Critical state in type-II superconductors with order-disorder transition

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An explicit analytical solution of the critical-state problem is obtained for long thin superconducting strips in a perpendicular magnetic field H_a when a sharp order-disorder transition occurs in the vortex lattice. We model this transition from the quasiordered Bragg glass into the disordered amorphous vortex phase by a jump of the critical current density occurring at a value $B_{\rm dis}$ of the local magnetic induction. Explicit expressions are presented for the magnetic-field profiles at the surface of the strip, for the distribution of the sheet current, and for the magnetization loop.

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I. INTRODUCTION

Most experiments on high-T_c superconductors deal with thin flat samples in a perpendicular magnetic field H_a—for example c-axis-oriented single crystals or films. In recent years the problem of the critical state of pinned vortices in such samples has attracted considerable interest; see, e.g., Ref. 1 and references cited therein. Explicit analytic solutions were obtained for a circular disk[2] and a thin infinitely long strip.[3–5] A very accurate approximate two-dimensional solution, which reproduces the solutions for the disk and for the strip in the limiting cases, was also obtained for an elliptic-shaped platelet.[6] In all these solutions the simple Bean model was implied; i.e., the critical value of the sheet current, J_c, was assumed to be constant (the sheet current is the current density integrated over the thickness d of a flat superconductor). An analytical solution for the case when J_c depends on the magnetic induction B was derived only for a strip with a particular dependence $J_c(B)$.[7] This special dependence can be used to describe, for example, anisotropic flux-line pinning in superconductors with columnar defects.[8] In this paper we obtain an *explicit* analytical solution of the critical-state problem for one more case of the B dependence of J_c. Namely, we shall deal with a piecewise-constant function $J_c(B)$. Such a dependence of J_c on B enables one to describe the critical state in superconductors with a sharp order-disorder transition.[9–11]

At the order-disorder transition, which is induced by quenched disorder in the vortex system at a certain value $B_{\rm dis}$ of the local magnetic induction, a transformation of a quasi-ordered Bragg glass[12] into the disordered amorphous vortex phase occurs via the proliferation of dislocations in the flux-line lattice; see, e.g., Ref. 13 and papers cited therein. Under this transformation the flux-line pinning increases, and this leads to an abrupt increase of the critical value of the sheet current, J_c, at an induction $B_{\rm dis}$. At present the peak effect in low-T_c superconductors[14–19] and the second magnetization peak (or fishtail effect) in high-T_c superconductors[20–25] are frequently associated with this proliferation of dislocations.[14–18,26–34] In local magnetization experiments which use arrays of small Hall sensors placed on the surface of a flat superconductor[9] or in magneto-optic investigations of magnetic-field profiles on the surface,[35,36] this transition is identified with a sharp change in the gradient of the local magnetic field. However, as was noticed in Ref. 37, in a thin sample the assumption of a sharp jump of J_c at the front where this change occurs is not compatible with the macroscopic electrodynamics of such superconductors. A sharp boundary between two different currents in a thin sample would produce a similar singular peak (or dip) in the magnetic field as it occurs at its edges, rather than the observed jump in the gradient. Numerical analysis[37] of this problem showed that near the front where the change of the gradient occurs there is a *region* in which the sheet current continuously changes from one critical value to another and the front marks only one of the two boundaries of this region. Inside the region a *mixture of the two vortex phases* inevitably exists. Our analytical solution supports this view and gives a full description of the critical state in a thin superconducting strip.

Of course, in analyzing experiments it is necessary to take into account that in high-T_c superconductors flux creep plays an important role. This creep affects the magnetization and the magnetic field profiles measured on the surface of flat superconductors. Moreover, in experiments with $NbSe_2$ (Ref. 16) and with $Bi_2Sr_2CaCu_2O_8$ (Ref. 35 and 36), metastable states of the amorphous vortex phase, so-called transient vortex states, were observed, and the existence of these states in the superconductors can explain a number of findings.[16,38,39] Beside this, the edge barrier[40,41] is essential for understanding the vortex dynamics in superconductors.[42,43] In our paper we disregard all these effects: the flux creep, the edge barrier, and the metastable states of the disordered vortex phase. But our simple critical-state model and its analytical solution obtained below take properly into account the geometry of thin flat superconductors and thus provide a basis for analyzing various phenomena in such superconductors with an order-disorder transition.

