero is very illusive. If one approaches the zero from the upper half-plane and crosses the real axis it makes a jump to the upper half-plane. This clearly shows that we cannot use Eq. (2), directly.

Before we continue let us discuss the general analytical properties of a dielectric function. The physical dielectric function, the one that can be measured in experiments, exists on the real frequency axis, only. It is retarded, which means that it obeys causality. In theoretical treatments one obtains a function that is analytical in the whole complex frequency plane except on the real axis, where all the poles are situated. To obtain the retarded version one either shifts all the poles downwards to an infinitesimal distance below the real axis and perform the calculation on the real axis. Alternatively one lets the poles stay put and perform the calculation just above the axis. For the discussion in this work it is better to use this last method. There are other versions of the function, advanced, time ordered and antitime ordered. The time ordered is often used in many-body calculations since it allows some very useful theorems to be used. With this version one

calculates the function just above the positive real axis and below the negative real axis. All different versions are identical everywhere except at the real frequency axis. From now on, if not stated otherwise, when we discuss the dielectric function we mean the function with its poles on the real axis. The function has the following properties:

$$\begin{aligned} \varepsilon(-\omega) &= \varepsilon(\omega), \\ \varepsilon(\omega^*) &= \varepsilon(\omega)^*. \end{aligned} \quad (7)$$

From these equations follows that the relation between the different forms of the function in the lower and upper half-plane is

$$\varepsilon_l(\omega) = \varepsilon_u(\omega^*)^*. \quad (8)$$

For the full dielectric function with frequency dependent η these two analytical expressions are the same but not when η is treated as a constant. Then we have $\varepsilon_u(\omega) = 1 - \omega_{pl}^2/\omega(\omega + i\eta)$ and $\varepsilon_l(\omega) = 1 - \omega_{pl}^2/\omega(\omega - i\eta)$, respectively. In the full treatment we have

$$i\eta(\omega^*) = [i\eta(\omega)]^* = -i[\eta(\omega)]^* = -i\text{Re}[\eta(\omega)] - \text{Im}[\eta(\omega)], \quad (9)$$

and we may identify

$$\text{Re}[\eta(\omega^*)] = -\text{Re}[\eta(\omega)], \quad \text{Im}[\eta(\omega^*)] = \text{Im}[\eta(\omega)]. \quad (10)$$

We see that the dominating real part changes sign when we cross the real axis while the imaginary part does not.

As an illustration, let us look at the expression for the dielectric function of an impure metal in the so-called generalized Drude approach.⁹ Let n_i , n , $S(\mathbf{q})$, and $\omega_0(\mathbf{q})$ be the density of impurities, electron density, structure factor for the impurities and impurity potential, respectively. Then the dielectric function in the small momentum limit is

$$\varepsilon(\omega) = 1 + \frac{4\pi i\sigma(\omega)}{\omega} = 1 + \frac{4\pi i}{\omega\rho(\omega)} = 1 - \frac{\omega_{pl}^2}{\omega \left[\omega + \frac{1}{24\pi e^2 m_e \omega} \frac{n_i}{n} \frac{1}{\Omega} \sum_{\mathbf{q}} S(\mathbf{q}) q^4 |\omega_0(\mathbf{q})|^2 \left(\frac{1}{\varepsilon(\mathbf{q}, \omega)} - \frac{1}{\varepsilon(\mathbf{q}, 0)} \right) \right]}, \quad (11)$$

where $\sigma(\omega)$, $\rho(\omega)$, and Ω are the dynamical conductivity, the dynamical resistivity and volume of the system, respectively. The second term within the parentheses in the denominator is $i\eta$. For a large system the summation over momentum use to be replaced by an integral over a continuous momentum variable. Let us now keep the discrete summation. The derivation of the dielectric function of the pure metal at finite momentum also contains a discrete summation, now over the electron momentum. The function in RPA (random phase approximation) is