II. CRITICAL-STATE PROBLEM AND ITS SOLUTION

Consider an infinitely long and thin strip of thickness d and width $2w$ with $d \ll w$. Let this strip fill the space $|x| \le w$, $|y| < \infty$, and $|z| \le d/2$, and be in a constant and uniform external magnetic field H_a directed along the z axis—i.e., perpendicularly to the strip plane, Fig. 1. We shall model the

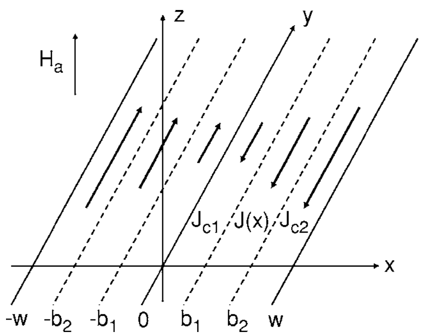

FIG. 1. Schematic distribution of the sheet current J in a thin strip with the order-disorder transition described by model (1). The lines $x=\pm b_2$ and $x=\pm b_1$ mark the boundaries of the disordered and ordered vortex phases, respectively. At $b_1<|x|<b_2$ a mixture of the two vortex phases occurs with $J_{c1}\leqslant J(x)\leqslant J_{c2}$. Shown is the case of increasing applied field H_a when $b_2>b_1$.

order-disorder transition with the following critical-state model:

$$J_c(B_z)=J_{c1} \quad \text{for } B_z<B_{\text{dis}},$$

$$J_c(B_z)=J_{c2} \quad \text{for } B_z>B_{\text{dis}}, \tag{1}$$

where J_{c1} and J_{c2} are constants and $J_{c2}>J_{c1}$. We shall assume that B_{dis} exceeds $\mu_0 H_{c1}$, where H_{c1} is the lower critical field, and so we use $B=\mu_0 H$ everywhere below. Since we are interested in the critical state at the fields near the order-disorder transition, we shall disregard the effect of this transition on the initial penetration of the applied magnetic field into the sample and on the reversal of the magnetization of the strip when the hysteresis loop is considered. For this to be justified, it is sufficient that H_{dis} exceed the penetration field $H_p\approx(J_{c1}/\pi)\ln(2ew/d)\sim J_{c1}$ (Ref. 44) and the maximum amplitude $H_{\max}$ of the applied magnetic field be larger than $H_{\text{dis}}+(J_{c2}/\pi)\ln(2ew/d)$.

According to the Biot-Savart law, the magnetic field in the strip is expressed in terms of the sheet current $J(x)$ as follows:

$$H_z(x)=H_a+\frac{1}{2\pi}\int_{-w}^{w}\frac{J(t)dt}{t-x}. \tag{2}$$

The symmetry of this problem gives

$$J(x)=-J(-x), \tag{3}$$

and one has $J<0$ at $x>0$ for increasing applied field H_a and $J>0$ at $x>0$ for decreasing field. Taking into account this symmetry, it is convenient to rewrite Eq. (2) in the form

$$H_z(x)=H_a\mp\frac{1}{2\pi}\int_0^{w^2}\frac{|J(u)|du}{u-x^2}, \tag{4}$$

where the minus and plus signs correspond to the cases of the increasing and decreasing field H_a, respectively.

For increasing H_a the critical-state problem reads

$$|J(x)|=J_{c1} \quad \text{for } 0\leqslant x^2\leqslant b_1^2, \tag{5}$$

$$H_z(x)=H_{\text{dis}} \quad \text{for } b_1^2\leqslant x^2\leqslant b_2^2, \tag{6}$$

$$|J(x)|=J_{c2} \quad \text{for } b_2^2\leqslant x^2\leqslant w^2. \tag{7}$$

Here $|x|=b_1$ defines the boundary of the region where $|J(x)|=J_{c1}$ and $H_z(x)<H_{\text{dis}}$—i.e., where the ordered vortex phase exists—while $|x|=b_2$ describes the boundary of the amorphous-vortex-phase region where $|J(x)|=J_{c2}$ and $H(x)>H_{\text{dis}}$. At the boundaries the field H reaches H_{dis}, and at $b_1\leqslant|x|\leqslant b_2$ one has $H(x)=H_{\text{dis}}$, while the sheet current lies in the interval $J_{c1}\leqslant|J|\leqslant J_{c2}$. Note that these critical-state equations include, as a special case, the situation when $b_1=b_2$—i.e., when immediate contact of the above-mentioned regions occurs. But we shall see that the parameters b_1 and b_2 cannot be chosen arbitrarily but *are found* self-consistently from Eqs. (5)–(7), and hence immediate contact of the regions is forbidden. In a similar manner the critical-state problem can be formulated for the case of decreasing H_a (in this case $b_2<b_1$).

Taking into account formula (4), Eq. (6) for the case of increasing H_a can be rewritten in the form

$$\frac{1}{2\pi}\int_{b_1^2}^{b_2^2}\frac{|J(u)|du}{u-x^2}=F(x^2), \tag{8}$$

where

$$F(x^2)=H_a-H_{\text{dis}}+\frac{J_{c1}}{2\pi}\ln\left|\frac{x^2}{x^2-b_1^2}\right|+\frac{J_{c2}}{2\pi}\ln\left|\frac{b_2^2-x^2}{w^2-x^2}\right|. \tag{9}$$

The last two terms in function (9) are the magnetic fields $H_z(x)$ generated by the sheet currents J_{c1} and J_{c2} flowing in the regions $0<x^2<b_1^2$ and $b_2^2<x^2<w^2$, respectively. Equation (8) is a linear singular integral equation with Cauchy-type kernel. The theory of such equations[45] yields for the sheet current $J(x)$ in the interval $b_1^2\leqslant x^2\leqslant b_2^2$

$$J(x)=-\frac{2}{\pi}Y(x^2)\int_{b_1^2}^{b_2^2}\frac{F(u)du}{Y(u)(u-x^2)}, \tag{10}$$

where $Y(u)\equiv(b_2^2-u)^{1/2}(u-b_1^2)^{1/2}$ and the integral is in the sense of the Cauchy principal value. This solution of the critical-state equations exists only under the following condition, which, in fact, is a relationship between b_1, b_2, and H_a:

$$\int_{b_1^2}^{b_2^2}\frac{F(u)du}{Y(u)}=0. \tag{11}$$

Considering x^2 and u as complex variables, making appropriate cuts in this complex plane, and using the residue theory, we calculate explicitly integrals (10) and (11); see the Appendix. Knowing the sheet current in the whole interval $0<x^2<w^2$, we then find the magnetic-field profile $H_z(x)$ in the strip. But for this solution of the critical-state problem to be self-consistent, three simultaneous conditions should be fulfilled: the obtained sheet current $J(x)$ lie in the interval

$J_{c1} \leqslant J(x) \leqslant J_{c2}$, $H_z(x)$ at $x^2 < b_1^2$ be less than $H_{\rm dis}$, and $H_z(x)$ at $b_2^2 < x^2 < w^2$ exceed $H_{\rm dis}$. These requirements lead to the second relationship between b_1 and b_2,

$$J_{c2}(w^2 - b_2^2)^{1/2} b_1 - J_{c1}(w^2 - b_1^2)^{1/2} b_2 = 0. \tag{12}$$

We now present the final results.

A. Solution for the case of increasing H_a

The sheet current at $b_1^2 \leqslant x^2 \leqslant b_2^2$ has the form

$$|J(x)| = \frac{2J_{c1}}{\pi} \arctan \frac{b_1\sqrt{b_2^2 - x^2}}{b_2\sqrt{x^2 - b_1^2}} + \frac{2J_{c2}}{\pi} \arctan \frac{\sqrt{w^2 - b_2^2}\sqrt{x^2 - b_1^2}}{\sqrt{w^2 - b_1^2}\sqrt{b_2^2 - x^2}} \tag{13}$$

[while $|J(x)| = J_{c1}$ at $x^2 \leqslant b_1^2$, and $|J(x)| = J_{c2}$ at $b_2^2 \leqslant x^2 \leqslant w^2$]. The magnetic field outside the interval $b_1^2 \leqslant x^2 \leqslant b_2^2$ is given by

$$H_z(x) = H_a + \frac{J_{c1}}{\pi} \ln \frac{|x|(b_1 + b_2)}{b_2\sqrt{|b_1^2 - x^2|} + b_1\sqrt{|b_2^2 - x^2|}} + \frac{J_{c2}}{\pi} \ln \frac{\sqrt{w^2 - b_2^2}\sqrt{|b_1^2 - x^2|} + \sqrt{w^2 - b_1^2}\sqrt{|b_2^2 - x^2|}}{\sqrt{|w^2 - x^2|}(\sqrt{w^2 - b_1^2} + \sqrt{w^2 - b_2^2})} \tag{14}$$

[while $H_z(x) = H_{\rm dis}$ at $b_1^2 \leqslant x^2 \leqslant b_2^2$]. The region boundaries b_1 and b_2 are found from the equations

$$\frac{b_1^2}{w^2} = \frac{J_{c1}^2 b_2^2}{J_{c1}^2 b_2^2 + J_{c2}^2(w^2 - b_2^2)}, \tag{15}$$

$$H_{\rm dis} = H_a + \frac{J_{c1}}{\pi} \ln \frac{\sqrt{b_1 + b_2}}{\sqrt{b_2 - b_1}} + \frac{J_{c2}}{\pi} \ln \frac{\sqrt{b_2^2 - b_1^2}}{\sqrt{w^2 - b_1^2} + \sqrt{w^2 - b_2^2}}. \tag{16}$$

Formulas (13)–(16) completely describe the critical state for the case of an increasing H_a. Using these formulas, we find the two equivalent expressions for $M_z = \int_{-w}^{w} xJ(x)dx$, the magnetic moment of the strip per unit length,

$$M_z(H_a) = -J_{c1}b_1b_2 - J_{c2}\sqrt{w^2 - b_1^2}\sqrt{w^2 - b_2^2} = -J_{c1}w^2\frac{b_2}{b_1}, \tag{17}$$

where $b_1(H_a)$ and $b_2(H_a)$ are found from Eqs. (15) and (16). Note that in the limiting case $J_{c1} = H_{\rm dis} = 0$ formulas (13)–(17) reproduce the known result[3–5] for the penetration of the magnetic field into a strip with constant sheet current, $J_c = J_{c2}$.