$$\begin{aligned} \varepsilon(\mathbf{q}, \omega) &= 1 + \frac{v_q}{\Omega} \sum_{\mathbf{k}, \sigma} n(\mathbf{k}) [1 - n(\mathbf{k} + \mathbf{q})] \left(\frac{1}{\hbar\omega + (\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\mathbf{k}})} \right. \\ &\quad \left. - \frac{1}{\hbar\omega - (\varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{\mathbf{k}})} \right), \end{aligned} \quad (12)$$

where $n(\mathbf{k})$ is the Fermi-Dirac occupation number. We see that, if we make the calculation just off the real frequency

axis, the imaginary part consists of a sum of δ -functions, infinitesimally spaced when the volume of the system goes to infinity. These form the single particle continuum in the ωq plane. The real part passes through zero between each neighboring pairs of δ functions. When the volume goes to infinity one can replace the summation by an integral. The imaginary part then turns into a smooth continuous function and the real part no longer passes through zero inside the continuum. When one wants to find the zeros and poles of the function $\varepsilon(\omega)$, in Eq. (11), one should keep the discrete summations everywhere. Then one realizes that this function, also, has its poles and zeros on the real axis and that they are in the form of a continuum. So in principle Eq. (2) may be used even in presence of dissipation. It is however impractical.

Thus, the actual poles and zeros of the physical dielectric function of the system are situated on the real frequency axis, not at a finite distance above or below the axis. The poles and zeros form a continuum of points on the axis. This has some resemblance with the case of mirror charges at an in-

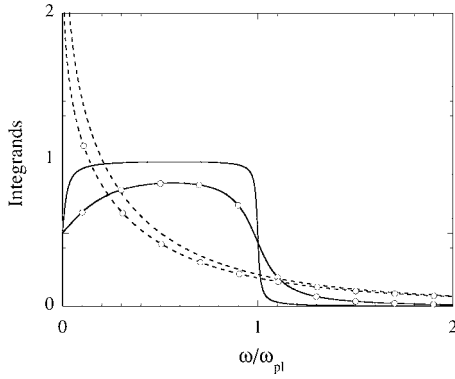


FIG. 1. The solid and dashed curves are the integrand of Eqs. (17) and (18), respectively. The curves with circles are for $x=0.1$ and the others for $x=0.01$.

terface in electromagnetism in the presence of an actual point charge. On the side with actual charge the resulting field can be viewed as the field from the actual charge plus a mirror charge on the other side; the resulting field on the opposite side can be viewed as the field from the actual charge plus the field from another mirror charge; the actual induced charge is in the form of an induced surface-charge density on the interface. Here, in the present problem, the dielectric function on one side of the real axis looks as if there were a pole on the opposite side of the axis; on the other side it looks as if there were a pole on the first side; the actual poles form a continuum on the axis itself.

If we now instead let all summations turn into integrals the parameter η is a smooth complex valued function of frequency. However its real part dominates and is almost constant for frequencies below the plasma frequency. From the expression in Eq. (11) we can easily verify that the real part of η changes sign when one crosses the real axis. For higher frequencies the contribution to the interaction energy quickly drops off, as is seen in Fig. 1, so Eq. (4) is good enough for our purpose here.

Let us now instead use Eq. (3) and choose our contour to encircle the positive real axis. The contour then consists of two parts; an integration from minus to plus infinity performed just below the real axis; an integration from plus to minus infinity performed just above the axis. We may first perform an integration by parts in both contributions and end up with

$$E = -\frac{1}{2\pi i} \frac{\hbar}{2} \oint dz \ln f(z). \quad (13)$$

Then the integration below and above the axis are combined into one integral

$$\begin{aligned} E &= -\frac{1}{i} \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} \ln \frac{\varepsilon(\omega)^*}{\varepsilon(\omega)} = \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \arg[\varepsilon(\omega)] \\ &= \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \tan^{-1}\{\text{Im}[\varepsilon(\omega)]/\text{Re}[\varepsilon(\omega)]\} \\ &= \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \tan^{-1}[\eta\omega_{pl}^2/\omega(\omega^2 + \eta^2 - \omega_{pl}^2)], \end{aligned} \quad (14)$$

where the function \tan^{-1} is taken from the branch where $0 \leq \tan^{-1} \leq \pi$. The dielectric function in this equation should be on retarded form, on the form just above the real frequency axis.