B. Solution for the case of decreasing H_a

For the case of decreasing field H_a, all the calculations are similar to those of the increasing field. The sheet current at $b_2^2 \leqslant x^2 \leqslant b_1^2$ has the form (note that now $b_2 < b_1$)

$$|J(x)| = \frac{2J_{c1}}{\pi} \arctan \frac{\sqrt{w^2 - b_1^2}\sqrt{x^2 - b_2^2}}{\sqrt{w^2 - b_2^2}\sqrt{b_1^2 - x^2}} + \frac{2J_{c2}}{\pi} \arctan \frac{b_2\sqrt{b_1^2 - x^2}}{b_1\sqrt{x^2 - b_2^2}} \tag{18}$$

[and $|J(x)| = J_{c2}$ at $x^2 \leqslant b_2^2$, and $|J(x)| = J_{c1}$ at $b_1^2 \leqslant x^2 \leqslant w^2$]. The magnetic field outside the interval $b_2^2 \leqslant x^2 \leqslant b_1^2$ is given by

$$H_z(x) = H_a - \frac{J_{c2}}{\pi} \ln \frac{|x|(b_1 + b_2)}{b_2\sqrt{|b_1^2 - x^2|} + b_1\sqrt{|b_2^2 - x^2|}} - \frac{J_{c1}}{\pi} \ln \frac{\sqrt{w^2 - b_1^2}\sqrt{|b_2^2 - x^2|} + \sqrt{w^2 - b_2^2}\sqrt{|b_1^2 - x^2|}}{\sqrt{|w^2 - x^2|}(\sqrt{w^2 - b_1^2} + \sqrt{w^2 - b_2^2})} \tag{19}$$

[and $H_z(x) = H_{\rm dis}$ at $b_2^2 \leqslant x^2 \leqslant b_1^2$]. The parameters b_1 and b_2 are found from the equations

$$\frac{b_1^2}{w^2} = \frac{J_{c2}^2 b_2^2}{J_{c2}^2 b_2^2 + J_{c1}^2(w^2 - b_2^2)}, \tag{20}$$

$$H_{\rm dis} = H_a - \frac{J_{c2}}{\pi} \ln \frac{\sqrt{b_1 + b_2}}{\sqrt{b_1 - b_2}} - \frac{J_{c1}}{\pi} \ln \frac{\sqrt{b_1^2 - b_2^2}}{\sqrt{w^2 - b_1^2} + \sqrt{w^2 - b_2^2}}. \tag{21}$$

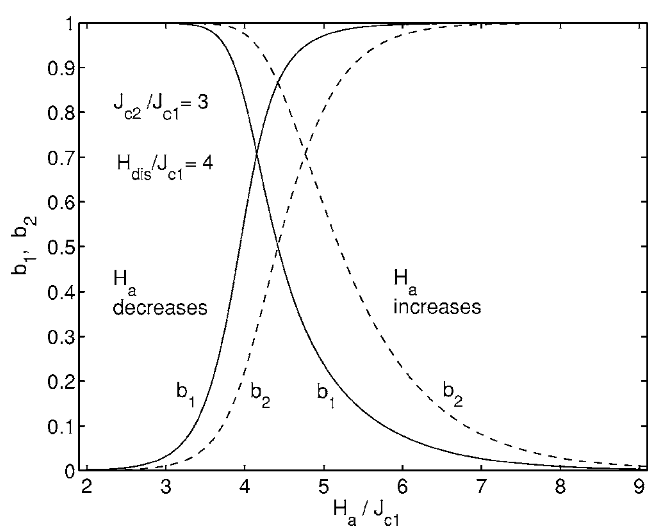

FIG. 2. The boundary positions b_1 (solid lines) and b_2 (dashed lines) as functions of the applied magnetic field H_a for increasing and decreasing H_a, Eqs. (15), (16), (20), and (21), respectively. Here $J_{c2}/J_{c1} = 3$, $H_{\rm dis}/J_{c1} = 4$, and b_1 and b_2 are in units of the half-width w.

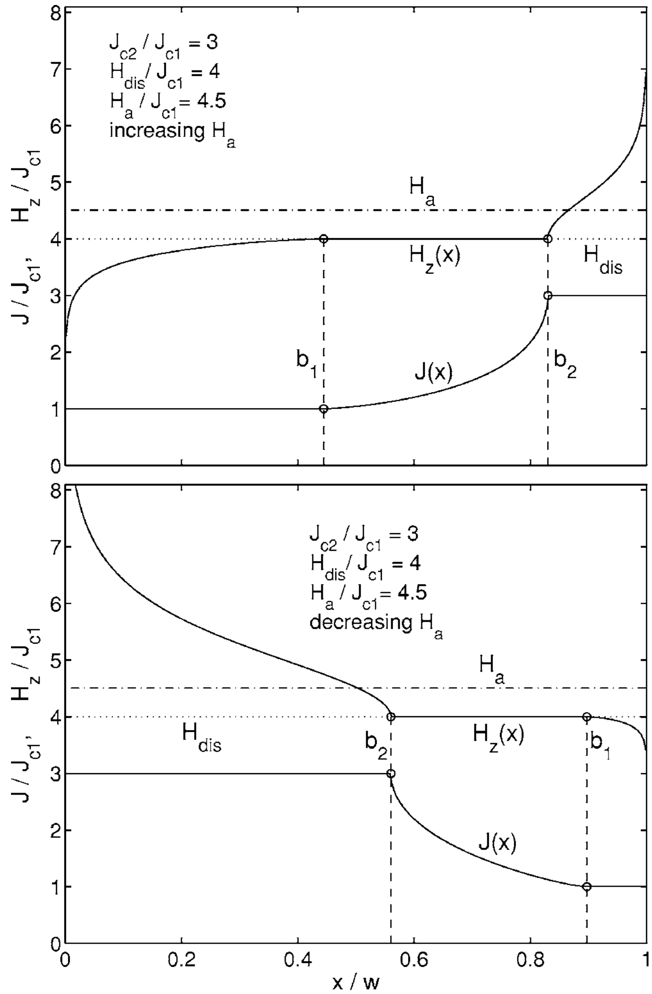

FIG. 3. Magnetic-field and sheet-current profiles $H_z(x)$ and $|J(x)|$ in the strip at $J_{c2}/J_{c1}=3$, $H_{\rm dis}/J_{c1}=4$, and $H_a/J_{c1}=4.5$ for the cases of increasing H_a, Eqs. (13)–(16) (top), and decreasing H_a, Eqs. (18)–(21) (bottom).

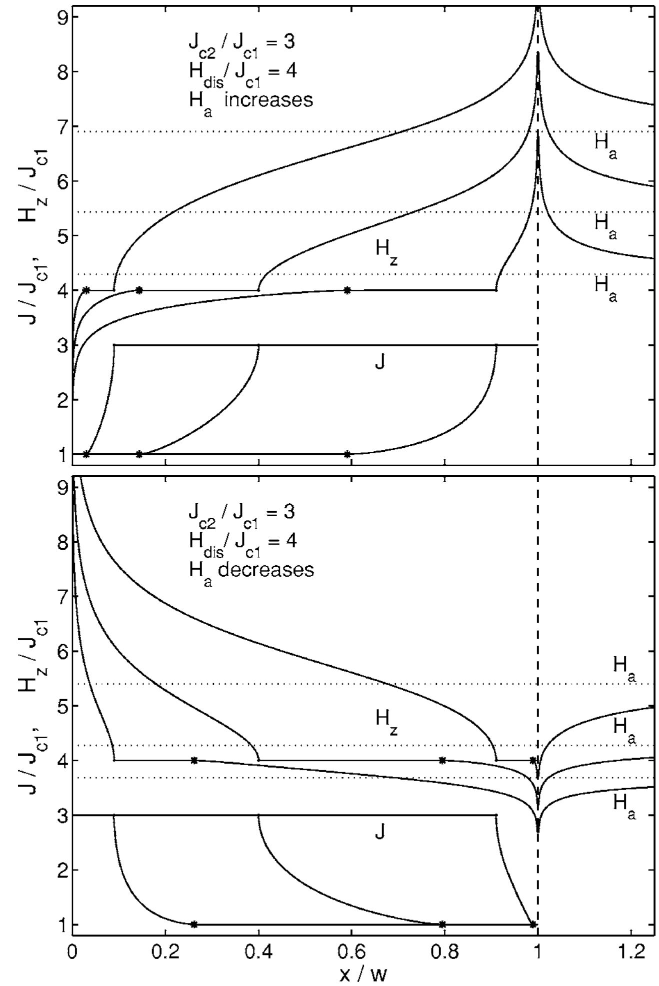

FIG. 4. Evolution of magnetic-field and sheet-current profiles $H_z(x)$ and $|J(x)|$ in the strip at $J_{c2}/J_{c1}=3$, $H_{\rm dis}/J_{c1}=4$ when the applied field H_a increases, $H_a/J_{c1}=4.30$, 5.44, 6.91 (top), and decreases, $H_a/J_{c1}=5.40$, 4.28, 3.68 (bottom).