Alternatively, we deform the integration contour into a semicircle in the right half-plane with the center of the circle at the origin and the straight part parallel with and just to the right of the imaginary frequency axis. We let the radius tend to infinity and end up with an integral that can be solved analytically,

$$\begin{aligned} E &= \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \ln \varepsilon'(\omega) = \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \ln \varepsilon(i\omega) \\ &= \frac{\hbar}{2} \int_0^\infty \frac{d\omega}{2\pi} 2 \ln[1 + \omega_{pl}^2/\omega(\omega + \eta)] = \frac{\hbar}{2} \left[\sqrt{\omega_{pl}^2 - (\eta/2)^2} \right. \\ &\quad \left. + \frac{2}{\pi} \left((\eta/2) \ln \frac{\eta}{\omega_{pl}} - \sqrt{\omega_{pl}^2 - (\eta/2)^2} \tan^{-1} \frac{\eta/2}{\sqrt{\omega_{pl}^2 - (\eta/2)^2}} \right) \right]. \end{aligned} \quad (15)$$

We have made use of the fact that η is odd on the imaginary axis which causes the integrand to be an even function. The integration along the curved part of the contour vanishes when the radius goes to infinity. Thus, we end up with two integrals for the same thing; one along the real frequency axis; one along the imaginary axis. Now, we may simplify the analytical result by letting $x = (\eta/2)/\omega_{pl}$,

$$\frac{E}{\hbar\omega_{pl}/2} = \sqrt{1-x^2} + \frac{2}{\pi} \left[x \ln(2x) - \sqrt{1-x^2} \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) \right]. \quad (16)$$

The first term is the result one would get if one were to use Eq. (2) and just neglect the imaginary parts of the zeros and poles of the function $f(\omega)$. This might seem to be a good and simple short cut to an approximate result. However, to do this turns out not to be such a good idea. This is illustrated in Fig. 2, where the solid curve is the exact result and the dashed curve shows the short cut. The circles are the result from another approximation that will be discussed later.

Let us rewrite Eqs. (14) and (15) with the same scaling as we just used. We find

$$\frac{E}{\hbar\omega_{pl}/2} = \int_0^\infty \frac{d\omega}{2\pi} 2 \tan^{-1}[2x/\omega(\omega^2 + 4x^2 - 1)] \quad (17)$$

and

$$\frac{E}{\hbar\omega_{pl}/2} = \int_0^\infty \frac{d\omega}{2\pi} 2 \ln[1 + 1/\omega(\omega + 2x)], \quad (18)$$

respectively. In Fig. 1 the two integrands are shown for the two parameter choices $x=0.1$ and 0.01 .

The question is now whether the mode-summation method can be used at all in the case of dissipative materials. To investigate this we make a Lehmann representation of the dielectric function,

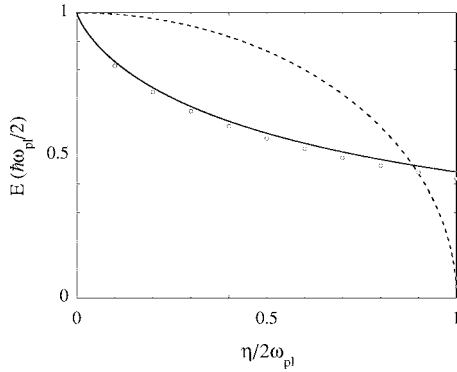


FIG. 2. The contribution to the interaction energy from a bulk mode. The solid curve is the exact result. The dashed curve is from neglecting the imaginary parts of the poles and zeros. The circles are the result from using Eq. (20) and the parameters given in the text.