As to the magnetic moment, we have

$$M_z(H_a) = J_{c2}b_1b_2 + J_{c1}\sqrt{w^2-b_1^2}\sqrt{w^2-b_2^2} = J_{c2}w^2\frac{b_2}{b_1}, \tag{22}$$

where $b_1(H_a)$ and $b_2(H_a)$ are found from Eqs. (20) and (21). Interestingly, it follows from a comparison of Eqs. (20) and (21) with Eqs. (15) and (16) that these b_i $(i=1,2)$ for decreasing H_a are equal to $(w^2-b_i^2)^{1/2}$ calculated for the case of increasing H_a. Hence, $M_z(H_a)$ in Eqs. (22) and (17) differ only in their signs.

III. ANALYSIS OF THE SOLUTION

The obtained solutions, Eqs. (13)–(22), essentially depend only on the two parameters: J_{c2}/J_{c1} and $(H_a-H_{\rm dis})/J_{c1}$. A change of $H_{\rm dis}$ leads to a trivial shift of the profiles $H_z(x)$ and of the functions $M_z(H_a)$ along the H_z and H_a axes, respectively. In Figs. 2–4 we show an example of the solution for the cases of increasing and decreasing field H_a. It is seen that the width $|b_2-b_1|$ of the region where the magnetic-field profiles are flat, $H_z(x)=H_{\rm dis}$, can be a noticeable part of w. This width increases with increasing J_{c2}/J_{c1}, and at $J_{c2}\gg J_{c1}$ and $H_a\sim H_{\rm dis}$ the field profiles are practically composed of a nearly flat part and a steep part with $J=J_{c2}$. We also emphasize that in agreement with the numerical results of Ref. 37, a sharp change of the gradient of $H_z(x)$ and of $J(x)$ occurs only at $|x|=b_2$—i.e., at the boundary of the amorphous vortex phase—while at the boundary of the ordered phase, $|x|=b_1$, the profiles are smooth.

This smoothness of the profiles at the point $|x|=b_1$ results from condition (15) [or (20)]. Consider, for example, the case of increasing field H_a. Formulas (13), (14), and (16) with b_1 different from that of Eq. (15) also describe a solution of Eq. (8). However, this formal solution is not self-consistent since it does not satisfy the physical requirements $H_z(x)<H_{\rm dis}$ at $|x|<b_1$ and $J_{c1}\le|J(x)|\le J_{c2}$ at $b_1\le|x|\le b_2$; see Fig. 5. It is straightforwardly verified that the derivative $d|J(x)|/dx$ at the point x tending to b_1 from above is proportional to the left-hand side of Eq. (12), while the derivative $dH_z(x)/dx$ at the point x tending to this b_1 from below is proportional to the same expression but with opposite sign.

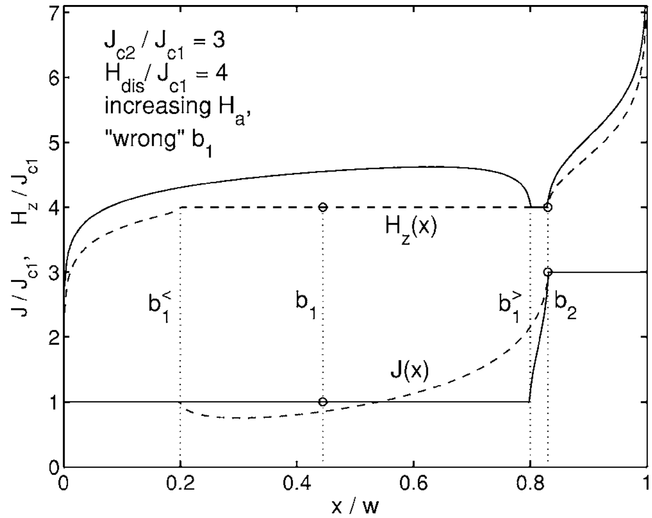

FIG. 5. The magnetic-field and sheet-current profiles $H_z(x)$ and $|J(x)|$ in the strip at $J_{c2}/J_{c1}=3$, $H_{\rm dis}/J_{c1}=4$, for increasing field H_a, Eqs. (13), (14), and (16), when b_1 is larger than the b_1 given by Eq. (15) ($b_1=b_1^>$, solid lines) and smaller than this value ($b_1=b_1^<$, dashed lines).

To fulfill the above physical requirements, both these derivatives should not be negative. Thus, condition (12), equivalent to Eq. (15), has to be valid. This condition means that these derivatives are equal to zero at $x=b_1$, and hence they are continuous at this point. Figure 5 also explains why the regions of the ordered and disordered phases have no immediate contact in the strip and what is the nature of the vortex state in the interval $b_1<x<b_2$. If during the formation of the critical state in the strip the two regions practically touch (see the profiles with $b_1^> \approx b_2$ in Fig. 5), then the magnetic field $H_z(x)$ at $x<b_1^>$ exceeds $H_{\rm dis}$ and nucleation of the disordered vortex phase begins there. This nucleation increases J and decreases $b_1^>$ until this $b_1^>$ reaches the value b_1 determined by Eq. (15). A further decrease of b_1 stops since, if $b_1^< < b_1$, the sheet current $J(x)$ becomes less than the critical value J_{c1} (see Fig. 5) and any vortex motion is impossible.