$$\begin{aligned}
 \varepsilon(\omega) &= 1 + \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\omega' - 2 \operatorname{Im} \varepsilon(\omega')}{\omega - \omega'} \\
 &= 1 - 2\eta\omega_{pl}^2 \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{(\omega - \omega')\omega'(\omega'^2 + \eta^2)} \\
 &= 1 - 4\eta\omega_{pl}^2 \int_0^{\infty} \frac{d\omega'}{2\pi} \frac{1}{(\omega^2 - \omega'^2)(\omega'^2 + \eta^2)} \\
 &= 1 - 8x \int_0^{\infty} \frac{d\omega'}{2\pi} \frac{1}{[(\omega/\omega_{pl})^2 - \omega'^2](\omega'^2 + 4x^2)}. \quad (19)
 \end{aligned}$$

The dielectric function in the first appearing integral should be on retarded form or calculated just above the axis. The integral over frequency can be viewed as a limit of discrete frequency summations where the step size goes towards zero. In each summation the dielectric function has its poles and zeros on the axis. These points come closer and closer when we take the limit. The integrations over frequency and momentum should always be viewed as limits of discrete summations since the system is always finite in size and possible energy and momentum transfers are discrete.

The derivation with an example is illustrated. In approximating the dielectric function with

$$\varepsilon(\omega) \approx 1 - \frac{8x}{2\pi} \frac{\omega_{\max}}{i_{\max}} \sum_{i=1}^{i_{\max}} \frac{1}{[(\omega/\omega_{pl})^2 - \omega_i^2](\omega_i^2 + 4x^2)}, \quad (20)$$

$$\omega_i = \frac{\omega_{\max}}{i_{\max}}(i - 1/2),$$

finding the zeros and poles and using Eq. (2) we find an approximate result which asymptotically becomes the exact

result when we let ω'_{\max} and i_{\max} go towards infinity. In Fig. 2 the circles indicate the result when we have included 50 equidistant poles in the region below $5\omega_{pl}$, i.e., $\omega'_{\max}=5$ and $i_{\max}=50$. Thus, we must find the frequency of 50 poles and 50 zeros. This is feasible and the result is in much better agreement with the exact result than what one obtains when just neglecting the imaginary part of the zeros and poles. Of course, it is always better to use Eq. (3) instead and then preferably use a contour that partly is made up by the imaginary axis. If we do not have the analytic expression of the dielectric functions but experimental values, these are valid on the real axis, or rather just above the real axis. Then we may obtain the functions on the imaginary axis through a type of Kramers-Kronig dispersion relation,¹

$$\varepsilon(i\omega) = 1 + \frac{2}{\pi} \int_0^{\infty} d\omega' \frac{\omega' \varepsilon_2(\omega')}{(\omega')^2 + \omega^2}, \quad (21)$$

or

$$\varepsilon(i\omega) = 1 + \frac{2}{\pi} \int_0^{\infty} d\omega' \frac{\omega' [\varepsilon_1(\omega') - 1]}{(\omega')^2 + \omega^2}. \quad (22)$$

One chooses one or the other of the above equations depending on which of the real or imaginary parts on the real axis one knows or has greatest confidence in. The obtained results are used in Eq. (15).

In summary, this paper has demonstrated that the energy of the electromagnetic normal modes in dissipative media are real valued and lead to real-valued interaction energies and forces. The modes appear to be distinct with complex valued energies, but they are not. The modes form a continuum on the real frequency axis and the direct use of the mode summation method is no longer feasible. The functions appearing in the condition for modes contain integrals over momentum or frequency. Since the system is finite in size these integrals should be considered as discrete summations; the possible momentum and energy transfers in a finite system is discrete. In doing so all zeros and poles end up on the real axis. When one takes the limit when the volume goes to infinity the zeros and poles form a continuum of points on the real axis. One may use an approximation in which this continuum of modes is replaced by a finite number of distinct modes and obtain reasonably good results. This gives a great improvement compared to the result from just neglecting the imaginary parts of the complex valued energies of the apparent modes.

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