It follows from the above consideration that in the interval $b_1<x<b_2$ a mixture of the ordered and disordered vortex phases exists and the concentration of these phases, $n_{\rm ord}$ and $n_{\rm dis}$, at a point x is determined by $J(x)$, $n_{\rm dis}(x)=[J(x)-J_{c1}]/[J_{c2}-J_{c1}]$ and $n_{\rm ord}(x)=1-n_{\rm dis}(x)$. Probably, this mixture can be visualized as drops or islands of one of the phases in the other phase, with the concentration or the size of the drops (islands) changing with x. Alternatively, if the sample is not too thin and the nucleation of the disordered phase begins at the upper and lower surfaces of the sample, it is also possible that there are two fronts $z=\pm z_{\rm front}(x)$ separating the phases across the thickness of the strip and these fronts are determined by $z_{\rm front}=(d/2)n_{\rm ord}(x)$. Of course, one cannot decide between these scenarios within our static approach that disregards kinetics of the nucleation process.

In Fig. 6 we show an example of the magnetization loop for the strip. As was explained in Sec. II B, this loop is symmetric relative to the axis $M_z=0$. The characteristic width of the region of H_a where $|M_z(H_a)|$ changes from its minimum value $J_{c1}w^2$ to its maximum value $J_{c2}w^2$ is finite even at

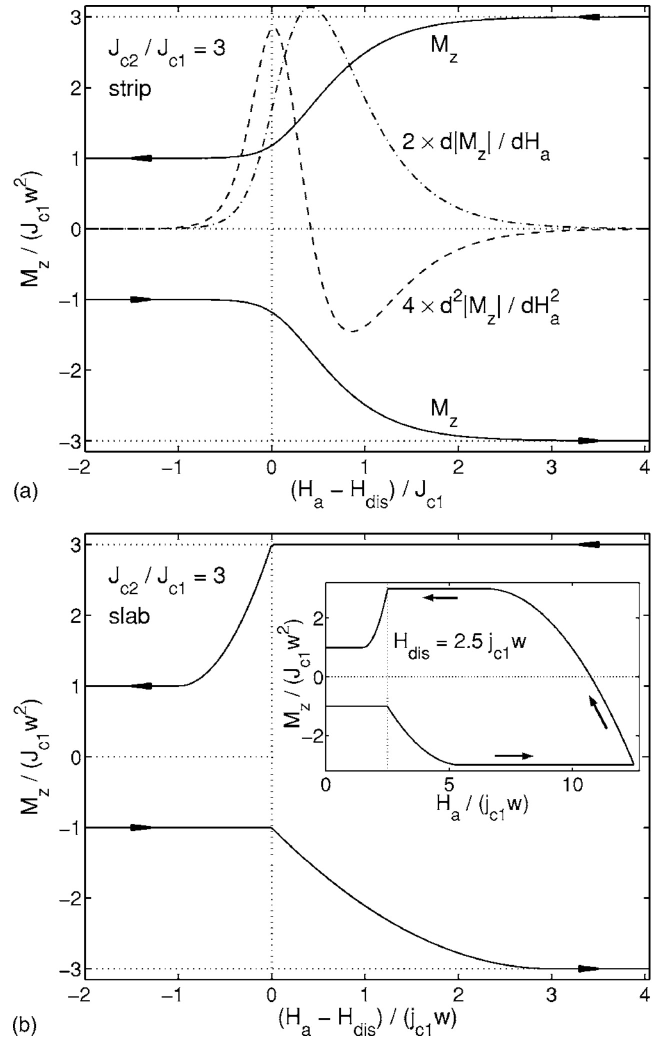

FIG. 6. Magnetic hysteresis loop (solid lines) in the strip, Eqs. (17) and (22) (top), and in the slab, Eqs. (23) and (24) (bottom), for $J_{c2}/J_{c1}=3$. Note the different scales of the H_a axes which are due to the different fields of full flux penetration into the strip and the slab. The arrows mark the increase and decrease of H_a. The dashed lines show the first and second derivatives of the dimensionless $|M_z|$ over H_a/J_{c1}. In the strip the maximum of the second derivative practically coincides with $H_{\rm dis}$. The magnetization loop in the strip is symmetric relative to the axis $M_z=0$, while in the slab this loop is asymmetric. The inset shows an example for the half loop of the slab when $H_{\rm dis}=2.5j_{c1}w$ and $H_{\max}=5H_{\rm dis}$.

$J_{c2}\to J_{c1}$ and is of the order of J_{c1}. With increasing J_{c2}/J_{c1}, this width increases, too. Interestingly, the maximum of $d|M_z(H_a)|/dH_a$ is always slightly above the point $H_a=H_{\rm dis}$, while the position of the maximum of the second derivative $d^2|M_z(H_a)|/dH_a^2$ practically coincides with this point. In other words, the point $H_a=H_{\rm dis}$ is, in fact, the point of the *maximum curvature* for the curve $|M_z(H_a)|$. This finding may be useful in analyzing the fishtail or the peak effect in thin flat type-II superconductors.

It is instructive to compare the obtained magnetization loop with the appropriate result for a slab of thickness $2w$ in the magnetic field H_a applied parallel to its surface, Fig. 6. The magnetic moment of this slab (calculated per length d

along H_a and per unit length in the perpendicular direction) in the case of increasing H_a is

$$M_z(H_a) = -J_{c1}w^2 \quad \text{for } h \leqslant 0,$$

$$M_z(H_a) = -J_{c1}w^2 - w^2(J_{c2}-J_{c1})\left[2h\frac{J_{c1}}{J_{c2}} - \frac{J_{c1}^2}{J_{c2}^2}h^2\right]$$

$$\text{for } 0 \leqslant h \leqslant \frac{J_{c2}}{J_{c1}},$$

$$M_z(H_a) = -J_{c2}w^2 \quad \text{for } \frac{J_{c2}}{J_{c1}} \leqslant h, \tag{23}$$

while in the case of decreasing H_a one has

$$M_z(H_a) = J_{c1}w^2 \quad \text{for } h \leqslant -1,$$

$$M_z(H_a) = J_{c2}w^2 + w^2(J_{c2}-J_{c1})[2h+h^2] \quad \text{for } -1 \leqslant h \leqslant 0,$$

$$M_z(H_a) = J_{c2}w^2 \quad \text{for } 0 \leqslant h. \tag{24}$$

Here $h \equiv (H_a - H_{\rm dis})/(j_{c1}w)$, $J_{c1}=j_{c1}d$, and $J_{c2}=j_{c2}d$ where j_{c1} and j_{c2} are the critical current densities in the ordered and disordered vortex phases, respectively. In contrast to the case of the strip, the magnetization loop for the slab is *asymmetric* relative to the axis, $M_z=0$. This asymmetry is mainly caused by the fact that with increasing H_a a sharp boundary between the ordered and disordered vortex phases appears in the slab at $H_a > H_{\rm dis}$, while with decreasing H_a the boundary exists at $H_a < H_{\rm dis}$. On the other hand, in the strip the boundaries b_1 and b_2 exist both above and below the field $H_{\rm dis}$ when H_a increases or decreases, Fig. 2. These b_1 and b_2 for decreasing and increasing H_a are related to each other, and this relationship leads to a *symmetric* loop; see Sec. II B.

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APPENDIX: CALCULATION OF THE INTEGRALS

As an example, let us calculate the integral

$$I(x^2) = \int_{b_1^2}^{b_2^2} \frac{du}{Y(u)(u-x^2)} \ln\left|\frac{b_2^2-u}{w^2-u}\right|, \tag{A1}$$

where $Y(u) \equiv (b_2^2-u)^{1/2}(u-b_1^2)^{1/2}$ and the integration is in the sense of the Cauchy principal value. For this integral one has $I(x^2)=[I(x^2+i\epsilon)+I(x^2-i\epsilon)]/2$ where $\epsilon \to 0$ and $I(Z)$ is the integral depending on the complex parameter Z,

$$I(Z) = \int_{b_1^2}^{b_2^2} \frac{du}{Y(u)(u-Z)} \ln \frac{b_2^2-u}{w^2-u}. \tag{A2}$$

To calculate $I(Z)$, we cut the complex plane from b_1^2 to w^2 and consider the closed contour Γ consisting of the upper and lower sides of this cut. The integrand $f(u)$ in Eq. (A2) is an analytical function outside the cut and the point $u=Z$ at which $f(u)$ has a pole. Integrating $f(u)$ over Γ, we obtain the relation

$$2I(Z) + \int_{b_2^2}^{w^2} \frac{2\pi du}{\sqrt{(u-b_2^2)(u-b_1^2)}(u-Z)} = 2\pi i R(Z), \tag{A3}$$

where $R(Z)$ is the residue of the function $f(u)$ at the point Z and we have taken into account that the difference of $\ln[(b_2^2-u)/(w^2-u)]$ calculated on the upper and lower sides of the part of the cut from b_2^2 to w^2 is $-2\pi i$. In Eq. (A3) we may put $Z=x^2$ in the integral from b_2^2 to w^2. This integral is calculated by standard methods. Allowing for $R(x^2+i\epsilon)+R(x^2-i\epsilon)=0$, we eventually obtain

$$I(x^2) = -\frac{2\pi}{Y(x^2)} \arctan \frac{\sqrt{w^2-b_2^2}\sqrt{x^2-b_1^2}}{\sqrt{w^2-b_1^2}\sqrt{b_2^2-x^2}}. \tag{A4}$$

The other integrals in Eqs. (10) and (11) are calculated in a similar manner.

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